# On a class of circulas: copulas for circular distributions

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Supplementary Material

This supplementary material consists of a derivation of formula (2.1) followed by four figures, each referred to in the main text.

**Derivation of (2.1)**

The circula distribution function $C_q$ can be expressed as

$$C_q(\theta_1, \theta_2) = \int_0^{\theta_1} \int_0^{\theta_2} c_q(t_1, t_2) dt_2 dt_1 = \frac{1}{2\pi} \int_0^{\theta_1} \int_0^{\theta_2} g(t_2 - q t_1) dt_2 dt_1.$$

Consider the case when $q = 1$. Putting $a = t_2 - t_1$ and $b = t_1$, it follows that

$$C_1(\theta_1, \theta_2) = \frac{1}{2\pi} \int_0^{\theta_1} \int_{-b}^{\theta_2 - b} g(a)\, da\, db.$$

The double integral in this equation can be decomposed as

$$\int_0^{\theta_1} \int_{-b}^{\theta_2 - b} = \int_0^{\theta_1} \int_{-b}^{0} + \int_0^{\theta_1} \int_0^{\theta_2 - b}. \tag{1}$$

The first double integral on the right-hand side of (1) is

$$\int_0^{\theta_1} \int_{-b}^{0} g(a)\, da\, db = -\int_0^{\theta_1} \int_0^{-b} g(a)\, da\, db = \int_0^{-\theta_1} \int_0^{b'} g(a)\, da\, db' = W(-\theta_1),$$

the change-of-variable in the second equality being $b' = -b$. The second double integral on the right-hand side of (1) can be expressed as

$$\begin{aligned} \int_0^{\theta_1} \int_0^{\theta_2 - b} g(a)\, da\, db &= -\int_{\theta_2}^{\theta_2 - \theta_1} \int_0^{b''} g(a)\, da\, db'' \\ &= -\left( \int_{\theta_2}^{0} + \int_0^{\theta_2 - \theta_1} \right) \int_0^{b''} g(a)\, da\, db'' \\ &= W(\theta_2) - W(\theta_2 - \theta_1), \end{aligned}$$

where $b'' = \theta_2 - b$. Therefore we have

$$C_1(\theta_1, \theta_2) = \frac{1}{2\pi} \left\{ W(\theta_2) + W(-\theta_1) - W(\theta_2 - \theta_1) \right\}.$$

The case when $q = -1$ can be dealt with in a similar manner. □

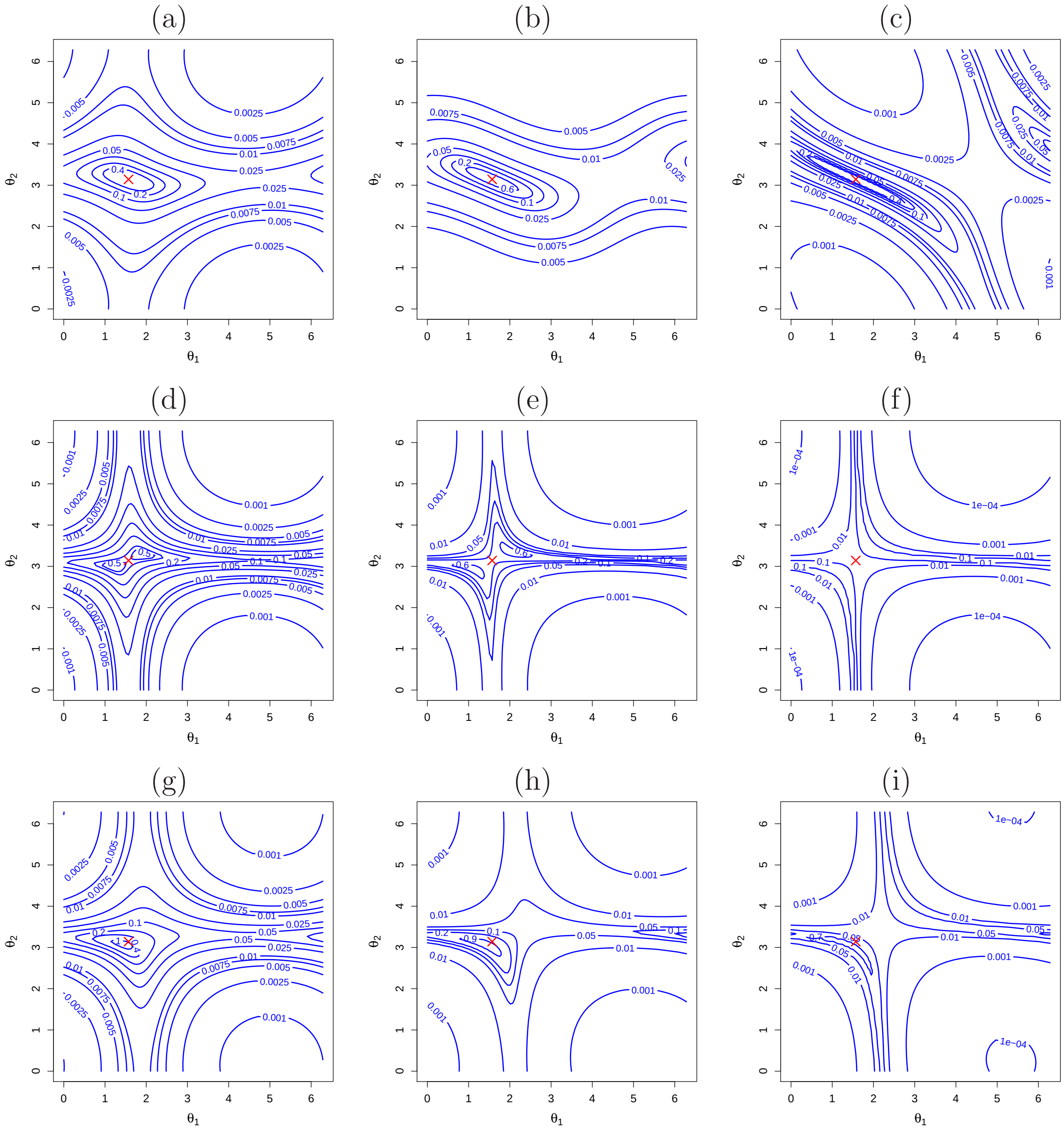


Figure S1: Contour plots of wC-wC-wC$(-1, \pi/2, 0.6, \pi, 0.8, \mu_g, \rho_g)$ densities with: first row, $\mu_g = 0$; second row, $\mu_g = \pi$; third row, $\mu_g = 5$. From left to right, the columns correspond to: $\rho_g = 0.3$, $\rho_g = 0.6$ and $\rho_g = 0.9$. The cross in each panel identifies $(\mu_1 = \pi/2, \mu_2 = \pi)$.

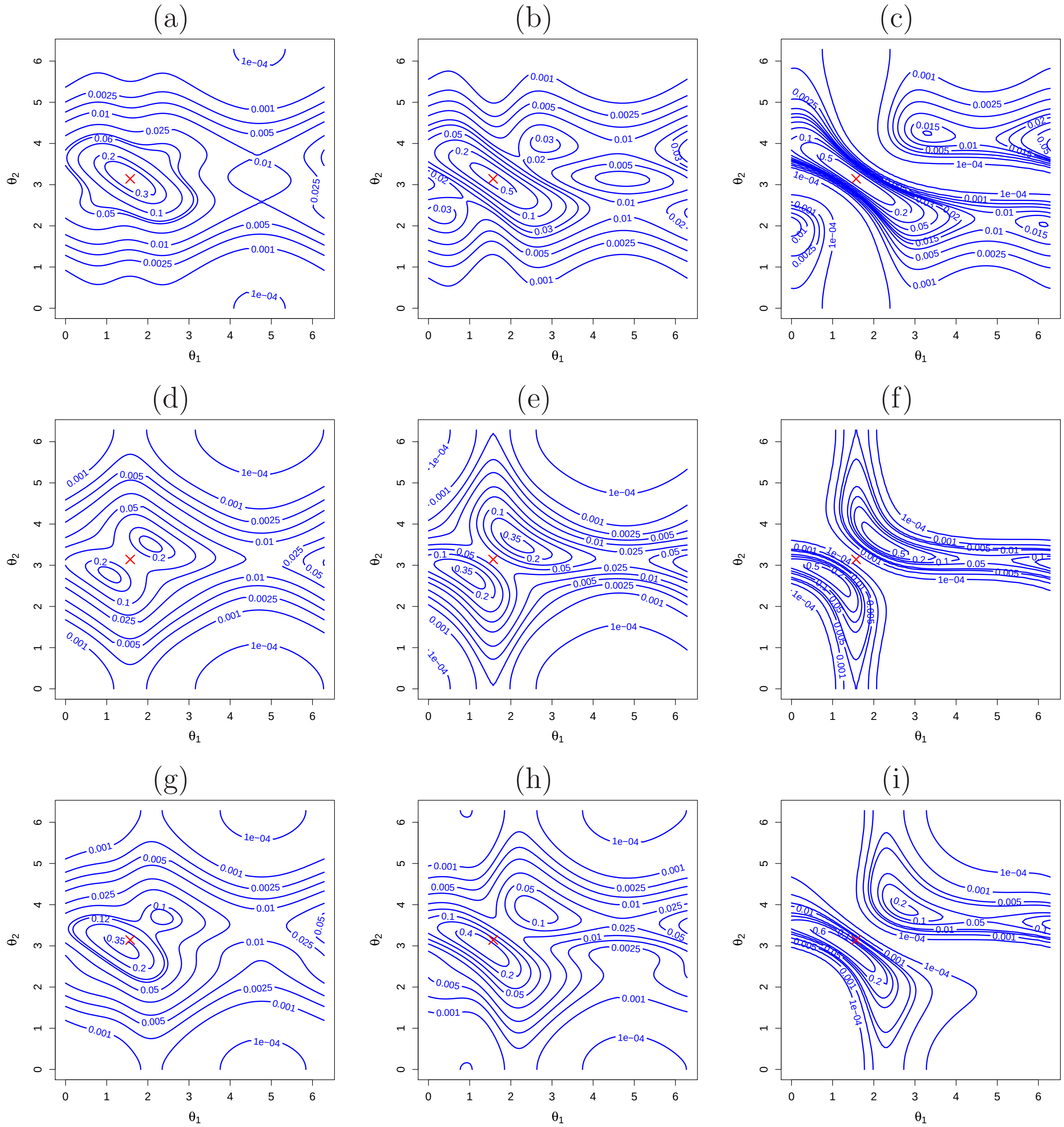


Figure S2: Contour plots of vM-vM-vM$(-1, \pi/2, 1.509, \pi, 2.862, \mu_g, \kappa_g)$ densities with: first row, $\mu_g = 0$; second row, $\mu_g = \pi$; third row, $\mu_g = 5$. From left to right, the columns correspond to: $\kappa_g = 0.629$ $(\rho_g = 0.3)$, $\kappa_g = 1.509$ $(\rho_g = 0.6)$ and $\kappa_g = 5.291$ $(\rho_g = 0.9)$. The cross in each panel identifies $(\mu_1 = \pi/2, \mu_2 = \pi)$.

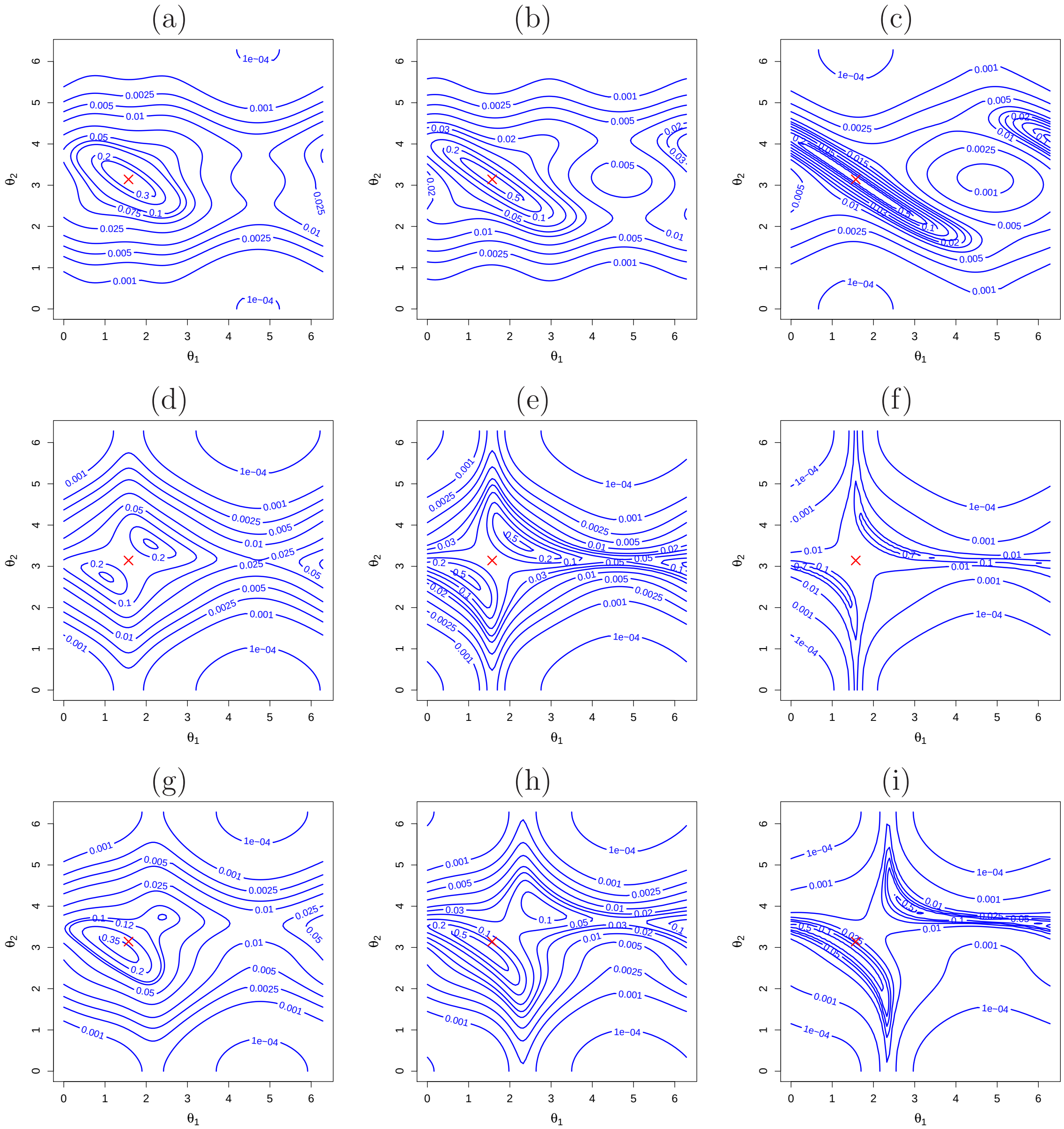


Figure S3: Contour plots of vM-vM-wC$(-1, \pi/2, 1.509, \pi, 2.862, \mu_g, \rho_g)$ densities with: first row, $\mu_g = 0$; second row, $\mu_g = \pi$; third row, $\mu_g = 5$. From left to right, the columns correspond to: $\rho_g = 0.3$, $\rho_g = 0.6$ and $\rho_g = 0.9$. The cross in each panel identifies $(\mu_1 = \pi/2, \mu_2 = \pi)$.

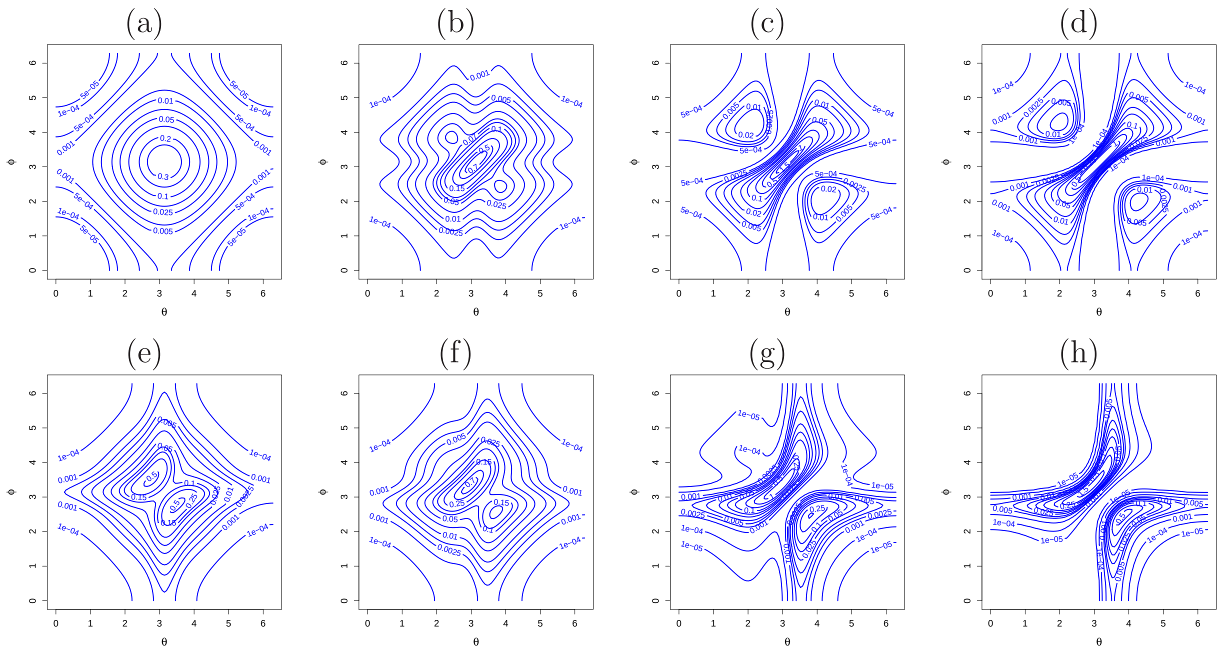

Figure S4: Contour plots of vM-vM-vM$(1, \pi, 3, \pi, 3, \mu_g, \kappa_g)$ densities. In the top row, $\mu_g = 0$ and: (a) $\kappa_g = 0$; (b) $\kappa_g = 1$; (c) $\kappa_g = 4$; (d) $\kappa_g = 7$. In the bottom row: (e) $\mu_g = \pi$ and $\kappa_g = 1$; (f) $\mu_g = 3\pi/2$ and $\kappa_g = 1$; (g) $\mu_g = 3\pi/2$ and $\kappa_g = 4$; (h) $\mu_g = 3\pi/2$ and $\kappa_g = 7$.