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Proportional odds frailty model and stochastic comparisons

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Abstract In this paper, we present some distributional properties of the survival and frailty distribution involved in the proportional odds (PO) frailty model. Stochastic orderings are studied for this proportional odds frailty model. It is showed that negative dependence arises in the PO frailty model as opposed to the proportional hazard frailty model.

Keywords Likelihood ratio ordering · Failure rate ordering · Stochastic ordering · $TP_2(RR_2)$ functions · Frailty distributions

1 Introduction

In survival analysis, Cox proportional hazard model is the most popular, but in certain situations Cox model is inappropriate. The Cox model postulates that the covariates have a fixed multiplicative effect on their hazard. Zucker and Yang (2006) note that often it is more reasonable to suppose that the effect of the covariates on the hazard diminishes over time. As an alternative, Bennett (1983a,b) introduced proportional odds model defined by

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$$\frac{1 - S(t|z)}{S(t|z)} = \frac{1 - S_0(t)}{S_0(t)} \exp(\beta^T z)$$

where $S(t|z)$ denotes the survival function under the covariates z and $S_0(t)$ is the unspecified baseline survival function. Various authors including Cheng et al. (1995), Cuzick (1988), Dabrowska and Doksum (1988), Lam and Leung (2001), Murphy et al. (1997), Pettitt (1984), Shen (1998), Yang and Prentice (1999) and Wu (1995) consider the estimation of the parameter in a semiparametric proportional odds model. Kirmani and Gupta (2001) explored the structure, implication and properties of the proportional odds model. Using a nonparametric rank-based empirical likelihood approach, Guan and Peng (2011) studied the two sample proportional odds model and proposed a simultaneous procedure to estimate the model parameter and assess the goodness-of-fit. Murphy et al. (1997) remarked that the proportional odds model can be viewed as a proportional hazard model with unobserved heterogeneity.

The proportional hazard model with unobserved heterogeneity has been studied in the literature under the umbrella of proportional hazard (PH) frailty models. The PH frailty model is given by

$$\lambda(t|v) = v\lambda_0(t), \quad t > 0 \tag{1}$$

where $\lambda_0(t)$ is the baseline hazard rate independent of v and v is the unobserved heterogeneity, known as the frailty. Frailty models have been used when groups of subjects have responses that are likely to be dependent in some general way. For example, in an animal carcinogenicity study, the responses of members of the same litter are not likely to be independent. Liang et al. (1995) discuss the use of frailty models when multiple events have been observed on the same subjects. The PH frailty model has been extensively used in modeling survival data. We refer the reader to two recent books of Hanagal (2011) and Duchateau and Janssen (2008) and the references therein.

Economou and Caroni (2007) proposed maximum likelihood and nonlinear least square methods to estimate the parameters in the purely parametric proportional odds frailty model to real life survival data with right censoring observations using different frailty distributions including Gamma, uniform on $(1-\theta, 1+\theta)$ with $\theta \in (0, 1)$, and Triangular distribution on $(1-\theta, 1+\theta)$ with $\theta \in (0, 1)$. In the least square method, Economou and Caroni (2007) proposed using stabilized probability plot method to reduce the variability of the survival probabilities obtained from the well-known Kaplan–Meier estimator and produced a data set with complete lifetimes and their corresponding survival probabilities. A hypothetical distribution is then fit to this derived data set using the least square procedure. Bootstrap confidence intervals were used to make inferences on the model parameters. The numerical examples show that the choice of Gamma frailty improves the fit of the simple log-logistic model (which is close to the proportional frailty model). Their simulation and numerical examples indicate the feasibility of implementing this computationally intensive proportional odds frailty model.

As indicated in the numerical example in Economou and Caroni (2007), the choice of frailty distribution affects the estimate of the baseline hazard as well as that of conditional probabilities; see also Hougaard (1984, 1991, 1995, 2000), Heckman and Singer (1984) and Agresti et al. (2004). Agresti et al. (2004) have demonstrated that a considerable loss of efficiency can result from assuming a parameter distribution for a random effect that is substantially different from the true distribution. These authors observed that misspecification of the random effect has the potential for a serious drop of the efficiency in the prediction of the random effects and the estimation of other parameters. In this context, Gupta and Kirmani (2006), and Gupta and Gupta (2009, 2010) studied some general frailty model and their stochastic comparisons.

Similar to the PH frailty model, the purpose of this paper is to study the stochastic orderings in the proportional odds (PO) frailty model. The PO frailty model is defined as follows.

Assume that $F(t|v) = P[T \leq t|V = v]$ is the cumulative distribution of the lifetime T given frailty $V = v$ and F_0 is the baseline cumulative distribution function of the lifetime T. Assume further that the frailty random variable has density function $h(v)$. We define two odds functions based on $F(t|v)$ and $F_0(t)$ as follows:

$$\phi(t|v) = \frac{\bar{F}(t|v)}{1 - \bar{F}(t|v)}, \quad \phi_0(t) = \frac{\bar{F}_0(t)}{1 - \bar{F}_0(t)}, \tag{2}$$

$\bar{F}(t|v)$ and $\bar{F}_0(t)$ are corresponding survival functions associated with $F(t|v)$ and $F_0(t)$. The proportional odds frailty model is defined by

$$\phi(t|v) = v\phi_0(t). \tag{3}$$

Note that Marshall and Olkin (1997) is a PO model. Also, see Marshall and Olkin (2007). Its extensions and modifications have been studied by various authors including Gupta and Peng (2009).

Since different distributions of frailty give rise to different population-level distribution for analyzing survival data, it is appropriate to investigate how the comparative effect of two frailties translates into the comparative effect on the resulting survival distribution. The stochastic ordering, on various characteristics, of this model studied in this paper address this problem. The comparisons are studied with respect to failure rates, survival distributions and the mean residual life functions.

Our chief aim in this paper is to develop the properties of the model (3) and obtain some results for the stochastic comparisons.

The following definitions will be used for various stochastic comparisons. Let X and Y be non-negative absolutely continuous random variables with density functions $f(x)$ and $g(x)$ and survival functions $\bar{F}(x)$ and $\bar{G}(x)$, respectively.

Then

1. X is said to be smaller than Y in the likelihood ratio ordering, written as $X \leq^{\text{LR}} Y$, if $f(x)/g(x)$ is non-increasing in x.
2. X is said to be smaller than Y in the failure (hazard) rate ordering, written as $X \leq^{\text{FR}} Y$ or $X \leq^{\text{hr}} Y$, if $r_F(x) \geq r_G(x)$ for all x. This means that $\bar{G}(x)/\bar{F}(x)$ increases in x.

3. X is said to be smaller than Y in the stochastic ordering, written as $X \leq^{\text{st}} Y$ if $\bar{F}(x) \leq \bar{G}(x)$ for all x.
4. X is said to be smaller than Y in the mean residual life ordering, written as $X \leq^{\text{MRL}} Y$, if $\mu_F(x) \leq \mu_G(x)$ for all x. Deshpande et al. (1990) show that $X \leq^{\text{MRL}} Y$ if and only if $\int_x^\infty \bar{F}(u)\mathrm{d}u / \int_x^\infty \bar{G}(u)\mathrm{d}u$ increases in x.

It is well known that

$$\begin{array}{c} X \leq^{\text{LR}} Y \ \Rightarrow X \leq^{\text{FR}} Y \Rightarrow X \leq^{\text{MRL}} Y \\ \Downarrow \\ X \leq^{\text{st}} Y. \end{array}$$

To conclude this section, we present the following definition to be used in the subsequent discussion.

Definition Let $f(t, v)$ be a real valued function defined on $[0, \infty) \times [0.\infty)$. If, for all $0 < t_1 < t_2$ and $0 < v_1 < v_2$,

$$f(t_1, v_1) f(t_2, v_2) \leq (\geq) f(t_1, v_2) f(t_2, v_1).$$

Then $f(t, v)$ is called $RR_2(TP_2)$.

It is useful to notice the following equivalent conditions:

1. A real value function $f(t, v)$ is $RR_2(TP_2)$ in $[0, \infty) \times [0.\infty)$.
2. $f(t, v_1)/f(t, v_2)$ is increasing (decreasing) in $t > 0$, $0 < v_1 < v_2$.
3. $\partial^2[\ln f(t, v)]/\partial t \partial v < (>)0$.
4. If $f(t, v)$ and $f(t|v)$ are the joint and conditional densities, $f(t|v)$ is $RR_2(TP_2)$.

The organization of this paper is as follows. In Sect. 2, we present some distributional properties of the survival and frailty distribution involved in the PO frailty model. Stochastic orderings are studied in Sect. 3. Some conclusions and comments are provided in Sect. 4.

2 Distributional properties of proportional odds frailty models

Before we study the properties of lifetime distribution associated with proportional odds frailty model, we define some unconditional (population level) and conditional (on frailty) survival functions. First, we define survival and failure rate functions.

$$\frac{\bar{F}(t|v)}{1 - \bar{F}(t|v)} = v \times \frac{\bar{F}_0(t)}{1 - \bar{F}_0(t)}. \tag{4}$$

From (4), we can also calculate the the conditional survival function

$$\bar{F}(t|v) = \frac{v\bar{F}_0(t)}{1 - (1 - v)\bar{F}_0(t)} = \frac{v\phi_0(t)}{1 + v\phi_0(t)}. \tag{5}$$

The corresponding conditional density is

$$f(t|v) = \frac{vf_0(t)}{[1-(1-v)\bar{F}_0(t)]^2} = \frac{v\phi_0(t)\lambda_0(t)}{F_0(t)[1+v\phi_0(t)]^2}, \tag{6}$$

where $\lambda_0(t)$ is the baseline hazard rate function. The conditional hazard rate is given by

$$\lambda(t|v) = \frac{f(t|v)}{\bar{F}(t|v)} = \frac{\lambda_0(t)}{F_0(t)[1+v\phi_0(t)]}. \tag{7}$$

Note that the hazard rate is defined to be the instantaneous rate which involves the conditional probability $\lim_{h\to 0} P(t \le T < t+h|T \ge t)/h$. Therefore, the population-level hazard rate can be expressed as

$$\lambda(t) = \int_0^\infty \lambda(t|v)h(v|T \ge t)\mathrm{d}v,$$

where the conditional density function $h(v|T \ge t)$ for conditional random variable $V|T \ge t$ can be found in the next few steps. We calculate the unconditional distribution functions. The unconditional density function is given by

$$f(t) = \int_0^\infty f(t|v)h(v)\mathrm{d}v = \int_0^\infty \frac{v\phi_0(t)\lambda_0(t)h(v)}{F_0(t)[1+v\phi_0(t)]^2}\mathrm{d}v. \tag{8}$$

From (5), we can find the unconditional survival function

$$\bar{F}(t) = \int_0^\infty \frac{v\phi_0(t)h(v)}{1+v\phi_0(t)}\mathrm{d}v. \tag{9}$$

Therefore, the population-level hazard rate can be expressed in terms of the conditional hazard rate as follows

$$\begin{aligned}
\lambda(t) &= \frac{f(t)}{\bar{F}(t)} = \int_0^\infty \frac{v\phi_0(t)\lambda_0(t)h(v)}{F_0(t)[1+v\phi_0(t)]^2}\mathrm{d}v \Big/ \int_0^\infty \frac{v\phi_0(t)h(v)}{1+v\phi_0(t)}\mathrm{d}v \\
&= \int_0^\infty \frac{\lambda_0(t)}{F_0(t)[1+v\phi_0(t)]^2}\left(\frac{vh(v)/[1+v\phi_0(t)]}{\int_0^\infty \{vh(v)/[1+v\phi_0(t)]\}\mathrm{dv}}\right)\mathrm{d}v \\
&= \int_0^\infty \lambda(t|v)\left(\frac{vh(v)/[1+v\phi_0(t)]}{\int_0^\infty \{vh(v)/[1+v\phi_0(t)]\}\mathrm{d}v}\right)\mathrm{d}v.
\end{aligned}$$

Therefore,

$$h(v|T > t) = \frac{vh(v)/[1+v\phi_0(t)]}{\int_0^\infty \{vh(v)/[1+v\phi_0(t)]\}\mathrm{d}v} \tag{10}$$

is the probability density function of random variable $V|T > t$. The population-level hazard rate can be expressed as a weighted average of the conditional hazard rate as follows

$$\lambda(t) = \int_0^\infty \lambda(t|v)h(v|T > t)\mathrm{d}v = \int_0^\infty \lambda(t|v)\mathrm{d}H(v|T > t) = E_{V|T>t}[\lambda(t|V)]. \tag{11}$$

The cumulative distribution of frailty V conditioning on $T > t$ is given by

$$H(v|T > t) = \frac{\int_0^v \{sh(s)/[1 + s\phi_0(t)]\}\mathrm{d}s}{\int_0^\infty \{vh(v)/[1 + v\phi_0(t)]\}\mathrm{d}v}. \tag{12}$$

With the above notations and descriptions of the conditional and unconditional distributions and hazard functions, we summarize some related properties in the following theorem.

Theorem 1 *Assume the proportional odds frailty model (3) holds. Then, for* $0 < v_1 < v_2$*, we have*

1. $\lambda(t|v)$ *is a decreasing function of* v.
2. *The joint density function* $f(t, v)$ *and the conditional density* $f(t|v)$ *of* T *and* V *are* TP_2.
3. $F(t|v_1)/F(t|v_2)$ *decreases in* t*, i.e.,* $F(t|v)$ *is* TP_2.
4. $\bar{F}(t|v_1)/\bar{F}(t|v_2)$ *decreases in* t*, i.e.,* $\bar{F}(t|v)$ *is* TP_2.

Proof (1) It is obvious from the expression given in (7).
(2) Note that the joint density function of T and V is given by

$$f(t, v) = f(t|v)h(v) = \frac{v\phi_0(t)\lambda_0(t)h(v)}{F_0(t)[1 + v\phi_0(t)]^2}. \tag{13}$$

Therefore,

$$\frac{f(t, v_1)}{f(t, v_2)} = \frac{v_1 h(v_1)}{v_2 h(v_2)}\left[\frac{1 + v_2\phi_0(t)}{1 + v_1\phi_0(t)}\right]^2 = \frac{v_1 h(v_1)}{v_2 h(v_2)}\left[1 + \frac{v_2 - v_1}{v_1 + [\phi_0(t)]^{-1}}\right]^2$$

and

$$\frac{f(t|v_1)}{f(t|v_2)} = \frac{v_1}{v_2}\left[1 + \frac{v_2 - v_1}{v_1 + [\phi_0(t)]^{-1}}\right]^2.$$

Since $\phi_0(t)$ is a decreasing function of t and $v_1 < v_2$, $f(t, v_1)/f(t, v_2)$ and $f(t|v_1)/f(t|v_2)$ are decreasing functions of t. Hence, $f(t, v)$ and $f(t|v)$ are TP_2 as functions of t and v.
(3) Note that

$$\frac{F(t|v_1)}{F(t|v_2)} = \frac{1 - \bar{F}(t|v_1)}{1 - \bar{F}(t|v_2)} = 1 + \frac{v_2 - v_1}{v_1 + [\phi_0(t)]^{-1}}.$$

Decreasing function $\phi_0(t)$ implies that $F(t|v_1)/F(t|v_2)$ decreases in t.

(4) We follow the same arguments as used in the proof of (3) to show the TP_2 property of $\bar{F}(t|v)$ as a function of t and v by observing the following

$$\frac{\bar{F}(t|v_1)}{\bar{F}(t|v_2)} = \frac{v_1}{v_2}\left[\frac{1+v_2\phi_0(t)}{1+v_1\phi_0(t)}\right] = \frac{v_1}{v_2}\left[1+\frac{v_2-v_1}{v_1+[\phi_0(t)]^{-1}}\right].$$

The proof of the theorem is complete. □

The behavior of survival function on the right tail is of practical importance. In proportional odds frailty models, there are two distributions involved. We next see the monotonicity of both lifetime variable and frailty variable on the right tail. From (12), we can easily get

$$\bar{H}(v|T>t) = 1 - H(v|T>t) = \frac{\int_v^\infty \{sh(s)/[1+s\phi_0(t)]\}\mathrm{d}s}{\int_0^\infty \{vh(v)/[1+v\phi_0(t)]\}\mathrm{d}v}. \tag{14}$$

Note also that

$$\bar{F}(t|V>v) = \frac{P(T>t, V>v)}{P(V>v)} = \frac{\int_v^\infty \{sh(s)/[1+s\phi_0(t)]\}\mathrm{d}s}{\int_v^\infty h(s)\mathrm{d}s}. \tag{15}$$

The monotonicity of T and V on the right tail is summarized in the following.

Theorem 2 *Assume the proportional odds frailty model (3) holds. We have*

1. *V stochastically increases in the right tail with respect to T. That is, $\bar{H}(v|T>t)$ is increasing in t for $t>0$.*
2. *T stochastically increases in the right tail with respect to V. That is, $\bar{F}(t|V>v)$ is increasing in v for $v>0$.*

Proof (1) We take the first-order derivative of $\bar{H}(v|T>t)$ with respect to t, The numerator of the resulting first-order derivative $\partial\bar{H}(v|T>t)/\partial t$ is

$$\phi_0'(t)\left[\int_v^\infty \frac{sh(s)}{1+s\phi_0(t)}\mathrm{d}s\int_0^\infty \frac{v^2h(v)}{[1+v\phi_0(t)]^2}\mathrm{d}v - \int_0^\infty \frac{vh(v)}{1+v\phi_0(t)}\mathrm{d}v\int_v^\infty \frac{v^2h(v)}{[1+v\phi_0(t)]^2}\mathrm{d}v\right]$$

$$= \phi_0'(t)\int_v^\infty \frac{sh(s)}{1+s\phi_0(t)}\mathrm{d}s\int_0^\infty \frac{vh(v)}{1+v\phi_0(t)}\mathrm{d}v\left[\frac{\int_0^\infty \frac{v^2h(v)}{[1+v\phi_0(t)]^2}\mathrm{d}v}{\int_0^\infty \frac{vh(v)}{[1+v\phi_0(t)]}\mathrm{d}v} - \frac{\int_v^\infty \frac{s^2h(s)}{[1+s\phi_0(t)]^2}\mathrm{d}s}{\int_v^\infty \frac{sh(s)}{[1+s\phi_0(t)]}\mathrm{d}v}\right].$$

Since $\phi_0(t)$ is a decreasing function of t, that is, $\phi_0'(t)<0$, we only need to prove that

$$A(v) = \int_v^\infty \frac{s^2h(s)}{[1+s\phi_0(t)]^2}\mathrm{d}s\Big/\int_v^\infty \frac{sh(s)}{[1+s\phi_0(t)]}\mathrm{d}s$$

is an increasing function of v. To see this, we take first-order derivative of $A(v)$ with respect to v. Since $s > v$, the corresponding numerator, denoted by $D(v)$, is

$$D(v) = \frac{vh(v)}{1+v\phi_0(t)}\left[\int_v^\infty \frac{s^2 h(s)}{[1+s\phi_0(t)]^2}\mathrm{d}s - \frac{v}{1+v\phi_0(t)}\int_v^\infty \frac{sh(s)}{1+s\phi_0(t)}\mathrm{d}s\right] > 0$$

which implies that $A(v)$ is an increasing function of v. Therefore, $\bar{H}(v|T > t)$ increases in t.

(2) We take the first-order derivative of $\bar{F}(t|V > v)$ with respect to v and obtain

$$\frac{\partial \bar{F}(t|V > v)}{\partial v} = \frac{h(v)}{[\int_0^\infty h(s)\mathrm{d}s]^2}\left[\int_v^\infty \frac{sh(s)}{1+s\phi_0(t)}\mathrm{d}s - \frac{v}{1+v\phi_0(t)}\int_v^\infty h(s)\mathrm{d}s\right] > 0$$

since $s/[1+s\phi_0(t)] > v/[1+v\phi_0(t)]$ for all $s > v$. Therefore $\bar{F}(t|V > v)$ is increasing in v.

□

Corollary 1 *If proportional odds frailty model (3) holds, then $E[V|T > t]$ is an increasing function of t and $E[T|V > v]$ is an increasing function of v.*

Proof The above conclusion follows immediately by looking at the facts that $E[V|T > t] = \int_0^\infty \bar{H}(v|T > t)\mathrm{d}v$ and $E[T|V > v] = \int_0^\infty \bar{H}(t|V > v)\mathrm{d}v$. □

The implication of Corollary 1 is obvious. Since proportional odds frailty model has a decreasing conditional hazard $\lambda(t|v)$ in v, less frail individuals die earlier than others so that the remaining individuals are less robust (more frail). On the other hand, as frailty increases, the hazard rate decreases so that the individual survival time increases in the rest of the population.

Next, we study the mean residual life of the distributions. Let

$$\mu_0(t) = \frac{\int_t^\infty \bar{F}_0(s)\mathrm{d}s}{\bar{F}_0(t)}$$

be the mean residual life function at baseline. The mean residual life function conditioning on $V = v$ is given by

$$\mu(t|v) = \frac{\int_t^\infty \bar{F}_0(s|v)\mathrm{d}s}{\bar{F}_0(t|v)}.$$

The next theorem characterizes the monotonicity of the conditional mean residual lifetime of $T|V = v$.

Theorem 3 *Assume the proportional odds frailty model (3) holds. Then, for $0 < v_1 < v_2$, we have $\mu(t|v_1) < \mu(t|v_2)$.*

Proof We only need to show that $\mu(t|v)$ is an increasing function of v. Since survival odds function $\phi_0(t)$ decreases, i.e., $\phi_0(s) < \phi_0(t)$ for all $s > t$, therefore

$$\frac{\phi_0(s)}{1 + v\phi_0(s)} < \frac{\phi_0(t)}{1 + v\phi_0(t)}.$$

Next, we take the first-order derivative of $\mu(t|v)$ with respect to v and obtain

$$\frac{\partial \mu(t|v)}{\partial v} = \frac{1 + v\phi_0(t)}{\phi_0(t)} \left[\frac{\phi_0(t)}{1 + v\phi_0(t)} \int_t^\infty \frac{\phi_0(s)}{1 + v\phi_0(s)} \mathrm{d}s - \int_t^\infty \frac{\phi_0^2(s)}{[1 + v\phi_0(s)]^2} \mathrm{d}s \right] > 0$$

which implies that $\mu(t|v)$ increases in $v > 0$. □

3 Stochastic orderings in proportional odds frailty models

Our main objective, in this section, is to see how some of the well-known stochastic orderings between V_1 and V_2 translate into the orderings of T_1 and T_2. The first result compares the frailty distribution of two groups, one surviving up to time t_1, and the other surviving up to time t_2.

Theorem 4 *Assume that the proportional odds frailty model (3) holds. Then, for* $0 < t_1 < t_2$, *we have* $V|T > t_2 \geq^{\mathrm{LR}} V|T > t_1$.

Proof We only need to show that $h(v|T > t_2)/h(v|T > t_1)$ decreases in v. Let $c(t) = \int_0^\infty \{vh(v)/[1 + v\phi_0(t)]\}\mathrm{d}v$. From (10), we have

$$\frac{h(v|T > t_2)}{h(v|T > t_1)} = \frac{c(t_1)}{c(t_2)} \times \frac{1 + v\phi_0(t_1)}{1 + v\phi_0(t_2)} = \frac{c(t_1)}{c(t_2)} \left[1 + \frac{\phi_0(t_1) - \phi_0(t_2)}{v^{-1} + \phi_0(t_2)} \right]. \tag{16}$$

Since $\phi_0(t_1) > \phi_0(t_2)$ for $0 < t_1 < t_2$, $c(t_1)/c(t_2)$ are positive function of t_1 and t_2. Therefore, (16) is an increasing function of v which implies that $V|T > t_2 \geq^{\mathrm{LR}} V|T > t_1$. □

The next result compares two frailties in a population that has survived up to t.

Theorem 5 *Assume that the proportional odds frailty model (3) holds and furthermore* $V_1 \leq^{\mathrm{LR}} V_2$. *Then, we have* $V_1|T > t \leq^{\mathrm{LR}} V_2|T > t$.

Proof Let $h_1(v|T > t)$ and $h_2(v|T > t)$ be the *pdf* of $V_1|T > t$ and $V_2|T > t$ respectively. The unconditional density functions of V_1 and V_2 are denoted by $h_1(v)$ and $h_2(v)$, respectively. We will show that $h_1(v|T > t)/h_2(v|T > t)$ is decreasing in v. To this end,

$$\frac{h_1(v|T > t)}{h_2(v|T > t)} = \frac{h_1(v)}{h_2(v)} \times \frac{\int_0^\infty \{vh_2(v)/[1 + v\phi_0(t)]\}\mathrm{d}v}{\int_0^\infty \{vh_2(v)/[1 + v\phi_0(t)]\}\mathrm{d}v}. \tag{17}$$

Note that the last fraction of two integrals are independent on v. $V_1 \leq^{\text{LR}} V_2$ implies that $h_1(v)/h_2(v)$ is decreasing in v. Therefore (17) is decreasing in v meaning that $V_1|T > t \leq^{\text{LR}} V_2|T > t$. □

Although the choice of frailty affects the estimate of baseline hazard, there are no guidelines for selecting appropriate frailty distribution available in literature. The common practice is to select the frailty distribution based on ease of mathematical manipulation. From this perspective, we may be interested in what and how the stochastic orderings in frailty variables translate into the lifetime variables under proportional odds frailty model framework. The following theorems provide some insights into these transitions of stochastic orders.

Theorem 6 *Under proportional odds frailty model (3), if* $V_1 \leq^{\text{LR}} V_2$. *Then, we have* $T_1 \leq^{\text{FR}} T_2$.

Proof Let $\lambda_1(t)$ and $\lambda_2(t)$ be the population-level hazard rates corresponding to T_1 and T_2. We need to show that $\lambda_1(t) - \lambda_2(t) > 0$. From (11), we have

$$
\begin{aligned}
\lambda_1(t) - \lambda_2(t) &= \int_0^\infty \lambda(t|v)h_1(v|T > v)dv - \int_0^\infty \lambda(t|v)h_2(v|T > v)\mathrm{d}v \\
&= \int_0^\infty \lambda(t|v)[h_1(v|T > v) - h_2(v|T > t)]\mathrm{d}v \\
&= \int_0^\infty \lambda(t|v)\mathrm{d}[H_1(v|T > v) - H_2(v|T > t)] \\
&= \int_0^\infty \frac{\partial\lambda(t|v)}{\partial v}[H_2(v|T > v) - H_1(v|T > t)]\mathrm{d}v.
\end{aligned}
$$

We know from Theorem 1(1) that $\partial\lambda(t|v)/\partial v < 0$. Next we prove that $D(v|T > t) = H_2(v|T > v) - H_1(v|T > t) > 0$. From (12), we have

$$
\begin{aligned}
D(v|T > t) &= \frac{\int_0^v \{sh_2(s)/[1 + s\phi_0(t)]\mathrm{d}s\}}{\int_0^\infty \{vh_2(v)/[1 + v\phi_0(t)]\}\mathrm{d}s} - \frac{\int_0^v \{sh_1(s)/[1 + s\phi_0(t)]\mathrm{d}s\}}{\int_0^\infty \{vh_1(v)/[1 + v\phi_0(t)]\}} \\
&= R(t)\left[\frac{\int_0^v \{sh_1(s)/[1 + s\phi_0(t)]\}\mathrm{d}s}{\int_0^v \{sh_2(s)/[1 + s\phi_0(t)]\}\mathrm{d}s} - \frac{\int_0^\infty \{vh_1(v)/[1 + v\phi_0(t)]\mathrm{d}v\}}{\int_0^\infty \{vh_2(v)/[1 + v\phi_0(t)]\}\mathrm{d}v}\right],
\end{aligned}
$$

where

$$
R(t) = \frac{\int_0^v \{sh_2(s)/[1 + s\phi_0(t)]\mathrm{d}s\}}{\int_0^\infty \{vh_1(v)/[1 + v\phi_0(t)]\}\mathrm{d}v} > 0.
$$

Observe that $D(v|T > t) > 0$ is equivalent to the fact that

$$
B(v) = \frac{\int_0^v \{sh_1(s)/[1 + s\phi_0(t)]\}\mathrm{d}s}{\int_0^v \{sh_2(s)/[1 + s\phi_0(t)]\}\mathrm{d}s}
$$

is a decreasing function of v. To see this, let $B_b(v)$ be the numerator of the first-order derivative of $B(v)$ with respect to v. After some algebra, we have

$$N_b(v) = \frac{vh_1(v)}{1+v\phi_0(t)} \left\{ \int_0^v \frac{sh_2(s)}{1+s\phi_0(s)} \mathrm{d}s - \int_0^v \frac{h_2(v)}{h_1(v)} \frac{sh_1(s)}{1+s\phi_0(s)} \mathrm{d}s \right\}. \tag{18}$$

Since $V_1 \le^{\mathrm{LR}} V_2$ implies that $h_2(v)/h_1(v)$ is an increasing function of v. Therefore,

$$\frac{h_2(v)}{h_1(v)} > \frac{h_2(s)}{h_1(s)} \quad \text{for} \quad s < v. \tag{19}$$

$N_b(v) < 0$ follows immediately from (19). That is, $D(v|T > t) > 0$ which completes the proof. □

Next, we will show that the likelihood ratio order between the two frailty variables can be transmitted into the likelihood ratio order between the two unconditional time variables.

Theorem 7 *Under proportional odds frailty model (3), if $V_1 \le^{\mathrm{LR}} V_2$, then we have $T_1 \le^{\mathrm{LR}} T_2$.*

Proof Let $A = B = [0, \infty)$ and $I = \{1, 2\}$. From Theorem 4(2), we know that $f(t|v)$ is TP_2 on $A \times B$. Let $g(v, i) = h_i(v)$ be the *p.d.f* of V_1 and V_2, respectively, for $i = 1, 2$. The given condition $V_1 \le^{\mathrm{LR}} V_2$ implies that $g(v, i) = h_i(v)$ is TP_2 on $B \times I$. The *basic composition formula* (B.1. Theorem, p. 697) in Marshall and Olkin (2007) implies that

$$f_i(t) = \int_0^\infty f(t|v)h_i(v)\mathrm{d}v$$

is TP_2 on $A \times I$, which furthermore implies that $f_1(t)/f_2(t)$ is decreasing in t. Therefore, $T_1 \le^{\mathrm{LR}} T_2$. □

Remark The basic composition formula was generalized from the well-known *Binet–Cauchy formula* in matrix theory and the proof is outlined in Karlin (1968). It is one of the very useful properties of totally positive functions.

Comparison of failure rates is of practical importance in survival analysis. The next result concerns the translation of failure rate ordering between the lifetime and the frailty.

Theorem 8 *Under proportional odds frailty model (3), if $V_1 \le^{\mathrm{FR}} V_2$, then we have $T_1 \le^{\mathrm{FR}} T_2$.*

Proof Let $A = B = [0, \infty)$ and $I = \{1, 2\}$. Define

$$g(t, v) = \frac{\phi_0(t)}{1+v\phi_0(t)}$$

where $\phi_0(t)$ is the baseline survival odds function in the proportional odds frailty model (3).

$$\frac{g(t, v_1)}{g(t, v_2)} = \left[\frac{1 + v2\phi_0(t)}{1 + v_1\phi_0(t)}\right]^2 = \left[1 + \frac{v_2 - v_1}{v_1 + \phi_0^{-1}(t)}\right]^2$$

is a decreasing function of t for $0 < v_1 < v_2$. That is, $g(t, v)$ is TP_2 on $A \times B$. Note that the given condition $V_1 \leq^{\mathrm{FR}} V_2$ implies that $\bar{H}_1(v)/\bar{H}_2(v)$ is decreasing in v for $v > 0$, which implies that $H(v, i) = \bar{H}_i(v)$ is TP_2 on $B \times I$. Let $h_1(v)$ and $h_2(v)$ be density functions of frail variable V_1 and V_2 respectively. Observe that, under proportional odds frailty model (3), the cumulative failure rate function can be expressed as, for $i = 1, 2$,

$$\begin{aligned}\bar{F}_i(t) &= \int_0^\infty \bar{F}(t|v)h_i(v)\mathrm{d}v = -\int_0^\infty \bar{F}(t|v)\mathrm{d}\bar{H}_i(v) \\ &= \int_0^\infty \frac{\phi_0(t)}{[1 + v\phi_0(t)]^2}\bar{H}_i(v)\mathrm{d}v = \int_0^\infty g(t, v)H(v, i)\mathrm{d}v.\end{aligned}$$

The facts that $g(t, v)$ is TP_2 on $A \times B$ and $H(v, i)$ is TP_2 on $B \times I$, by the basic composition formula, imply that $\bar{F}_i(t)$ is TP_2 on $A \times I$. Therefore, $\bar{F}_1(t)/\bar{F}_2(t)$ is decreasing in t meaning that $T_1 \leq^{\mathrm{FR}} T_2$. The proof is complete. □

The stochastic order is used to compare the cumulative rates of two lifetime variables at given time t. In frailty models, we are interested in whether this order can be transmitted from frailty distributions to lifetime distributions. The following result shows that the stochastic ordering is transmittable.

Theorem 9 *Under proportional odds frailty model (3), if $V_1 \leq^{\mathrm{st}} V_2$ then we have $T_1 \leq^{\mathrm{st}} T_2$.*

Proof On one hand, we re-express (5) as

$$\bar{F}(t|v) = \frac{v\phi_0(t)}{1 + v\phi_0(t)} = 1 - \frac{1}{1 + v\phi_0(t)}.$$

Clearly, $\bar{F}(t|v)$ is an increasing function of v. Hence, $\partial\bar{F}(t|v)/\partial v > 0$. Note that $V_1 \leq^{\mathrm{st}} V_2$ implies that $\bar{H}_1(v) - \bar{H}_2(v) < 0$. On the other hand,

$$\begin{aligned}\bar{F}_1(t) - \bar{F}_2(t) &= \int_0^\infty \bar{F}_(t|v)[h_1(v) - h_2(v)]\mathrm{d}v \\ &= \int_0^\infty \bar{F}(t|v)\mathrm{d}[\bar{H}_1(v) - \bar{H}_2(v)] \\ &= \int_0^\infty \frac{\partial\bar{F}(t|v)}{\partial v}\left[\bar{H}_1(v) - \bar{H}_2(v)\right] < 0\end{aligned}$$

for all $t > 0$. Hence, $T_1 \leq^{\mathrm{st}} T_2$. □

We next present the following result showing, in proportional odds frailty model, how the failure rate ordering between two frailty variables will be shifted to mean residual lifetime ordering between the two associated lifetime variables. To be more specific, we have

Theorem 10 *Under proportional odds frailty model (3), if* $V_1 \le^{\mathrm{FR}} V_2$*, then we have* $T_1 \le^{\mathrm{MRL}} T_2$.

Proof Let $h_1(v)$ and $h_2(v)$ be the probability density functions of frailty variables V_1 and V_2, respectively, and

$$\bar{F}_i(t) = \int_0^\infty \frac{v\phi_0(t)h_i(v)}{1 + v\phi_0(t)} \text{ for } i = 1, 2,$$

be the CDFs of the two lifetime variables with associated frailties V_1 and V_2. By Theorem 2.1(2) of Deshpande et al. (1990), $T_1 \le^{\mathrm{MRL}} T_2$ is equivalent to the fact that $\int_t^\infty \bar{F}_1(s)\mathrm{d}s / \int_t^\infty \bar{F}_2(s)\mathrm{d}s$ is decreasing in t. In other words, we only need to show that $\int_t^\infty \bar{F}_i(s)\mathrm{d}s$ is TP_2 on $[0, \infty) \times \{1, 2\}$. To this end, let

$$\begin{aligned}
\Phi_i(t) &= \int_t^\infty \bar{F}_i(s)\mathrm{d}s = \int_t^\infty \left[\int_0^\infty \frac{v\phi_0(s)h_i(v)}{1 + v\phi_0(s)}\mathrm{d}v\right]\mathrm{d}s \\
&= \int_0^\infty \left[\int_t^\infty \frac{v\phi_0(s)}{1 + v\phi_0(s)}\mathrm{d}s\right] h_i(v)\mathrm{d}v \\
&= -\int_0^\infty \left[\int_t^\infty \frac{v\phi_0(s)}{1 + v\phi_0(s)}\mathrm{d}s\right] \mathrm{d}\bar{H}_i(v) \\
&= \int_0^\infty \left[\int_t^\infty \frac{\phi_0(s)}{[1 + v\phi_0(s)]^2}\mathrm{d}s\right] \bar{H}_i(v)\mathrm{d}v.
\end{aligned}$$

Since $V_1 \le^{\mathrm{FR}} V_2$ implies that $\bar{H}_i(v)$ is TP_2 on $[0, \infty) \times \{1, 2\}$, according to the basic composition formula, the theorem is proved if

$$g(t, v) = \int_t^\infty \frac{\phi_0(s)}{[1 + v\phi_0(s)]^2}\mathrm{d}s$$

is TP_2 on $[0, \infty) \times [0, \infty)$. For $0 < v_1 < v_2$, define

$$G(t, v_1, v_2) = \frac{g(t, v_1)}{g(t, v_2)} = \int_t^\infty \frac{\phi_0(s)}{[1 + v_1\phi_0(s)]^2}\mathrm{d}s \Big/ \int_t^\infty \frac{\phi_0(s)}{[1 + v_2\phi_0(s)]^2}\mathrm{d}s.$$

Let $N_G(t)$ be the numerator of the first-order derivative of $G(t, v_1, v_2)$ with respect to t, that is,

$$N_G(t) = \frac{\phi_0(t)}{[1 + v_1\phi_0(t)]^2}\left[\left(\frac{1 + v_1\phi_0(t)}{1 + v_2\phi_0(t)}\right)^2 \int_t^\infty \frac{\phi_0(s)}{[1 + v_1\phi_0(s)]^2}\mathrm{d}s - \int_t^\infty \frac{\phi_0(s)}{[1 + v_2\phi_0(s)]^2}\mathrm{d}s\right].$$

Since $\partial\phi_0(t)/\partial t < 0$,

$$\frac{\partial}{\partial t}\left(\frac{1+v_1\phi_0(t)}{1+v_2\phi_0(t)}\right)^2 = 2\left(\frac{1+v_1\phi_0(t)}{1+v_2\phi_0(t)}\right)\frac{\partial\phi_0(t)}{\partial t}[v_1 - v_2] > 0,$$

that is,

$$\left(\frac{1+v_1\phi_0(t)}{1+v_2\phi_0(t)}\right)^2$$

is an increasing function of t. So, we have

$$\left(\frac{1+v_1\phi_0(t)}{1+v_2\phi_0(t)}\right)^2\int_t^\infty \frac{\phi_0(s)}{[1+v_1\phi_0(s)]^2}\mathrm{d}s < \int_t^\infty \frac{\phi_0(s)}{[1+v_2\phi_0(s)]^2}\mathrm{d}s$$

which implies that $N_G(t) < 0$. Therefore, $G(t, v_1, v_2) = g(t, v_1)/g(t, v_2)$ is decreasing in t for $0 < v_1 < v_2$, or equivalently, $g(t, v)$ is TP_2 on $[0, \infty) \times [0, \infty)$ which completes the proof. □

As a final note, we point out that all the theorems in this section can be illustrated by choosing specific distributions for frailty variables V_1 and V_2 and lifetime distributions for T_1 and T_2 satisfying the conditions specified in the proportional odds frailty model (3). To conclude this section, we present an example that illustrates the result in Theorem 9 with specific distributions from the exponential family.

Example Let V_1 and V_2 be the two frailty variables having probability density distributions $h_1(v) = \lambda_1 \exp(-\lambda_1 v)$ and $h_2(v) = \lambda_2 \exp(-\lambda_2 v)$, respectively. Assume further that $\lambda_1 \geq \lambda_2$ implying that $V_1 \leq^{\mathrm{st}} V_2$. We choose baseline odds $\phi_0(t) = \exp(-\lambda_0 t)/[1 - \exp(-\lambda_0 t)]$ with $\lambda_0 > 0$. The two lifetime variables T_1 and T_2 with associated frailty variables V_1 and V_2 are given by

$$\bar{F}_1(t) = \int_0^\infty \frac{v\lambda_1 \exp(-\lambda v)}{\exp(\lambda_0 t) - 1 + v}\mathrm{d}v \quad \text{and } \bar{F}_2(t) = \int_0^\infty \frac{v\lambda_2 \exp(-\lambda v)}{\exp(\lambda_0 t) - 1 + v}\mathrm{d}v.$$

Using integral by parts, we have

$$\begin{aligned}\bar{F}_1(t) - \bar{F}_2(t) &= \int_0^\infty \frac{v[\lambda_1 \exp(-\lambda_1 v) - \lambda_2 \exp(-\lambda_2 v)]}{\exp(\lambda_0 t) - 1 + v}\mathrm{d}v \\ &= -\int_0^\infty \frac{[\exp(\lambda_0 t) - 1][\exp(-\lambda_2 v) - \exp(-\lambda_1 v)]}{[\exp(\lambda_0 t) - 1 + v]^2}\mathrm{d}v \leq 0,\end{aligned}$$

since $\exp(-\lambda_2 v) - \exp(-\lambda_1 v) \geq 0$ and $\exp(\lambda_0 t) \geq 1$. Therefore, $T_1 \leq^{\mathrm{st}} T_2$.

4 Conclusion and comments

In the classical frailty model, the random frailty variable appears as a multiplicative factor of the baseline hazard. In a similar manner, we have studied a proportional odds frailty model where the proportionality parameter plays the role of the frailty. We have presented some distributional properties of the survival and frailty distribution involved in the PO frailty model. Stochastic orderings are investigated for this PO frailty model. It is observed that the conditional failure rate, in the case of PO frailty model, is a decreasing function of the frailty as opposed to the PH frailty model where the conditional failure rate is an increasing function of the frailty. It is hoped that this work will be useful for analyzing survival data.

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