Supplementary material for "On the Tail Index Inference for Heavy-Tailed GARCH-type Innovations"

1 Proofs of Lemmas

In this section, we provide the proofs of the lemmas in Section 5.

Lemma 1.

$$\lim_{n\to\infty}\sup_{0\le y_1<y\le y_*}\left|\frac{n}{k}\frac{P\left(y^{-1/\alpha}\le U_1/b(n/k)<y_1^{-1/\alpha}\right)}{y-y_1}-1\right|=0.$$

Proof. Let $\tilde{\ell}(y)=\ell(y^{-1/\alpha}b(n/k))/\ell(b(n/k))$ (cf. (4)). Then, we have $\tilde{\ell}(y)=1+o(1)$ uniformly in $0<y<y_*$. Thus,

$$\begin{aligned}
&\left|\frac{n}{k}\frac{\bar{F}\left(y^{-1/\alpha}b(n/k)\right)-\bar{F}\left(y_1^{-1/\alpha}b(n/k)\right)}{y-y_1}-1\right|=\left|\frac{n}{k}\bar{F}(b(n/k))\frac{y\tilde{\ell}(y)-y_1\tilde{\ell}(y_1)}{y-y_1}-1\right|\\
&=\left|\frac{(y-y_1)\tilde{\ell}(y)+y_1\left(\tilde{\ell}(y)-\tilde{\ell}(y_1)\right)}{y-y_1}-1\right|\\
&=\left|\tilde{\ell}(y)\left\{1+\frac{1}{y/y_1-1}\left(1-\frac{\ell\left((y_1/y)^{-1/\alpha}y^{-1/\alpha}b(n/k)\right)}{\ell\left(y^{-1/\alpha}b(n/k)\right)}\right)\right\}-1\right|.
\end{aligned}$$

On the other hand, we have from (5) that

$$\begin{aligned}
&\frac{1}{z-1}\left(1-\frac{\ell(zx)}{\ell(x)}\right)=\frac{1}{z-1}\left(1-\frac{\ell(zx)}{\ell(x)}\right)=\frac{x^\gamma}{z-1}\left[D\left\{1-z^\gamma\right\}+\Delta\left(x\right)-z^\gamma\Delta(zx)\right]\\
&=Dx^\gamma\frac{1-z^\gamma}{z-1}\{1+o(1)\},\quad \text{uniformly in } z>1, \text{ as } x\to\infty.
\end{aligned}$$

Since $\lim_{n\to\infty}\sup_{0<y\le y_*}\left\{y^{-1/\alpha}b(n/k)\right\}^\gamma=0$, we have

$$\lim_{n\to\infty}\sup_{0<y_1<y\le y_*}\left|\frac{1}{y/y_1-1}\left(1-\frac{\ell\left((y_1/y)^{-1/\alpha}y^{-1/\alpha}b(n/k)\right)}{\ell\left(y^{-1/\alpha}b(n/k)\right)}\right)\right|=0.$$

Thus,

$$\lim_{n\to\infty}\sup_{0<y_1<y\le y_*}\left|\frac{n}{k}\frac{P\left(y^{-1/\alpha}\le U_1/b(n/k)<y_1^{-1/\alpha}\right)}{y-y_1}-1\right|=0.$$

Further,

$$\lim_{n\to\infty}\sup_{0<y\le y_*}\left|\frac{n}{k}\frac{\bar{F}\left(y^{-1/\alpha}b(n/k)\right)}{y}-1\right|=0.$$

This completes the proof. $\square$

Lemma 2. *Let $y_* > 1$. Then, as $n \to \infty$,*

$$\mathrm{E}\{Z_{ni}(y_2) - Z_{ni}(y_1)\} \sim \frac{k}{\alpha n}(y_2 - y_1), \tag{S.1}$$

$$\mathrm{E}\{Z_{ni}(y_2) - Z_{ni}(y_1)\}^2 \sim \frac{2k}{\alpha^2 n}\left(y_2 - y_1 - y_1 \log \frac{y_2}{y_1}\right), \tag{S.2}$$

$$\mathrm{E}\{Z_{ni}(y_2) - Z_{ni}(y_1)\}^3 \sim \frac{3k}{\alpha^3 n}\left(2y_2 - y_1 - y_1\left(\log \frac{y_2}{y_1} + 1\right)^2\right), \tag{S.3}$$

$$\mathrm{E}\{Z_{ni}(y_4) - Z_{ni}(y_3)\}\{Z_{ni}(y_2) - Z_{ni}(y_1)\} \sim \frac{k}{\alpha^2 n}\left(\log \frac{y_4}{y_3}\right)(y_2 - y_1), \tag{S.4}$$

$$\mathrm{E}\{Z_{ni}(y_4) - Z_{ni}(y_3)\}^2\{Z_{ni}(y_2) - Z_{ni}(y_1)\}^2 \sim \frac{2k}{\alpha^2 n}\left(\log \frac{y_4}{y_3}\right)^2\left(y_2 - y_1 - y_1 \log \frac{y_2}{y_1}\right) \tag{S.5}$$

uniformly in $0 \le y_1 < y_2 \le y_3 < y_4 < y_*$.

Proof. Let p be an positive integer. Notice that

$$\mathrm{E}\{Z_{ni}(y_2) - Z_{ni}(y_1)\}^p = \left(\frac{1}{\alpha}\log \frac{y_2}{y_1}\right)^p \bar{F}\left(y_1^{-1/\alpha} b(n/k)\right) + \mathrm{E}Z_{ni}^p(y_2) I\left(U_i \le y_1^{-1/\alpha} b(n/k)\right)$$
$$= \bar{F}(y_1^{-1/\alpha} b(n/k)) \int_0^{\left(\frac{1}{\alpha}\log \frac{y_2}{y_1}\right)^p} \exp\left(-\alpha u^{1/p}\right) \frac{y_2}{y_1} \frac{\ell(\exp\left(u^{1/p}\right) y_2^{-1/\alpha} b(n/k))}{\ell(y_1^{-1/\alpha} b(n/k))} du,$$

since

$$\begin{aligned}
&\mathrm{E}Z_{ni}^p(y_2) I\left(U_i \le y_1^{-1/\alpha} b(n/k)\right) \\
&= \bar{F}(y_1^{-1/\alpha} b(n/k)) \int_0^{\left(\frac{1}{\alpha}\log \frac{y_2}{y_1}\right)^p} \exp\left(-\alpha u^{1/p}\right) \frac{y_2}{y_1} \frac{\ell(\exp\left(u^{1/p}\right) y_2^{-1/\alpha} b(n/k))}{\ell(y_1^{-1/\alpha} b(n/k))} du \\
&\quad - \bar{F}(y_1^{-1/\alpha} b(n/k)) \left(\frac{1}{\alpha}\log \frac{y_2}{y_1}\right)^p.
\end{aligned}$$

Further, note that

$$\frac{\ell(\exp\left(u^{1/p}\right) y_2^{-1/\alpha} b(n/k))}{\ell(y_1^{-1/\alpha} b(n/k))} = 1 + o(1) \quad \text{as } n \to \infty$$

uniformly in $0 \le u < \infty$ and $0 \le y_1 < y_2 \le y_*$,

$$\bar{F}(y_1^{-1/\alpha} b(n/k)) \int_0^{\frac{1}{\alpha}\log \frac{y_2}{y_1}} e^{-\alpha u} \frac{y_2}{y_1} du = \frac{1}{\alpha} \bar{F}(y_1^{-1/\alpha} b(n/k)) \left(\frac{y_2}{y_1} - 1\right) \sim \frac{k}{\alpha n}(y_2 - y_1),$$

$$\begin{aligned}
\bar{F}(y_1^{-1/\alpha} b(n/k)) \int_0^{\left(\frac{1}{\alpha}\log \frac{y_2}{y_1}\right)^2} e^{-\alpha u^{1/2}} \frac{y_2}{y_1} du &= \frac{2}{\alpha^2} \bar{F}(y_1^{-1/\alpha} b(n/k)) \left(\frac{y_2}{y_1} - 1 - \log \frac{y_2}{y_1}\right) \\
&\sim \frac{2k}{\alpha^2 n}\left(y_2 - y_1 - y_1 \log \frac{y_2}{y_1}\right),
\end{aligned}$$

and

$$\bar{F}(y_1^{-1/\alpha}b(n/k))\int_0^{\left(\frac{1}{\alpha}\log\frac{y_2}{y_1}\right)^3} e^{-\alpha u^{1/3}}\frac{y_2}{y_1}du = \frac{3}{\alpha^3}\bar{F}(y_1^{-1/\alpha}b(n/k))\left\{2\frac{y_2}{y_1}-1-\left(\log\frac{y_2}{y_1}+1\right)^2\right\}$$
$$\sim \frac{3k}{\alpha^2 n}\left\{2y_2-y_1-y_1\left(\log\frac{y_2}{y_1}+1\right)^2\right\}$$

uniformly in $0 \le y_1 < y_2 \le 1$. Thus, (S.1)-(S.3) hold. Since (S.4) and (S.5) are easily seen owing to the fact that

$$\mathrm{E}\{Z_{ni}(y_4)-Z_{ni}(y_3)\}^p\{Z_{ni}(y_2)-Z_{ni}(y_1)\}^p = \left(\log\frac{y_4}{y_3}\right)^p \mathrm{E}\{Z_{ni}(y_2)-Z_{ni}(y_1)\}^p,$$

the lemma is asserted. □

Lemma 3. *Let $\rho_* > 0$. Then,*

$$\log b\left(\frac{n}{\rho k}\right) = \log b(n/k) - \frac{1}{\alpha}\log\rho + \frac{MD}{\alpha\sqrt{k}}(\rho^{-\gamma/\alpha}-1) + o\left(\frac{1}{\sqrt{k}}\right)$$

uniformly in $0 < \rho \le \rho_*$.

Proof. The lemma is a direct result of (8) and (6) with $M \in [0,\infty)$. □

Lemma 4. *Let $\zeta \in \mathbb{R}$ and $0 < \rho_\circ < \rho_* < \infty$. Then,*

(S.6) $$W_n(\rho,s,t) \ge \zeta \quad \textit{if and only if} \quad M_n(\rho,\zeta,s,t) \ge \zeta\rho(t-s)+o(1)$$

uniformly in $\rho_\circ < \rho < \rho_*$ and $0 \le s < t \le 1$ with $t-s > 1/\sqrt{k}$. Further, for any $K > 0$ and $\epsilon > 0$,

(S.7) $$\lim_{n\to\infty} P\left(\sup_{\rho_\circ\le\rho\le\rho_*}\sup_{-K\le\zeta\le K}\sup_{t\in[0,1]}|L_n(\rho,\zeta,0,t)-L_n^*(\rho,t)| > \epsilon\right) = 0,$$

(S.8) $$\lim_{n\to\infty} P\left(\sup_{\rho_\circ\le\rho\le\rho_*}\sup_{-K\le\zeta\le K}\sup_{t\in[0,1]}\left|M_n(\rho,\zeta,0,t)-\alpha^{-1}M_n^*(\rho,t)\right| > \epsilon\right) = 0,$$

and

(S.9) $$\lim_{\delta\to 0}\limsup_{n\to\infty} P\left(\sup_{\delta}|L_n(\rho_1,0,0,s)-L_n(\rho_2,0,0,t)| > \epsilon\right) = 0,$$

where the supremum is taken over $|(\rho_1,s)-(\rho_2,t)| \le \delta$ with $\rho_\circ < \rho_1 \le \rho_2 < \rho_*$. Hence, $\{L_n(\rho,0,0,t) : (\rho,t) \in [\rho_\circ,\rho_*]\times[0,1]\}$ is tight.

Proof. Note that (S.7)-(S.9) is directly obtained from Lemma 3, since $\{L_n^*\}$ and $\{M_n^*\}$ are asymptotically uniformly equicontinuous. Observe that $W_n(\rho,s,t) \ge \zeta$ if and only if

$$\begin{aligned}
&\frac{1}{\sqrt{k}}\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\left\{I\left(U_i \ge e^{\zeta/\sqrt{k}}b\left(\frac{n}{\rho k}\right)\right) - P\left(U_i \ge e^{\zeta/\sqrt{k}}b\left(\frac{n}{\rho k}\right)\right)\right\}\\
\ge\ & \sqrt{k}\left\{\rho(t-s) - \frac{n(t-s)}{k}P\left(U_1 \ge e^{\zeta/\sqrt{k}}b\left(\frac{n}{\rho k}\right)\right) + o\left(\frac{1}{k}\right)\right\}\\
=\ & \sqrt{k}(t-s)\left\{\rho - \rho\left(1-\frac{\alpha\zeta}{\sqrt{k}}+o\left(\frac{1}{\sqrt{k}}\right)\right) + o\left(\frac{1}{\sqrt{k}}\right)\right\} = \alpha\zeta\rho(t-s)+o(1),
\end{aligned}$$

uniformly in $\rho_\circ \le \rho \le \rho_*$ and $0 \le s < t \le 1$ with $t - s > 1/\sqrt{k}$, since

$$\begin{aligned}
&\frac{n}{k} P\left(U_1 \ge e^{\zeta/\sqrt{k}} b\left(\frac{n}{\rho k}\right)\right) = \frac{n}{k} P\left(U_1 \ge b\left(\frac{n}{\rho k}\right)\right) \frac{P\left(U_1 \ge e^{\zeta/\sqrt{k}} b\left(\frac{n}{\rho k}\right)\right)}{P\left(U_1 \ge b\left(\frac{n}{\rho k}\right)\right)} \\
&= \rho e^{-\alpha\zeta/\sqrt{k}} \left[1 + D\left\{b\left(\frac{n}{\rho k}\right)\right\}^{\gamma} \left(e^{\gamma\zeta/\sqrt{k}} - 1\right) + o\left(\frac{1}{\sqrt{k}}\right)\right] = \rho\left(1 - \frac{\alpha\zeta}{\sqrt{k}} + o\left(\frac{1}{\sqrt{k}}\right)\right)
\end{aligned}$$

uniformly in $\rho_\circ \le \rho \le \rho_*$. Hence, (S.6) holds, which completes the proof. □

Lemma 5. *Let $0 < \rho_\circ < \rho_* < \infty$ and $t_0 \in (0,1)$. Then, $\{\rho t W_n(\rho, 0, t) : (\rho, t) \in [\rho_\circ, \rho_*] \times [t_0, 1]\}$ and $\{\rho(1-t) W_n(\rho, t, 1) : (\rho, t) \in [\rho_\circ, \rho_*] \times [0, 1 - t_0]\}$ are tight.*

Proof. We only provide the proof of the first since the latter can be verified similarly. From (31), (S.6) and (S.8), we have

$$\sup_{t_0 < t \le 1} \sup_{\rho_\circ < \rho < \rho_*} |\rho t W_n(\rho, 0, t)| = O_P(1). \tag{S.10}$$

Let $\epsilon > 0$ and $K > 0$. Take

$$-K = \zeta_0 < \zeta_1 < \cdots < \zeta_l = K$$

with $\epsilon/4 < \zeta_i - \zeta_{i-1} < \epsilon/2$ for $i = 1, \ldots, l$, and put

$$\begin{aligned}
A_n(K) &= \left\{ \sup_{t_0 < t \le 1} \sup_{\rho_\circ < \rho < \rho_*} |\rho t W_n(\rho, 0, t)| < K \right\}, \\
B_n(\delta) &= \left\{ \sup_{\delta} |\rho_1 t_1 W_n(\rho_1, 0, t_1) - \rho_2 t_2 W_n(\rho_2, 0, t_2)| > \epsilon \right\}, \quad \delta > 0,
\end{aligned}$$

where $\sup_\delta$ is taken over $|(\rho_1, t_1) - (\rho_2, t_2)| < \delta$ with $t_0 < t_1 \wedge t_2$ and $\rho_\circ < \rho_1 \le \rho_2 < \rho_*$.

Note that $A_n(K) \cap B_n(\delta)$ implies

$$\begin{aligned}
&\bigcup_{i=1}^{l} \bigcup_{|(\rho_1,t_1)-(\rho_2,t_2)|<\delta} \{\rho_1 t_1 W_n(\rho_1, 0, t_1) < \zeta_{i-1},\ \rho_2 t_2 W_n(\rho_2, 0, t_2) \ge \zeta_i\} \\
&\subset \bigcup_{i=1}^{l} \bigcup_{|(\rho_1,t_1)-(\rho_2,t_2)|<\delta} \left\{ M_n\left(\rho_1, \frac{\zeta_1}{\rho_1 t_1}, 0, t_1\right) < \zeta_{i-1} + o(1),\ M_n\left(\rho_2, \frac{\zeta_2}{\rho_2 t_2}, 0, t_2\right) \ge \zeta_i + o(1) \right\} \\
&= \bigcup_{i=1}^{l} \bigcup_{|(\rho_1,t_1)-(\rho_2,t_2)|<\delta} \{M_n(\rho_1, 0, 0, t_1) < \zeta_{i-1} + o_P(1),\ M_n(\rho_2, 0, 0, t_2) \ge \zeta_i + o_P(1)\} \\
&\subset \bigcup_{|(\rho_1,t_1)-(\rho_2,t_2)|<\delta} \left\{ |M_n(\rho_1, 0, 0, t_1) - M_n(\rho_2, 0, 0, t_2)| \ge \frac{\epsilon}{4} + o_P(1) \right\}
\end{aligned}$$

uniformly in $\rho_1, \rho_2 \in [\rho_\circ, \rho_*], t_1, t_2 \in [t_0, 1]$. Thus, we have from (5.31) and (S.8) that

$$\begin{aligned}
\lim_{\delta \to 0} \limsup_{n \to \infty} P(B_n(\delta)) &\le \lim_{\delta \to 0} \limsup_{n \to \infty} P(B_n(\delta) \cap A_n(K)) + \limsup_{n \to \infty} \{1 - P(A_n(K))\} \\
&= \limsup_{n \to \infty} \{1 - P(A_n(K))\}.
\end{aligned}$$

Hence, owing to (S.10), the lemma is validated by letting $K \to \infty$. □

Lemma 6. *Let $0 < \rho_\circ < \rho_* < \infty$ and $K > 0$. Then,*

$$\text{(S.11)} \qquad \mathrm{E}\left(\log U_i - \log b\left(\frac{n}{\rho k}\right) - \frac{\zeta}{\sqrt{k}}\right)_+ = \frac{\rho k}{n}\left\{\frac{1}{\alpha} - \frac{\zeta}{\sqrt{k}} + \frac{\gamma DM}{\sqrt{k}\alpha(\alpha-\gamma)}\rho^{-\gamma/\alpha} + o\left(\frac{1}{\sqrt{k}}\right)\right\}$$

uniformly in $\zeta \in [-K, K]$ and $\rho_\circ \le \rho \le \rho_*$.

Proof. The lemma is a direct result of (7) $\square$

Lemma 7. *Let $0 < \rho_\circ < \rho_* < \infty$. Then,*

$$L_n(\rho,0,0,t) - \rho t W_n(\rho,0,t) \xrightarrow{d} \frac{1}{\alpha}\mathrm{B}(\rho,t) \quad \text{in } D([\rho_\circ,\rho_*]\times[t_0,1]),$$

$$\begin{pmatrix} L_n(\rho,0,0,t) - \rho t W_n(\rho,0,t) \\ L_n(\rho,0,t,1) - \rho(1-t)W_n(\rho,t,1) \end{pmatrix} \xrightarrow{d} \frac{1}{\alpha}\begin{pmatrix} \mathrm{B}(\rho,t) \\ \mathrm{B}(\rho,1)-\mathrm{B}(\rho,t)\end{pmatrix}, \quad \text{in } D^2([\rho_\circ,\rho_*]\times[t_0,1]).$$

Proof. We only provide the proof of the first. Since $\{L_n(\rho,0,0,t) - \rho t W_n(\rho,0,t) : (\rho,t) \in [\rho_\circ,\rho_*]\times[t_0,1]\}$ is tight (cf. Lemma 4-5), it suffices to show that every finite dimensional distribution of $L_n(\rho,0,0,t) - \rho t W_n(\rho,0,t)$ converges weakly to the corresponding finite dimensional distribution of $\alpha^{-1}\mathrm{B}(\rho,t)$. Let $\rho_\circ \le \rho_1 < \cdots, \rho_m \le \rho_*$ ($m \in \mathbb{N}$). Then, from (S.6)-(S.8), the distributional limit of

$$(L_n(\rho_1,0,0,t), \rho_1 t W_n(\rho_1,0,t), \cdots, L_n(\rho_m,0,0,t), \rho_m t W_n(\rho_m,0,t))$$

is equal to that of

$$A_n(t) := \left(L_n^*(\rho_1,t), \alpha^{-1}M_n^*(\rho_1,t), \cdots, L_n^*(\rho_m,t), \alpha^{-1}M_n^*(\rho_m,t)\right).$$

Subsequently, since the distributional limit of $A_n(t)$ has stationary and independent increments, the lemma is validated by (42). $\square$

Lemma 8. *There exists $s > 0$ such that $\mathrm{E}|\varepsilon_0|^s < \infty$.*

Proof. See the proof of Theorem 6 of Pan et al. (2008). $\square$

Lemma 9. *Let $\eta > 0$. Then, there exist $r \in (0,1)$ and $K > 0$, such that for large n,*

$$\text{(S.12)} \qquad \sup_{\boldsymbol{\theta}\in N_n(\eta)} B^j(1,b) \le K r^j, \quad \text{for } j \in \mathbb{N} \text{ and } b = 1,2,\ldots,q,$$

and there exists a sequence of positive $\mathcal{F}_i$-measurable r.v.s $\{V_i\}$ such that $\mathrm{E}V_i^\nu < \infty$ for some $\nu > 0$, and

$$\text{(S.13)} \qquad \sup_{\boldsymbol{\theta}\in N_n(\eta)} h_i(\boldsymbol{\theta}) \le V_i.$$

Proof. The proof is rather standard and is omitted for brevity. $\square$

Lemma 10. *There exists $r \in (0,1)$ and a $\mathcal{F}_0$-measurable r.v. $V \geq 0$, such that $\mathrm{E}V^{\nu} < \infty$ for some $\nu > 0$ and for large n,*

$$\sup_{\boldsymbol{\theta} \in N_n(\eta)} \left| \hat{h}_i(\boldsymbol{\theta}) - h_i(\boldsymbol{\theta}) \right| \leq r^i V.$$

Proof. The proof is rather standard and is omitted for brevity. $\square$

Lemma 11. *Let $\epsilon > 0$. For each $j \in \mathbb{N}$,*

$$\{B((1+\epsilon)\boldsymbol{\theta}^\circ)\}^j(1,1) \leq (1+\epsilon)^j B_\circ^j(1,1), \tag{S.14}$$

and there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and $j \in \mathbb{N}$,

$$B_\circ^j(1,1) \leq (1+\epsilon)^j \inf_{\boldsymbol{\theta} \in N_n(\eta)} B^j(1,1). \tag{S.15}$$

Proof. First, we prove (S.14). Let A and C be any square matrices of the same dimension with nonnegative entries. Define $A \leq C$ if every entry of A is less than or equal to the corresponding entry of C. Observe that $A \leq C$ implies that $A^j \leq C^j$ for every $j \in \mathbb{N}$. Obviously, $B((1+\epsilon)\boldsymbol{\theta}^\circ) \leq (1+\epsilon)B_\circ$. Thus, (S.14) is verified. Note that there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $B_\circ \leq (1+\epsilon)B(\boldsymbol{\theta})$, whenever $\boldsymbol{\theta} \in N_n(\eta)$. Thus, (S.15) is verified. $\square$

Lemma 12. *Let $\eta > 0$. Then, for every $w > 0$,*

$$\limsup_{n\to\infty} \mathrm{E} \left| \sup_{\boldsymbol{\theta} \in N_n^-(\eta)} \frac{|h_1 - h_1(\boldsymbol{\theta})|}{|\boldsymbol{\theta}^\circ - \boldsymbol{\theta}| h_1(\boldsymbol{\theta})} \right|^w < \infty.$$

Proof. We have

$$\begin{aligned} \frac{|h_1 - h_1(\boldsymbol{\theta})|}{h_1(\boldsymbol{\theta})} \leq\ & \frac{\left|\omega^\circ \sum_{j=0}^\infty B_\circ^j(1,1) - \omega \sum_{j=0}^\infty B^j(1,1)\right|}{\omega \sum_{j=0}^\infty B^j(1,1)} \\ & + \frac{\sum_{l=1}^p \sum_{j=0}^\infty |B_\circ^j(1,1) - B^j(1,1)| f_l(\varepsilon_{1-j-l})}{\omega \sum_{j=0}^\infty B^j(1,1) + \sum_{l=1}^p \sum_{j=0}^\infty B^j(1,1) f_l(\varepsilon_{1-j-l})} \\ & + \frac{\sum_{l=1}^p \sum_{j=0}^\infty B_\circ^j(1,1) |f_l^\circ(\varepsilon_{1-j-l}) - f_l(\varepsilon_{1-j-l})|}{\omega \sum_{j=0}^\infty B^j(1,1) + \sum_{l=1}^p \sum_{j=0}^\infty B^j(1,1) f_l(\varepsilon_{1-j-l})}. \end{aligned} \tag{S.16}$$

First, we deal with the second term in the righthand side of the above inequality. Let $r > 0$ fulfill (S.12). Take $\epsilon > 0$ and $s \in (0,1)$ such that $\mathrm{E}|\varepsilon_0|^{2cws} < \infty$ (cf. Lemma 8) and

$$\left(1 + 2\epsilon + \epsilon^2\right) r^s < 1,$$

where c is a constant such that $\delta^\circ < c$. Observe that for positive integers $a_1, \cdots, a_q$,

$$\left| \prod_{i=1}^q \{\beta_i^\circ\}^{a_i} - \prod_{i=1}^q \beta_i^{a_i} \right| \leq \frac{|\boldsymbol{\theta}^\circ - \boldsymbol{\theta}|(a_1 + \cdots + a_q)}{\min_{1\leq i\leq q} \beta_i^\circ \vee \beta_i} \prod_{i=1}^q (\beta_i^\circ \vee \beta_i)^{a_i},$$

and $B^j(1,1)$ and $B_\circ^j(1,1)$ are sums of finitely many products of $\beta_1,\cdots,\beta_q$ and $\beta_1^\circ,\cdots,\beta_q^\circ$ of which the degrees are at most j, respectively. By virtue of this and (12), we can take $n_0\in\mathbb{N}$ and $K_0>0$ such that for $n\geq n_0$,

$$\sup_{\boldsymbol{\theta}\in N_n(\eta)}\left|B_\circ^j(1,1)-B^j(1,1)\right|\leq|\boldsymbol{\theta}^\circ-\boldsymbol{\theta}|K_0 j\{B((1+\epsilon)\boldsymbol{\theta}^\circ)\}^j(1,1).$$

Further, by (S.12) and Lemma 11, there exist $n_1\in\mathbb{N}$ and $K_1>0$, such that whenever $n\geq n_1$ and $\boldsymbol{\theta}\in N_n(\eta)$,

$$\begin{aligned}\frac{[B((1+\epsilon)\boldsymbol{\theta}^\circ)]^j(1,1)}{\{B^j(1,1)\}^{1-s}} &\leq \left(1+2\epsilon+\epsilon^2\right)^j\left\{B^j(1,1)\right\}^s\\ &\leq K_1\left[\left(1+2\epsilon+\epsilon^2\right)r^s\right]^j.\end{aligned}$$

Put $n_2=n_0\vee n_1$. Then, owing to (12) and the inequality $x/(1+x)\leq x^s$ for every $x\geq 0$, there exists $K_2>0$ such that whenever $n\geq n_2$ and $\boldsymbol{\theta}\in N_n(\eta)$,

$$\begin{aligned}&\frac{\sum_{l=1}^p\sum_{j=0}^\infty|B_\circ^j(1,1)-B^j(1,1)|f_l(\varepsilon_{1-j-l})}{\omega\sum_{j=0}^\infty B^j(1,1)+\sum_{l=1}^p\sum_{j=0}^\infty B^j(1,1)f_l(\varepsilon_{1-j-l})}\leq K_2\sum_{l=1}^p\sum_{j=0}^\infty\frac{|B_\circ^j(1,1)-B^j(1,1)|f_l(\varepsilon_{i-j-l})}{1+B^j(1,1)f_l(\varepsilon_{1-j-l})}\\ &\leq|\boldsymbol{\theta}^\circ-\boldsymbol{\theta}|K_0K_2\sum_{l=1}^p\sum_{j=0}^\infty\frac{j\{B((1+\epsilon)\boldsymbol{\theta}^\circ)\}^j(1,1)f_l(\varepsilon_{1-j-l})}{1+B^j(1,1)f_l(\varepsilon_{1-j-l})}\\ &\leq|\boldsymbol{\theta}^\circ-\boldsymbol{\theta}|K_0K_1K_2\sum_{l=1}^p\sum_{j=0}^\infty j\left[\left(1+2\epsilon+\epsilon^2\right)r^s\right]^j\left(|\varepsilon_{1-j-l}|^{2cs}+1\right),\end{aligned}$$

where the infinite sum in the last term has a finite w-th moment.

To deal with the third term in the righthand side of (S.16), observe that

$$f_l^\circ(x)-f_l(x)=(\psi_{1,l}^\circ-\psi_{1,l})(x)_+^{2\delta}+\psi_{1,l}^\circ\{(x)_+^{2\delta_\circ}-(x)_+^{2\delta}\}+(\psi_{2,l}^\circ-\psi_{2,l})(x)_-^{2\delta}+\psi_{2,l}^\circ\{(x)_-^{2\delta_\circ}-(x)_-^{2\delta}\},$$

where both $\left|(x)_+^{2\delta_\circ}-(x)_+^{2\delta}\right|$ and $\left|(x)_-^{2\delta_\circ}-(x)_-^{2\delta}\right|$ are bounded by

$$\begin{aligned}\left||x|^{2\delta_\circ}-|x|^{2\delta}\right| &\leq 2|\delta_\circ-\delta|\left\{I(|x|<1,x\neq 0)|x|^{2(\delta\wedge\delta_\circ)}\left|\log|x|\right|+I(|x|\geq 1)|x|^{2(\delta\vee\delta_\circ)}\left|\log|x|\right|\right\}\\ &=: 2|\delta_\circ-\delta|g(x;\delta).\end{aligned}$$

We take $s\in(0,1)$ such that $\mathrm{E}|\varepsilon_0|^{2cws}|\log|\varepsilon_0||^w<\infty$. From (S.15), we have that for large n,

$$\begin{aligned}&\frac{\sum_{l=1}^p\sum_{j=0}^\infty B_\circ^j(1,1)|\left||\varepsilon_{1-j-l}|^{2\delta_\circ}-|\varepsilon_{1-j-l}|^{2\delta}\right|}{\omega\sum_{j=0}^\infty B^j(1,1)+\sum_{l=1}^p\psi_l\sum_{j=0}^\infty B^j(1,1)f_l(\varepsilon_{1-j-l})}\leq K\sum_{l=1}^p\sum_{j=0}^\infty\frac{B_\circ^j(1,1)|\left||\varepsilon_{1-j-l}|^{2\delta_\circ}-|\varepsilon_{1-j-l}|^{2\delta}\right|}{1+B^j(1,1)|\varepsilon_{1-j-l}|^{2\delta}}\\ &\leq 2|\boldsymbol{\theta}^\circ-\boldsymbol{\theta}|K\sum_{l=1}^p\sum_{j=0}^\infty\frac{B_\circ^j(1,1)g(\varepsilon_{1-j-l})}{1+B^j(1,1)|\varepsilon_{1-j-l}|^{2\delta}}\\ &\leq 2|\boldsymbol{\theta}^\circ-\boldsymbol{\theta}|K\sum_{l=1}^p\sum_{j=0}^\infty\frac{(1+\epsilon)^jB^j(1,1)|\varepsilon_{1-j-l}|^{2\delta}}{1+B^j(1,1)|\varepsilon_{1-j-l}|^{2\delta}}\frac{g(\varepsilon_{1-j-l};\delta)}{|\varepsilon_{1-j-l}|^{2\delta}}\\ &\leq 2|\boldsymbol{\theta}^\circ-\boldsymbol{\theta}|K\sum_{l=1}^p\sum_{j=0}^\infty\{(1+\epsilon)r^s\}^j|\varepsilon_{1-j-l}|^{2\delta(s-1)}g(\varepsilon_{1-j-l};\delta),\end{aligned}$$

when $\boldsymbol{\theta} \in N_n(\eta)$. Moreover, for large n, $|\varepsilon_0|^{2\delta(s-1)} g(\varepsilon_0; \delta)$ is bounded by

$$I(|\varepsilon_0| < 1, |\varepsilon_0| \neq 0)|\varepsilon_0|^{2c_1 s}|\log|\varepsilon_0|| + I(|\varepsilon_0| \geq 1)|\varepsilon_0|^{2cs}|\log|\varepsilon_0||, \quad 0 < c_1 < \delta^\circ,$$

and thus has a finite w-th moment, since $x \mapsto |x|^{2c_1 s}|\log|x||$ is bounded in $|x| < 1$ and $x \neq 0$. Since other terms can be handled similarly, the lemma is validated. □

Lemma 13. *Let $\eta > 0$ and $m_n \to \infty$ with $m_n < n$ as $n \to \infty$. Then, there exist*

(S.17) $$r_0 \in [0, 1),\ \nu_0 > 0,\ \text{a } \mathcal{F}_{i-1}\text{-measurable r.v. } V_i \geq 0, \text{ and } \Delta_{n,i} = \Delta_{n,i}(\eta) \geq 0$$

such that $\sup_{i\in\mathbb{N}} \mathrm{E}|V_i|^{\nu_0} < \infty$ *and*

(S.18) $$\frac{1}{n}\sum_{i=1}^{n} \mathrm{E}\Delta_{n,i} = O(1), \quad \max_{1\leq i\leq n} \frac{\Delta_{n,i}}{n^\kappa} = o_P(1) \quad \text{as } n \to \infty,$$

Moreover, there exists $\eta_0 > 0$, such that with probability tending to 1 as $n \to \infty$,

(S.19) $$A_i(y, \eta, -\eta_0) \leq A_i(y, \boldsymbol{\theta}) \leq A_i(y, \eta, \eta_0) \quad \text{for each } \boldsymbol{\theta} \in N_n(\eta) \text{ and } i = m_n, \ldots, n,$$

where

(S.20) $$A_i(y, \eta, \eta_0) := I\left(U_i \left\{1 + r_0^i \eta_0 V_i\right\} \left\{1 + \frac{\eta_0 \Delta_{n,i}}{n^\kappa}\right\} \left\{1 + \frac{\eta_0 h_i^{\frac{\eta_0}{n^\kappa}} |\log h_i|}{n^\kappa}\right\} > y^{-\frac{1}{\alpha}} b_n \right).$$

Proof. Put

$$\Delta_{n,i} := \sup_{\boldsymbol{\theta} \in N_n^-(\eta)} \left| \frac{h_i - h_i(\boldsymbol{\theta})}{|\boldsymbol{\theta} - \boldsymbol{\theta}^\circ| h_i(\boldsymbol{\theta})} \right| \in \mathcal{F}_{i-1}, \quad i = 1, \ldots, n,$$

which satisfies (S.18) owing to Lemma 12. Further, owing to Lemma 10, we can take $r_0 \in [0, 1)$, $\nu_0 > 0$, and a $\mathcal{F}_0$-measurable r.v. $V_0 \geq 0$, such that $\mathrm{E}V_0^{\nu_0} < \infty$ and

$$\sup_{\boldsymbol{\theta} \in N_n(\eta)} \left| \frac{h_i(\boldsymbol{\theta}) - \hat{h}_i(\boldsymbol{\theta})}{\hat{h}_i(\boldsymbol{\theta})} \right| \leq r_0^i V_0 \quad \text{for large } n.$$

If we take $V_i = V_0$, $V_i \in \mathcal{F}_{i-1}$ and $\sup_i \mathrm{E}V_i^{\nu_0} < \infty$, and thus, by mean value theorem, we can take $\eta_0 > 0$ such that (S.19) holds with probability tending to 1. Hence, the proof is completed. □

Lemma 14. *Let $\eta > 0$, $\epsilon_0 \in (0, 1)$, and $r_1 \in (r_0, 1)$, where r_0 is the one in (S.17). If $m_n < n$ and $m_n \to \infty$ as $n \to \infty$ and $\eta_0 \in \mathbb{R}$, we have that for $i = m_n, \ldots, n$,*

$$\frac{n}{k} w_i \left|\mathrm{E}_{i-1}\left\{A_i(y, \eta, \eta_0) - A_i(y)\right\}\right| \leq K y \max\left\{ r_1^i, \frac{|\eta_0|\Delta_{n,i}}{n^\kappa}, \frac{|\eta_0| h_i^{\frac{\eta_0}{n^\kappa}} |\log h_i|}{n^\kappa} \right\},$$

where

(S.21) $$w_i = I\left(|\eta_0| \max\left\{ \frac{\Delta_{n,i}}{n^\kappa}, \frac{h_i^{\frac{\eta_0}{n^\kappa}} |\log h_i|}{n^\kappa} \right\} \leq \epsilon_0,\ r_0^i |\eta_0| V_i < r_1^i \right).$$

Proof. If $w_i = 1$,

$$A_i(y,\eta,\eta_0) = I\left(U_i > y^{-\frac{1}{\alpha}}\left\{1 + r_0^i\eta_0 V_i\right\}^{-1}\left\{1 + \frac{\eta_0\Delta_{n,i}}{n^\kappa}\right\}^{-1}\left\{1 + \frac{\eta_0 h_i^{\frac{\eta_0}{n^\kappa}}|\log h_i|}{n^\kappa}\right\}^{-1} b_n\right),$$

so that from the $\mathcal{F}_{i-1}$-measurability of V_i, $\Delta_{n,i}$, and h_i,

$$\frac{n}{k}\left|\mathrm{E}_{i-1}\left\{A_i(y,\eta,\eta_0) - A_i(y)\right\}\right| \le Ky\left|\left\{1 + r_0^i\eta_0 V_i\right\}^{\alpha}\left\{1 + \frac{\eta_0\Delta_{n,i}}{n^\kappa}\right\}^{\alpha}\left\{1 + \frac{\eta_0 h_i^{\frac{\eta_0}{n^\kappa}}\log h_i}{n^\kappa}\right\}^{\alpha} - 1\right|.$$

The lemma is then verified by the mean value theorem. $\square$

Lemma 15. *Let $\eta > 0$, $\eta_0 \in \mathbb{R}$, $0 < \underline{y} < \bar{y} < \infty$ and $B_i(y,\eta,\eta_0) = A_i(y,\eta,\eta_0) - A_i(y)$. Then,*

$$\sup_{\underline{y}\le y\le\bar{y}}\sup_{0\le t\le 1}\left|\frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor} B_i(y,\eta,\eta_0)\right| = o_P(1).$$

Proof. Let $R_n = \lfloor\sqrt{k}\log n\rfloor$. Then,

$$R_n = o(n^\kappa) \tag{S.22}$$

and

$$y_u = \underline{y} + \frac{(\bar{y} - \underline{y})u}{R_n}, \quad u = 0, 1, 2, \ldots, R_n.$$

Suppose that $y_u < y \le y_{u+1}$. Then, we have

$$\begin{aligned}
&\frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor} B_i(y,\eta,\eta_0) \le \frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\{A_i(y_{u+1},\eta,\eta_0) - A_i(y_u)\}\\
&\le \frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\{B_i(y_{u+1},\eta,\eta_0) - \mathrm{E}_{i-1}B_i(y_{u+1},\eta,\eta_0)\} + \frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\mathrm{E}_{i-1}B_i(y_{u+1},\eta,\eta_0)\\
&+ \frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\{A_i(y_{u+1}) - A_i(y_u) - \mathrm{E}\{A_i(y_{u+1}) - A_i(y_u)\}\} + \frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\mathrm{E}\{A_i(y_{u+1}) - A_i(y_u)\}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor} B_i(y,\eta,\eta_0) \ge \frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\{A_i(y_u,\eta,\eta_0) - A_i(y_{u+1})\}\\
&\ge \frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\{B_i(y_u,\eta,\eta_0) - \mathrm{E}_{i-1}B_i(y_u,\eta,\eta_0)\} + \frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\mathrm{E}_{i-1}B_i(y_u,\eta,\eta_0)\\
&+ \frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\{A_i(y_u) - A_i(y_{u+1}) - \mathrm{E}\{A_i(y_u) - A_i(y_{u+1})\}\} + \frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\mathrm{E}\{A_i(y_u) - A_i(y_{u+1})\}.
\end{aligned}$$

Thus, it suffices to show that

(S.23)
$$\max_{0\le u\le R_n}\sup_{0\le t\le 1}\left|\frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\{B_i(y_u,\eta,\eta_0)-\mathrm{E}_{i-1}B_i(y_u,\eta,\eta_0)\}\right| = o_P(1),$$

(S.24)
$$\max_{0\le u\le R_n}\sup_{0\le t\le 1}\left|\frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\{\mathrm{E}_{i-1}B_i(y_u,\eta,\eta_0)\}\right| = o_P(1),$$

(S.25)
$$\max_{0\le u\le R_n}\frac{1}{\sqrt{k}}\sum_{i=1}^{n}\mathrm{E}\{A_i(y_{u+1})-A_i(y_u)\} = o(1),$$

(S.26)
$$\max_{0\le u\le R_n}\left|\frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\{A_i(y_{u+1})-A_i(y_u)-\mathrm{E}\{A_i(y_{u+1})-A_i(y_u)\}\}\right| = o_P(1).$$

First, we verify (S.23). We take $m_n\in\mathbb{N}$ such that $m_n\to\infty$ and $R_n m_n/k\to 0$ as $n\to\infty$. Then, it follows from the subadditivity, Doob's inequality, Lemma 14, (S.22), (S.18), and Lemma 8 that

$$\begin{aligned}
&P\left(\max_{0\le u\le R_n}\sup_{0\le t\le 1}\left|\frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}w_i\{B_i(y_u,\eta,\eta_0)-\mathrm{E}_{i-1}B_i(y_u,\eta,\eta_0)\}\right|>\epsilon\right)\\
&\le \sum_{u=0}^{R_n}P\left(\sup_{0\le t\le 1}\left|\frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}w_i\{B_i(y_u,\eta,\eta_0)-\mathrm{E}_{i-1}B_i(y_u,\eta,\eta_0)\}\right|>\epsilon\right)\\
&\le \sum_{u=0}^{R_n}\frac{1}{\epsilon^2 k}\sum_{i=1}^{n}\mathrm{E}w_i\{B_i(y_u,\eta,\eta_0)\}\\
&\le \sum_{u=0}^{R_n}\frac{2}{\epsilon^2 k}\sum_{i=m_n+1}^{n}\frac{k}{n}K|\eta_0|\mathrm{E}\max\left\{r_1^i,\frac{\Delta_{n,i}}{n^\kappa},\frac{h_i^{\frac{\eta_0}{n^\kappa}}|\log h_i|}{n^\kappa}\right\}+o(1)=o(1),
\end{aligned}$$

where w_is are defined in (S.21). Thus, since $w_{m_n}=\cdots=w_n=1$ with probability tending to 1, (S.23) is asserted. On the other hand, (S.24) is readily verified since owing to (S.18), (S.22), and Lemma 8,

$$\begin{aligned}
&\max_{0\le u\le R_n}\sup_{0\le t\le 1}\left|\frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\{w_i\mathrm{E}_{i-1}B_i(y_u,\eta,\eta_0)\}\right|\\
&\le \frac{1}{\sqrt{k}}\sum_{i=m_n+1}^{n}\frac{k}{n}K|\eta|\max\left\{r_1^i,\frac{\Delta_{n,i}}{n^\kappa},\frac{h_i^{\frac{\eta}{n^\kappa}}|\log h_i|}{n^\kappa}\right\}+o_P(1)=o_P(1).
\end{aligned}$$

Meanwhile, (S.25) follows from (S.22) and the fact that

$$\mathrm{E}\{A_i(y_{u+1})-A_i(y_u)\}=\frac{k}{n}(y_{u+1}-y_u)(1+o(1))$$

uniformly in $u=0,1,2,\ldots,R_n$. Finally, note that (S.26) is dominated by

$$\sup\left\{|M_n^*(w_1,t)-M_n^*(w_2,t)|:t\in[0,1],|w_1-w_2|\le\xi,\underline{y}\le w_1<w_2\le\bar{y}\right\}.$$

Since $\{M_n^*\}$ is asymptotically uniformly equicontinuous (Proposition 1), the proof is completed. $\square$

Lemma 16. $\mathrm{E}|\varepsilon_0|^\nu < \infty$ *for every* $\nu \in (0,2)$.

Proof. The proof is rather standard and is omitted for brevity. $\square$

Lemma 17. *There exists* $r \in (0,1)$ *and* $\mathcal{F}_{i-1}$*-measurable r.v.s* $V_i \geq 0$*, such that* $\mathrm{E}V_i^\nu < \infty$ *for some* $\nu > 0$*, and*

$$\sup_{\boldsymbol{\theta} \in N_n(\eta)} |\hat{\varepsilon}_i(\boldsymbol{\theta}) - \varepsilon_i(\boldsymbol{\theta})| \vee \left|\hat{\sigma}_i^2(\boldsymbol{\theta}) - \sigma_i^2(\boldsymbol{\theta})\right| \leq r^i V_i \quad \textit{for large } n \textit{ and each } i \in \mathbb{N}.$$

Proof. The proof is rather standard and is omitted for brevity. $\square$

Lemma 18. *Let* $\eta > 0$ *and* $v \in (0,2)$*. Then, for every* $w > 0$*,*

$$\limsup_{n\to\infty} \mathrm{E}\left\{\sup_{\boldsymbol{\theta}\in N_n^-(\eta)} \frac{|\tilde{\sigma}_1^2(\boldsymbol{\theta}) - \sigma_1^2(\boldsymbol{\theta})|}{|\boldsymbol{\theta} - \boldsymbol{\theta}^\circ|\sigma_1^2(\boldsymbol{\theta})}\right\}^w < \infty, \tag{S.27}$$

and there exist $\Pi_{1,n,i} = \Pi_{1,n,i}(\eta, v) \geq 0$ *and* $\Pi_{2,n,i} = \Pi_{2,n,i}(\eta, v) \geq 0$*, such that*

$$\frac{|\sigma_i^2 - \tilde{\sigma}_i^2(\boldsymbol{\theta})|}{\tilde{\sigma}_i^2(\boldsymbol{\theta})} \leq |\boldsymbol{\theta} - \boldsymbol{\theta}^\circ|\Pi_{1,n,i} + |\boldsymbol{\theta} - \boldsymbol{\theta}^\circ|^2\Pi_{2,n,i}, \quad \textit{when } \boldsymbol{\theta} \in N_n(\eta), \tag{S.28}$$

and $\limsup_n \mathrm{E}\{\Pi_{1,n,1}\}^v < \infty$ *and* $\limsup_n \mathrm{E}\{\Pi_{2,n,1}\}^{v/2} < \infty$.

Proof. One can easily check that the argument in (S.27) can be proven in essentially the same manner to prove Lemma 12. Below, we deal with (S.28). Note that

$$\begin{aligned}
\frac{|\sigma_i^2 - \tilde{\sigma}_i^2(\boldsymbol{\theta})|}{\tilde{\sigma}_i^2(\boldsymbol{\theta})} \quad &\leq \quad \frac{\sum_{j=0}^\infty B_\circ^j(1,1) \sum_{l=1}^p \psi_l^\circ \left|\varepsilon_{i-j-l}^2 - \varepsilon_{i-j-l}^2(\boldsymbol{\theta})\right|}{\omega^\circ \sum_{j=0}^\infty B_\circ^j(1,1) + \sum_{j=0}^\infty B_\circ^j(1,1) \sum_{l=1}^p \psi_l^\circ \varepsilon_{i-j-l}^2(\boldsymbol{\theta})} \\
&\leq \quad \frac{2\sum_{j=0}^\infty B_\circ^j(1,1) \sum_{l=1}^p \psi_l^\circ |\varepsilon_{i-j-l}(\boldsymbol{\theta})|\, |\varepsilon_{i-j-l} - \varepsilon_{i-j-l}(\boldsymbol{\theta})|}{\omega^\circ \sum_{j=0}^\infty B_\circ^j(1,1) + \sum_{j=0}^\infty B_\circ^j(1,1) \sum_{l=1}^p \psi_l^\circ \varepsilon_{i-j-l}^2(\boldsymbol{\theta})} \\
&\quad + \frac{\sum_{j=0}^\infty B_\circ^j(1,1) \sum_{l=1}^p \psi_l^\circ \,|\varepsilon_{i-j-l} - \varepsilon_{i-j-l}(\boldsymbol{\theta})|^2}{\omega^\circ \sum_{j=0}^\infty B_\circ^j(1,1) + \sum_{j=0}^\infty B_\circ^j(1,1) \sum_{l=1}^p \psi_l^\circ \varepsilon_{i-j-l}^2(\boldsymbol{\theta})} \\
&=: \quad I_1 + I_2.
\end{aligned}$$

First, we deal with I_1. Let

$$Y_i = Y_{n,i}(\eta) = \sup_{\boldsymbol{\theta}\in N_n^-(\eta)} \frac{|\varepsilon_i - \varepsilon_i(\boldsymbol{\theta})|}{|\boldsymbol{\theta} - \boldsymbol{\theta}^\circ|}.$$

Then, Y_i is bounded by an infinite sum of X_j, $j < i$, with exponentially decaying coefficients, and

$$\begin{aligned} I_1 &\leq K|\boldsymbol{\theta} - \boldsymbol{\theta}^\circ| \frac{\sum_{j=0}^{\infty} B_\circ^j(1,1) \sum_{l=1}^{p} |\varepsilon_{i-j-l}(\boldsymbol{\theta})| Y_{i-j-l}}{1 + \sum_{j=0}^{\infty} B_\circ^j(1,1) \sum_{l=1}^{p} \varepsilon_{i-j-l}^2(\boldsymbol{\theta})} \\ &\leq K|\boldsymbol{\theta} - \boldsymbol{\theta}^\circ| \sum_{l=1}^{p} \sum_{j=0}^{\infty} \frac{B_\circ^j(1,1) |\varepsilon_{i-j-l}(\boldsymbol{\theta})| Y_{i-j-l}}{1 + B_\circ^j(1,1) \varepsilon_{i-j-l}^2(\boldsymbol{\theta})} \\ &\leq K|\boldsymbol{\theta} - \boldsymbol{\theta}^\circ| \left\{ \sum_{l=1}^{p} \sum_{j=0}^{\infty} \frac{B_\circ^j(1,1) \varepsilon_{i-j-l}^2(\boldsymbol{\theta}) Y_{i-j-l}}{1 + B_\circ^j(1,1) \varepsilon_{i-j-l}^2(\boldsymbol{\theta})} + \sum_{l=1}^{p} \sum_{j=0}^{\infty} B_\circ^j(1,1) Y_{i-j-l} \right\}. \end{aligned}$$

Here, we take $s > 0$ such that $\mathrm{E}\sup_{\boldsymbol{\theta} \in N_n(\eta)} |\varepsilon_1(\boldsymbol{\theta})|^{2sv} |Y_1|^v < \infty$ and $\mathrm{E}|Y_1|^v < \infty$. Then,

$$\sum_{l=1}^{p} \sum_{j=0}^{\infty} \frac{B_\circ^j(1,1) \varepsilon_{i-j-l}^2(\boldsymbol{\theta}) Y_{i-j-l}}{1 + B_\circ^j(1,1) \varepsilon_{i-j-l}^2(\boldsymbol{\theta})} \leq \sum_{l=1}^{p} \sum_{j=0}^{\infty} |B_\circ(1,1)|^{sj} \sup_{\boldsymbol{\theta} \in N_n(\eta)} |\varepsilon_{i-j-l}(\boldsymbol{\theta})|^{2s} Y_{i-j-l},$$

where the r.v. in the righthand of the inequality has a finite v-th moment. Since this is true for $\sum_{l=1}^{p} \sum_{j=0}^{\infty} B_\circ^j(1,1) Y_{i-j-l}$, there exists a r.v. $\Pi_{1,n,i} \geq 0$, such that $I_1 \leq |\boldsymbol{\theta} - \boldsymbol{\theta}^\circ| \Pi_{1,n,i}$ and $\mathrm{E}\{\Pi_{1,n,i}\}^v < \infty$. Accordingly, we have

$$I_2 \leq K|\boldsymbol{\theta} - \boldsymbol{\theta}^\circ|^2 \sum_{l=1}^{p} \sum_{j=0}^{\infty} B_\circ^j(1,1) Y_{i-j-l}^2,$$

and the lemma is validated. □

Lemma 19. *Let $\eta > 0$, $\epsilon_0 \in (0, 0.1)$, $r_1 \in (r_0, 1)$, and $\{m_n\}$ be a sequence such that $m_n < n$ and $m_n \to \infty$ as $n \to \infty$. Then, there exists $\eta_0 > 0$ such that when $\boldsymbol{\theta} \in N_n(\eta)$,*

$$w_i A_i(y, \eta, -\eta_0) \leq w_i A_i(y, \boldsymbol{\theta}) \leq w_i A_i(y, \eta, \eta_0) \quad \text{for each } i = m_n, \ldots, n,$$

where

$$w_i := I\left(\max\left\{ \frac{|\eta_0| \Pi_{1,n,i}^*}{\sqrt{n}}, \frac{|\eta_0|^2 \Pi_{2,n,i}^*}{n}, \frac{|\eta_0| \Pi_{3,n,i}}{\sqrt{n} b_n} \right\} < \epsilon_0,\ r_0^i |\eta_0| V_i < r_1^i \right) \tag{S.29}$$

and

$$\frac{n}{k} w_i |\mathrm{E}_{i-1}\left\{A_i(y, \eta, \eta_0) - A_i(y)\right\}| \leq K w_i \max\left\{ r_1^i, \frac{|\eta_0| \Pi_{1,n,i}^*}{\sqrt{n}}, \frac{|\eta_0|^2 \Pi_{2,n,i}^*}{n}, \frac{|\eta_0| \Pi_{3,n,i}}{\sqrt{n} b_n} \right\}. \tag{S.30}$$

In particular,

$$\frac{1}{\sqrt{k}} \sum_{i=m_n}^{n} (1 - w_i) = o_P(1), \tag{S.31}$$

$$\sqrt{k} \log n \cdot \mathrm{E} \frac{|\eta_0|^2 \Pi_{2,n,1}^*}{n} I\left(\frac{|\eta_0|^2 \Pi_{2,n,1}^*}{n} < \epsilon_0 \right) = o(1) \quad \text{as } n \to \infty. \tag{S.32}$$

Proof. Note that (S.31) can be easily derived from (67) and the fact that $\sup_i \mathrm{E}V_i^{\nu} < \infty$ for some $\nu > 0$. Further, since $0 < \epsilon_0 < 1$ and $v_0/2 > 1/2$, we have

$$\begin{aligned}
\sqrt{k}\log n \cdot \mathrm{E}\frac{|\eta_0|^2\Pi^*_{1,n,1}}{n} I\left(\frac{|\eta_0|^2\Pi^*_{2,n,1}}{n} < \epsilon_0\right) &\leq \sqrt{k}\log n \cdot \mathrm{E}\left(\frac{|\eta_0|^2\Pi^*_{2,n,1}}{n}\right)^{v_0/2} \\
&\leq \frac{\sqrt{k}\log n \cdot |\eta_0|^{v_0}}{n^{v_0/2}} \mathrm{E}\left(\Pi^*_{2,n,1}\right)^{v_0/2} = o(1).
\end{aligned}$$

Therefore, (S.32) is verified. Since the remaining statement can be proven in a similar manner to prove Lemma 13, we complete the proof without detailing algebras. □

Lemma 20. *Let $\eta > 0$, $\eta_0 \in \mathbb{R}$, $0 < \underline{y} < \bar{y} < \infty$, and $B_i(y, \eta, \eta_0) = A_i(y, \eta, \eta_0) - A_i(y)$. Then, we have*

$$\sup_{\underline{y}\leq y\leq \bar{y}} \sup_{0\leq t\leq 1} \left| \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor nt \rfloor} B_i(y, \eta, \eta_0) \right| = o_P(1).$$

Proof. Let R_n and $\{y_u\}$ be the ones in the proof of Lemma 15, and let $\{m_n\}$ be such that $m_n \to \infty$ and $m_n = o(\sqrt{k})$ as $n \to \infty$. In view of the proof of Lemma 15, it suffices to show that

$$\max_{0\leq u\leq R_n} \sup_{0\leq t\leq 1} \left| \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor nt \rfloor} \{B_i(y_u, \eta, \eta_0) - \mathrm{E}_{i-1}B_i(y_u, \eta, \eta_0)\} \right| = o_P(1), \tag{S.33}$$

$$\max_{0\leq u\leq R_n} \sup_{0\leq t\leq 1} \left| \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor nt \rfloor} \{\mathrm{E}_{i-1}B_i(y_u, \eta, \eta_0)\} \right| = o_P(1). \tag{S.34}$$

Defining w_i as the one in (S.29), we can verify (S.33) in the same manner to prove (S.23), owing to (65), (66), (S.30), (S.31), and (S.32). Moreover, (S.34) can be easily seen, since owing to (65), (66) and (S.32),

$$\begin{aligned}
&\max_{0\leq u\leq R_n} \sup_{0\leq t\leq 1} \left| \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor nt \rfloor} \{w_i \mathrm{E}_{i-1}B_i(y_u, \eta, \eta_0)\} \right| \\
&\leq \frac{1}{\sqrt{k}} \sum_{i=m_n}^{n} \frac{k}{n} K w_i \max\left\{ r_1^i, \frac{|\eta_0|\Pi^*_{1,n,i}}{\sqrt{n}}, \frac{|\eta_0|^2\Pi^*_{2,n,i}}{n}, \frac{|\eta_0|\Pi_{3,n,i}}{\sqrt{n}b_n} \right\} + o_P(1) = o_P(1).
\end{aligned}$$

This asserts the lemma. □

References

Pan, J., Wang, H. and Tong, H. (2008). Estimation and tests for power-transformed and threshold GARCH models. *J. Econometrics*, **142** 352–378.