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Maximum likelihood estimator for the sub-fractional Brownian motion approximated by a random walk

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Abstract We estimate the drift parameter in a simple linear model driven by sub-fractional Brownian motion. We construct a maximum likelihood estimator (MLE) for the drift parameter by using a random walk approximation of the sub-fractional Brownian motion and study the asymptotic behaviors of the estimator. Simulations confirm the theoretical results and indicate superiority of the new proposed estimator.

Keywords Maximum likelihood estimator · Sub-fractional Brownian motion · Random walk

1 Introduction

The self-similar processes are of interest for various applications, such as economics, internet traffic or hydrology. The sub-fractional Brownian motion besides fractional Brownian motion is the usual candidate to model phenomena in which the self-similarity property can be observed from the empirical data. Recall that the sub-fractional Brownian motion $(S_t^H)_{t\in[0,T]}$ is a centered Gaussian process with the covariance function

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$$C_H(s,t) = \mathbf{E}(S_s^H S_t^H) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right],$$
$$t \geq 0, s \geq 0, H \in (0,1). \tag{1}$$

The sub-fractional Brownian motion has properties analogous to those of fractional Brownian motion (self-similarity, long- range dependence, Hölder paths), and satisfies the following estimates:

$$\left[\left(2 - 2^{2H-1}\right) \wedge 1\right] |t-s|^{2H} \leq \mathbf{E}|S_t^H - S_s^H|^2 \leq \left[\left(2 - 2^{2H-1}\right) \vee 1\right] |t-s|^{2H}.$$

The main properties of a sub-fractional Brownian motion were studied by Bojdecki et al. (2004).

The stochastic analysis of the sub-fractional Brownian motion naturally led to the statistical inference for diffusion processes with the sub-fractional Brownian motion as the driving noise. For the problem of the estimation of the drift parameter in the model

$$\mathrm{d}X_t = \theta b(X_t)\mathrm{d}t + \mathrm{d}W_t^H, \quad t \in [0,T], \tag{2}$$

where $(W_t^H)_{t\in[0,T]}$ is a fractional Brownian motion with a Hurst index $H \in (0,1)$ and b is a deterministic function satisfying some regularity conditions, and assume that the parameter $\theta \in \mathbf{R}$ has to be estimated. Such questions have been treated in several papers (see Le Breton 1998; Kleptsyna and Le Breton 2002; Prakasa Rao 2008 for the case $H \in (\frac{1}{2}, 1)$ and b linear or Tudor and Viens 2007 for the general case or Sottinen and Tudor 2008 for the two-parameter case): in general, the techniques used to construct MLE for the drift parameter θ are based on Girsanov transforms for fractional Brownian motion and depend on the properties of the deterministic fractional operators related to the fractional Brownian motion. Another possibility is to use Euler-type approximations of the above equation and to construct a MLE based on the density of the observations given "the past", as in e.g. Prakasa Rao (1999), Section 3.4.

In this work, our purpose is to discuss the simple linear model

$$X_t = \theta t + S_t^H, \tag{3}$$

where $\left(S_t^H\right)_{t\in[0,T]}$ is a sub-fractional Brownian motion with a Hurst index $H \in (\frac{1}{2}, 1)$ and $\theta \in \mathbf{R}$ is the parameter to be estimated. We consider an approximated model in which we replace the noise S^H by a disturbed random walk $S^{H,N}$. We will prove that $S^{H,N}$ converges weakly in the Skorohod topology to S^H as $N \to \infty$. Note that this approximated model still keeps the main properties of the original model since the noise is asymptotically self-similar and it exhibits long-range dependence. We then construct a MLE estimator using a Euler scheme method and we prove that this estimator is L^p-consistent ($p \geq 1$) and strong consistent.

Note that the approaches previously used in the literature (Kleptsyna and Le Breton 2002; Tudor and Viens 2007) to treat the problem of drift parameter estimation in models such as (2), are based on the following idea: first, one constructs an estimator (via Girsanov transform) in the continuous time model and then the estimator is discretized. We propose here an alternative point of view: first we discretize the equation and then we construct the estimator.

Our technics used in this work are inspired from Bertin et al. (2009, 2011) and Sottinen (2001). Our paper is organized as follows. In Sect. 2, we recall some facts on the MLE for the drift parameter in the model driven by sub-fractional Brownian motion. In Sect. 3, we introduce a statistical model driven by a disturbed random walk that converges weakly to the sub-fractional Brownian motion. We construct an estimator for the drift parameter and we prove its L^p-consistency ($p \geq 1$) and strong consistency under the condition $\alpha > 1$ where N^α is the number of observations at our disposal and the step of the Euler scheme is $\frac{1}{N}$. This condition extends the usual hypothesis in the standard Wiener case (see Prakasa Rao 1999, paragraph 3.4). Section 4 is devoted to comparison of three parameter estimators by numerical simulations which illustrate the efficiency of the proposed estimator.

2 Preliminaries

Let us start by recalling some known facts on maximum likelihood estimation in the simple case. Let $(S_t^H)_{t\in[0,T]}$ be a sub-fractional Brownian motion with a Hurst parameter $H \in (\frac{1}{2}, 1)$ and let us consider the simple model (3).

Recall that the sub-fractional Brownian process $(S_t^H)_{t\in[0,T]}$ with a Hurst parameter $H \in (\frac{1}{2}, 1)$ can be written (see, Mendy 2013)

$$S_t^H = c(H)\int_0^t n_H(t,s)\mathrm{d}W_s,$$

where $(W_t, t \in [0,T])$ is a standard Wiener process,

$$n_H(t,s) = \frac{2^{1-H}\sqrt{\pi}s^{\frac{3}{2}-H}}{\Gamma(H-\frac{1}{2})}\left(\int_s^t (x^2-s^2)^{H-\frac{3}{2}}\mathrm{d}x\right)\mathbf{I}_{(0,t)}(s), \tag{4}$$

and

$$c^2(H) = \frac{\Gamma(1+2H)\sin(\pi H)}{\pi}.$$

For $t > s$, we have

$$\frac{\partial n_H}{\partial t}(t,s) = \frac{2^{1-H}\sqrt{\pi}s^{\frac{3}{2}-H}}{\Gamma\left(H-\frac{1}{2}\right)}\left(t^2-s^2\right)^{H-\frac{3}{2}}.$$

In the model (3), we aim to estimate the drift parameter θ by assuming that H is known and on the basis on discrete observations $X_1, \ldots, X_{N^\alpha}$ (the condition on α will be clarified later). We use the Euler type method with $t_j = \frac{j}{N}$ and we denote $X_{t_j} = X_j$.

We can easily find the following expression for the observations X_j, $j = 1, \ldots, N^\alpha$,

$$X_j = j\frac{\theta}{N} + S^H_{\frac{j}{N}}.$$

We need to compute the density of the vector $(X_1, \dots, X_{N^\alpha})$. Since the covariance matrix of this vector is given by $\Sigma = (\Sigma_{ij})_{i,j=1,\dots,N^\alpha}$ with $\Sigma_{ij} = \mathrm{Cov}\left(S^H_{\frac{i}{N}}, S^H_{\frac{j}{N}}\right)$, the density of $(X_1, \dots, X_{N^\alpha})$ will be given by

$$(2\pi)^{-\frac{N^\alpha}{2}} \frac{1}{\sqrt{\det \Sigma}} \exp\left(-\frac{1}{2}\left(x_1 - \frac{\theta}{N}, \dots, x_{N^\alpha} - N^\alpha \frac{\theta}{N}\right)' \times \Sigma^{-1}\left(x_1 - \frac{\theta}{N}, \dots, x_{N^\alpha} - N^\alpha \frac{\theta}{N}\right)\right),$$

and by maximizing the above expression with respect to the variable θ, we obtain the following MLE estimator

$$\widetilde{\theta}_N = N \frac{\sum_{i,j=1}^{N^\alpha} (j \Sigma_{ij}^{-1} X_i)}{\sum_{i,j=1}^{N^\alpha} (ij \Sigma_{ij}^{-1})}, \tag{5}$$

where the Σ_{ij}^{-1} are the coordinates of the matrix Σ^{-1}. Then,

$$\widetilde{\theta}_N - \theta = N \frac{\sum_{i,j=1}^{N^\alpha} \left(j \Sigma_{ij}^{-1} S^H_{\frac{i}{N}}\right)}{\sum_{i,j=1}^{N^\alpha} \left(ij \Sigma_{ij}^{-1}\right)}. \tag{6}$$

Thus,

$$\mathbf{E}|\widetilde{\theta}_N - \theta|^2 = N^2 \frac{\sum_{i,j,k,l=1}^{N^\alpha} \left(jl \Sigma_{ij}^{-1} \Sigma_{kl}^{-1} \mathbf{E}\left(S^H_{\frac{i}{N}} S^H_{\frac{k}{N}}\right)\right)}{\left(\sum_{i,j=1}^{N^\alpha} \left(ij \Sigma_{ij}^{-1}\right)\right)^2} = N^2 \frac{\sum_{i,j,k,l=1}^{N^\alpha} \left(jl \Sigma_{ij}^{-1} \Sigma_{kl}^{-1} \Sigma_{ik}\right)}{\left(\sum_{i,j=1}^{N^\alpha} \left(ij \Sigma_{ij}^{-1}\right)\right)^2}.$$

Note that

$$\begin{aligned}
\sum_{i,j,k,l=1}^{N^\alpha} \left(jl \Sigma_{ij}^{-1} \Sigma_{kl}^{-1} \Sigma_{ik}\right) &= \sum_{j,k,l=1}^{N^\alpha} \left(jl \Sigma_{kl}^{-1} \left(\sum_{i=1}^{N^\alpha} \Sigma_{ij}^{-1} \Sigma_{ik}\right)\right) \\
&= \sum_{j,k,l=1}^{N^\alpha} \left(jl \Sigma_{kl}^{-1} \delta_{jk}\right) \\
&= \sum_{j,l=1}^{N^\alpha} \left(jl \Sigma_{jl}^{-1}\right),
\end{aligned}$$

and consequently

$$\begin{aligned}
\mathbf{E}|\widetilde{\theta}_N - \theta|^2 &= \frac{N^2}{\sum_{i,j=1}^{N^\alpha} \left(ij \Sigma_{ij}^{-1}\right)} \\
&= \frac{N^{2-2H}}{\sum_{i,j=1}^{N^\alpha} \left(ij m_{ij}^{-1}\right)},
\end{aligned}$$

where m_{ij}^{-1} are the coefficients of the matrix M^{-1} with $M = (m_{ij})_{i,j=1,\dots,N^\alpha}$, $m_{ij} = i^{2H} + j^{2H} - \frac{1}{2}[(i+j)^{2H} + |i-j|^{2H}]$. Let x be the vector $(1, 2, \dots, N^\alpha) \in \mathbf{R}^{N^\alpha}$. We use the inequality

$$x' M^{-1} x \geq \frac{||x||_2^2}{\lambda_{\max}},$$

where $\lambda_{\max}$ is the largest eigenvalue of the matrix M. Thus, we have

$$\mathbf{E}|\widetilde{\theta}_N - \theta|^2 \leq \frac{N^{2-2H}\lambda_{\max}}{||x||_2^2}.$$

Since $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$, we know that $||x||_2^2 \approx \frac{1}{3}N^{3\alpha}$. On the other hand, by the Gerschgorin circle theorem (see Golub and van Loan 1996, Theorem 8.1.3), we have

$$\lambda_{\max} \leq \max_{i=1,2,\dots,N^\alpha} \sum_{j=1}^{N^\alpha} |m_{ij}| \leq C_1 N^{\alpha(2H+1)},$$

with C_1 a positive constant. Finally,

$$\mathbf{E}|\widetilde{\theta}_N - \theta|^2 \leq C_1 N^{(2-2H)(1-\alpha)}, \tag{7}$$

and this goes to zero if and only if $\alpha > 1$.

Let us summarize the above discussion.

Theorem 1 *Let $(S_t^H)_{t\in[0,T]}$ be a sub-fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ and let $\alpha > 1$. Then, the estimator (5) is L^p-consistent for any $p \geq 1$.*

Proof Since for every N the random variable $\widetilde{\theta}_N - \theta$ is a centered random variable, it holds that, for some positive constant c_p depending on p,

$$\mathbf{E}|\widetilde{\theta}_N - \theta|^p \leq c_p \left(\mathbf{E}|\widetilde{\theta}_N - \theta|^2\right)^{\frac{p}{2}} \leq c_p C_1^{\frac{p}{2}} N^{p(1-H)(1-\alpha)},$$

and this converges to zero as $N \to \infty$ since $\alpha > 1$. □

It is also possible to obtain the almost sure convergence of the estimator to the true parameter from the the estimate (7).

Theorem 2 *Let $(S_t^H)_{t\in[0,T]}$ be a sub-fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ and let $\alpha > 1$. Then, the estimator (5) is strong consistent, that is $\widetilde{\theta}_N \xrightarrow{a.s.} \theta$, as $N \to \infty$.*

Proof Using Chebyshev's inequality,

$$\mathbf{P}\left(|\widetilde{\theta}_N - \theta| > \frac{1}{N^\gamma}\right) \leq c_p C_1^{\frac{p}{2}} N^{p\gamma} N^{p(1-H)(1-\alpha)},$$

for some $\gamma > 0$. To apply the Borel–Cantelli lemma, we need to find a strictly positive γ such that

$$\sum_{N \geq 1} N^{p\gamma} N^{p(1-H)(1-\alpha)} < \infty.$$

One needs $p\gamma + p(1-H)(1-\alpha) < -1$, and this is possible if and only if $\alpha > 1$. □

Remark 1 Let us comment on the problem of estimation of the diffusion parameter in the model (3). Assume that the sub-fractional Brownian motion is replaced by σS^H in (3), with $\sigma \in \mathbf{R}$. In this case, it is known that the sequence

$$N^{2H-1} \sum_{i=0}^{N-1} \left(X_{\frac{i+1}{N}} - X_{\frac{i}{N}} \right)^2,$$

converges (in L^2 and almost surely) to σ^2. Thus. we easily obtain an estimator for the diffusion parameter using such quadratic variations. For this reason, we assume throughout this paper that the diffusion coefficient is equal to 1.

3 MLE based on random walk

In this section, we propose a new model: we replace in (3) the sub-fractional Brownian motion S^H by its associated disturbed random walk

$$S_t^{H,N} := \sum_{i=1}^{[Nt]} \sqrt{N} \left(\int_{\frac{i-1}{N}}^{\frac{i}{N}} c(H) n_H \left(\frac{[Nt]}{N}, s \right) \mathrm{d}s \right) \xi_i^{(N)}, \quad t \in [0, T],$$

where the $\xi_i^{(N)}$ are i.i.d. random variables with zero-mean and variance equal to 1 and $n_H(t, s)$ is defined by (4) and $[x]$ denotes the greatest integer not exceeding x and $c(H) = \sqrt{\frac{\Gamma(1+2H)\sin(\pi H)}{\pi}}$. Now, we prove that the sequence $S^{H,N}$ converges weakly as $N \to \infty$ in the Skorohod topology to the sub-fractional Brownian motion S^H.

Theorem 3 *The random walk $S^{H,N}$ converges weakly to the sub-fractional Brownian motion S^H in the Skorohod space.*

Proof The proof is adapted from that of Theorem 1 in Sottinen (2001). The proof consists of showing that the finite-dimensional distributions of $S^{H,N}$ converge to those of S^H and then showing that $S^{H,N}$ is tight.

Let us consider the limiting finite-dimensional distributions. For arbitrary $a_1, \ldots, a_d \in \mathbf{R}$ and $t_1, \ldots, t_d \in [0, T]$, we want to show that

$$Y^{(N)} := \sum_{k=1}^{d} a_k S_{t_k}^{H,N}$$

converges to a normal distribution with variance $\mathbf{E}\left(\sum_{k=1}^{d} a_k S_{t_k}^H\right)^2$. Let us calculate the limiting variance of $Y^{(N)}$. Denote $(\sigma^{(N)})^2 := DY^{(N)}$ which is the variance of $Y^{(N)}$. Now,

$$\left(\sigma^{(N)}\right)^2 = c^2(H) \sum_{k,l=1}^{d} a_k a_l N \sum_{i=1}^{[NT]} \int_{\frac{i-1}{N}}^{\frac{i}{N}} n_H\left(\frac{[Nt_k]}{N}, s\right) \mathrm{d}s \int_{\frac{i-1}{N}}^{\frac{i}{N}} n_H\left(\frac{[Nt_l]}{N}, s\right) \mathrm{d}s. \quad (8)$$

By the mean value theorem, (8) is equal to

$$c^2(H) \sum_{k,l=1}^{d} a_k a_l \frac{1}{N} \sum_{i=1}^{[NT]} n_H\left(\frac{[Nt_k]}{N}, s_{i,k}^{(N)}\right) n_H\left(\frac{[Nt_l]}{N}, s_{i,l}^{(N)}\right), \quad (9)$$

for some $s_{i,k}^{(N)}, s_{i,l}^{(N)} \in \left(\frac{i-1}{N}, \frac{i}{N}\right]$. Since the functions $n_H(t, \cdot)$ are continuous and decreasing in $(0, T]$, we obtain that the inner sum in Formula (9) is equal to

$$\frac{1}{N} \sum_{i=1}^{[NT]} n_H\left(\frac{[Nt_k]}{N}, u_i^{(N)}\right) n_H\left(\frac{[Nt_l]}{N}, u_i^{(N)}\right), \quad (10)$$

for some

$$u_i^{(N)} \in \left[\min\left(s_{i,k}^{(N)}, s_{i,l}^{(N)}\right), \max\left(s_{i,k}^{(N)}, s_{i,l}^{(N)}\right)\right] \subseteq \left(\frac{i-1}{N}, \frac{i}{N}\right].$$

Using the fact that the kernel $n_H(\cdot, \cdot)$ is continuous with respect to both arguments and that the maps $t \mapsto \frac{[Nt]}{N}$ converge uniformly to the identity map in $[0, T]$, we see that (10) is a Riemann type sum. It follows that (9), and hence $(\sigma^{(N)})^2$, converges to

$$c^2(H) \sum_{k,l=1}^{d} a_k a_l \int_0^T n_H(t_k, s) n_H(t_l, s) \mathrm{d}s = \mathbf{E}\left(\sum_{k=1}^{d} a_k S_{t_k}^H\right)^2.$$

Let us now write $Y^{(N)}$ as a sum in i.

$$Y^{(N)} = \sum_{i=1}^{[NT]} \sqrt{N} \xi_i^{(N)} \sum_{k=1}^{d} a_k \int_{\frac{i-1}{N}}^{\frac{i}{N}} c(H) n_H\left(\frac{[Nt_k]}{N}, s\right) \mathrm{d}s =: \sum_{i=1}^{[NT]} Y_i^{(N)}.$$

Lindeberg's condition is satisfied if for all $\epsilon > 0$ we have

$$\lim_{N\to\infty} \frac{1}{(\sigma^{(N)})^2} \sum_{i=1}^{[NT]} \mathbf{E}\left(Y_i^{(N)}\right)^2 \mathbf{I}_{\left\{\left|Y_i^{(N)}\right| > \epsilon\sigma^{(N)}\right\}} = 0. \quad (11)$$

We give an upper bound for the random variables $\left(Y_i^{(N)}\right)^2$. By Cauchy–Schwartz inequality and the facts that $n_H(\cdot, s)$ is increasing and $n_H(t, \cdot)$ is decreasing, we obtain

$$\begin{aligned}
\left(Y_i^{(N)}\right)^2 &= Nc^2(H)\left(\xi_i^{(N)}\right)^2\left(\sum_{k=1}^{d} a_k \int_{\frac{i-1}{N}}^{\frac{i}{N}} n_H\left(\frac{[Nt_k]}{N}, s\right) \mathrm{d}s\right)^2 \\
&\leq Nc^2(H)\left(\xi_i^{(N)}\right)^2 A\left(\int_{\frac{i-1}{N}}^{\frac{i}{N}} n_H\left(T, s\right) \mathrm{d}s\right)^2 \\
&\leq c^2(H)(\xi_i^{(N)})^2 A \int_{\frac{i-1}{N}}^{\frac{i}{N}} n_H^2\left(T, s\right) \mathrm{d}s \\
&\leq c^2(H)(\xi_i^{(N)})^2 A \int_{0}^{\frac{1}{N}} n_H^2\left(T, s\right) \mathrm{d}s \\
&= c^2(H)(\xi_i^{(N)})^2 A\delta^{(N)},
\end{aligned} \tag{12}$$

where $A := \left(\sum_{k=1}^{d} a_k\right)^2$ and $\delta^{(N)} := \int_0^{\frac{1}{N}} n_H^2\left(T, s\right) \mathrm{d}s$. We obtain

$$\left\{\left|Y_i^{(N)}\right| > \epsilon\sigma^{(N)}\right\} \subseteq \left\{c^2(H)\left(\xi_i^{(N)}\right)^2 A\delta^{(N)} > \epsilon^2\left(\sigma^{(N)}\right)^2\right\} =: D^{(N)}\left(\xi_i^{(N)}\right). \tag{13}$$

Using inequality (12) and the inclusion (13), we obtain

$$\begin{aligned}
\mathbf{E}\left(Y_i^{(N)}\right)^2 \mathbf{I}_{\left\{\left|Y_i^{(N)}\right| > \epsilon\sigma^{(N)}\right\}} &\leq \left(\sigma_i^{(N)}\right)^2 c^2(H)\mathbf{E}\left(\xi_i^{(N)}\right)^2 \mathbf{I}_{D^{(N)}\left(\xi_i^{(N)}\right)} \\
&=: \left(\sigma_i^{(N)}\right)^2 c^2(H)\mathbf{E}\xi^2\mathbf{I}_{D^{(N)}},
\end{aligned}$$

where $\xi := \xi_1^{(1)}$, $D^{(N)} := D^{(N)}\left(\xi_1^{(1)}\right)$ and $\left(\sigma_i^{(N)}\right)^2 := D\left(Y_i^{(N)}\right)$. Using this upper bound to the Lindeberg's condition (11), we obtain

$$\begin{aligned}
\frac{1}{(\sigma^{(N)})^2}\sum_{i=1}^{[NT]} \mathbf{E}\left(Y_i^{(N)}\right)^2 \mathbf{I}_{\left\{\left|Y_i^{(N)}\right| > \epsilon\sigma^{(N)}\right\}} &\leq \frac{\left(\sigma_1^{(N)}\right)^2 + \cdots + \left(\sigma_{[NT]}^{(N)}\right)^2}{\left(\sigma^{(N)}\right)^2} c^2(H)\mathbf{E}\xi^2\mathbf{I}_{D^{(N)}} \\
&= c^2(H)\mathbf{E}\xi^2\mathbf{I}_{D^{(N)}}.
\end{aligned}$$

Since $n_H^2(T, \cdot)$ is integrable, $\delta^{(N)}$, and consequently $c^2(H)\mathbf{E}\xi^2\mathbf{I}_{D^{(N)}}$, tends to zero. Hence, (11) holds and the convergence of the finite-dimensional distributions follows.

It remains to prove the tightness. Let $s < t$ be arbitrary time points. Using Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}
\mathbf{E}\left(S_t^{H,N} - S_s^{H,N}\right)^2 &= c^2(H)\mathbf{E}\left(\sum_{i=1}^{[Nt]} \sqrt{N} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(n_H\left(\frac{[Nt]}{N}, u\right) - n_H\left(\frac{[Ns]}{N}, u\right)\right) \mathrm{d}u \xi_i^{(N)}\right)^2 \\
&= c^2(H) \sum_{i=1}^{[Nt]} \left(\sqrt{N} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(n_H\left(\frac{[Nt]}{N}, u\right) - n_H\left(\frac{[Ns]}{N}, u\right)\right) \mathrm{d}u\right)^2 \\
&\leq c^2(H) \sum_{i=1}^{[Nt]} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(n_H\left(\frac{[Nt]}{N}, u\right) - n_H\left(\frac{[Ns]}{N}, u\right)\right)^2 \mathrm{d}u \\
&\leq c^2(H) \int_0^t \left(n_H\left(\frac{[Nt]}{N}, u\right) - n_H\left(\frac{[Ns]}{N}, u\right)\right)^2 \mathrm{d}u \\
&\leq \left[\left(2 - 2^{2H-1}\right) \vee 1\right] \left|\frac{[Nt]}{N} - \frac{[Ns]}{N}\right|^{2H}. \qquad (14)
\end{aligned}$$

Let now $s < t < u$ be arbitrary. Using Cauchy–Schwartz inequality again and the bound (14), we obtain

$$\begin{aligned}
\mathbf{E}\left|S_t^{H,N} - S_s^{H,N}\right| \left|S_u^{H,N} - S_t^{H,N}\right| &\leq \left(\mathbf{E}\left(S_t^{H,N} - S_s^{H,N}\right)^2\right)^{\frac{1}{2}} \left(\mathbf{E}\left(S_u^{H,N} - S_t^{H,N}\right)^2\right)^{\frac{1}{2}} \\
&\leq \left[\left(2 - 2^{2H-1}\right) \vee 1\right] \left|\frac{[Nt]}{N} - \frac{[Ns]}{N}\right|^H \left|\frac{[Nu]}{N} - \frac{[Nt]}{N}\right|^H \\
&\leq \left[\left(2 - 2^{2H-1}\right) \vee 1\right] \left|\frac{[Nu]}{N} - \frac{[Ns]}{N}\right|^{2H}.
\end{aligned}$$

If now $u - s \geq \frac{1}{N}$, we have

$$\mathbf{E}\left|S_t^{H,N} - S_s^{H,N}\right| \left|S_u^{H,N} - S_t^{H,N}\right| \leq \left[\left(2 - 2^{2H-1}\right) \vee 1\right] |2(u-s)|^{2H}. \qquad (15)$$

If on the other hand $u - s < \frac{1}{N}$, then either s and t or t and u lie in the same subinterval $\left[\frac{m}{N}, \frac{m+1}{N}\right)$ for some m. Thus, the left hand side of (15) is zero. Therefore, (15) holds for all $s < t < u$. Recalling now that $H > \frac{1}{2}$ and by Theorem 15.6 of Billingsley (1968), we have the tightness of $S^{H,N}$. Therefore, we finish the proof of this theorem. □

Now, we turn to estimate the drift parameter θ on the basis of the observations

$$X_{t_{j+1}} = X_{t_j} + \theta(t_{j+1} - t_j) + \left(S_{t_{j+1}}^{H,N} - S_{t_j}^{H,N}\right),$$

where $t_j = \frac{j}{N}$, $j = 0, 1, \ldots, N^\alpha - 1$ and $X_0 = 0$. We will assume again that we have at our disposal a number N^α of observations and we use a discretization of order $\frac{1}{N}$ of the model. Denoting $X_j = X_{t_j}$, we can write

$$X_{j+1} = X_j + \frac{\theta}{N} + \left(S^{H,N}_{\frac{j+1}{N}} - S^{H,N}_{\frac{j}{N}} \right), \quad j = 0, 1, \ldots, N^{\alpha} - 1,$$

and

$$X_{j+1} = X_j + \frac{\theta}{N} + \sum_{i=1}^{j} f_{ij} \xi_i^{(N)} + F_j \xi_{j+1}^{(N)}, \tag{16}$$

where

$$F_j = \sqrt{N} c(H) \int_{\frac{j}{N}}^{\frac{j+1}{N}} n_H \left(\frac{j+1}{N}, s \right) \mathrm{d}s,$$

and

$$f_{ij} = \sqrt{N} c(H) \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(n_H \left(\frac{j+1}{N}, s \right) - n_H \left(\frac{j}{N}, s \right) \right) \mathrm{d}s.$$

Using (16), for $j \in \{1, \ldots, N^{\alpha}\}$, each $\xi_j^{(N)}$ can be expressed explicitly in terms of $X_1, \ldots, X_j$ and θ. More precisely, we have

$$(X_1, \ldots, X_j)' = \frac{\theta}{N} (1, 2, \ldots, j)' + B(\xi_1^{(N)}, \ldots, \xi_j^{(N)})',$$

where $B = (b_{ij})$ with $b_{ij} = \sqrt{N} c(H) \int_{\frac{i-1}{N}}^{\frac{i}{N}} n_H \left(\frac{j}{N}, s \right) \mathrm{d}s$. Then

$$\left(\xi_1^{(N)}, \ldots, \xi_j^{(N)} \right)' = B^{-1} (X_1, \ldots, X_j)' - \frac{\theta}{N} B^{-1} (1, 2, \ldots, j)'. \tag{17}$$

Using (16) and (17), we can write for $j \in \{0, 1, \ldots, N^{\alpha} - 1\}$,

$$X_{j+1} = X_j + \frac{\theta}{N} (1 + \alpha_j) + h_j (X_1, \ldots, X_j) + F_j \xi_{j+1}^{(N)}, \tag{18}$$

where the functions h_j and the α_j depend on the b_{ij}, and the f_{ij}.

The α_j satisfy

$$\begin{aligned} \alpha_0 &= 0, \\ \alpha_1 &= -\frac{f_{11}}{F_0}, \\ \alpha_j &= -\sum_{i=1}^{j} \frac{f_{ij}}{F_{i-1}} (1 + \alpha_{i-1}), \end{aligned} \tag{19}$$

since

$$X_{j+1} = X_j + \frac{\theta}{N} + \sum_{i=1}^{j} \frac{f_{ij}}{F_{i-1}} \Big(X_i - X_{i-1} - h_{i-1}(X_1, \ldots, X_{i-1}) - \frac{\theta}{N}(1+\alpha_{i-1}) \Big) + F_j \xi_{j+1}^{(N)}.$$

From now on, we will assume that the r.v. $\xi_i^{(N)}$ follows a standard normal law $N(0,1)$. Then, given $X_1, \ldots, X_j$ the random variable X_{j+1} is conditionally Gaussian and the conditional density of X_{j+1} given $X_1, \ldots, X_j$ can be written as

$$f_{X_{j+1}/X_1,\ldots,X_j}(x_{j+1}/x_1, \ldots, x_j) = \frac{1}{\sqrt{2\pi F_j^2}} \exp\left(-\frac{1}{2} \frac{\Big(x_{j+1} - x_j - h_j(x_1, \ldots, x_j) - \frac{\theta(1+\alpha_j)}{N} \Big)^2}{F_j^2} \right).$$

The likelihood function of $X_1, \ldots, X_{N^\alpha}$ can be expressed as

$$\begin{aligned} L(\theta, x_1, \ldots, x_{N^\alpha}) &= f_{X_1}(x_1) f_{X_2/X_1}(x_2/x_1) \cdots f_{X_{N^\alpha}/X_1,\ldots,X_{N^\alpha-1}}(x_{N^\alpha}/x_1, \ldots, x_{N^\alpha-1}) \\ &= \prod_{j=0}^{N^\alpha-1} \frac{1}{\sqrt{2\pi F_j^2}} \exp\left(-\frac{1}{2} \frac{\Big(x_{j+1} - x_j - h_j(x_1, \ldots, x_j) - \frac{\theta(1+\alpha_j)}{N} \Big)^2}{F_j^2} \right). \end{aligned}$$

This leads to the expression of the MLE

$$\hat{\theta}_N = N \frac{\sum_{j=0}^{N^\alpha-1} \frac{(1+\alpha_j)\left(X_{j+1}-X_j-h_j(X_1,\ldots,X_j)\right)}{F_j^2}}{\sum_{j=0}^{N^\alpha-1} \frac{(1+\alpha_j)^2}{F_j^2}}, \tag{20}$$

with

$$\hat{\theta}_N - \theta = \frac{N \sum_{j=0}^{N^\alpha-1} \frac{\xi_{j+1}^{(N)}(1+\alpha_j)}{F_j}}{\sum_{j=0}^{N^\alpha-1} \frac{(1+\alpha_j)^2}{F_j^2}}. \tag{21}$$

By the independence of $\xi_i^{(N)}$, we can write

$$\mathbf{E} \left| \hat{\theta}_N - \theta \right|^2 = N^2 \mathbf{E} \left(\frac{\sum_{j=0}^{N^\alpha-1} \frac{\xi_{j+1}^{(N)}(1+\alpha_j)}{F_j}}{\sum_{j=0}^{N^\alpha-1} \frac{(1+\alpha_j)^2}{F_j^2}} \right)^2 = \frac{N^2}{\sum_{j=0}^{N^\alpha-1} \frac{(1+\alpha_j)^2}{F_j^2}}. \tag{22}$$

Remark 2 Clearly the estimator (20) is unbiased.

Remark 3 From (21), we obtain

$$\frac{\sqrt{\sum_{j=0}^{N^\alpha-1}\frac{(1+\alpha_j)^2}{F_j^2}}}{N}(\hat{\theta}_N-\theta)\sim N(0,1),$$

since $\xi_{j+1}^{(N)}$, $j=0,\ldots,N^\alpha-1$ are i.i.d $N(0,1)$. But from (6), we cannot get the similar result because $S_{\frac{i}{N}}^H$, $i=1,\ldots,N^\alpha$, are not independent, even though they follow normal distribution.

Let us study the L^p-consistency of (20). We prove here the following.

Theorem 4 *Assume that* $\alpha>1$. *Then, the estimator* $\hat{\theta}_N$ *given in (20) is* L^p*-consistent for any* $p\geq 1$.

To prove Theorem 4, we need the following two lemmas.

Lemma 1 *There exists a positive constant* C, *such that*

$$\int_0^{\frac{N^\alpha}{N}} n_H^2\left(\frac{N^\alpha}{N},s\right)\mathrm{d}s\leq C\left(\frac{N^\alpha}{N}\right)^{2H}. \tag{23}$$

Proof Using the definition of $n_H(t,s)$ defined by (4), we have

$$\int_0^{\frac{N^\alpha}{N}} n_H^2\left(\frac{N^\alpha}{N},s\right)\mathrm{d}s=\int_0^{\frac{N^\alpha}{N}}\frac{2^{2-2H}\pi s^{3-2H}}{\Gamma^2(H-\frac{1}{2})}\left(\int_s^{\frac{N^\alpha}{N}}(x^2-s^2)^{H-\frac{3}{2}}\mathrm{d}x\right)^2\mathrm{d}s.$$

Since

$$\begin{aligned}\int_s^{\frac{N^\alpha}{N}}(x^2-s^2)^{H-\frac{3}{2}}\mathrm{d}x&=\frac{1}{2H-1}\int_s^{\frac{N^\alpha}{N}}\frac{1}{x}\mathrm{d}(x^2-s^2)^{H-\frac{1}{2}}\\&=\frac{1}{2H-1}\left\{\frac{N}{N^\alpha}\left[\left(\frac{N^\alpha}{N}\right)^2-s^2\right]^{H-\frac{1}{2}}+\int_s^{\frac{N^\alpha}{N}}\frac{(x^2-s^2)^{H-\frac{1}{2}}}{x^2}\mathrm{d}x\right\}\\&\leq\frac{1}{2H-1}\left\{\frac{N}{N^\alpha}\left[\left(\frac{N^\alpha}{N}\right)^2-s^2\right]^{H-\frac{1}{2}}+\left[\left(\frac{N^\alpha}{N}\right)^2-s^2\right]^{H-\frac{1}{2}}\int_s^{\frac{N^\alpha}{N}}\frac{1}{x^2}\mathrm{d}x\right\}\\&=\frac{1}{(2H-1)s}\left[\left(\frac{N^\alpha}{N}\right)^2-s^2\right]^{H-\frac{1}{2}},\end{aligned}$$

we obtain

$$
\begin{aligned}
\int_0^{\frac{N^\alpha}{N}} n_H^2\left(\frac{N^\alpha}{N}, s\right) \mathrm{d}s &\le \frac{2^{2-2H}\pi}{(2H-1)^2\Gamma^2(H-\frac{1}{2})} \int_0^{\frac{N^\alpha}{N}} s^{1-2H}\left[\left(\frac{N^\alpha}{N}\right)^2 - s^2\right]^{2H-1} \mathrm{d}s \\
&= \frac{2^{2-2H}\pi}{(2H-1)^2\Gamma^2(H-\frac{1}{2})}\left(\frac{N^\alpha}{N}\right)^{2H} \int_0^1 (1-s^2)^{2H-1} s^{1-2H} \mathrm{d}s \\
&= \frac{2^{1-2H}\pi\beta(2H, 1-H)}{(2H-1)^2\Gamma^2(H-\frac{1}{2})}\left(\frac{N^\alpha}{N}\right)^{2H},
\end{aligned}
$$

where $\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ is the usual Beta function. Thus, (23) holds with

$$
C = \frac{2^{1-2H}\pi\beta(2H, 1-H)}{(2H-1)^2\Gamma^2(H-\frac{1}{2})}. \tag{24}
$$

□

Lemma 2 *We have*

$$
\mathbf{E}\left|\hat{\theta}_N - \theta\right|^2 \le c^2(H) C N^{(2-2H)(1-\alpha)}, \tag{25}
$$

where a positive constant C is defined by (24) and $c^2(H) = \frac{\Gamma(1+2H)\sin(\pi H)}{\pi}$.

Proof By taking the sum from 1 to $N^\alpha - 1$ in the recurrence formula (19), we can write

$$
\begin{aligned}
\sum_{j=1}^{N^\alpha-1} \alpha_j &= -\sum_{j=1}^{N^\alpha-1}\sum_{i=1}^{j} \frac{f_{ij}}{F_{i-1}}(1+\alpha_{i-1}) \\
&= -\sum_{i=1}^{N^\alpha-1} \frac{1+\alpha_{i-1}}{F_{i-1}} \sum_{j=i}^{N^\alpha-1} f_{ij} \\
&= -\sum_{i=1}^{N^\alpha-1} \frac{1+\alpha_{i-1}}{F_{i-1}}\left[\sqrt{N}c(H)\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(n_H\left(\frac{N^\alpha}{N}, s\right) - n_H\left(\frac{i}{N}, s\right)\right)\mathrm{d}s\right] \\
&= -\sum_{i=1}^{N^\alpha-1} \frac{1+\alpha_{i-1}}{F_{i-1}}\left[\sqrt{N}c(H)\int_{\frac{i-1}{N}}^{\frac{i}{N}} n_H\left(\frac{N^\alpha}{N}, s\right)\mathrm{d}s - F_{i-1}\right].
\end{aligned}
$$

So,

$$\sum_{j=1}^{N^\alpha-1}\alpha_j=\sum_{i=1}^{N^\alpha-1}(1+\alpha_{i-1})-\sum_{i=1}^{N^\alpha-1}\frac{1+\alpha_{i-1}}{F_{i-1}}\sqrt{N}c(H)\int_{\frac{i-1}{N}}^{\frac{i}{N}}n_H\left(\frac{N^\alpha}{N},s\right)\mathrm{d}s$$

$$=N^\alpha-1+\sum_{i=1}^{N^\alpha-1}\alpha_i-\alpha_{N^\alpha-1}-\sum_{i=1}^{N^\alpha-1}\frac{1+\alpha_{i-1}}{F_{i-1}}\sqrt{N}c(H)\int_{\frac{i-1}{N}}^{\frac{i}{N}}n_H\left(\frac{N^\alpha}{N},s\right)\mathrm{d}s,$$

which gives

$$\alpha_{N^\alpha-1}=N^\alpha-1-\sum_{i=1}^{N^\alpha-1}\frac{1+\alpha_{i-1}}{F_{i-1}}\sqrt{N}c(H)\int_{\frac{i-1}{N}}^{\frac{i}{N}}n_H\left(\frac{N^\alpha}{N},s\right)\mathrm{d}s. \tag{26}$$

On the other hand, by the recurrence relation (19), we get

$$\alpha_{N^\alpha-1}=-\sum_{i=1}^{N^\alpha-1}\frac{1+\alpha_{i-1}}{F_{i-1}}\sqrt{N}c(H)\int_{\frac{i-1}{N}}^{\frac{i}{N}}n_H\left(\frac{N^\alpha}{N},s\right)\mathrm{d}s$$

$$+\sum_{i=1}^{N^\alpha-1}\frac{1+\alpha_{i-1}}{F_{i-1}}\sqrt{N}c(H)\int_{\frac{i-1}{N}}^{\frac{i}{N}}n_H\left(\frac{N^\alpha-1}{N},s\right)\mathrm{d}s. \tag{27}$$

From (26) and (27), we will deduce (changing $N^\alpha-1$ into N^α)

$$N^\alpha=\sum_{i=1}^{N^\alpha}\frac{1+\alpha_{i-1}}{F_{i-1}}\sqrt{N}c(H)\int_{\frac{i-1}{N}}^{\frac{i}{N}}n_H\left(\frac{N^\alpha}{N},s\right)\mathrm{d}s$$

$$=\sum_{i=0}^{N^\alpha-1}\frac{1+\alpha_i}{F_i}\sqrt{N}c(H)\int_{\frac{i}{N}}^{\frac{i+1}{N}}n_H\left(\frac{N^\alpha}{N},s\right)\mathrm{d}s,$$

and from the bound $\left(\sum_{i=1}^n a_ib_i\right)^2\le\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$, we obtain

$$N^{2\alpha}\le\sum_{i=0}^{N^\alpha-1}\left(\frac{1+\alpha_i}{F_i}\right)^2\sum_{i=0}^{N^\alpha-1}\left(\sqrt{N}c(H)\int_{\frac{i}{N}}^{\frac{i+1}{N}}n_H\left(\frac{N^\alpha}{N},s\right)\mathrm{d}s\right)^2$$

$$\le c^2(H)\sum_{i=0}^{N^\alpha-1}\left(\frac{1+\alpha_i}{F_i}\right)^2\int_0^{\frac{N^\alpha}{N}}n_H^2\left(\frac{N^\alpha}{N},s\right)\mathrm{d}s$$

$$\le c^2(H)C\left(\frac{N^\alpha}{N}\right)^{2H}\sum_{i=0}^{N^\alpha-1}\left(\frac{1+\alpha_i}{F_i}\right)^2\qquad\text{(by Lemma 1).}$$

Then,

$$\sum_{i=0}^{N^{\alpha}-1}\left(\frac{1+\alpha_i}{F_i}\right)^2 \geq \frac{N^{2\alpha}}{c^2(H)C}\left(\frac{N^{\alpha}}{N}\right)^{-2H},$$

and this and (22) will imply

$$\mathbf{E}\left|\hat{\theta}_N-\theta\right|^2 \leq c^2(H)CN^{(2-2H)(1-\alpha)},$$

where C is defined by (24). □

Proof of Theorem 4 Note that, by Lemma 2,

$$\mathbf{E}\left|\hat{\theta}_N-\theta\right|^p \leq c_p\left(\mathbf{E}\left|\hat{\theta}_N-\theta\right|^2\right)^{\frac{p}{2}} \leq c_p c^p(H)C^{\frac{p}{2}}N^{p(1-H)(1-\alpha)},$$

then, the proof of Theorem 4 is easily obtained since $\alpha > 1$. □

By Chebyshev's inequality and the Borel–Cantelli lemma, we can get the following result.

Theorem 5 *Assume that* $\alpha > 1$. *The estimator (20) is strong consistent, that is* $\hat{\theta}_N \xrightarrow{a.s.} \theta$, *as* $N \to \infty$.

Proof It is similar to the proof of Theorem 2, therefore we omit it here. □

4 Simulations and comparison

It is obvious that $\overline{\theta}_t := \frac{X_t}{t}$ is also the consistent estimator of θ and has the asymptotic normality. In fact,

$$\overline{\theta}_t = \frac{X_t}{t} = \theta + \frac{S_t^H}{t},$$

thus,

$$\frac{t^{1-H}}{\sqrt{2-2^{2H-1}}}(\overline{\theta}_t-\theta) \sim N(0,1).$$

In this section, we compare $\widehat{\theta}_N$ given by (20), $\widetilde{\theta}_N$ given by (5) with $\overline{\theta}_t$ by numerical simulations.

We have simulated the observations $X_1, \cdots, X_{N^{\alpha}}$ for different values of H : 0.55, 0.75 and 0.90 and for the parameter of discretization $N = 30$ and the true value of $\theta = 1$. For each case, we calculate 1,000 estimations of $\widehat{\theta}_N$, $\widetilde{\theta}_N$ and $\overline{\theta}_t$(take $t = N^{\alpha-1}$) and Table 1 shows the mean and the standard deviation (SD) of these estimations. In all of the cases, we use $\alpha = 2$. From the data in Table 1, it seems that

Table 1 Mean and standard deviation of three parameter estimators

H	$\widehat{\theta}_N$		$\widetilde{\theta}_N$		$\overline{\theta}_t\left(t=N^{\alpha-1}\right)$	
	Mean	SD	Mean	SD	Mean	SD
0.55	1.0025	0.0452	0.9897	0.2559	0.9910	0.1967
0.75	0.9966	0.1771	0.9638	1.0633	1.0149	0.3373
0.90	0.9941	0.1671	0.9215	1.9010	1.0064	0.3711

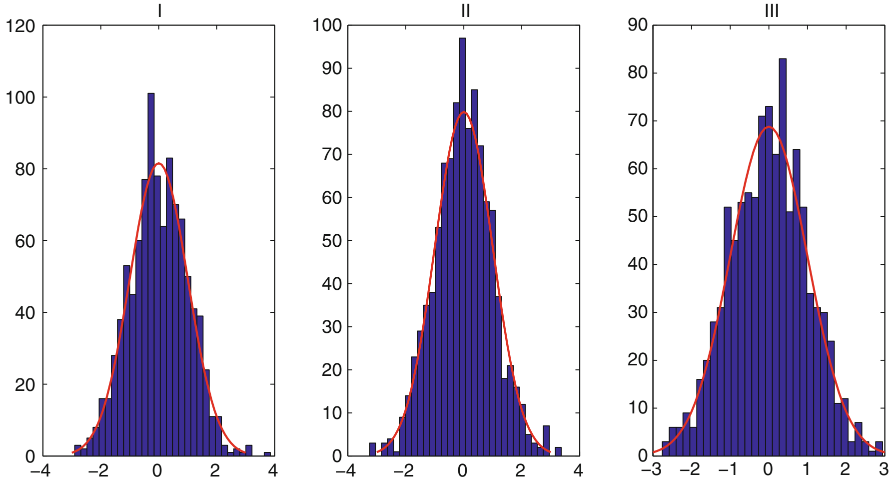

Fig. 1 Histogram of three normalized estimators(I: $\widehat{\theta}_N$, II: $\widetilde{\theta}_N$ and III: $\overline{\theta}_t$)

$\widehat{\theta}_N$ is better than $\overline{\theta}_t$ and $\overline{\theta}_t$ is better than $\widetilde{\theta}_N$. Therefore the new proposed estimator $\widehat{\theta}_N$ has superiority.

Figure 1 depicts the histogram of normalized estimators $\widehat{\theta}_N$, $\widetilde{\theta}_N$ and $\overline{\theta}_t$ with $N=30$, $H=0.55, \alpha=2$ and $t=N^{\alpha-1}$. We can draw a conclusion that $\widehat{\theta}_N$, $\widetilde{\theta}_N$ and $\overline{\theta}_t$ have asymptotic normalities from the Fig. 1, though we cannot prove the asymptotic normality of $\widetilde{\theta}_N$ in theory. The simulations confirm the theoretical results proved in Theorem 1, Theorem 4 and Remark 3.

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References

Bertin, K., Torres, S., Tudor, C.A. (2009). Maximum likelihood estimators and random walks in long memory models. arXiv:0711.0513v2.

Bertin, K., Torres, S., Tudor, C. A. (2011). Drift parameter estimation in fractional diffusions driven by perturbed random walks. *Statistics and Probability Letters*, *81*, 243–249.

Billingsley, P. (1968). *Convergence of Probability Measures*. New York: Chapman and Hall.

Bojdecki, T. L. G., Gorostiza, L. G., Talarczyk, A. (2004). Sub-fractional Brownian motion and its relation to occupation times. *Statistics and Probability Letters*, *69*, 405–419.

Golub, G. H., van Loan, C. F. (1996). *Matrix Computations* (3rd ed.). Baltimore/London: Hopkins University Press.

Kleptsyna, M., Le Breton, A. (2002). Statistical analysis of the fractional Ornstein-Uhlenbeck process. *Statistical Inference for Stochastic Processes*, *5*, 229–248.

Le Breton, A. (1998). Filtering and parameter estimation in a simple linear system driven by a fractional Brownian motion. *Statistics and Probability Letters*, *38*, 263–274.

Mendy, I. (2013). Parametric estimation for sub-fractional Ornstein-Uhlenbeck process. *Journal of Statistical Planning and Inference*, *143*, 663–674.

Prakasa Rao, B. L. S. (1999). *Statistical Inference for Diffusion Type Processes*. London: Arnold, Hodder Headlone Group.

Prakasa Rao, B. L. S. (2008). Parametric estimation for linear stochastic delay differential equations driven by fractional Brownian motion. *Random Operators and Stochastic Equations*, *16*(1), 27–38.

Sottinen, T. (2001). Fractional Brownian motion, random walks and binary market models. *Finance and Stochastics*, *5*, 343–355.

Sottinen, T., Tudor, C. A. (2008). Parameter estimation for stochastic equations with additive fractional Brownian sheet. *Statistical Inference for Stochastic Processes*, *11*(3), 221–236.

Tudor, C. A., Viens, F. (2007). Statistical aspects of the fractional stochastic calculus. *The Annals of Statistics*, *35*(3), 1183–1212.