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Quantile residual lifetime with right-censored and length-biased data

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Abstract Right-censored length-biased data are commonly encountered in many applications such as cancer screening trials, prevalent cohort studies and labor economics. Such data have a unique structure that is different from traditional survival data. In this paper, we propose an estimator of the quantile residual lifetime (QRL) with this kind of data based on the nonparametric maximum likelihood estimation method. In addition, we develop two tests by taking difference and ratio of the QRL from two independent samples. We also establish the asymptotic properties of the proposed estimator and the test statistics. Simulation studies are performed to demonstrate that the proposed estimator works well in finite-sample situations. We illustrate its application using two data examples: one is the Oscars Award data, the other is the Channing house data.

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1 Introduction

In many practical situations such as medical studies, we are often interested in the residual life (or residual survival lifetime), which is defined as the remaining survival time given an individual surviving to a known time point $t(t \geq 0)$, or more formally $T - t_0 | T \geq t_0$. The residual lifetime distribution is a useful quantity in practice, for example, an insurance company may be interested in how long a cancer patient is expected to survive given that this patient has survived so far; a hospital may be concerned about the percentage of patients who have cancer for three years will still live for two more years. Questions such as these can be formulated in terms of residual life; also, when a new treatment is used for a patient after he/she has taken an old one for a period of time, residual life can be applied to judge whether the new treatment is better or not. It will be more straightforward to illustrate it as "Given another person shares the similar genetic and environment with you, if you takes the drug, then your life will be prolong 10 years on average" rather than simply explain it as "this drug will reduce 25 % hazard". When $t = 0$, then the residual life is actually the survival time from disease onset to failure. As a result, estimate a survival function can be considered as a special case of estimate a residual lifetime distribution by setting $t = 0$.

Much work had been done on estimating the residual lifetime distribution, most of which focused on the mean residual lifetime and the median residual lifetime (see Chen et al. 1983; Berger et al. 1988; Oakes and Dasu 1990; Chen and Cheng 2005; Chen 2007; Sun and Zhang 2009; Chan et al. 2012 and Sun et al. 2012, among others), which are defined as the mean and median of the residual life random variable, respectively. The quantile residual lifetime, defined as the quantile of the residual life variable, provided a more complete description of residual life than median residual life; moreover, when the distribution is skewed or non-symmetric, the mean residual life is sensitive to outliers (Jeong 2014), and it may not be calculated, in this situation, the QRL still works. However, references on this topic is rare until recent, including Jeong et al. (2008), Jeong and Fine (2009), Jung et al. (2009), Ma and Wei (2012) and Lin et al. (2014). As far as we know, there has been no study on quantile residual life under right-censored length-biased data. Our work fills this gap.

Length-biased data often arise in many practical situations such as prevalent cohort studies, cancer screening, labor economics, and so on. It comes from length-biased sampling where the observed failure times are not randomly selected from the distribution of interest, instead, they are chosen with probability proportional to its length of the failure time interval. Under this sampling scheme, only the individuals who experience initial event (such as disease onset) and have not experienced terminal event (such as failure) can be observed, thus the collected data are left truncated. Length-biased data are a special case of left truncated data where the left truncation variable is assumed to follow a uniform distribution. This assumption is called the stationarity assumption in the literature (Addona 2005; Asgharian et al. 2002, 2006; Asgharian and Wolfson 2005). In medical studies, the stationary assumption is equivalent to saying that the disease incidence rate is a constant. In addition to length-bias, the patients may die from causes not related to the disease or lost follow-up, thus the observed data are often right-censored.

Our research is motivated by the need to properly analyze the Oscars Award data (Redelmeier and Singh 2001). This dataset contains the survival information (date of birth, nomination, death or censoring) of the nominees for Oscars Award from 1929 to 2001. The motivation of analyzing this dataset is to explore whether the increased statue from winning an academy award is associated with a longer life expectancy among actors and actresses. Those winning the Oscar Award must first being nominated. Obviously, only those who have survived long enough to be nominated can be recruited, thus these data are left truncated. Moreover, Addona and Wolfson's (2006) test shows that these data are length-biased. We denote the initial event as birth of the nominees, the terminal event as death, the age at nomination as truncation time (Wolkewitz et al. 2010). Redelmeier and Singh (2001) used the Kaplan–Meier curve to analyze these data and conclude that the winners have a longer life expectancy than the nonwinning nominees, with a 25 % relative reduction in death rates. However, this conclusion based on the Kaplan–Meier estimator is questionable because the data are length-biased. Here, we will use quantile residual lifetime to analyze these data taking into account the length-bias. Compared with the classical mean approach (life expectancy), quantile approach can be more informative. For instance, patients may be more concerned about the 90 and 75 % quantiles he/she will live rather than average residual time, because the former means that they can survive to a time point with 90 or 75 % probability, while the median residual lifetime means they can survive to that point with 50 % probability, and the mean residual life is less useful for a single person.

In this paper, a quantile residual lifetime estimator is proposed based on an estimating equation for right-censored and length-biased data. The estimating equation is constructed using a nonparametric maximum likelihood estimation (NPMLE) approach (Asgharian et al. 2002; Asgharian and Wolfson 2005; Vardi 1989). We establish the asymptotic properties of the proposed estimator. Furthermore, we propose two tests for two-sample comparisons. The asymptotic properties of the proposed test statistics are studied. To the best of our knowledge, this is the first time quantile residual lifetime being studied under right-censored length-biased data.

The remainder of the paper is organized as follows. In Sect. 2 we introduce the sampling model, propose an estimator for the quantile residual life function and state the asymptotic results for the proposed estimator. In Sect. 3 we apply the proposed method to two-sample comparison problems, two tests are proposed based on the difference and the ratio of the QRL estimators, respectively. Simulation studies are conducted in Sect. 4, which demonstrate that the proposed method has excellent finite-sample performance. In Sect. 5 we apply the proposed method to two real data sets to illustrate its application. Concluding remarks are given in Sect. 6. Proofs of the theoretical results are given in the Appendix.

2 Estimation method and main results

2.1 Notation and model

For the population being studied, $\tilde{T}$ is the full survival time variable, the support of the distribution function F of $\tilde{T}$ is $[0, \tau)$, let the distribution of length-biased variable be G, and the corresponding density functions of F and G are f and g, respectively. Let $S(\cdot) = 1 - F(\cdot)$ be the survival function of the unbiased data.

In the following, we will denote the truncation variable by $\tilde{A}$, which obeys the uniform distribution on $[0, \kappa]$. Suppose $\kappa > \tau$ so that the stationarity assumption is satisfied. Suppose that $\tilde{A}$ is independent of $\tilde{T}$. Thus the truncation probability is $P(\tilde{T} \geq \tilde{A})$. We assume $0 < P(\tilde{T} \geq \tilde{A}) < 1$. In the presence of length-biased sampling, an individual can be observed if and only if $\tilde{T} \geq \tilde{A}$. Denote T, A as the observed failure time and observed truncation time, thus the joint distribution of (T, A) is the same as $(\tilde{T}, \tilde{A})|\tilde{T} \geq \tilde{A}$. Let $R = T - A$. Here R is called the residual lifetime (forward recurrence time), and A is the current lifetime (backward recurrence time) (Zelen and Feinleib 1969). Also, we denote $A + C$ as the censoring time, where C is the time span from recruitment to censoring. Following Asgharian and Wolfson (2005), we assume that C and (A, R) are independent. Under the above mechanism, we observe the i.i.d. triple (Y_i, A_i, δ_i), $i = 1, \ldots, n$, where $Y_i = \min(A_i + R_i, A_i + C_i)$, $\delta_i = I(R_i \leq C_i)$. Conditionally on $\tilde{T} \geq \tilde{A}$, g (the p.d.f. of length-biased variable T) is related to f (the p.d.f. of unbiased data $\tilde{T}$) by (Shen et al. 2009)

$$g(t) = \frac{tf(t)}{\mu}, \qquad \mu = \int_0^\tau uf(u)\mathrm{d}u, \mu < \infty.$$

Due to the structure of the observed sample, the total censoring time $A + C$ and the failure time $A + R$ are dependent because they share a common part A. Thus the observed failure time T is subject to informative censoring (Asgharian and Wolfson 2005, pp. 2113). In the following, due to technical reason, we only consider the estimate in $(\gamma, \psi) \subset (0, \tau)$, where γ is defined in Assumption 5 and stands for the lower boundary for the support of $S(t)$, $\psi < \tau$ is an arbitrary number close to τ.

For the unbiased population, the αth quantile residual life time function at time t is defined as

$$\theta_\alpha(t) = \text{quantile}(\tilde{T} - t|\tilde{T} \geq t). \tag{1}$$

Obviously, $\theta_\alpha(\cdot)$ in (1) satisfies

$$P\{\tilde{T} - t \geq \theta_\alpha(t)|\tilde{T} \geq t\} = \alpha,$$

which implies that

$$P\{\tilde{T} - t \geq \theta_\alpha(t)\} = \alpha P(\tilde{T} \geq t).$$

Thus given t and α, the quantile residual life function $\theta_\alpha(t)$ satisfies

$$S(t+\theta_\alpha(t)) = \alpha S(t), \tag{2}$$

where $\gamma \le t < t+\theta_\alpha(t) \le \psi$.

To estimate $\theta_\alpha(t)$, it is natural to estimate the function $S(\cdot)$ first. Then, the estimator $\hat{\theta}_\alpha^{(n)}(t)$ is the solution to the estimating equation

$$\hat{M}_n(\theta_\alpha(t)) \equiv \hat{S}_n(t+\theta_\alpha(t)) - \alpha \hat{S}_n(t) = 0, \tag{3}$$

where $\hat{S}_n(\cdot)$ is an estimator of $S(\cdot)$. Here, we choose

$$\hat{S}_n(t) = 1 - \frac{\int_0^t s^{-1}\mathrm{d}\hat{G}_n(s)}{\int_0^\tau s^{-1}\mathrm{d}\hat{G}_n(s)}, \tag{4}$$

which is proposed by Asgharian and Wolfson (2005). The nonparametric maximum likelihood estimator $\hat{G}_n(\cdot)$ in (4) is obtained by maximizing

$$L_R(G) = \prod_{i=1}^n \mathrm{d}G(Y_i)^{\delta_i} \left\{ \int_{s \ge Y_i} s^{-1}\mathrm{d}G(s) \right\}^{1-\delta_i} \tag{5}$$

with respect to $G(\cdot)$. However, $\hat{G}_n(\cdot)$ does not have a closed-form expression, and an EM algorithm is proposed by Vardi (1989) for computation.

A referee pointed out that an equivalent way is to define $\hat{\theta}_\alpha^{(n)}(t)$ as the αth quantile of the estimated conditional distribution of $\tilde{T} - t$ given $\tilde{T} \ge t$. That is, $\hat{\theta}_\alpha^{(n)}(t) = \hat{S}_n^{-1}(\alpha \hat{S}_n(t)) - t$, where $\hat{S}_n^{-1}(p) = \inf\{t > 0 : \hat{S}_n(t) < p\}$. This is a solution to (3) and more intuitive. We note that the solutions to (3) generally are not unique due to the fact that $\hat{S}_n$ is not continuous. Here the reason we define $\hat{\theta}_\alpha^{(n)}(t)$ as a solution to (3) is that it is easier to carry out the theoretical analysis. The main difficulty is to show that $\hat{S}_n$ is stochastically equicontinous. Since $\hat{S}_n$ does not have a closed-form expression, this is a highly nontrivial problem, see Theorem 2 and its proof given in the Appendix.

2.2 Main results

We now state the asymptotic properties of $\hat{\theta}_\alpha^{(n)}(t)$.

Theorem 1 *Under the Assumptions 1–5 given in the Appendix, when $n \to \infty$*

$$\hat{\theta}_\alpha^{(n)}(t) \xrightarrow{a.s.} \theta_\alpha(t)$$

uniformly for any $\gamma \le t < t+\theta_\alpha(t) \le \psi < \tau$.

Theorem 2 *Under the Assumptions 1–5 given in the Appendix, for any $\gamma \le t < t + \theta_\alpha(t) \le \psi < \tau$, we have*

$$\sqrt{n}\left[\hat{\theta}_\alpha^{(n)}(t) - \theta_\alpha(t)\right] \xrightarrow{\mathscr{D}} [f(t+\theta_\alpha(t))]^{-1}\mu \int_0^\tau \left[\mathscr{L}_{t+\theta_\alpha(t)}(x) - \alpha \mathscr{L}_t(x)\right] \mathrm{d}U(x)$$

on $D_0[\gamma, \psi]$ (the space of cadlag functions on $[\gamma, \psi]$), where $U(\cdot)$ is the process defined in Eq. (11) in the Appendix, $\mathscr{L}_t(x) = [I_{[t,\infty)}(x) - S(t)]/x$, and the process on the right-hand side has covariance function

$$r(y, z) = \{[f(t+\theta_\alpha(t))]^{-1}\}^{\mathrm{T}} r_1(y, z)\{[f(t+\theta_\alpha(t))]^{-1}\}$$

and $r_1(y, z)$ is defined in Lemma 5 in the Appendix.

The proofs of these two theorems are given in the Appendix. These theorems state that the proposed estimator is uniformly consistent and asymptotically normal. In particular, Theorem 2 can be used to construct confidence intervals for $\hat{\theta}_\alpha^{(n)}(t)$. It can also be used to study the distributions of the two-sample test statistics.

3 The two-sample problem

In practice, we are often interested in examining whether there is any difference between the residual lifetimes under two different treatments. In this section we develop two methods for two-sample comparison problems. For jth individual in ith group, let T_{ij} be the observed failure time, let A_{ij} be the observed truncation time, and let C_{ij} be the time span from recruitment to censoring. We observe the triple $(Y_{ij}, A_{ij}, \delta_{ij})$, where $Y_{ij} = \min(A_{ij} + R_{ij}, A_{ij} + C_{ij})$, $\delta_{ij} = I(R_{ij} \le C_{ij})$, $j = 1, \ldots, n_i$, $i = 1, 2$. Within the ith group, we assume that C_{ij} is independent of (A_{ij}, R_{ij}), Y_{ij}s are i.i.d., the density functions of unbiased data and length-biased data are f_i and g_i, the corresponding distribution functions are F_i and G_i, the survival function of the unbiased data is $S_i(\cdot)$, the support for unbiased and length-biased data is $[0, \tau_i]$, and A_{ij} obeys uniform distribution on $[0, \kappa_i]$, where $\kappa_i \ge \tau_i$. However, we can only compare their quantile residual lifetimes in $[0, \rho]$, where $\rho = \min(\tau_1, \tau_2)$. Suppose $n_1/(n_1 + n_2) \to q$, where $0 < q < 1$ is a constant. Let $\theta_{i,\alpha}(t)$ be the αth quantile residual lifetime at time t, the corresponding estimator is $\hat{\theta}_{i,\alpha}^{(n_i)}(t)$. We have $g_i(x) = x f_i(x)/\mu_i$, $\mu_i = \int_0^{\tau_i} t f_i(t)\mathrm{d}t$, $0 < x < \tau_i$. The estimates of the length-biased distribution and unbiased survival function are $\hat{G}_{i,n_i}(\cdot)$ and $\hat{S}_{i,n_i}(\cdot)$, and $\hat{F}_{i,n_i}(\cdot) = 1 - \hat{S}_{i,n_i}(\cdot)$, $i = 1, 2$. The estimate quantile residual lifetimes $\hat{\theta}_{i,\alpha}^{(n_i)}(t)$ are the solutions to the equations

$$\hat{M}_{i,n_i}(\theta_{i,\alpha}(t)) = \hat{S}_{i,n_i}(t + \theta_{i,\alpha}(t)) - \alpha \hat{S}_{i,n_i}(t) = 0, \quad i = 1, 2. \tag{6}$$

3.1 Ratio of quantile residual lifetimes

First we consider the ratio of the quantile residual lifetimes. Recall in the one sample case, the quantile residual lifetime $\theta_\alpha(t)$ satisfies the equation

$$S(t+\theta_\alpha(t)) = \alpha S(t).$$

Thus

$$\theta_\alpha(t) = S^{-1}(\alpha S(t)) - t,$$

where $S^{-1}(q) = \inf\{t \geq 0 : S(t) < q\}$. Consequently, the ratio of the two quantile lifetimes $\tau_\alpha(t)$ is

$$\tau_\alpha(t) = \frac{S_1^{-1}(\alpha S_1(t)) - t}{S_2^{-1}(\alpha S_2(t)) - t}.$$

Where $S_i^{-1}(q) = \inf\{t \geq 0 : S_i(t) < q\}$, $i = 1, 2$. This can be rewritten as

$$\tau_\alpha(t) = 1 - \frac{S_1^{-1}(\alpha S_1(t)) - S_2^{-1}(\alpha S_2(t))}{t - S_2^{-1}(\alpha S_2(t))}. \tag{7}$$

Hence, a consistent estimator of $\tau_\alpha(t)$ is

$$\hat{\tau}_\alpha(t) = 1 - \frac{\hat{S}_{1,n_1}^{-1}(\alpha \hat{S}_{1,n_1}(t)) - S_{2,n_2}^{-1}(\alpha \hat{S}_{2,n_2}(t))}{t - \hat{S}_{2,n_2}^{-1}(\alpha \hat{S}_{2,n_2}(t))}. \tag{8}$$

where $\hat{S}_{i,n_i}^{-1}(q) = \inf\{t \geq 0 : S_{i,n_i}(t) < q\}$, $i = 1, 2$. To state the asymptotic distributional results for $\hat{\tau}_\alpha(t)$, we first introduce some notations. Let $\mathscr{L}_y^i(x) = [I_{[y,\infty)}(x) - S_i(y)]/x$. Define

$$\begin{aligned}
r_{1,i}(y,z) = \mu_i^2 &\int_0^{\tau_i}\int_0^{\tau_i} \psi_i(s,t)\mathrm{d}\mathscr{L}^i_{y+\theta_{i,\alpha}(y)}(t)\mathrm{d}\mathscr{L}^i_{z+\theta_{i,\alpha}(z)}(s)\\
&- \mu_i^2\alpha \int_0^{\tau_i}\int_0^{\tau_i} \psi_i(s,t)\mathrm{d}\mathscr{L}^i_{y+\theta_{i,\alpha}(y)}(t)\mathrm{d}\mathscr{L}^i_{z}(s)\\
&- \mu_i^2\alpha \int_0^{\tau_i}\int_0^{\tau_i} \psi_i(s,t)d\mathscr{L}^i_{y}(t)\mathrm{d}\mathscr{L}^i_{z+\theta_{i,\alpha}(z)}(s)\\
&+ \mu_i^2\alpha^2 \int_0^{\tau_i}\int_0^{\tau_i} \psi_i(s,t)\mathrm{d}\mathscr{L}^i_{y}(t)\mathrm{d}\mathscr{L}^i_{z}(s),
\end{aligned}$$

where $\psi_i(s,t) = \mathrm{cov}(U_i(t), U_i(s))$, $i = 1, 2$.

Theorem 3 *Under the conditions in Theorem 2, we have*

$$
\begin{aligned}
&\sqrt{n_1+n_2}(\hat{\tau}_\alpha(t)-\tau_\alpha(t))\\
&\xrightarrow{\mathscr{D}} \frac{\mu_1}{\sqrt{q} f_1(t+\theta_{1,\alpha}(t))\theta_{1,\alpha}(t)} \int_0^{\tau_1} \left[\mathscr{L}^1_{t+\theta_{1,\alpha}(t)}(x) - \alpha \mathscr{L}^1_t(x)\right] \mathrm{d}U_1(x)\\
&\quad - \frac{\mu_2 \theta_{1,\alpha}(t)}{\sqrt{1-q} f_2(t+\theta_{2,\alpha}(t))[\theta_{2,\alpha}(t)]^2} \int_0^{\tau_2} \left[\mathscr{L}^2_{t+\theta_{2,\alpha}(t)}(x) - \alpha \mathscr{L}^2_t(x)\right] \mathrm{d}U_2(x),
\end{aligned}
$$

where $U_1(x)$ and $U_2(x)$ are independent Gaussian processes, defined the same way as Eq. (11) in the Appendix. And the covariance function of the right hand is

$$
\begin{aligned}
&\frac{1}{q\theta_{1,\alpha}(y) f_1(y+\theta_{1,\alpha}(y))} r_{1,1}(y,z) \frac{1}{\theta_{1,\alpha}(z) f_1(z+\theta_{1,\alpha}(z))}\\
&\quad + \frac{\theta_{1,\alpha}(y)}{(1-q)[\theta_{2,\alpha}(y)]^2 f_2(y+\theta_{2,\alpha}(y))} r_{1,2}(y,z) \frac{\theta_{1,\alpha}(z)}{[\theta_{2,\alpha}(z)]^2 f_2(z+\theta_{2,\alpha}(z))}.
\end{aligned}
$$

The proof of this theorem is given in the Appendix.

3.2 Difference of quantile residual lifetimes

Let $d_\alpha(t) = \theta_{1,\alpha}(t) - \theta_{2,\alpha}(t)$ be the difference between two quantile residual lifetimes. We have

$$
d_\alpha(t) = S_1^{-1}(\alpha S_1(t)) - S_2^{-1}(\alpha S_2(t)).
$$

A natural estimator of $d_\alpha(t)$ is

$$
\hat{d}_\alpha(t) = \hat{S}^{-1}_{1,n_1}(\alpha \hat{S}_{1,n_1}(t)) - \hat{S}^{-1}_{2,n_2}(\alpha \hat{S}_{2,n_2}(t)). \tag{9}
$$

We can also derive the asymptotic consistency and normality properties for this estimator.

Theorem 4 *Under the same assumptions as in Theorem 2, for given t and α, we have*

$$
\begin{aligned}
&\sqrt{n_1+n_2}(\hat{d}_\alpha(t)-d_\alpha(t))\\
&\xrightarrow{\mathscr{D}} \frac{\mu_1}{\sqrt{q} f_1(t+\theta_{1,\alpha}(t))} \int_0^{\tau_1} \left[\mathscr{L}^1_{t+\theta_{1,\alpha}(t)}(x) - \alpha \mathscr{L}^1_t(x)\right] \mathrm{d}U_1(x)\\
&\quad - \frac{\mu_2}{\sqrt{1-q} f_2(t+\theta_{2,\alpha}(t))} \int_0^{\tau_2} \left[\mathscr{L}^2_{t+\theta_{2,\alpha}(t)}(x) - \alpha \mathscr{L}^2_t(x)\right] \mathrm{d}U_2(x),
\end{aligned}
$$

where $U_1(x)$ and $U_2(x)$ are independent Gaussian processes, defined the same way as Eq. (11) in the Appendix. And the covariance function of the right hand is

$$\frac{1}{qf_1(y+\theta_{1,\alpha}(y))}r_{1,1}(y,z)\frac{1}{f_1(z+\theta_{1,\alpha}(z))} + \frac{1}{(1-q)f_2(y+\theta_{2,\alpha}(y))}r_{1,2}(y,z)\frac{1}{f_2(z+\theta_{2,\alpha}(z))}.$$

We defer the proof to the Appendix. The above theorems can be used to construct the confidence intervals for the ratio and difference.

4 Simulation studies

In this section, we conduct simulation studies to evaluate the finite-sample performance of the proposed estimator, and also compare the performance of our methods based on the ratio and difference of quantile residual lifetimes for two-sample problems. We generate right-censored length-biased data in a way similar to that in Shen et al. (2009). First we generate the independent pairs $(\tilde{A}, \tilde{T})$, where $\tilde{T}$ obeys the truncated standard exponential distribution from 0 to 5 (unbiased data), and $\tilde{A}$ is generated from the uniform distribution $U(0, a)$ $(a \geq 5)$, we choose different a to obtain different truncation probabilities. Then, we select n pairs that conform the condition $\tilde{T} \geq \tilde{A}$. The resulting $\tilde{T}_i$s are the length-biased samples, and we denote it as $T = A + V$. The censoring variable $C \sim U(0, W)$, where W is used to control censoring rate. So the censoring indicator is $I(A_i + V_i \leq A_i + C_i)$, where $A_i + C_i$ is the total censoring time.

In the simulation, we set sample size $n = 100$ and repeat 500 times to obtain independent estimates, then the mean of the 500 estimates is used in the evaluation. To calculate the standard deviation (SD), we use the bootstrap with bootstrap sample size $B = 100$. When we estimate the quantile residual lifetime, we compute $\hat{\theta}_\alpha^{(n)}(t)$ each time at a fixed time point t with quantiles α ranging from 0.3 to 0.9 with step 0.1. Table 1 displays the simulation results under different truncation and censoring probabilities. The SE and Cov represent the standard error of the estimator and the empirical coverage probability with the nominal level 95 %, respectively. The simulation results in Table 1 show the proposed method works well. To be specific, the bias is small and reasonably distributed around zero. In each case, SE and SD is very close. When the time point t is fixed, the SD and SE decrease as quantile level α increase. Furthermore, the empirical coverage probabilities are all close to the nominal coverage probability 95 %. For fixed t and α, when we fix the censoring rate, it seems that lower truncation probability will lead to lower SD and SE as we expect.

To verify that ignoring length bias will cause problem, we conduct the second simulation. The results are shown in Table 2. In this table, we ignore length bias and treat the observed sample as standard right-censored data, then the popular Kaplan–Meier estimator is employed to estimate the survival function, that is, in Eq. (3), we set $\hat{S}_n(t)$ as Kaplan–Meier estimator, the resulting estimator of $\theta_\alpha(t)$ is denoted as $\tilde{\theta}_\alpha^{(n)}(t)$. We can see that $\tilde{\theta}_\alpha^{(n)}(t)$ are all much larger than the true value $\theta_\alpha(t)$, which means that ignoring length bias will lead to overestimation of the parameter, while the proposed estimates $\hat{\theta}_\alpha^{(n)}(t)$ are still close to $\theta_\alpha(t)$, i.e., unbiased. The results indicate that if we ignore length bias, the estimator will be biased. In practical situation such

Table 1 Simulation results under different truncation and censoring probability

t	α	$\theta_\alpha(t)$	TRUN prob ≈19.3				TRUN prob ≈9.6			
			Bias	SE	SD	Cov	Bias	SE	SD	Cov
1.62 (CnsR = 19)	0.3	1.127	0.054	0.290	0.280	0.948	0.042	0.276	0.272	0.964
	0.4	0.866	−0.060	0.229	0.230	0.948	−0.070	0.207	0.224	0.958
	0.5	0.660	−0.090	0.158	0.182	0.924	−0.089	0.152	0.176	0.946
	0.6	0.488	−0.076	0.115	0.144	0.940	−0.074	0.113	0.138	0.938
	0.7	0.342	−0.048	0.099	0.111	0.946	−0.051	0.090	0.110	0.942
	0.8	0.215	−0.022	0.083	0.087	0.944	−0.026	0.071	0.084	0.942
	0.9	0.102	−0.004	0.057	0.062	0.944	−0.002	0.054	0.060	0.930
2.5 (CnsR = 48)	0.3	1.029	0.009	0.313	0.320	0.914	−0.020	0.311	0.314	0.912
	0.4	0.800	0.026	0.282	0.283	0.916	0.002	0.284	0.280	0.938
	0.5	0.614	0.044	0.244	0.250	0.940	0.011	0.237	0.246	0.924
	0.6	0.458	0.049	0.200	0.220	0.936	0.021	0.201	0.211	0.930
	0.7	0.322	0.053	0.169	0.189	0.930	0.045	0.177	0.179	0.938
	0.8	0.202	0.050	0.144	0.158	0.934	0.037	0.144	0.149	0.958
	0.9	0.096	0.049	0.106	0.123	0.932	0.039	0.104	0.113	0.908

Quantiles for a fix point under different truncation and censoring probability. In the table, bias = $\hat{\theta}_\alpha^{(n)}(t) - \theta_\alpha(t)$, *SE* standard error, *SD* standard deviation, *cov* empirical coverage, *CnsR* censoring rate, *TRUN prob* truncation probability

Table 2 Simulation results when length bias is ignored

t	α	$\theta_\alpha(t)$	Kaplan–Meier			Length bias		
			$\tilde{\theta}_\alpha^{(n)}(t)$	Bias	SE	$\hat{\theta}_\alpha^{(n)}(t)$	Bias	SE
1.2	0.3	1.153	1.679	0.526	0.216	1.158	0.005	0.162
	0.4	0.883	1.345	0.462	0.198	0.892	0.009	0.141
	0.5	0.671	1.051	0.380	0.173	0.676	0.005	0.122
	0.6	0.496	0.805	0.309	0.157	0.505	0.009	0.111
	0.7	0.347	0.589	0.242	0.132	0.354	0.007	0.092
	0.8	0.218	0.380	0.162	0.104	0.228	0.010	0.075
	0.9	0.103	0.192	0.089	0.074	0.110	0.007	0.052
2.5	0.3	1.029	1.207	0.178	0.244	1.019	−0.010	0.212
	0.4	0.800	0.962	0.162	0.209	0.805	0.005	0.189
	0.5	0.614	0.754	0.140	0.187	0.622	0.008	0.170
	0.6	0.458	0.580	0.122	0.169	0.476	0.018	0.146
	0.7	0.322	0.428	0.106	0.139	0.343	0.021	0.126
	0.8	0.202	0.281	0.079	0.115	0.230	0.028	0.109
	0.9	0.096	0.148	0.052	0.087	0.122	0.026	0.079

Quantiles for a fix point when length bias is ignored.‘$\tilde{\theta}_\alpha^{(n)}(t)$’ means the estimate calculated from KM estimator treating observed sample as normal right-censored data. *bias* estimate-true value, *SE* standard error

as clinical studies, this will lead to overoptimistic about a bad disease and may cause serious consequences.

To examine the performance of the proposed two-sample statistics, we perform the third simulation. The first sample is generated same as in Table 1, $\tilde{T}_1 \sim \text{TEXP}(0, 5)$, $\tilde{A}_1 \sim U(0, 5)$, $C_1 \sim U(0, W_1)$, with different censoring rate 0, 19, 48 %. In the second sample, we generate unbiased data $\tilde{T}_2$ from truncated Weibull distribution with scale parameter 1 and shape parameter 0.5, the support is $(0, 5]$. We generate $\tilde{A}_2 \sim U(0, 5)$, $C_2 \sim U(0, W_2)$, and choose different censoring rates 0, 0 and 48 %. We compare the quantile residual lifetime at the same fixed point t and the same quantile α. The simulation results are reported in Table 3. From Table 3, we can see that the bias for the ratio statistic is larger than for the difference statistic. For example, when $t = 1.62$, for $\alpha = 0.5, \ldots, 0.9$, the biases for ratio are larger than 0.2, even reach 0.3 when $\alpha = 0.6$, while the biases for difference are smaller than 0.1. Also, the confidence interval of ratio can contain negative values, see Table 3. We found that it only happens in two situations: t is large or α is large. When t is large, the calculation is limited in the right tail of the distribution. When α is large, $\hat{S}_n(t)$ and $\hat{S}_n(t+\hat{\theta}_\alpha^{(n)}(t))$ are too close to be distinguished in Eq. (3). In both situations, the sample is rare and this leads to large standard deviation. However, to avoid negative value, we can use the similar method in Peng and Fine (2007). We redefine the confidence interval as $[\max\{0, L_\alpha(t)\}, \min\{U_\alpha(t), \rho\}]$, where $[L_\alpha(t), U_\alpha(t)]$ is the original confidence interval and $\rho = \tau_1 \wedge \tau_2$.

In the following, we will study the size and power of the proposed two tests. Jeong (2014) proposed a two-sample test statistic for difference of two QRL estimators and

Table 3 Two-sample case

t	α	Difference			Ratio		
		$d_\alpha(t)$	Bias	CI for $d_\alpha(t)$	$\tau_\alpha(t)$	Bias	CI for $\tau_\alpha(t)$
1.2	0.3	0.682	−0.004	(0.093, 1.271)	1.592	0.003	(1.036, 2.148)
	0.4	0.541	0.004	(−0.007, 1.090)	1.613	0.009	(0.947, 2.279)
	0.5	0.413	−0.005	(−0.114, 0.939)	1.615	−0.008	(0.815, 2.416)
	0.6	0.301	0.011	(−0.183, 0.785)	1.607	0.017	(0.647, 2.566)
	0.7	0.205	0.013	(−0.207, 0.618)	1.591	0.027	(0.417, 2.766)
	0.8	0.124	0.010	(−0.198, 0.447)	1.571	0.026	(0.100, 3.042)
	0.9	0.056	0.012	(−0.179, 0.292)	1.548	0.080	(−0.726, 3.823)
1.62	0.3	0.638	−0.058	(0.096, 1.179)	1.566	−0.075	(0.706, 2.426)
	0.4	0.529	0.056	(−0.005, 1.062)	1.610	0.116	(0.668, 2.552)
	0.5	0.419	0.095	(−0.065, 0.903)	1.635	0.266	(0.709, 2.561)
	0.6	0.316	0.079	(−0.126, 0.758)	1.646	0.312	(0.654, 2.638)
	0.7	0.222	0.052	(−0.179, 0.623)	1.649	0.285	(0.402, 2.897)
	0.8	0.139	0.039	(−0.204, 0.482)	1.647	0.276	(−0.041, 3.334)
	0.9	0.065	0.023	(−0.199, 0.329)	1.640	0.267	(−0.935, 4.215)
2.5	0.3	0.436	−0.020	(−0.293, 1.165)	1.424	−0.023	(0.437, 2.410)
	0.4	0.391	−0.016	(−0.290, 1.072)	1.489	−0.036	(0.297, 2.680)
	0.5	0.330	−0.022	(−0.301, 0.961)	1.537	−0.070	(0.138, 2.937)
	0.6	0.263	−0.018	(−0.275, 0.801)	1.575	−0.093	(−0.013, 3.163)
	0.7	0.194	−0.016	(−0.280, 0.668)	1.604	−0.129	(−0.344, 3.551)
	0.8	0.127	−0.011	(−0.247, 0.501)	1.626	−0.164	(−0.987, 4.239)
	0.9	0.062	−0.010	(−0.202, 0.326)	1.643	−0.287	(−2.411, 5.697)

Simulation results in comparing two residual lifetime distributions by difference and ratio, respectively. '$d_\alpha(t)$' stands for the true value of the difference of two quantile residual lifetimes. '$\hat{d}_\alpha(t)$' stands for the its estimate. 'bias' stands for the bias of the estimator, 'CI for $d_\alpha(t)$' stands for the 95 % confidence interval for $d_\alpha(t)$. The same meaning for '$\tau_\alpha(t)$', '$\hat{\tau}_\alpha(t)$' and 'CI for $\tau_\alpha(t)$'

showed that the limiting distribution follows a standard normal distribution. Jeong et al. (2008) proposed an inference procedure based on ratio of the median residual life and proved that the limiting distribution is a χ^2 distribution with degree of freedom 1. Unfortunately, we cannot directly use their results because the limiting distribution for the QRL under right-censored length-biased data is complicated and the variance does not have a closed form. Instead, we will use a simple bootstrap procedure to study the size and power of the proposed two tests, the detail of the procedure is as follows:

Step 1 Assume that we observe the sample $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_n)$, and the test statistic is $T(X_1, X_2, \ldots, X_m; Y_1, Y_2, \ldots, Y_n)$. Mixed the two samples.

Step 2 Generate a random sample (bootstrap) with sample size $(n + m)$ from the mixed sample in Step 1, denote the bootstrapped sample as $(Z_1^*, \ldots, Z_m^*; Z_{m+1}^*, \ldots, Z_{m+n}^*)$, split the sample into two parts $(Z_1^*, \ldots, Z_m^*)$ and $(Z_{m+1}^*, \ldots, Z_{m+n}^*)$. (In our simulation we just set the first m sample as $Z_1^*, \ldots, Z_m^*$.)

Table 4 Size for the two tests

		α						
		0.3	0.4	0.5	0.6	0.7	0.8	0.9
Difference	*A*	0.046	0.040	0.032	0.040	0.038	0.058	0.042
	B	0.062	0.056	0.046	0.044	0.042	0.036	0.044
	C	0.052	0.042	0.050	0.048	0.046	0.036	0.032
	D	0.046	0.050	0.042	0.052	0.050	0.050	0.050
Ratio	*A*	0.056	0.042	0.042	0.034	0.038	0.052	0.042
	B	0.060	0.064	0.050	0.042	0.038	0.030	0.036
	C	0.050	0.042	0.038	0.040	0.048	0.034	0.024
	D	0.038	0.044	0.030	0.034	0.032	0.038	0.038

A: $n_1 = 100, n_2 = 100, t = 1.62$, censoring rate 19 %. B: $n_1 = 50, n_2 = 100, t = 1.62$, censoring rate 19 %. C: $n_1 = 100, n_2 = 100, t = 2.5$, censoring rate 48 %. D: $n_1 = 50, n_2 = 100, t = 2.5$, censoring rate 48 %

Step 3 Calculate the statistic: $T^* = T(Z_1^*, \ldots, Z_m^*; Z_{m+1}^*, \ldots, Z_{m+n}^*)$.
Step 4 Repeat Steps 2 and 3 for B times to obtain $T_j^*, j = 1, \ldots, B$.
Step 5 Given the observed test statistic $T(X_1, X_2, \ldots, X_m; Y_1, Y_2, \ldots, Y_n)$, reject or accept the null hypothesis based on the distribution of $T_j^*, j = 1, \ldots, B$.

We calculated the size of the two tests under 19 and 48 % censoring rate with failure time distribution following the truncated standard exponential distribution from 0 to 5. And we consider two circumstances: the first one with $n_1 = 100, n_2 = 100$ and the second with $n_1 = 50, n_2 = 100$. The same as Table 1, we set $t = 1.62$ under 19 % censoring rate, $t = 2.5$ under 48 % censoring rate, and the quantiles equal 0.3–0.9 with step size 0.1. The sizes are shown in Table 4. From the table, we can see that under all the circumstances, the sizes are very close to the nominal level 0.05; this means that the proposed two test procedures are all valid.

To study the power, we consider the local alternative, in our simulation, we consider the test problem as

H_0: The failure distribution follows a truncated standard exponential distribution from 0 to 5.
H_1: The distribution for the second sample follows a truncated exponential distribution from 0 to 5, with the mean of the distribution $1/\lambda = 1/(1 + C/\sqrt{n_2})$.

We select four points in Table 4 to plot the power curve, the results are shown in Fig. 1. From the figure, we can see that the ratio inference procedure is more sensitive than the difference, though the confidence interval may contains negative value for ratio way sometimes as we discussed before. One of the referees suggests that the ratio differences are often preferred than the absolute differences in practical settings; our results for the power support this viewpoint.

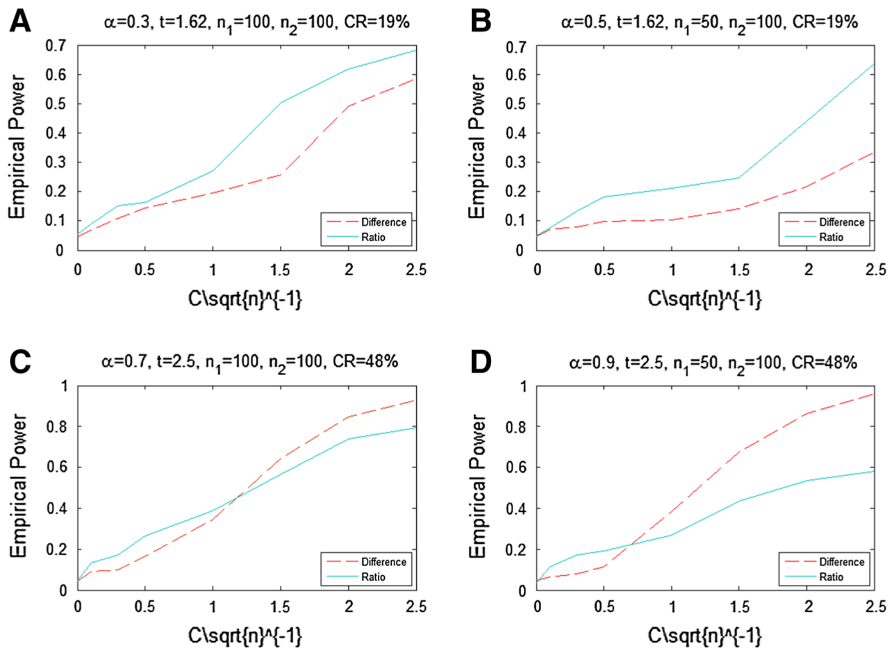

Fig. 1 The power curve for difference and ratio inference procedure

5 Real data examples

In this section, we illustrate the proposed method on two datasets: the Oscard Award data and the Channing House data.

5.1 The Oscars Award data

The Oscars Award data were analyzed by Redelmeier and Singh (2001), Sylvestre et al. (2006) and Han et al. (2011). Their analyses suggest that the Academy Award-winning actors and actresses live longer than the less successful performers. Redelmeier and Singh (2001) considered an interesting question whether winning an Oscar Award (Academy Award) would cause the actors'/actress' expected lifetime to increase among Hollywood actors and actresses. They stated that life expectancy was 3.9 years longer for Oscar Award winners than for other less recognized performers (those only nominated without further received Oscar Award) and that this difference corresponded to a 28 % mortality rate reduction for winners compared to less recognized performers. Sylvestre et al. (2006) pointed out that this analysis suffers from immortal time bias, that is, performers who live longer have more opportunities to win Oscar Awards. They improved the methods of Redelmeier and Singh (2001) by eliminating immortal time bias and stated that winning an Oscar Award had a positive effect on lifetime, but the estimated effect was not significant. However, a more reasonable method is to consider this Academy Award dataset as length-biased data because of immortal time bias (Wolkewitz et al. 2010).

Table 5 Analysis results of Oscars Award data

t	$\hat{\theta}^{(n)}_{0.05}(t)$	$\hat{\theta}^{(n)}_{0.25}(t)$	$\hat{\theta}^{(n)}_{0.5}(t)$	$\hat{\theta}^{(n)}_{0.75}(t)$	$\hat{\theta}^{(n)}_{0.95}(t)$
25	67 (66, 68)	54 (52, 57)	35 (33, 37)	22 (20, 24)	7 (4, 9)
30	63 (61, 63)	50 (47, 52)	31 (30, 32)	19 (17, 21)	6 (4, 8)
35	58 (56, 59)	46 (43, 47)	27 (26, 29)	16 (14, 18)	4 (3, 6)
40	53 (51, 54)	42 (40, 43)	24 (22, 25)	14 (12, 15)	3 (3, 5)
45	48 (46, 49)	38 (37, 39)	20 (18, 22)	11 (10, 12)	2 (1, 4)
50	43 (42, 44)	34 (33, 36)	17 (16, 19)	9 (8, 11)	3 (2, 3)
55	39 (37, 39)	31 (29, 32)	20 (15, 22)	8 (7, 9)	2 (2, 3)
60	34 (33, 36)	27 (26, 28)	20 (17, 21)	6 (6, 7)	2 (1, 2)
63	31 (30, 33)	24 (24, 25)	19 (17, 20)	6 (5, 11)	1(1, 2)
65	29 (28, 33)	23 (22, 24)	18 (16, 19)	10 (5, 12)	1 (1, 2)

Calculated quantile residual lifetime for the nominees of Oscars Award data under quantiles 0.05, 0.25, 0.5, 0.75 and 0.95. 't' stands for a pre-specified 't' years to which those nominees survived. The pair numbers in the square brackets stands for the calculated 95 % confidence interval for each quantile residual lifetimes (in year)

This dataset contains 766 nominees from 1929 to 2000 (72 years), among the 766 nominees, only 238 win the award. The survival time is the time span from the date of birth to death. The censoring rate is about 57.3 %. The time span from the date of birth to nomination is the truncation time denoted by A; length bias occurs because the actors must live long enough to get a nomination (Wolkewitz et al. 2010). Those live longer have more opportunities to win the awards. First we calculate the quantile residual lifetime by the proposed method; the results are given in Table 5.

Table 5 shows that for a fixed year t, the quantile residual lifetime $\hat{\theta}^{(n)}_{\alpha}(t)$ decreases as α increases. And for the same α, as t increases, $\hat{\theta}^{(n)}_{\alpha}(t)$ decreases. Bandos (2007) shows that theoretically $t + \theta_{\alpha}(t)$ is always non-decreasing; in this table, we can see that $t+\hat{\theta}^{(n)}_{\alpha}(t)$ is also increasing as t increases. As expected, these results are consistent with the usual results based on survival analysis.

It is widely recognized that being nominated for an Academy Award is due to talent, however, winning one is due to luck (Redelmeier and Singh 2001). As a result, the following analysis is based on the assumption that winner of each award is selected randomly from nominees (Han et al. 2011). In the next, we will compare the quantile residual lifetime between two groups: one is the group of nonwinning nominees, the other is the group of winners. We use difference and ratio tests developed in Sect. 3, and the results are shown in Table 6.

In Table 6, we denote those awarded as the first sample and nonwinning nominees as the second sample, with sample sizes $n_1 = 238$ and $n_2 = 528$, respectively. We can see that in most situations, the winners survive longer than nonwinning nominees, but the result is not significant; our result is consistent with those obtained by Sylvestre et al. (2006) and Han et al. (2011). The conclusion of insignificant difference is based on the fact that all confidence intervals for difference include zero and all confidence intervals for ratio include 1.

Table 6 Comparing winners and nominees in the Oscar Award data

t	α	Difference			Ratio		
		$\hat{d}_\alpha(t)$	CI for $\hat{d}_\alpha(t)$	SD_1	$\tau_\alpha(t)$	CI for $\hat{\tau}_\alpha(t)$	SD_2
35	0.05	2	(−3.312, 7.312)	2.710	1.035	(0.945, 1.125)	0.046
	0.25	3	(−0.679, 6.679)	1.877	1.064	(0.990, 1.138)	0.038
	0.50	5	(−4.155, 14.155)	4.671	1.200	(0.843, 1.557)	0.182
	0.75	5	(−1.450, 11.450)	3.291	1.357	(0.943, 1.771)	0.211
40	0.05	2	(−3.974, 7.974)	3.048	1.039	(0.927, 1.151)	0.057
	0.25	3	(−0.648, 6.648)	1.861	1.071	(0.987, 1.155)	0.043
	0.50	−2	(−14.813, 10.813)	6.537	0.929	(0.455, 1.403)	0.242
	0.75	3	(−2.204, 8.204)	2.655	1.250	(0.852, 1.648)	0.203
45	0.05	1	(−5.058, 7.058)	3.091	1.021	(0.898, 1.144)	0.063
	0.25	3	(−0.644, 6.644)	1.859	1.079	(0.985, 1.173)	0.048
	0.50	2	(−9.995, 13.995)	6.120	1.074	(0.653, 1.495)	0.215
	0.75	3	(−2.845, 8.845)	2.982	1.300	(0.796, 1.804)	0.257
50	0.05	1	(−5.000, 7.000)	3.061	1.023	(0.886, 1.160)	0.070
	0.25	1	(−2.610, 4.610)	1.842	1.029	(0.925, 1.133)	0.053
	0.50	1	(−8.484, 10.484)	4.839	1.039	(0.686, 1.392)	0.180
	0.75	2	(−4.721, 8.721)	3.429	1.222	(0.577, 1.867)	0.683
55	0.05	1	(−5.876, 7.876)	3.508	1.026	(0.848, 1.204)	0.091
	0.25	0	(−3.552, 3.552)	1.812	1.000	(0.888, 1.112)	0.057
	0.50	1	(−5.515, 7.515)	3.324	1.042	(0.777, 1.307)	0.135
	0.75	0	(−9.853, 9.853)	5.027	1.000	(−0.096, 2.096)	0.559
60	0.05	0	(−9.968, 6.968)	3.555	1.000	(0.794, 1.206)	0.105
	0.25	−1	(−4.695, 2.695)	1.885	0.964	(0.829, 1.099)	0.069
	0.50	1	(−3.336, 5.336)	2.212	1.046	(0.817, 1.275)	0.117
	0.75	−5	(−18.710, 8.710)	6.995	0.667	(−0.693, 2.027)	0.694
65	0.05	2	(−7.008, 11.008)	4.596	1.069	(0.773, 1.365)	0.151
	0.25	0	(−6.198, 6.198)	3.162	1.000	(0.731, 1.269)	0.137
	0.50	3	(−0.904, 6.904)	1.992	1.177	(0.959, 1.395)	0.111
	0.75	3	(−3.991, 9.991)	3.567	1.273	(0.736, 1.810)	0.274

Two-sample comparison between winners and nonwinning nominees in the Oscar Awards data. '$\hat{d}_\alpha(t)$' and '$\hat{\tau}_\alpha(t)$' stand for the estimated difference and ratio of the two quantile residual lifetimes, respectively. 'CI for $\hat{d}_\alpha(t)$' stands for the 95 % confidence interval for $\hat{d}_\alpha(t)$. 'SD_1' stands for standard deviation of $\hat{d}_\alpha(t)$. The same meaning for 'CI for $\hat{\tau}_\alpha(t)$' and 'SD_2'

5.2 The Channing house data

The second dataset is the well-known Channing House Data (Hyde 1980). The goal of this study is to assess the impact of the Channing House medical program on survival. This dataset contains 462 individuals, including 97 men and 365 women. Channing House is a retirement center located in Palo Alto, California. The study began at the

Table 7 Analysis results of Channing House Data

t	$\hat{\theta}_{0.05}^{(n)}(t)$	$\hat{\theta}_{0.25}^{(n)}(t)$	$\hat{\theta}_{0.5}^{(n)}(t)$	$\hat{\theta}_{0.75}^{(n)}(t)$	$\hat{\theta}_{0.95}^{(n)}(t)$
850	322 (289, 350)	230 (215, 238)	170 (162, 179)	94 (84, 102)	41 (23, 45)
900	272 (234, 300)	185 (171, 194)	131 (123, 141)	55 (48, 61)	14 (11, 26)
950	242 (197, 250)	147 (135, 164)	104 (93, 112)	63 (46, 70)	7 (5, 13)
1000	200 (172, 207)	114 (93, 132)	70 (57, 83)	37 (30, 44)	11 (8, 14)
1050	150 (122, 157)	89 (65, 103)	44 (35, 65)	21 (14, 33)	5 (3, 9)
1100	107 (92, 107)	86 (47, 100)	39 (28, 53)	18 (7, 31)	5 (2, 7)
1150	57 (42, 57)	50 (22, 57)	42 (22, 50)	22 (2, 50)	2 (2, 36)

Calculated quantile residual lifetime for Channing House Data under quantiles 0.05, 0.25, 0.5, 0.75 and 0.95. 't' stands for a pre-specified 't' months to which those nominees survived. The pair numbers in the square brackets stand for the calculated 95 % confidence intervals for each quantile residual lifetime (months)

opening of house in 1965 and ended on July 1, 1975. The residents are being recruited and then they lived in the Channing House until the study ends or death, those being recruited are charged a fixed fee every month; the fee covers any medical care they require, except for physical therapy. Those died before 1965 cannot be recruited, thus they are left truncated. Further study indicates that if we select those whose entry age is larger than 786 months (totally 448 individuals, including 94 men and 354 women), then the sub-sample is a length-biased datum (Chen and Zhou 2012).

In this dataset, the current lifetime is from birth to recruitment, and the residual lifetime is from recruitment to death. A large number of participants remain alive at the end of the study so they are right censored. The censoring rate is about 61.9 %. We calculate the quantile residual lifetime using the proposed method, and the results are in Table 7. In this table, we use month as the measurement unit. Just like the results in Table 5, for a fixed t, the residual lifetime $\hat{\theta}_{\alpha}^{(n)}(t)$ decreases with the increase of α. In addition, for the same α, as t increases, $\hat{\theta}_{\alpha}^{(n)}(t)$ decreases but the expected survival time increases. This is probably due to the reason that those living longer are with higher status, actually, a large number of the residents are retired professors or their spouses, and this is a group with lower mortality (Hyde 1980).

We use the difference and ratio tests developed in Sect. 3 to study whether there exist differences in residual lifetime between men and women, with sample sizes $n_1 = 94$ and $n_2 = 354$, respectively. The results are summarized in Table 8. From Table 8, we find that in early age ($t = 900, 950$), men will have chance to survive longer than women. However, when $t \geq 1050$, women survive much longer than men. Most of the results are not significant, this is consistent with former results such as Chen and Zhou (2012).

6 Concluding remarks

We proposed a nonparametric estimator of the quantile residual lifetime with right-censored length-biased data and studied its asymptotic properties. We also applied

Table 8 Comparing men and women in the Channing House data

t	α	Difference			Ratio		
		$\hat{d}_\alpha(t)$	CI for $\hat{d}_\alpha(t)$	SD_1	$\hat{\tau}_\alpha(t)$	CI for $\hat{\tau}_\alpha(t)$	SD_2
900	0.05	−47	(−118.434, 24.434)	36.446	0.836	(0.562, 1.110)	0.140
	0.25	17	(−43.905, 77.905)	31.074	1.101	(0.766, 1.436)	0.171
	0.50	11	(−49.883, 71.883)	31.063	1.117	(0.535, 1.699)	0.297
	0.75	5	(−34.225, 44.225)	20.013	1.093	(0.421, 1.765)	0.343
950	0.05	−39	(−101.679, 23.679)	31.979	0.839	(0.553, 1.125)	0.146
	0.25	9	(−55.688, 73.688)	33.004	1.067	(0.671, 1.463)	0.202
	0.50	34	(−31.491, 99.491)	33.414	1.567	(0.775, 2.359)	0.404
	0.75	−1	(−45.533, 43.533)	22.721	0.971	(−0.336, 2.278)	0.667
1000	0.05	−47	(−99.514, 5.514)	26.793	0.765	(0.430, 1.100)	0.171
	0.25	−4	(−69.501, 61.501)	33.419	0.967	(0.418, 1.516)	0.280
	0.50	7	(−49.713, 63.713)	28.935	1.096	(0.373, 1.819)	0.369
	0.75	9	(−51.147, 69.147)	30.687	1.257	(−0.254, 2.768)	0.771
1050	0.05	−17	(−69.689, 35.689)	26.882	0.687	(0.191, 1.183)	0.253
	0.25	−24	(−99.776, 51.776)	38.661	0.765	(−0.109, 1.639)	0.446
	0.50	−11	(−66.343, 44.343)	28.236	0.800	(−0.398, 1.998)	0.611
	0.75	0	(−42.154, 42.154)	21.507	1.000	(−0.833, 2.833)	0.935
1100	0.05	−54	(−101.060, −6.940)	24.01	0.495	(−0.375, 1.365)	0.444
	0.25	−53	(−111.888, 5.888)	30.045	0.424	(−0.879, 1.727)	0.665
	0.50	−19	(−81.312, 43.312)	31.792	0.596	(−1.340, 2.532)	0.988
	0.75	−12	(−59.203, 35.203)	24.083	0.368	(−5.872, 6.609)	3.184

Two-sample comparison between men and women in the Channing House data. '$\hat{d}_\alpha(t)$' and '$\hat{\tau}_\alpha(t)$' stand for the estimated difference and ratio of the two quantile residual lifetimes, respectively. 'CI for $\hat{d}_\alpha(t)$' stands for the 95 % confidence interval for $\hat{d}_\alpha(t)$. 'SD_1' stands for standard deviation of $\hat{d}_\alpha(t)$. The same meaning for 'CI for $\hat{\tau}_\alpha(t)$' and 'SD_2'

the proposed estimator to two-sample comparison problems. Our simulation studies indicate that the proposed estimator has good performance in finite-sample situations.

There are several problems that can be considered. The proposed estimator is based on seeking the zero solution to (3). Since the nonparametric estimator is not smooth, the numerical solutions to this equation may not be stable with small to moderate sample sizes. A possible remedy is to first smooth the nonparametric estimator $\hat{S}_n(\cdot)$ using the kernel method and uses the smoothed estimator to substitute $\hat{S}_n(t)$ in (3). Another interesting question in the two-sample comparison problems with right-censored length-biased data is to consider a semiparametric model for the underlying population hazard functions. Then, we can derive a score test statistic for testing whether the quantiles of the residual lifetime distributions are equal.

The phenomenon of high status low mortality is commonly encountered in many fields see for example, Smith et al. (1992), Syme and Balfour (1997) and Marmot (2004), among others. In the original data of Oscars Award, data on non-nominees are also collected with a matching procedure to receive a further 'control group', it is beyond the scope to analyze these data by the proposed method. In the future, we intend to mine additional information contained in the Oscar Awards data.

The class of decreasing percentile residual life is denote as DPRL, a random variable $\tilde{T}$ is said to have DPRL(α) if $\theta_\alpha(t)$ is decreasing in t. Recently, Franco-Pereira et al. (2012) and Franco-Pereira and de Uña-Álvarez (2013) considered the properties of the class DPRL(α) and estimated the problem under this monotone restriction in the presence of censoring. The simulation results suggest that under right censoring, using the auxiliary information will lead to a more efficient estimator. By examining the results in Tables 5 and 7, it appears reasonable to assume that they belong to the DPRL(α) class. We can also use this information to construct a more efficient estimator. We hope to study this problem in the future.

7 Appendix

In proving our results, we require several lemmas and assumptions. Let $\psi < \tau$ denote an arbitrary positive number that is close to τ. We require the following assumptions.

Assumption 1 $F(\cdot)$ and $G(\cdot)$ are absolute continuous distribution functions.

Assumption 2 The operator $\mathcal{F}^{-1}$ is Lipschitz continuous.

Assumption 3 For $0 < t < \tau$, $g(t) > 0$, $f(t) > 0$, in addition, $g(t)$ and $f(t)$ are continuous in $[\gamma, \psi]$.

Assumption 4 $\left\{\frac{2\tau}{\int_0^\tau S_c(u)\mathrm{d}u} - 1\right\}\{1 - S_c(\tau)\} < 1$, where $S_c(\cdot)$ is the survival function of the residual censoring time C.

Assumption 5 Suppose there exists some $\gamma > 0$ such that $S(t) = 1$ for $t < \gamma$ and $\mu = \int_0^\tau uf(u)du < \infty$.

Assumption 1 is a common condition to facilitate mathematical calculation when integration and differentiation are involved. Assumption 2 is an unpleasant technical condition. It would be interesting to weaken this condition. Assumption 3 is a regular condition to assure the consistency of the estimator. Assumption 4 essentially requires that the support of censoring distribution contains the support of the failure time distribution. Assumption 5 is a standard condition in studying the nonparametric estimation with length-biased and right-censored data, see for example, Asgharian et al. (2002).

Let

$$
\begin{aligned}
G^*(t) &= P(A + R \le t|\delta = 1), \\
F^*(t) &= P(A + C \le t|\delta = 0), \\
p &= P(\delta = 1) = P(R \le C).
\end{aligned}
$$

Let f^* and g^* be the density function of $F^*(\cdot)$ and $G^*(\cdot)$, that is, $f^*(t) = (\mathrm{d}/\mathrm{d}t)F^*(t)$, $g^*(t) = (\mathrm{d}/\mathrm{d}t)G^*(t)$. Define the linear operator $\mathcal{F}$

$$\begin{aligned}\mathcal{F}(u)(t) = p\int_0^t \frac{g^*(x)}{g(x)}\mathrm{d}u(x)\\ + (1-p)\int_0^t y\left(\int_{y\le z<\infty}\frac{u(z)}{z^2}\mathrm{d}z\right)\mathrm{d}\left\{\left(\frac{h(t)}{h(y)}-1\right)\frac{f^*(y)}{h(y)}\right\},\end{aligned} \tag{10}$$

where $h(t) = \int_t^\tau z^{-1}\mathrm{d}G(z)$. Let $m = \sum_{i=1}^n I(R_i \le C_i) = \sum_{i=1}^n \delta_i$, $k = n - m$. Define

$$W_m^{(1)}(t) = \sqrt{m}\left(\frac{1}{m}\sum_{i=1}^n I(T_i \le t, \delta_i = 1) - G^*(t)\right),$$
$$W_k^{(2)}(t) = \sqrt{k}\left(\frac{1}{k}\sum_{i=1}^n I(T_i \le t, \delta_i = 0) - F^*(t)\right).$$

And

$$\begin{aligned}V_{m,k}(t) = p^{1/2}W_m^{(1)}(t) + (1-p)^{1/2}h(t)\int_{0<z\le t} W_k^{(2)}(z)\mathrm{d}\frac{1}{h(z)}\\ + \left(\frac{p}{1-p}\right)\{G^*(t) - G(t)\}\sqrt{n}(\hat{p} - p).\\ V(t) = p^{1/2}B_1(G^*(t)) + (1-p)^{1/2}h(t)\int_{0<z\le t} B_2(F^*(z))\mathrm{d}\frac{1}{h(z)}\\ + \left(\frac{p}{1-p}\right)\{G^*(t) - G(t)\}Z\end{aligned}$$

is the limiting process of $V_{m,k}(t)$, where B_1 and B_2 are independent Browian bridge processes, independent of $Z \sim N(0,1)$. Furthermore, we define the process U as

$$U(\cdot) = \mathcal{F}^{-1}(V)(\cdot). \tag{11}$$

Before prove the theorem, we present several lemmas.

Lemma 1 *We have the following expansion*

$$\hat{G}_n(t) - G(t) = \mathcal{F}^{-1}\left(\frac{V_{m,k}(t)}{\sqrt{n}}\right) + o_p(n^{-1/2}). \tag{12}$$

Proof It is straightforward result of Asgharian and Wolfson (2005). □

We also need the following lemma, which is Lemma 2.17 in Pakes and Pollard (1989).

Lemma 2 *If $\{f(\cdot,\theta) : \theta \in \Theta\}$ is a Euclidean class with envelop F for which $\int F^2 \mathrm{d}P < \infty$, and if the parameterization is $\mathscr{L}^2(P)$ continuous at θ_0, define $\nu_n = \sqrt{n}(P_n - P)$ as the empirical process, then, for each sequence of positive numbers $\{\delta_n\}$ converging to zero,*

$$\sup_{\|\theta-\theta_0\|<\delta_n} |\nu_n f(\cdot,\theta) - \nu_n f(\cdot,\theta_0)| = o_p(1).$$

Proof Please see Lemma 2.17 in Pakes and Pollard (1989, pp. 1036). □

The lemma below is a variation of the Theorem 1 in Asgharian et al. (2002).

Lemma 3 *Suppose Assumptions 4 and 5 hold. Define:*

$$\mathscr{L}_y(x) = \frac{I_{[y,\infty)}(x) - S(y)}{x}.$$

Then

(1)

$$\sup_{\gamma\le t\le\tau} |\hat{S}_n(t) - S(t)| \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty.$$

(2)

$$\sqrt{n}\left\{\hat{S}_n(t) - S(t)\right\} \xrightarrow{\mathscr{D}} \mu \int_0^\tau \mathscr{L}_t(x)\mathrm{d}U(x),$$

where $U(\cdot)$ is defined in Eq. (11) in the Appendix, $\gamma \le t \le \psi$, and the process on the right side has covariance function:

$$r_S(y,z) = \mu^2 \int_0^\tau \int_0^\tau \psi(s,t)\mathrm{d}\mathscr{L}_y(t)\mathrm{d}\mathscr{L}_z(s)$$

and

$$\psi(s,t) = \mathrm{cov}(U(s), U(t)).$$

Proof Please see Theorem 1 in Asgharian et al. (2002, pp. 203). □

Lemma 4 *Under Assumptions 1–4, for any $\hat{\theta}_\alpha(t)$ satisfies $|\hat{\theta}_\alpha(t) - \theta_\alpha(t)| \le \delta_n$, where $\{\delta_n\}$ is a sequence of positive numbers that approaches to zero, define $\mathcal{G}_n(\cdot) = \sqrt{n}[\hat{G}_n(\cdot) - G(\cdot)]$, $\mathcal{S}_n(\cdot) = \sqrt{n}[\hat{S}_n(\cdot) - S(\cdot)]$, then*

$$\sup_t \sup_{|\hat{\theta}_\alpha(t)-\theta_\alpha(t)|\le\delta_n} |\mathcal{G}_n(t+\hat{\theta}_\alpha(t)) - \mathcal{G}_n(t+\theta_\alpha(t))| = o_p(1),$$

for any $0 \le t < t+\theta_\alpha(t) \le \psi < \tau$. *And under Assumption 5, the following expression also holds*

$$\sup_t \sup_{|\hat{\theta}_\alpha(t)-\theta_\alpha(t)|\le\delta_n} |\mathcal{S}_n(t+\hat{\theta}_\alpha(t)) - \mathcal{S}_n(t+\theta_\alpha(t))| = o_p(1),$$

for $\gamma \le t < t+\theta_\alpha(t) \le \psi < \tau$. *That is, both* $\sqrt{n}[\hat{G}_n(t)-G(t)]$ *and* $\sqrt{n}[\hat{S}_n(t)-S(t)]$ *are asymptotic equicontinuous uniformly on* $[0,\psi]$ *and* $[\gamma,\psi]$*, respectively.*

Proof First, for the sake of convenient, for any known function $b(\cdot)$, we define

$$\|b(\hat{\theta}_\alpha(t)) - b(\theta_\alpha(t))\| = \sup_{|\hat{\theta}_\alpha(t)-\theta_\alpha(t)|\le\delta_n} |b(\hat{\theta}_\alpha(t)) - b(\theta_\alpha(t))|.$$

By Lemma 1 and Assumption 2 $\mathcal{F}^{-1}$ is Lipschitz continuous, for a fixed finite positive number M, we have

$$\begin{aligned}
&\left\| \sqrt{n}\left\{[\hat{G}_n(t+\hat{\theta}_\alpha(t)) - G(t+\hat{\theta}_\alpha(t))] - [\hat{G}_n(t+\theta_\alpha(t)) - G(t+\theta_\alpha(t))]\right\}\right\| \\
&= \left\| \mathcal{F}^{-1}[V_{m,k}(t+\hat{\theta}_\alpha(t)) - V_{m,k}(t+\theta_\alpha(t))]\right\| + o_p(1) \\
&\le M \left\| [V_{m,k}(t+\hat{\theta}_\alpha(t)) - V_{m,k}(t+\theta_\alpha(t))]\right\| + o_p(1).
\end{aligned}$$

As

$$\begin{aligned}
V_{m,k}(t) = {} & p^{1/2} W_m^{(1)}(t) + (1-p)^{1/2} h(t) \int_{0<z\le t} W_k^{(2)}(z) \mathrm{d}\frac{1}{h(z)} \\
& + \left(\frac{p}{1-p}\right) \{G^*(t) - G(t)\} \sqrt{n}(\hat{p} - p).
\end{aligned}$$

So in the following we only need to prove that $\left\|[V_{m,k}(t+\hat{\theta}_\alpha(t)) - V_{m,k}(t+\theta_\alpha(t))]\right\| = o_p(1)$. Notice that

$$\begin{aligned}
&\left\| [V_{m,k}(t+\hat{\theta}_\alpha(t)) - V_{m,k}(t+\theta_\alpha(t))]\right\| \\
&\le p^{1/2} \left\| W_m^{(1)}(t+\hat{\theta}_\alpha(t)) - W_m^{(1)}(t+\theta_\alpha(t))\right\| \\
&\quad + (1-p)^{1/2} \\
&\quad \times \left\| h(t+\hat{\theta}_\alpha(t)) \int_0^{t+\hat{\theta}_\alpha(t)} W_k^{(2)}(z) \mathrm{d}\frac{1}{h(z)} - h(t+\theta_\alpha(t)) \int_0^{t+\theta_\alpha(t)} W_k^{(2)}(z) \mathrm{d}\frac{1}{h(z)}\right\| \\
&\quad + \frac{\sqrt{n}p(\hat{p}-p)}{1-p} \\
&\quad \times \left\| [G^*(t+\hat{\theta}_\alpha(t)) - G(t+\hat{\theta}_\alpha(t))] - [G^*(t+\theta_\alpha(t)) - G(t+\theta_\alpha(t))]\right\| \\
&\triangleq I(t) + II(t) + III(t).
\end{aligned}$$

From Pakes and Pollard (1989), a class of indicator functions is Euclidean, we can know that for any $t \in [0, \tau]$, $I(t) = o_p(1)$. For $III(t)$, since $|\hat{\theta}_\alpha(t) - \theta_\alpha(t)| \le \delta_n$, and $G(\cdot)$, $G^*(\cdot)$ are absolute continuous, then

$$\left\| [G^*(t+\hat{\theta}_\alpha(t)) - G(t+\hat{\theta}_\alpha(t))] - [G^*(t+\theta_\alpha(t)) - G(t+\theta_\alpha(t))] \right\| = \zeta_n$$

for any $t \in [0, \tau]$, where $\zeta_n = O(\delta_n)$. From Asgharian and Wolfson (2005), $\frac{p}{1-p}\sqrt{n}(\hat{p} - p) \xrightarrow{P} p\left(\frac{p}{1-p}\right)^{1/2} Z$, where $Z \sim N(0, 1)$. Then $III(t) = o_p(1)$ uniformly on $[0, \tau]$. Thus to accomplish this proof, we only need to prove $II(t) = o_p(1)$. To see this, notice that

$$\begin{aligned}
II(t) &\le (1-p)^{1/2} \\
&\quad\times \left\| h(t+\hat{\theta}_\alpha(t)) \left[\int_0^{t+\hat{\theta}_\alpha(t)} W_k^{(2)}(z) \mathrm{d}\frac{1}{h(z)} - \int_0^{t+\theta_\alpha(t)} W_k^{(2)}(z) \mathrm{d}\frac{1}{h(z)} \right] \right\| \\
&\quad + (1-p)^{1/2} \left\| \left[h(t+\hat{\theta}_\alpha(t)) - h(t+\theta_\alpha(t)) \right] \int_0^{t+\theta_\alpha(t)} W_k^{(2)}(z) \mathrm{d}\frac{1}{h(z)} \right\| \\
&\triangleq II^{(1)}(t) + II^{(2)}(t).
\end{aligned}$$

Since $h(\cdot)$ is absolute continuous, then $\left\| h(t+\hat{\theta}_\alpha(t)) - h(t+\theta_\alpha(t)) \right\| = \eta_n$, where $\eta_n = O(\delta_n)$. And $\int_0^{t+\theta_\alpha(t)} W_k^{(2)}(z) \mathrm{d}\frac{1}{h(z)}$ is of order $O_p(1)$, then $II^{(2)}(t) = o_p(1)$ for any $t \in [0, \tau]$. For $II^{(1)}(t)$, first since $h(\cdot)$ is a decreasing function, then

$$\begin{aligned}
II^{(1)}(t) &\le (1-p)^{1/2} h(0) \left\| \int_0^{t+\hat{\theta}_\alpha(t)} W_k^{(2)}(z) \mathrm{d}\frac{1}{h(z)} - \int_0^{t+\theta_\alpha(t)} W_k^{(2)}(z) \mathrm{d}\frac{1}{h(z)} \right\| \\
&= (1-p)^{1/2} h(0) \left\| \int_{t+\theta_\alpha(t)}^{t+\hat{\theta}_\alpha(t)} W_k^{(2)}(z) \mathrm{d}\frac{1}{h(z)} \right\| \\
&\le (1-p)^{1/2} h(0) \\
&\qquad \times \left\| W_k^{(2)}(t+\hat{\theta}_\alpha(t)) \frac{1}{h(t+\hat{\theta}_\alpha(t))} - W_k^{(2)}(t+\theta_\alpha(t)) \frac{1}{h(t+\theta_\alpha(t))} \right\| \\
&\qquad + (1-p)^{1/2} h(0) \left\| \int_{t+\theta_\alpha(t)}^{t+\hat{\theta}_\alpha(t)} \frac{1}{h(z)} \mathrm{d}W_k^{(2)}(z) \right\| \\
&\triangleq II^{(1)\prime}(t) + II^{(1)\prime\prime}(t).
\end{aligned}$$

For $t+\theta_\alpha(t)\in[0,\psi]$,

$$\begin{aligned}
&II^{(1)\prime}(t)\\
&= (1-p)^{1/2}h(0)\left\|W_k^{(2)}(t+\hat\theta_\alpha(t))-W_k^{(2)}(t+\theta_\alpha(t))\right\|\frac{1}{h(t+\hat\theta_\alpha(t))}\\
&\quad+(1-p)^{1/2}h(0)\left\|\frac{1}{h(t+\hat\theta_\alpha(t))}-\frac{1}{h(t+\theta_\alpha(t))}\right\|\left\|W_k^{(2)}(t+\theta_\alpha(t))\right\|\\
&\le (1-p)^{1/2}h(0)\left\|W_k^{(2)}(t+\hat\theta_\alpha(t))-W_k^{(2)}(t+\theta_\alpha(t))\right\|\frac{1}{h(\psi)}\\
&+(1-p)^{1/2}h(0)\left\|\frac{1}{h(t+\hat\theta_\alpha(t))}-\frac{1}{h(t+\theta_\alpha(t))}\right\|\left\|W_k^{(2)}(t+\theta_\alpha(t))\right\|.
\end{aligned}$$

Since $\left\|W_k^{(2)}(t+\hat\theta_\alpha(t))-W_k^{(2)}(t+\theta_\alpha(t))\right\|=o_p(1)$, $\|W_k^{(2)}(t+\theta_\alpha(t))\|=O_p(1)$, and

$$\left\|\frac{1}{h(t+\hat\theta_\alpha(t))}-\frac{1}{h(t+\theta_\alpha(t))}\right\|=\phi_n,$$

where $\phi_n=O(\delta_n)$. Then $II^{(1)\prime}(t)=o_p(1)$. As for $II^{(1)\prime\prime}(t)$, we need to employ Lemma 2, here the class is $\{\frac{1}{h(z)}, z\in[0,\psi]\}$, this class is Euclidean class and has an envelope $\frac{1}{h(\psi)}$, obviously $\int\frac{1}{h(\psi)^2}\mathrm{d}P<\infty$, then the conditions in Lemma 2 is satisfied. Thus $II^{(1)\prime\prime}(t)=o_p(1)$ uniformly on $[0,\psi]$.

From above, we have proved that $\left\|[V_{m,k}(t+\hat\theta_\alpha(t))-V_{m,k}(t+\theta_\alpha(t))]\right\|=o_p(1)$, thus for sequence $\delta_n\downarrow 0$, and for any $0\le t<t+\hat\theta_\alpha(t)\le\psi$, $0\le t<t+\theta_\alpha(t)\le\psi$, the following equation holds

$$\sup_t\sup_{|\hat\theta_\alpha(t)-\theta_\alpha(t)|\le\delta_n}|\mathcal{G}_n(t+\hat\theta_\alpha(t))-\mathcal{G}_n(t+\theta_\alpha(t))|=o_p(1).$$

Since $S(t)=\mu\int_t^\infty z^{-1}\mathrm{d}G(z)$, $\hat S_n(t)=\mu\int_t^\infty z^{-1}\mathrm{d}\hat G_n(z)$, for any $\gamma\le t<t+\hat\theta_\alpha(t)\le\psi$, $\gamma\le t<t+\theta_\alpha(t)\le\psi$:

$$\begin{aligned}
&\left\|[\hat S_n(t+\hat\theta_\alpha(t))-S(t+\hat\theta_\alpha(t))]-[\hat S_n(t+\theta_\alpha(t))-S(t+\theta_\alpha(t))]\right\|\\
&=\mu\left\|\int_{t+\theta_\alpha(t)}^{t+\hat\theta_\alpha(t)}\frac{1}{z}\mathrm{d}\left[\hat G_n(z)-G(z)\right]\right\|\\
&\le\mu\gamma^{-1}\left\|[\hat G_n(t+\hat\theta_\alpha(t))-G(t+\hat\theta_\alpha(t))]-[\hat G_n(t+\theta_\alpha(t))-G(t+\theta_\alpha(t))]\right\|
\end{aligned}$$

thus for sequence $\delta_n\downarrow 0$, the following equation holds

$$\sup_t\sup_{|\hat\theta_\alpha(t)-\theta_\alpha(t)|\le\delta_n}|\mathcal{S}_n(t+\hat\theta_\alpha(t))-\mathcal{S}_n(t+\theta_\alpha(t))|=o_p(1).$$

Thus the proof is complete for Lemma 4. $\square$

Lemma 5 *Define:*

$$\mathscr{L}_y(x) = \frac{I_{[y,\infty)}(x) - S(y)}{x}.$$

Suppose Assumptions 4 and 5 hold. $\theta_\alpha(t)$ is solved from Eq. (2) in Sect. 2. Then, as $n \to \infty$, we have

(1)

$$\sup_{\gamma \le t \le \tau} |[\hat{S}_n(t + \theta_\alpha(t)) - S(t + \theta_\alpha(t))] - \alpha[\hat{S}_n(t) - S(t)]| \xrightarrow{a.s.} 0$$

(2)

$$\begin{aligned} &\sqrt{n}\left\{[\hat{S}_n(t + \theta_\alpha(t)) - S(t + \theta_\alpha(t))] - \alpha[\hat{S}_n(t) - S(t)]\right\} \\ &\xrightarrow{\mathscr{D}} \mu \int_0^\tau [\mathscr{L}_{t+\theta_\alpha(t)}(x) - \alpha \mathscr{L}_t(x)] \mathrm{d}U(x), \end{aligned}$$

where $U(\cdot)$ is defined in Eq. (11) in the Appendix, $\gamma \le t < t + \theta_\alpha(t) \le \psi$, and the process on the right side has covariance function:

$$\begin{aligned} &r_1(y, z) \\ &= \mu^2 \int_0^\tau \int_0^\tau \psi(s, t) \mathrm{d}\mathscr{L}_{y+\theta_\alpha(y)}(t) \mathrm{d}\mathscr{L}_{z+\theta_\alpha(z)}(s) \\ &\quad - \mu^2 \alpha \int_0^\tau \int_0^\tau \psi(s, t) \mathrm{d}\mathscr{L}_{y+\theta_\alpha(y)}(t) \mathrm{d}\mathscr{L}_z(s) \\ &\quad - \mu^2 \alpha \int_0^\tau \int_0^\tau \psi(s, t) \mathrm{d}\mathscr{L}_y(t) \mathrm{d}\mathscr{L}_{z+\theta_\alpha(z)}(s) \\ &\quad + \mu^2 \alpha^2 \int_0^\tau \int_0^\tau \psi(s, t) \mathrm{d}\mathscr{L}_y(t) \mathrm{d}\mathscr{L}_z(s) \end{aligned}$$

and

$$\psi(s, t) = \mathrm{cov}(U(s), U(t)).$$

Proof (1) This lemma is an easy extension of Lemma 3, so we only give a sketch of the proof, for the details, please refer to Asgharian et al. (2002). Since

$$\begin{aligned} &\sup_{\gamma \le t \le \tau} |[\hat{S}_n(t + \theta_\alpha(t)) - S(t + \theta_\alpha(t))] - \alpha[\hat{S}_n(t) - S(t)]| \\ &\le \sup_{\gamma \le t \le \tau} |\hat{S}_n(t + \theta_\alpha(t)) - S(t + \theta_\alpha(t))| + \alpha \sup_{\gamma \le t \le \tau} |\hat{S}_n(t) - S(t)| \end{aligned}$$

From Lemma 3, we can know that the results holds indeed.

(2) As for the part 2, the proof is an analog of Asgharian et al. (2002), but in their paper there was a minor error, and this can be fixed if we change the operator $P(g) = \int_\gamma^\tau K_y(x)g(x)\mathrm{d}x$ to $Q(g) = -g(y)/y + \int_\gamma^\tau K_y(x)g(x)\mathrm{d}x$, then

$$\begin{aligned}\|Q(g)\| = \sup_g |Q(g)| &\leq \int_\gamma^\tau \frac{|g(x)|}{x^2}\mathrm{d}x + \frac{|g(y)|}{y} \\ &\leq -\|g\|_\infty \int_\gamma^\tau \mathrm{d}\frac{1}{x} + \frac{1}{y}\|g(y)\| = \frac{2}{\gamma}\|g\|_\infty.\end{aligned}$$

Where $\|g\|_\infty = \sup_t |g(t)|$. Consequently, the $Q(g)$ is also a bounded linear operator. So we can also use the same method in Theorem 1 in Asgharian et al. (2002) to obtain the results we desire, that is:

$$\begin{aligned}&\sqrt{n}\left\{[\hat{S}_n(t+\theta_\alpha(t)) - S(t+\theta_\alpha(t))] - \alpha[\hat{S}_n(t) - S(t)]\right\} \\ &\xrightarrow{\mathscr{D}} \mu \int_0^\tau [\mathscr{L}_{t+\theta_\alpha(t)}(x) - \alpha\mathscr{L}_t(x)]\mathrm{d}U(x).\end{aligned}$$

Concerning the covariance calculation, notice the fact that integrand $[\mathscr{L}_{t+\theta_\alpha(t)}(x) - \alpha\mathscr{L}_t(x)]$ is a function unrelated with the process U, thus we can easily calculate the $r_1(y, z)$ by directly using the results of Lemma 3. Consequently, we complete the proof. □

Proof of Theorem 1: We know that

$$\begin{aligned}0 &= \hat{M}_n(\hat{\theta}_\alpha^{(n)}(t)) - M(\theta_\alpha(t)) \\ &= \hat{S}_n(t+\hat{\theta}_\alpha^{(n)}(t)) - \hat{S}_n(t+\theta_\alpha(t)) + \hat{S}_n(t+\theta_\alpha(t)) - S(t+\theta_\alpha(t)) \\ &\quad - \alpha[\hat{S}_n(t) - S(t)]\end{aligned} \tag{13}$$

where $\theta_\alpha(t)$ and $\hat{\theta}_\alpha^{(n)}(t)$ are solved from Eqs. (2) and (3) in Sect. 2, i.e. $\hat{M}_n(\hat{\theta}_\alpha^{(n)}(t)) = 0$ and $M(\theta_\alpha(t)) = 0$, respectively. Because $\hat{S}_n(\cdot)$ is uniformly consistent, so

$$\begin{aligned}\hat{S}_n(t+\hat{\theta}_\alpha^{(n)}(t)) &= S(t+\hat{\theta}_\alpha^{(n)}(t)) + o_p(1), \\ \hat{S}_n(t+\theta_\alpha(t)) &= S(t+\theta_\alpha(t)) + o_p(1)\end{aligned}$$

uniformly holds for $\gamma \leq t < t+\theta_\alpha(t) \leq \psi < \tau$. Substituting then into Eq. (13), we have that

$$\begin{aligned}&[S(t+\hat{\theta}_\alpha^{(n)}(t))] - S(t+\theta_\alpha(t))] + [\hat{S}_n(t+\theta_\alpha(t)) - S(t+\theta_\alpha(t))] \\ &-\alpha[\hat{S}_n(t) - S(t)] = o_p(1)\end{aligned} \tag{14}$$

uniformly holds for $\gamma \leq t < t+\theta_\alpha(t) \leq \psi < \tau$. In Assumption 1 we have assumed that $S(\cdot) = 1 - F(\cdot)$ is absolute continuous, from the mean value theorem in basic mathematical analysis, and there exists a $\theta_\alpha^*(t)$ between $\theta_\alpha(t)$ and $\hat{\theta}_\alpha^{(n)}(t)$, such that

$$\begin{aligned}
&S(t+\hat{\theta}_\alpha^{(n)}(t)) - S(t+\theta_\alpha(t)) \\
&\quad = -f(t+\theta_\alpha^*(t))(\hat{\theta}_\alpha^{(n)}(t) - \theta_\alpha(t)).
\end{aligned} \tag{15}$$

Substituting (15) into (14), we have

$$\begin{aligned}
&f(t+\theta_\alpha^*(t))(\hat{\theta}_\alpha^{(n)}(t) - \theta_\alpha(t)) \\
&\quad = [\hat{S}_n(t+\theta_\alpha(t)) - S(t+\theta_\alpha(t))] - \alpha[\hat{S}_n(t) - S(t)] + o_p(1).
\end{aligned}$$

Thus

$$\begin{aligned}
&\hat{\theta}_\alpha^{(n)}(t) - \theta_\alpha(t) \\
&\quad = \frac{1}{f(t+\theta_\alpha^*(t))}\left\{[\hat{S}_n(t+\theta_\alpha(t)) - S(t+\theta_\alpha(t))] - \alpha[\hat{S}_n(t) - S(t)] + o_p(1)\right\}.
\end{aligned}$$

In Assumption 3 we assume $f(t), g(t) > 0$ in the interval $(0, \tau)$, thus combined with the consistency of $\hat{S}_n(t)$, the estimator $\hat{\theta}_\alpha^{(n)}(t)$ is pointwise a.s. convergence in $(0, \tau)$. Furthermore, if $f(\cdot)$ is continuous, then there exists a positive constant λ, such that for all $t \in [\gamma, \psi]$, we have $f(t) > \lambda$, so

$$\begin{aligned}
&\sup_t |\hat{\theta}_\alpha^{(n)}(t) - \theta_\alpha(t)| \\
&\quad \le \frac{1}{\lambda}\sup_t \left|\left\{[\hat{S}_n(t+\theta_\alpha(t)) - S(t+\theta_\alpha(t))] - \alpha[\hat{S}_n(t) - S(t)] + o_p(1)\right\}\right|.
\end{aligned}$$

As a result, the pointwise a.s. convergence can be strengthened to uniformly a.s. convergence on $[\gamma, \psi]$. □

Proof of Theorem 2: Since $\hat{\theta}_\alpha^{(n)}(t)$ is a uniformly consistency estimator of $\theta_\alpha(t)$ in $[\gamma, \psi]$, it follows from Lemma 2 that

$$\sqrt{n}\left[\hat{S}_n(t+\hat{\theta}_\alpha^{(n)}(t)) - \hat{S}_n(t+\theta_\alpha(t)) - S(t+\hat{\theta}_\alpha^{(n)}(t)) + S(t+\theta_\alpha(t))\right] = o_p(1)$$

and applying this formula, we have

$$\begin{aligned}
&\sqrt{n}\left[\hat{M}_n(\hat{\theta}_\alpha^{(n)}(t)) - \hat{M}_n(\theta_\alpha(t))\right] \\
&\quad = \sqrt{n}\left[\hat{S}_n(t+\hat{\theta}_\alpha^{(n)}(t)) - \alpha\hat{S}_n(t) - \hat{S}_n(t+\theta_\alpha(t)) + \alpha\hat{S}_n(t)\right] \\
&\quad = \sqrt{n}\left[\hat{S}_n(t+\hat{\theta}_\alpha^{(n)}(t)) - \hat{S}_n(t+\theta_\alpha(t)) - S(t+\hat{\theta}_\alpha^{(n)}(t)) + S(t+\theta_\alpha(t))\right] \\
&\qquad + \sqrt{n}\left[S(t+\hat{\theta}_\alpha^{(n)}(t)) - S(t+\theta_\alpha(t))\right] \\
&\quad = \sqrt{n}\left[S(t+\hat{\theta}_\alpha^{(n)}(t)) - S(t+\theta_\alpha(t))\right] + o_p(1) \\
&\quad = f(t+\theta_\alpha(t)) \times \sqrt{n}\left[\hat{\theta}_\alpha^{(n)}(t) - \theta_\alpha(t)\right] + o_p(1)
\end{aligned}$$

uniformly on $[\gamma, \psi]$. Divide the last equation by $f(t+\theta_\alpha(t))$ on the both sides, we obtain

$$\begin{aligned}
&\sqrt{n}\left[\hat{\theta}_\alpha^{(n)}(t) - \theta_\alpha(t)\right] \\
&= \frac{1}{f(t+\theta_\alpha(t))} \times \left\{\sqrt{n}\left[\hat{M}_n(\hat{\theta}_\alpha^{(n)}(t)) - \hat{M}_n(\theta_\alpha(t))\right] + o_p(1)\right\}.
\end{aligned}$$

Note that $\theta_\alpha(t)$ and $\hat{\theta}_\alpha^{(n)}(t)$ are solved from Eqs. (2) and (3), then

$$\hat{M}_n(\hat{\theta}_\alpha^{(n)}(t)) - \hat{M}_n(\theta_\alpha(t)) = M(\theta_\alpha(t)) - \hat{M}_n(\theta_\alpha(t)).$$

Therefore, we obtain that

$$\begin{aligned}
&= \frac{1}{f(t+\theta_\alpha(t))} \times \left\{\sqrt{n}\left[M(\theta_\alpha(t)) - \hat{M}_n(\theta_\alpha(t))\right] + o_p(1)\right\} \\
&= \frac{1}{f(t+\theta_\alpha(t))} \\
&\quad \times \left\{\sqrt{n}\left[(\hat{S}_n(t+\theta_\alpha(t)) - S(t+\theta_\alpha(t))) - \alpha(\hat{S}_n(t) - S(t))\right] + o_p(1)\right\}.
\end{aligned}$$

Then by direct use of functional delta method and Lemma 5, the proof is complete.

□

Proof of Theorems 3 and 4 Note that

$$\begin{aligned}
&\sqrt{n_1+n_2}(\hat{\tau}_\alpha(t) - \tau_\alpha(t)) \\
&= \sqrt{n_1+n_2}\left\{\frac{\hat{S}_{1,n_1}^{-1}(\alpha\hat{S}_{1,n_1}(t)) - t}{\hat{S}_{2,n_2}^{-1}(\alpha\hat{S}_{2,n_2}(t)) - t} - \frac{S_1^{-1}(\alpha S_1(t)) - t}{S_2^{-1}(\alpha S_2(t)) - t}\right\} \\
&= \sqrt{n_1+n_2}\left\{\frac{\hat{S}_{1,n_1}^{-1}(\alpha\hat{S}_{1,n_1}(t)) - S_1^{-1}(\alpha S_1(t))}{\hat{S}_{2,n_2}^{-1}(\alpha\hat{S}_{2,n_2}(t)) - t}\right\} \\
&\quad + \sqrt{n_1+n_2}\left\{\frac{S_1^{-1}(\alpha S_1(t)) - t}{\hat{S}_{2,n_2}^{-1}(\alpha\hat{S}_{2,n_2}(t)) - t} - \frac{S_1^{-1}(\alpha S_1(t)) - t}{S_2^{-1}(\alpha S_2(t)) - t}\right\} \\
&= \sqrt{n_1+n_2}\left\{\frac{\hat{S}_{1,n_1}^{-1}(\alpha\hat{S}_{1,n_1}(t)) - S_1^{-1}(\alpha S_1(t))}{\hat{S}_{2,n_2}^{-1}(\alpha\hat{S}_{2,n_2}(t)) - t}\right\} \\
&\quad + \frac{S_1^{-1}(\alpha S_1(t)) - t}{S_2^{-1}(\alpha S_2(t)) - t}\sqrt{n_1+n_2}\left\{\frac{S_2^{-1}(\alpha S_2(t)) - \hat{S}_{2,n_2}^{-1}(\alpha\hat{S}_{2,n_2}(t))}{\hat{S}_{2,n_2}^{-1}(\alpha\hat{S}_{2,n_2}(t)) - t}\right\} \\
&= \sqrt{n_1+n_2}\frac{\hat{S}_{1,n_1}^{-1}(\alpha\hat{S}_{1,n_1}(t)) - S_1^{-1}(\alpha S_1(t))}{\hat{S}_{2,n_2}^{-1}(\alpha\hat{S}_{2,n_2}(t)) - t}
\end{aligned}$$

$$+\tau(t)\sqrt{n_1+n_2}\frac{S_2^{-1}(\alpha S_2(t))-\hat{S}_{2,n_2}^{-1}(\alpha\hat{S}_{2,n_2}(t))}{\hat{S}_{2,n_2}^{-1}(\alpha\hat{S}_{2,n_2}(t))-t}$$
$$=\frac{1}{\hat{\theta}_{2,\alpha}^{(n_2)}(t)}\left\{\sqrt{n_1+n_2}[\hat{\theta}_{1,\alpha}^{(n_1)}(t)-\theta_{1,\alpha}(t)]-\tau(t)\sqrt{n_1+n_2}[\hat{\theta}_{2,\alpha}^{(n_2)}(t)-\theta_{2,\alpha}(t)]\right\}.$$

Note that we have

$$\hat{S}_{i,n_i}^{-1}(\alpha\hat{S}_{i,n_i}(t))-t\xrightarrow{P}\theta_{i,\alpha}(t),$$

where $i=1,2$. By Theorem 2 and Slutsky Theorem, we can directly obtain the results.

Finally, since $d_\alpha(\cdot)=\theta_{1,\alpha}(\cdot)-\theta_{2,\alpha}(\cdot)$ and $\hat{d}_\alpha(\cdot)=\hat{\theta}_{1,\alpha}^{(n_1)}(\cdot)-\hat{\theta}_{2,\alpha}^{(n_2)}(\cdot)$, and that the two samples are independent, Theorem 4 follows directly from Theorem 2. □

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