Supplementary Material of 'On estimation and inference in a partially linear hazard model with varying coefficients' by Ma, Wan, Chen and Zhou

This supplementary file contains the proofs of Lemmas 2 and 3, and Theorem 5.

1 Proof of Lemma 2

The proof of Lemma 2 requires the following lemmas.

Lemma 4 *Assume that $g(w, u, Z_j(t), V_j(t))$ is equicontinuous in w and u, and $E(g(w_0, u, Z_j(t), V_j(t))|W_j = w_0)$ is equicontinuous in w_0. Under conditions C.3 and C.4, we have*

$$\sup_{0\leq t\leq\tau}\sup_{w_0\in\mathcal{B}}|C_{nj}(t) - C_j(t)| \xrightarrow{P} 0,$$

where $\mathcal{B}$ is a compact set that satisfies $\inf_{w\in\mathcal{B}} f(w) > 0$, and $j = 1, \cdots, m$.

The proof of Lemma 4 is similar to that of Lemma 1, and is omitted here for brevity.

Lemma 5 *Let C and D be compact sets in $\mathbb{R}^d$ and $\mathbb{R}^p$ respectively, $f(x, \theta)$ be a continuous function in $\theta \in C$ and $x \in D$. Assume that $\theta_0(x)$ is continuous in $x \in D$, as well as being the unique maximizer of $f(x, \theta)$. Let $\widehat{\theta}_n(x) \in C$ be a maximizer of $f_n(x, \theta)$. If*

$$\sup_{\theta\in C, x\in D}|f_n(x, \theta) - f(x, \theta)| \to 0,$$

then

$$\sup_{x\in D}|\widehat{\theta}_n(x) - \theta_0(x)| \to 0.$$

The proof is given in Cai et al. (2000).

Lemma 6 *For any quantity ξ_n, let $\xi_n^{[-i]}$ be the same as ξ_n but with the i^{th} subject deleted. Then ξ_n is the sum of $\xi_n^{[-i]}$ and $\xi_n - \xi_n^{[-i]}$. Under conditions C.1 - C.8, if $nh_1^2 \to \infty$, then*

$\chi_{n1}(W_{ij})$

$$
\begin{aligned}
&= h_1^{-1}\int_0^{W_{ij}} e_1^T {\mathbb{A}^*}^{-1}(W)C_2(W)dW + \int_0^{W_{ij}} e_1^T {\mathbb{A}^*}^{-1}(W)C_3(W)dW + r_n^{[-i]}(W_{ij}) + o_p(\frac{1}{\sqrt{n}}) \\
&\triangleq h_1^{-1}\chi_{11}(W_{ij}) + \chi_{12}(W_{ij}) + r_n^{[-i]}(W_{ij}) + o_p(\frac{1}{\sqrt{n}}), \\
&= \chi_1(W_{ij}) + r_n^{[-i]}(W_{ij}) + o_p(\frac{1}{\sqrt{n}}),
\end{aligned}
$$

and

$$
\begin{aligned}
\chi_{n2}(W_{ij}) &= \mathbf{e}_p^T {\mathbb{A}^*}^{-1}(W_i)C_2(W_{ij}) + \mathbf{e}_p^T {\mathbb{A}^*}^{-1}(W_{ij})h_1C_3(W_{ij}) + r_{nn}^{[-i]}(W_{ij}) + o_p(\frac{1}{\sqrt{n}}) \\
&\triangleq \chi_{21}(W_{ij}) + \chi_{22}(W_{ij}) + r_{nn}^{[-i]}(W_{ij}) + o_p(\frac{1}{\sqrt{n}}), \\
&= \chi_2(W_{ij}) + r_{nn}^{[-i]}(W_{ij}) + o_p(\frac{1}{\sqrt{n}}),
\end{aligned}
$$

where $r_n^{[-i]}(w_0)$ *and* $r_{nn}^{[-i]}(w_0)$ *are independent of the i-th subject such that* $E[r_n^{[-i]}(w_0)]^2 = E[r_{nn}^{[-i]}(w_0)]^2 = o(1)$ *uniformly for* $i = 1, 2, \cdots, n$ *and* $w \in \cup_{j=1}^m supp(f_j)$.

Proof:. The proof consists of the following three parts:

(i) Expression of $\partial\widehat{\varphi}_p^*(\alpha_0)/\partial\alpha^T$ based on all subjects. By its definition, $\widehat{\varphi}_p^*(\alpha)$ satisfies $\partial\tilde{\ell}_n^*(\widehat{\varphi}_p^*, w_0, \tau)/\partial\varphi^* = 0$; that is,

$$
\frac{1}{n}\sum_{j=1}^m\sum_{i=1}^n\int_0^\tau K_{h_1}(W_{ij}-w_0)\left[\tilde{U_{ij}} - \frac{\Phi_{nj1}(u,\alpha,\widehat{\varphi}_p^*(\alpha))}{\Phi_{nj0}(u,\alpha,\widehat{\varphi}_p^*(\alpha))}\right]dN_{ij}(u) = 0. \tag{1}
$$

For $k = 0, 1, 2,$ and $j = 1, \cdots, m$, let

$$
T_{njk}(u,\alpha,\widehat{\varphi}_p^*(\alpha)) = \frac{1}{n}\sum_{i=1}^n \tilde{S}_{ij}(u,\alpha,\widehat{\varphi}_p^*(\alpha))[\tilde{U}_{ij}(u)]^{\otimes k}\otimes V_{ij}^T(u)K_{h_1}(W_{ij}-w_0).
$$

Then for $k = 0, 1$, we have

$$
\frac{\partial\Phi_{njk}(u,\alpha,\widehat{\varphi}_p^*(\alpha))}{\partial\alpha^T} = T_{njk}(u,\alpha,\widehat{\varphi}_p^*(\alpha)) + \Phi_{nj(k+1)}^T(u,\alpha,\widehat{\varphi}_p^*(\alpha))\cdot\frac{\partial\widehat{\varphi}_p^*(\alpha)}{\alpha^T}.
$$

Let

$$
\Phi_{nj}^*(u,\alpha,\widehat{\varphi}_p^*(\alpha)) = [\Phi_{nj0}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\Phi_{nj2}(u,\alpha,\widehat{\varphi}_p^*(\alpha)) - \Phi_{nj1}^{\otimes 2}(u,\alpha,\widehat{\varphi}_p^*(\alpha))]\Phi_{nj0}^{-2}(u,\alpha,\widehat{\varphi}_p^*(\alpha))
$$

and

$$
\begin{aligned}
& T^*_{nj}(u,\alpha,\widehat{\varphi}^*_p(\alpha)) \\
= \quad & [T_{nj1}(u,\alpha,\widehat{\varphi}^*_p(\alpha))\Phi_{nj0}(u,\alpha,\widehat{\varphi}^*_p(\alpha)) - \Phi_{nj1}(u,\alpha,\widehat{\varphi}^*_p(\alpha))\otimes T_{nj0}(u,\alpha,\widehat{\varphi}^*_p(\alpha))]\Phi^{-2}_{nj0}(u,\alpha,\widehat{\varphi}^*_p(\alpha)).
\end{aligned}
$$

Differentiating (1) with respect to α, and using some simple algebra, we obtain

$$
\begin{aligned}
& \frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^\tau K_{h_1}(W_{ij}-w_0)\Phi^*_{nj}(u,\alpha,\widehat{\varphi}^*_p(\alpha))dN_{ij}(u)\frac{\partial\widehat{\varphi}^*_p(\alpha)}{\partial\alpha^T} \\
= \quad & -\frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^\tau K_{h_1}(W_{ij}-w_0)T^*_{nj}(u,\alpha,\widehat{\varphi}^*_p(\alpha))dN_{ij}(u).
\end{aligned}
$$

Now, denote the above expression by

$$
M_{n1}(\alpha,\widehat{\varphi}^*_p(\alpha))\frac{\partial\widehat{\varphi}^*_p(\partial\alpha)}{\alpha^T} = -M_{n2}(\alpha,\widehat{\varphi}^*_p(\alpha)). \tag{2}
$$

Then we have $\frac{\partial\widehat{\varphi}^*_p(\alpha_0)}{\partial\alpha^T} = -M^{-1}_{n1}(\alpha_0,\widehat{\varphi}^*_p(\alpha_0))M_{n2}(\alpha_0,\widehat{\varphi}^*_p(\alpha_0))$ provided that the inverse of $M_{n1}(\alpha_0,\widehat{\varphi}^*_p(\alpha_0))$ exists.

(ii) Expression of $\frac{\partial\widehat{\varphi}^*_p(\alpha_0)}{\partial\alpha^T}$ with the ith subject deleted. Our aim is to show (2) holds approximately with the leave-one-out data. Note that since $\widehat{\varphi}^*_p(\alpha_0)\to\varphi^*_0$ in probability, $\Phi_{njk}(u,\alpha_0,\widehat{\varphi}^*_p(\alpha_0)) = \Phi_{njk}(u,\alpha_0,\varphi^*_0)(1+o_p(1))$ uniformly for $u\in[0,\tau]$. It follows from (A.7) and (A.15) that

$$
M_{n1}(\alpha_0,\widehat{\varphi}^*_p(\alpha_0)) = \mathbb{A}^*(w_0)+o_p(1). \tag{3}
$$

By Theorem 4, we have $\widehat{\varphi}^*_p(\alpha_0)-\varphi^*_0 = O_p(\frac{1}{\sqrt{nh_1}})$. By Taylor series expansion, we obtain the following approximation evaluated at $\alpha=\alpha_0$:

$$
T_{nj0}(u,\alpha_0,\widehat{\varphi}^*_p(\alpha_0)) = T_{nj0}(u,\alpha_0,\varphi^*_0)+[\widehat{\varphi}^*_p(\alpha_0)-\varphi^*_0]T_{nj1}(u,\alpha_0,\varphi^*_0)+O_p(\frac{1}{nh_1})
$$

uniformly for $u\in[0,\tau]$. Likewise, similar approximations can be obtained for T_{nj1} and Φ_{njk}. Using an argument similar to (A.14), we can write

$$
\widehat{U}(\varphi^*_0,w_0)+\frac{\partial\widehat{U}(\tilde{\varphi}^*,w_0)}{\partial\varphi^*}(\widehat{\varphi}^*_p(\alpha_0)-\varphi^*_0) = 0. \tag{4}
$$

Applying (A.15)-(A.17) in (4), we obtain

$$\begin{aligned}\widehat{\varphi}_p^*(\alpha_0) - \varphi_0^* &= \mathbb{A}^{*^{-1}}(w_0)(1+o_p(1))n^{-1}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^\tau K_{h_1}(W_{ij}-w_0) \\ &\qquad \left[\tilde{U}_{ij}(u) - \frac{\Phi_{nj1}(u,\alpha_0,\varphi_0^*)}{\Phi_{nj0}(u,\alpha_0,\varphi_0^*)}\right] dM_{ij}(u) + O_p(h_1^2).\end{aligned}$$

Hence,

$$\widehat{\varphi}_p^*(\alpha_0) - \varphi_0^* = \widehat{\varphi}_p^{*[-i]}(\alpha_0) - \varphi_0^* + O_p(\frac{1}{nh_1}) = O_p(\frac{1}{\sqrt{nh_1}}). \tag{5}$$

Note that

$$\Phi_{njk}(u,\alpha_0,\widehat{\varphi}_p^*(\alpha_0)) = \Phi_{njk}^{[-i]}(u,\alpha_0,\widehat{\varphi}_p^{*[-i]}(\alpha_0)) + O_p(\frac{1}{nh_1}) = O_p(1)$$

and

$$T_{njk}(u,\alpha_0,\widehat{\varphi}_p^*(\alpha_0)) = T_{njk}^{[-i]}(u,\alpha_0,\widehat{\varphi}_p^{*[-i]}(\alpha_0)) + O_p(\frac{1}{nh_1}) = O_p(1)$$

uniformly for $u \in [0,\tau]$. Let

$$\begin{aligned}& A_{nj}(u,\alpha_0,\widehat{\varphi}_p^*(\alpha_0)) \\ =\ & \Phi_{nj0}(u,\alpha_0,\widehat{\varphi}_p^*(\alpha_0))T_{nj1}(u,\alpha_0,\widehat{\varphi}_p^*(\alpha_0)) - \Phi_{nj1}(u,\alpha_0,\widehat{\varphi}_p^*(\alpha_0)) \otimes T_{nj0}(u,\alpha_0,\widehat{\varphi}_p^*(\alpha_0))\end{aligned}$$

and

$$\begin{aligned}& A_{nj}^{[-i]}(u,\alpha_0,\widehat{\varphi}_p^{*[-i]}(\alpha_0)) \\ =\ & \Phi_{nj0}^{[-i]}(u,\alpha_0,\widehat{\varphi}_p^{*[-i]}(\alpha_0))T_{nj1}^{[-i]}(u,\alpha_0,\widehat{\varphi}_p^{*[-i]}(\alpha_0)) - \Phi_{nj1}^{[-i]}(u,\alpha_0,\widehat{\varphi}_p^{*[-i]}(\alpha_0)) \\ & \qquad \otimes T_{nj0}^{[-i]}(u,\alpha_0,\widehat{\varphi}_p^{*[-i]}(\alpha_0)).\end{aligned}$$

It then follows that $A_{bj}(u,\alpha_0,\widehat{\varphi}_p^*(\alpha_0)) = A_{nj}^{[-i]}(u,\alpha_0,\widehat{\varphi}_p^{*[-i]}(\alpha_0)) + O_p(\frac{1}{nh_1})$ uniformly for $u \in [0,\tau]$. Similarly,

$$M_{n1}(\alpha_0,\widehat{\varphi}_p^*(\alpha_0)) = M_{n1}^{[-i]}(\alpha_0,\widehat{\varphi}_p^{*[-i]}(\alpha_0)) + O_p(\frac{1}{nh_1}). \tag{6}$$

Let

$$A_{nj1}^{[-i]}(u,\alpha_0) = \Phi_{nj0}^{[-i]}(u,\alpha_0,\varphi_0^*)T_{nj1}^{[-i]}(u,\alpha_0,\varphi_0^*) - \Phi_{nj1}^{[-i]}(u,\alpha_0,\varphi_0^*) \otimes T_{n0}^{[-i]}(u,\alpha_0,\varphi_0^*)$$

and

$$A_{nj2}^{[-i]}(u,\alpha_0) \;\; = \;\; \Phi_{nj0}^{[-i]}(u,\alpha_0,\varphi_0^*)T_{nj2}^{[-i]}(u,\alpha_0,\varphi_0^*) - T_{nj0}^{[-i]}(u,\alpha_0,\varphi_0^*)\otimes\Phi_{nj2}^{[-i]}(u,\alpha_0,\varphi_0^*).$$

Then by Taylor series expansion,

$$A_{nj}^{[-i]}(u,\alpha_0,\widehat{\varphi}_p^{*[-i]}(\alpha_0)) = A_{nj1}^{[-i]}(u,\alpha_0) + A_{nj2}^{[-i]}(u,\alpha_0)\left[(\widehat{\varphi}_p^{*[-i]}(\alpha_0)-\varphi_0^*)\otimes I_d\right] + r_{nj1}^{[-i]}, \qquad (7)$$

where $r_{nj1}^{[-i]}$ is an item independent of the i,j-th subject such that $r_{nj1}^{[-i]} = O(\|\widehat{\varphi}_p^{*[-i]}(\alpha_0)-\varphi_0^*\|^2)$ and $E[r_{nj1}^{[-i]}]^2 = O(\frac{1}{n^2h_1^2}) = o(h_1^2)$. Similar to (A.5), we have

$$\begin{aligned}
E[T_{nj0}^{[-i]}(u,\alpha_0,\varphi_0^*)] &= e^{-g(w_0)}t_{j0}(u,) + O(h_1^2),\\
E[T_{nj1}^{[-i]}(u,\alpha_0,\varphi_0^*)] &= e^{-g(w_0)}\begin{pmatrix} t_{j1}(u)\\ 0_p\\ 0\end{pmatrix} + h_1\mu_2 e^{-g(w_0)}\begin{pmatrix} 0_p\\ \mathcal{D}_w[t_{j1}(u)]\\ \mathcal{D}_w[t_{j0}(u)]\end{pmatrix}\} + O(h_1^2),
\end{aligned}$$

and

$$\begin{aligned}
E[T_{nj2}^{[-i]}(u,\alpha_0,\varphi_0^*)] &= e^{-g(w_0)}\begin{pmatrix} t_{j2}(u) & 0_{p\times pd} & 0_{p\times d}\\ 0_{p\times pd} & \mu_2 t_{j2}(u) & \mu_2 t_{j1}(u)\\ 0_{pd}^T & \mu_2 t_j^*(u) & \mu_2 t_{j0}(u)\end{pmatrix} + O(h_1^2)\\
&\quad + h_1\mu_2 e^{-g(w_0)}\begin{pmatrix} 0_{p\times pd} & \mathcal{D}_w[t_{j2}(u)] & \mathcal{D}_w[t_{j1}(u)]\\ \mathcal{D}_w[t_{j2}(u)] & 0_{p\times pd} & 0_{p\times d}\\ \mathcal{D}_w[t_j^*(u)] & 0_{pd}^T & 0_d^T\end{pmatrix}. \qquad (8)
\end{aligned}$$

Furthermore, by (A.5) and (8), we have

$$\begin{aligned}
E[A_{nj1}^{[-i]}(u,\alpha_0)] &= e^{-2g(w_0)}\begin{pmatrix} b_{j0}(u)t_{j1}(u) - b_{j1}(u)\otimes t_{j0}(u)\\ 0_{p\times d}\\ 0_d^T\end{pmatrix} + o(h_1)\\
&\quad - h_1\mu_2 e^{-2g(w_0)}\begin{pmatrix} 0_{p\times d}\\ b_{j0}(u)\mathcal{D}_w[t_{j1}(u)] - \mathcal{D}_w[b_{j1}(u)]\otimes t_{j0}(u)\\ b_{j0}(u)\mathcal{D}_w[t_{j0}(u)] - \mathcal{D}_w[b_{j0}(u)]\otimes t_{j0}(u)\end{pmatrix}. \qquad (9)
\end{aligned}$$

Since $\Phi_{nj0}^{[-i]}(u,\alpha_0,\varphi_0^*)$ and $\Phi_{nj1}^{[-i]}(u,\alpha_0,\varphi_0^*)$ are the sum of i.i.d. random variables, $\Phi_{njk}^{[-i]}(u,\alpha_0,\varphi_0^*) = E[\Phi_{njk}^{[-i]}(u,\alpha_0,\varphi_0^*)] + o_p(1)$ almost surely. Note that $Var[T_{njk}^{[-i]}(u,\alpha_0,\varphi_0^*)] = O(\frac{1}{nh_1})$. It follows that

$$
\begin{aligned}
Var[A_{n1}^{[-i]}(u,\alpha_0)] &= O(1)Var[\Phi_{n0}^{[-i]}(u,\alpha_0,\varphi_0^*)T_{n1}^{[-i]}(u,\alpha_0,\varphi_0^*)] \\
&= O(1)Var\left\{[E(\Phi_{n0}^{[-i]}(u,\alpha_0,\varphi_0^*))+o_p(1)]T_{n1}^{[-i]}(u,\alpha_0,\varphi_0^*)\right\} \\
&= O(\frac{1}{nh_1}). \qquad (10)
\end{aligned}
$$

By condition (C.6),

$$
\begin{aligned}
A_{nj1}^{[-i]}(u,\alpha_0) &= e^{-2g(w_0)}\begin{pmatrix} b_{j0}(u)t_{j1}(u)-b_{j1}(u)\otimes t_{j0}(u) \\ 0_{p\times d} \\ 0_d^T \end{pmatrix} + O_p(h_1+\frac{1}{\sqrt{nh_1}}) \\
&\triangleq A_{j1}(u,w_0)+O_p(h_1+\frac{1}{\sqrt{nh_1}}) = O_p(h_1+\frac{1}{\sqrt{nh_1}}).
\end{aligned}
$$

Similarly, $Var[A_{n2}^{[-i]}(u,\alpha_0)] = O(\frac{1}{nh_1})$, and

$$
A_{n2}^{[-i]}(u,\alpha_0) = A_{j2}(u,w_0)+O_p(h_1+\frac{1}{\sqrt{nh_1}}),
$$

where

$$
\begin{aligned}
A_{j2}(u,w_0) &= e^{-2g(w_0)}\cdot diag\Bigg(b_{j0}(u)t_{j2}(u)-b_{j2}(u)\otimes t_{j0}(u), \\
&\quad \mu_2\begin{pmatrix} b_{j0}(u)t_{j2}(u)-b_{j2}(u)\otimes t_{j0}(u) & b_{j0}(u)t_{j1}(u)-b_{j1}(u)\otimes t_{j0}(u) \\ b_{j0}(u)t_j^*(u)-b_{j1}^T(u)\otimes t_{j0}(u) & 0_d^T \end{pmatrix}\Bigg).
\end{aligned}
$$

Combining the above results with (5) and (7) yields

$$
A_{nj}^{[-i]}(u,\alpha_0) = O_p(h_1+\frac{1}{\sqrt{nh_1}}) \qquad (11)
$$

uniformly for $u\in[0,\tau]$. Since $nh^2\to\infty$, $M_{n2}(\alpha_0,\widehat{\varphi}_p^*(\alpha_0)) = M_{n2}^{[-i]}(\alpha_0,\widehat{\varphi}_p^{*[-i]}(\alpha_0))+o_p(\frac{h_1}{\sqrt{n}})$. Combining this with (2) and (6) leads to

$$
\frac{\partial\widehat{\varphi}_p^*(\alpha_0)}{\partial\alpha^T} = -\{M_{n1}^{[-i]}(\alpha_0,\widehat{\varphi}_p^{*[-i]}(\alpha_0))+O_p(\frac{1}{nh_1})\}^{-1}\{M_{n2}^{[-i]}(\alpha_0,\widehat{\varphi}_p^{*[-i]}(\alpha_0))+o_p(\frac{h_1}{\sqrt{n}})\}. \qquad (12)
$$

Note that $\Phi_{nj0}^{[-i]}(u, \alpha_0, \widehat{\varphi}_0^{*[-i]}(\alpha_0)) = e^{-2g(w_0)} b_{j0}(u, w_0) + o_p(1)$ uniformly for $u \in [0, \tau]$. It follows from (11) that

$$A_{nj}^{[-i]}(u, \alpha_0)[\Phi_{nj0}^{[-i]}(u, \alpha_0, \widehat{\varphi}_0^{*[-i]}(\alpha_0))]^{-2} = e^{2g(w_0)} A_{j1}(u, w_0) b_{j0}^{-2}(u, w_0) + O_p(h_1 + \frac{1}{\sqrt{nh_1}})$$

uniformly for $u \in [0, \tau]$. Then

$$\begin{aligned} M_{n2}^{[-i]}(\alpha_0, \widehat{\varphi}_p^{*[-i]}(\alpha_0)) &= \sum_{j=1}^{m} \int_0^\tau e^{2g(w_0)} A_{j1}(u, w_0) b_{j0}^{-2}(u, w_0) dF_{w,j}(u) + O_p(h_1 + \frac{1}{\sqrt{nh_1}}) \\ &= O_p(h_1 + \frac{1}{\sqrt{nh_1}}). \end{aligned}$$

Taking the above result and (12) together, we obtain

$$\frac{\partial \widehat{\varphi}_p^*(\alpha_0)}{\partial \alpha^T} = -\{M_{n1}^{[-i]}(\alpha_0, \widehat{\varphi}_p^{*[-i]}(\alpha_0))\}^{-1} M_{n2}^{[-i]}(\alpha_0, \widehat{\varphi}_p^{*[-i]}(\alpha_0)) + o_p(\frac{h_1}{\sqrt{n}}) \tag{13}$$

when $nh_1^2 \to \infty$.

(iii) Asymptotic expression for $\chi_{n1}(W_i)$ and $\chi_{n2}(W_i)$. By an argument similar to that used for (3), it can be shown that $E[M_{n1}^{[-i]}(\alpha_0, \widehat{\varphi}_p^{*[-i]}(\alpha_0))] = \mathbb{A}^*(w_0)(1 + o_p(1))$. Then

$$M_{n1}^{[-i]}(\alpha_0, \widehat{\varphi}_p^{*[-i]}(\alpha_0))] = \mathbb{A}^*(w_0)(1 + o_p(1)) + r_{n2}^{[-i]}, \tag{14}$$

where $r_{n2}^{[-i]}$ has a mean of zero, and $E[r_{n2}^{[-i]}] = o(1)$ uniformly for $i = 1, \cdots, n$, and $w_0 \in \cup_{j=1}^m supp(f_j)$. Let

$$M_{n21}^{[-i]} = \frac{1}{n} \sum_{j=1}^{m} \sum_{l \neq i}^{n} \int_0^\tau K_{h_1}(W_{lj} - w_0) \left[\frac{A_{nj1}^{[-i]}(w_0, \alpha_0)}{(\Phi_{nj0}^{[-i]}(w_0, \alpha_0, \widehat{\varphi}^{*[-i]}(\alpha_0)))^2} \right] dN_{lj}(u),$$

and

$$M_{n22}^{[-i]} = \frac{1}{n} \sum_{j=1}^{m} \sum_{l \neq i}^{n} \int_0^\tau K_{h_1}(W_{lj} - w_0) \left[\frac{A_{nj2}^{[-i]}(w_0, \alpha_0)}{(\Phi_{nj0}^{[-i]}(w_0, \alpha_0, \widehat{\varphi}^{*[-i]}(\alpha_0)))^2} \right] dN_{lj}(u).$$

Then by (7),

$$M_{n2}^{[-i]}(\alpha_0, \widehat{\varphi}_p^{*[-i]}(\alpha_0)) = M_{n21}^{[-i]} + M_{n22}^{[-i]}[(\widehat{\varphi}_p^{*[-i]}(\alpha_0) - \varphi_0^*) \otimes I_d] + r_{n3}^{[-i]}(w_0),$$

where $r_{n3}^{[-i]}(w_0)$ is of mean zero, and $E[r_{n3}^{[-i]}(w_0)]^2 = o(h_1^2)$ uniformly for $w_0 \in \cup_{j=1}^m supp(f_j)$, $i = 1, \cdots, n$. Therefore, by (13),

$$\begin{aligned}
&\frac{\partial \widehat{\beta}_p(w_0, \alpha)}{\partial \alpha}|_{\alpha=\alpha_0} \\
= \quad & -\mathbf{e}_p[M_{n1}^{[-i]}(\alpha_0, \widehat{\varphi}_p^{*[-i]}(\alpha_0))]^{-1} M_{n21}^{[-i]} + r_{n4}^{[-i]}(w_0) \\
& \qquad - \mathbf{e}_p[M_{n1}^{[-i]}(\alpha_0, \widehat{\varphi}_p^{*[-i]}(\alpha_0))]^{-1} M_{n22}^{[-i]}[(\widehat{\varphi}_p^{*[-i]}(\alpha_0) - \varphi_0^*) \otimes I_d] + O_p(\frac{1}{nh_1}) \\
\equiv \quad & \mathbf{e}_p L_{n1}^{[-i]}(w_0) + \mathbf{e}_p L_{n2}^{[-i]}(w_0)[(\widehat{\varphi}_p^{*[-i]}(\alpha_0) - \varphi_0^*) \otimes I_d] + r_{n4}^{[-i]}(w_0) + O_p(\frac{1}{nh_1}),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial \widehat{g}_p'(w_0, \alpha)}{\partial \alpha}|_{\alpha=\alpha_0} \\
= \quad & -e_1^T h_1^{-1}[M_{n1}^{[-i]}(\alpha_0, \widehat{\varphi}_p^{*[-i]}(\alpha_0))]^{-1} M_{n21}^{[-i]} + r_{n5}^{[-i]}(w_0) \\
& \qquad - e_1^T h_1^{-1}[M_{n1}^{[-i]}(\alpha_0, \widehat{\varphi}_p^{*[-i]}(\alpha_0))]^{-1} M_{n22}^{[-i]}[(\widehat{\varphi}_p^{*[-i]}(\alpha_0) - \varphi_0^*) \otimes I_d] + o_p(\frac{1}{\sqrt{n}}) \\
\equiv \quad & e_1^T h_1^{-1} L_{n1}^{[-i]}(w_0) + e_1^T h_1^{-1} L_{n2}^{[-i]}(w_0)[(\widehat{\varphi}_p^{*[-i]}(\alpha_0) - \varphi_0^*) \otimes I_d] + r_{n5}^{[-i]}(w_0) + o_p(\frac{1}{\sqrt{n}}),
\end{aligned}$$

where $r_{n4}^{[-i]}(w_0)$ and $r_{n5}^{[-i]}(w_0)$ are independent of the i-th subject such that $E[r_{n4}^{[-i]}(w_0)]^2 = E[r_{n5}^{[-i]}(w_0)]^2 = o(1)$ uniformly for $w_0 \in \cup_{j=1}^m supp(f_j)$. By (9), we obtain

$$\begin{aligned}
E\left[\frac{A_{n1}^{[-i]}(u, \alpha_0)}{\{\Phi_{nj0}^{[-i]}(w_0, \alpha_0, \widehat{\varphi}_p^*(\alpha_0))\}^2}\right] \quad &= \quad \begin{pmatrix} \frac{b_{j0}(u)t_{j1}(u) - b_{j1}(u) \otimes t_{j0}(u)}{b_{j0}^2(u)} \\ 0_{p\times d} \\ 0_d^T \end{pmatrix} \\
& \qquad + h_1 \mu_2 \begin{pmatrix} 0_{p\times d} \\ \frac{b_{j0}(u)\mathcal{D}_w[t_{j1}(u)] - \mathcal{D}_w[b_{j1}(u)] \otimes t_{j0}(u)}{b_{j0}^2(u)} \\ \frac{b_{j0}(u)\mathcal{D}_w[t_{j0}(u)] - \mathcal{D}_w[b_{j0}(u)] \otimes t_{j0}(u)}{b_{j0}^2(u)} \end{pmatrix} + o(h_1) \\
&= \quad C_{j2}(u, w_0) + h_1 C_{j3}(u, w_0) + o(h_1), \qquad (15)
\end{aligned}$$

which yields

$$M_{n21}^{[-i]} \quad = \quad \sum_{j=1}^m \int_0^\tau [C_{j1}(u, w_0) + h_1 C_{j2}(u, w_0)] b_{j0}(u, w_0) d\Lambda_{0j}(u) + o_p(\frac{h_1}{\sqrt{n}}).$$

Together with (14), we can show that

$$L_{n1}^{[-i]}(w_0) = \mathbb{A}^{*^{-1}}(w_0)[C_2(w_0) + h_1 C_3(w_0)] + r_{n6}^{[-i]}(w_0),$$

where $r_{n6}^{[-i]}(w_0)$ is independent of the i-th subject such that $E[r_{n6}^{[-i]}(w_0)]^2 = o(1)$ uniformly for $w_0 \in \cup_{j=1}^m supp(f_j)$, $i = 1, \cdots, n$. Similarly, $L_{n2}^{[-i]}(w_0) = O_p(1) + r_{n7}^{[-i]}(w_0)$, where $r_{n6}^{[-i]}(w_0)$ is independent of the i-th subject such that $E[r_{n7}^{[-i]}(w_0)]^2 = o(1)$ uniformly for $w_0 \in \cup_{j=1}^m supp(f_j)$, $i = 1, \cdots, n$. Based on the above and assuming that $nh_1^2 \to \infty$, Lemma 6 holds true. $\Box$

Lemma 7 *Assume that conditions C.1 - C.8 hold. If $nh^2 \to \infty$, then the following hold uniformly for $w \in \cup_{j=1}^m supp(f_j)$,*
(i)

$$\begin{aligned}
\kappa_{n1}(w_0) &= -e_1^T \sum_{j=1}^m \int_0^{w_0} \mathbb{A}^{*^{-1}}(w) \int_0^\tau [C_{j4}(u,w) - C_{j5}(u,w)] d\Lambda_{0j}(u) dw + O_p(h_1^2) \\
&\triangleq \kappa_1(w_0) + o_p(1), \\
\textit{and} \quad \kappa_{n2}(w_0) &= -h_1 \mathbf{e}_p \mathbb{A}^{*^{-1}}(w_0) \sum_{j=1}^m \int_0^\tau [C_{j4}(u,w_0) - C_{j5}(u,w_0)] d\Lambda_{0j}(u) + O_p(h_1^2) \\
&\triangleq \kappa_2(w_0) + o_p(1)
\end{aligned}$$

(ii) $\dfrac{\partial^3 \widehat{g}_p'(w_0,\alpha)}{\partial\alpha_j \partial\alpha_k \partial\alpha_l} = O_p(1)$ *and* $\dfrac{\partial^3 \widehat{\beta}_p(w_0,\alpha)}{\partial\alpha_j \partial\alpha_k \partial\alpha_l} = O_p(h_1)$ *for $\alpha \in \mathcal{A}$.*

Proof: (i) The proof of this lemma involves some very tedious algebra. Here, we employ the arguments in Bates and Watts (1988) to compute the second derivative of some vector functions with respect to a vector. Following the notations of Bates and Watts (1980, 1988), we use a square bracket to denote the multiplication of a matrix and an array. For an $N_1 \times N_2$ matrix B and an $N_2 \times N_3 \times N_4$ array C, $A = [B][C]$ is an $N_1 \times N_3 \times N_4$ array with the s-th face, p-th row, and q-th column element given by

$$\{A\}_{pqs} = \sum_{i=1}^{N_2} \{B\}_{pi} \{C\}_{ips}.$$

Similarly, for an $N_1 \times N_2 \times N_4$ array E and an $N_2 \times N_3$ matrix F, $D = [E][F]$ is an $N_1 \times N_3 \times N_4$ array with

$$\{D\}_{pqs} = \sum_{i=1}^{N_2} \{E\}_{pis} \{F\}_{ip}$$

as a typical element. For the matrices G and H, we also use $[G][H]$ to represent the matrix multiplication of G and H.

Differentiating (2) with respect to α, and by some algebraic manipulations, we obtain

$$\begin{aligned}
&\frac{\partial M_{n2}(\alpha_0, \widehat{\varphi}_p^*(\alpha_0))}{\partial \alpha} + \left[\frac{\partial \widehat{\varphi}_p^{*T}(\alpha_0)}{\partial \alpha}\right]\left[\frac{\partial M_{n2}(\alpha_0, \widehat{\varphi}_p^*(\alpha_0))}{\partial \widehat{\varphi}_p^*(\alpha_0)}\right] \\
&+ \left\{\frac{\partial M_{n1}(\alpha_0, \widehat{\varphi}_p^*(\alpha_0))}{\partial \alpha} + \left[\frac{\partial \widehat{\varphi}_p^{*T}(\alpha_0)}{\partial \alpha}\right]\left[\frac{\partial M_{n1}(\alpha_0, \widehat{\varphi}_p^*(\alpha_0))}{\partial \widehat{\varphi}_p^*(\alpha_0)}\right]\right\}\left[\frac{\partial \widehat{\varphi}_p^*(\alpha_0)}{\partial \alpha^T}\right] \\
&+ \left[\frac{\partial^2 \widehat{\varphi}_p^*(\alpha_0)}{\partial \alpha \partial \alpha^T}\right] M_{n1}(\alpha_0, \widehat{\varphi}_p^*(\alpha_0)) = 0. \qquad (16)
\end{aligned}$$

We can also show that $\frac{\partial M_{n1}(\alpha, \widehat{\varphi}_p^*(\alpha))}{\partial \widehat{\varphi}_p^*(\alpha)}|_{\alpha=\alpha_0} = O_p(h_1)$, $\frac{\partial M_{n1}(\alpha, \widehat{\varphi}_p^*(\alpha))}{\partial \alpha}|_{\alpha=\alpha_0} = O_p(h_1)$, and $\frac{\partial M_{n2}(\alpha, \widehat{\varphi}_p^*(\alpha))}{\partial \widehat{\varphi}_p^*(\alpha)}|_{\alpha=\alpha_0} = O_p(h_1)$. Now, by Lemma 6, $\frac{\partial \widehat{\varphi}_p^*(\alpha)}{\partial \alpha^T}|_{\alpha=\alpha_0} = O_p(h_1)$. Then by (16),

$$\frac{\partial M_{n2}(\alpha, \widehat{\varphi}_p^*(\alpha))}{\partial \alpha}|_{\alpha=\alpha_0} + \frac{\partial^2 \widehat{\varphi}_p^*(\alpha)}{\partial \alpha \partial \alpha^T}|_{\alpha=\alpha_0} M_{n1}(\alpha_0, \widehat{\varphi}_p^*(\alpha_0)) = O_p(h_1^2). \qquad (17)$$

It can be shown that for $k = 0, 1$,

$$\begin{aligned}
\frac{\partial T_{njk}(u, \alpha, \widehat{\varphi}_p^*(\alpha))}{\partial \alpha} &= \frac{1}{n}\sum_{i=1}^n \tilde{S}_{ij}(u, \alpha, \widehat{\varphi}_p^*(\alpha)) \tilde{U}_{ij}^{\otimes k}(w_0) \otimes V_{ij}^{\otimes 2}(u) K_{h_1}(W_{ij} - w_0) \\
&\quad + [T_{nj(k+1)}^T(u, \alpha, \widehat{\varphi}_p^*(\alpha))]\left[\frac{\partial \widehat{\varphi}_p^*(\alpha)}{\partial \alpha^T}\right].
\end{aligned}$$

Let

$$G_{njk}(u, \alpha, \widehat{\varphi}_p^*(\alpha)) = \frac{1}{n}\sum_{i=1}^n \tilde{S}_{ij}(u, \alpha, \widehat{\varphi}_p^*(\alpha)) \tilde{U}_{ij}^{\otimes k}(w_0) \otimes V_{ij}^{\otimes 2}(u) K_{h_1}(W_{ij} - w_0).$$

This yields, for $k = 0, 1$,

$$\frac{\partial T_{njk}(u, \alpha, \widehat{\varphi}_p^*(\alpha))}{\partial \alpha} = G_{njk}(u, \alpha, \widehat{\varphi}_p^*(\alpha)) + [T_{nj(k+1)}^T(u, \alpha, \widehat{\varphi}_p^*(\alpha))]\left[\frac{\partial \widehat{\varphi}_p^*(\alpha)}{\partial \alpha^T}\right].$$

Similar algebraic manipulations lead to the following $(2p+1) \times (2p+1) \times d$ array:

$$\frac{\partial M_{n2}(\alpha, \widehat{\varphi}_p^*(\alpha))}{\partial \alpha}$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^{\tau} K_{h_1}(W_{ij}-w_0) \\
&\quad \cdot \frac{\partial\Big[\frac{T_{nj1}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\Phi_{nj0}(u,\alpha,\widehat{\varphi}_p^*(\alpha)) - \Phi_{nj1}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\otimes T_{nj0}(u,\alpha,\widehat{\varphi}_p^*(\alpha))}{\Phi_{nj0}^2(u,\alpha,\widehat{\varphi}_p^*(\alpha))}\Big]}{\partial\alpha} dN_{ij}(u) \\
&\triangleq \frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^{\tau} K_{h_1}(W_{ij}-w_0)\,[P_{nj1}(u,\alpha) - 2P_{nj2}(u,\alpha)]\,dN_{ij}(u),
\end{aligned}
$$

where

$$
\begin{aligned}
&P_{nj1}(u,\alpha) \\
&= \frac{\partial\Big[T_{nj1}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\Phi_{nj0}(u,\alpha,\widehat{\varphi}_p^*(\alpha)) - \Phi_{nj1}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\otimes T_{nj0}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\Big]}{\partial\alpha}\Phi_{nj0}^{-2}(u,\alpha,\widehat{\varphi}_p^*(\alpha)) \\
&= \Phi_{nj0}^{-2}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\Big[G_{nj1}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\Phi_{nj0}(u,\alpha,\widehat{\varphi}_p^*(\alpha)) - \Phi_{nj1}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\otimes G_{nj0}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
&P_{nj2}(u,\alpha) \\
&= \Big[T_{nj1}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\Phi_{nj0}(u,\alpha,\widehat{\varphi}_p^*(\alpha)) - \Phi_{nj1}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\otimes T_{nj0}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\Big] \\
&\qquad \Phi_{nj0}^{-3}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\frac{\partial\Phi_{nj0}(u,\alpha,\widehat{\varphi}_p^*(\alpha))}{\partial\alpha} \\
&= \Phi_{nj0}^{-3}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\Big[(T_{nj1}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\otimes T_{nj0}^T(u,\alpha,\widehat{\varphi}_p^*(\alpha)))\Phi_{nj0}(u,\alpha,\widehat{\varphi}_p^*(\alpha)) \\
&\qquad - \Phi_{nj1}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\otimes T_{nj0}^{\otimes 2}(u,\alpha,\widehat{\varphi}_p^*(\alpha))\Big].
\end{aligned}
$$

By an argument similar to part (ii) of Lemma 6, we have

$$
G_{nj0}(u,\alpha_0,\widehat{\varphi}_p^*(\alpha_0)) \quad = \quad e^{-g(w_0)}\tilde{t}_{j0}(u) + O_p(h_1^2 + \frac{1}{\sqrt{nh_1}})
$$

and

$$
\begin{aligned}
&G_{nj1}(u,\alpha_0,\widehat{\varphi}_p^*(\alpha_0)) \\
&= e^{-g(w_0)}\{\begin{pmatrix} \tilde{t}_{j1}(u) \\ 0_{pd\times d} \\ 0_{d\times d}\end{pmatrix} + h_1\mu_2\begin{pmatrix} 0_{pd\times d} \\ \mathcal{D}_w[\tilde{t}_{j1}(u)] \\ \mathcal{D}_w[\tilde{t}_{j0}(u)]\end{pmatrix}\} + O_p(h_1^2 + \frac{1}{\sqrt{nh_1}}),
\end{aligned}
$$

where, for $k = 0, 1$,

$$\tilde{t}_{jk}(u) = \tilde{t}_{jk}(u, w_0) = \tilde{t}_{jk}(u, \alpha_0, w_0) = f_j(w_0) E[\rho(u, Z_j, V_j, w_0) \cdot Z_j^{\otimes k} \otimes V_j^{\otimes 2}(u) | w = w_0].$$

Since Z and V are independent conditional on W, we have,

$$P_{n1}(u,\alpha) \quad = \quad h_1 \mu_2 \begin{pmatrix} 0_{pd\times d} \\ \frac{\mathcal{D}_w[\tilde{t}_{j1}(u)] b_{j0}(u) - \mathcal{D}_w[b_{j1}(u)] \otimes \tilde{t}_{j0}(u)}{b_{j0}^2(u)} \\ \frac{\mathcal{D}_w[\tilde{t}_{j0}(u)] b_{j0}(u) - \mathcal{D}_w[b_{j0}(u)] \otimes \tilde{t}_{j0}(u)}{b_{j0}^2(u)} \end{pmatrix} + O_p(h_1^2),$$

and

$$P_{n2}(u,\alpha) \quad = \quad h_1 \mu_2 b_{j0}^{-2}(u) \begin{pmatrix} 0_{pd\times d} \mathcal{D}_w[t_{j1}(u)] \otimes t_{j0}^T(u) - \frac{\mathcal{D}_w[b_{j1}(u)] \otimes t_{j0}^{\otimes 2}(u)}{b_{j0}(u)} \\ \mathcal{D}_w[t_{j0}(u)] \otimes t_{j0}^T(u) - \frac{\mathcal{D}_w[b_{j0}(u)] \otimes t_{j0}^{\otimes 2}(u)}{b_{j0}(u)} \end{pmatrix} + O_p(h_1^2).$$

This leads to

$$\begin{aligned}
\frac{\partial M_{n2}(\alpha, \widehat{\varphi}_p^*(\alpha))}{\partial \alpha} \quad &= \quad h_1 \mu_2 \sum_{j=1}^m \int_0^\tau b_{j0}^{-1}(u) \{ \begin{pmatrix} 0_{pd\times d} \\ \mathcal{D}_w[\tilde{t}_{j1}(u)] b_{j0}(u) - \mathcal{D}_w[b_{j1}(u)] \otimes \tilde{t}_{j0}(u) \\ \mathcal{D}_w[\tilde{t}_{j0}(u)] b_{j0}(u) - \mathcal{D}_w[b_{j0}(u)] \otimes \tilde{t}_{j0}(u) \end{pmatrix} \\
&\quad -2 \begin{pmatrix} 0_{pd\times d} \\ \mathcal{D}_w[t_{j1}(u)] \otimes t_{j0}^T(u) - \frac{\mathcal{D}_w[b_{j1}(u)] \otimes t_{j0}^{\otimes 2}(u)}{b_{j0}(u)} \\ \mathcal{D}_w[t_{j0}(u)] \otimes t_{j0}^T(u) - \frac{\mathcal{D}_w[b_{j0}(u)] \otimes t_{j0}^{\otimes 2}(u)}{b_{j0}(u)} \end{pmatrix} \} d\Lambda_{0j}(u) + O_p(h_1^2) \\
&\triangleq \quad h_1 \sum_{j=1}^m \int_0^\tau [C_{j4}(u, w_0) - C_{j5}(u, w_0)] d\Lambda_{0j}(u) + O_p(h_1^2),
\end{aligned}$$

which, when combined with (3) and (17), yields

$$\frac{\partial^2 \widehat{\varphi}_p^*(\alpha)}{\partial \alpha \partial \alpha^T}|_{\alpha=\alpha_0} = -h_1 \mathbb{A}^{*^{-1}}(w_0) \sum_{j=1}^m \int_0^\tau [C_{j4}(u, w_0) - C_{j5}(u, w_0)] d\Lambda_{0j}(u) + O_p(h_1^2).$$

We then have

$$\frac{\partial^2 \widehat{\beta}_p(\alpha)}{\partial \alpha \partial \alpha^T}|_{\alpha=\alpha_0} = -h_1 \mathbf{e}_p \mathbb{A}^{*^{-1}}(w_0) \sum_{j=1}^m \int_0^\tau [C_{j4}(u, w_0) - C_{j5}(u, w_0)] d\Lambda_{0j}(u) + O_p(h_1^2),$$

and

$$\frac{\partial^2 \widehat{g}_p^{'}(\alpha)}{\partial\alpha\partial\alpha^T}|_{\alpha=\alpha_0} = -e_1^T \mathbb{A}^{*^{-1}}(w_0)\sum_{j=1}^m \int_0^\tau [C_{j4}(u,w_0) - C_{j5}(u,w_0)]d\Lambda_{0j}(u) + O_p(h_1^2).$$

From these results, it is easily seen that part (i) of Lemma 7 holds.

(ii) The proof is straightforward and is thus omitted.

Proof of Lemma 2. For $k = 0, 1, 2$, and $j = 1, \cdots, m$, let

$$\begin{aligned} R_{njk}(u,\alpha_0) &= \frac{1}{n}\sum_{i=1}^n Y_{ij}(u)\exp\{\alpha_0^T V_{ij}(u) + \widehat{\beta}_p^T(W_{ij},\alpha_0)Z_{ij} + \widehat{g}_p(W_{ij},\alpha_0)\} \\ &\quad \{V_{ij}(u) + [\frac{\partial\widehat{\beta}_p(W_{ij},\alpha_0)}{\partial\alpha^T}]^T Z_{ij} + \frac{\partial\widehat{g}_p(W_{ij},\alpha_0)}{\alpha}\}^{\otimes k} \\ &= \frac{1}{n}\sum_{i=1}^n Y_{ij}(u)\exp\{\alpha_0^T V_{ij}(u) + \widehat{\beta}_p^T(W_{ij},\alpha_0)Z_{ij} + \widehat{g}_p(W_{ij},\alpha_0)\} \\ &\quad [V_{ij}(u) + \chi_{n2}^T(W_{ij})Z_{ij} + \chi_{n1}^T(W_{ij})]^{\otimes k}. \end{aligned}$$

By (2.5),

$$\sqrt{n}\frac{\partial\ell_p(\alpha)}{\partial\alpha}|_{\alpha=\alpha_0} = \frac{1}{\sqrt{n}}\sum_{j=1}^m\sum_{i=1}^n \left\{[V_{ij}(u) + \chi_{n2}^T(W_{ij})Z_{ij} + \chi_{n1}^T(W_{ij})] - \frac{R_{nj1}(u,\alpha_0)}{R_{nj0}(u,\alpha_0)}\right\} dN_{ij}(u).$$

For $k = 0, 1$, define $R_{njk}^*(u,\alpha_0)$ similarly to $R_{njk}(u,\alpha_0)$, except with $\widehat{\beta}_p^T(W_{ij},\alpha_0)$, $\widehat{g}_p(W_{ij},\alpha_0)$, $\chi_{n2}(W_{ij})$ and $\chi_{n1}(W_{ij})$ replaced by $\beta_p^T(W_{ij})$, $g(W_{ij})$, $\chi_2(W_{ij})$ and $\chi_1(W_{ij})$ respectively. Let

$$\begin{aligned} L_{n1}^* &= \frac{1}{\sqrt{n}}\sum_{j=1}^m\sum_{i=1}^n\int_0^\tau \left\{V_{ij}(u) + \chi_2^T(W_{ij})Z_{ij} + \chi_1^T(W_{ij}) - \frac{R_{nj1}^*(u,\alpha_0)}{R_{nj0}^*(u,\alpha_0)}\right\} dN_{ij}(u), \\ L_{n2}^* &= \frac{1}{\sqrt{n}}\sum_{j=1}^m\sum_{i=1}^n\int_0^\tau \left\{\frac{R_{nj1}^*(u,\alpha_0)}{R_{nj0}^*(u,\alpha_0)} - \frac{R_{nj1}(u,\alpha_0)}{R_{nj0}(u,\alpha_0)}\right\} dN_{ij}(u), \end{aligned}$$

and

$$L_{n3}^* = \frac{1}{\sqrt{n}}\sum_{j=1}^m\sum_{i=1}^n\int_0^\tau \left\{\chi_{n2}^T(W_{ij})Z_{ij} + \chi_{n1}^T(W_{ij}) - [\chi_2^T(W_{ij})Z_{ij} + \chi_1^T(W_{ij})]\right\} dN_{ij}(u).$$

Then

$$\sqrt{n}\frac{\partial \ell_p(\alpha)}{\partial \alpha}|_{\alpha=\alpha_0} = L_{n1}^* + L_{n2}^* + L_{n3}^*. \tag{18}$$

Note that

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^\tau \left\{ V_i(u) + \chi_{22}^T(W_{ij})Z_{ij} + \chi_{12}^T(W_{ij}) - \frac{R_{nj1}^*(u,\alpha_0)}{R_{nj0}^*(u,\alpha_0)} \right\}$$
$$Y_{ij}(u)\exp\{\alpha_0^T V_{ij}(u) + \beta^T(W_{ij})Z_{ij} + g(W_{ij})\}\lambda_{0j}(u)du = 0.$$

This implies

$$\begin{aligned}
& L_{n1}^* \\
= & \frac{1}{\sqrt{n}}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^\tau \left\{ V_{ij}(u) + \chi_2^T(W_{ij})Z_{ij} + \chi_1^T(W_{ij}) - \frac{R_{nj1}^*(u,\alpha_0)}{R_{nj0}^*(u,\alpha_0)} \right\} dN_{ij}(u) + o_p \\
= & \frac{1}{\sqrt{n}}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^\tau \left\{ V_{ij}(u) + \chi_2^T(W_{ij})Z_{ij} + \chi_1^T(W_{ij}) - \frac{r_{j1}(u,\alpha_0)}{r_{j0}(u,\alpha_0)} \right\} dM_{ij}(u)(1). \qquad (19)
\end{aligned}$$

The last equality in the above results because $\frac{R_{nj1}^*(u,\alpha_0)}{R_{nj0}^*(u,\alpha_0)} - \frac{r_{j1}(u,\alpha_0)}{r_{j0}(u,\alpha_0)}$ is $\mathcal{F}_{u,ij}$-predictable and converges to zero in probability for each $j = 1, \cdots, m$, and uniformly for $u \in [0, \tau]$. Write

$$\begin{aligned}
\tilde{R}_{njk}^*(u,\alpha_0) &= \frac{1}{n}\sum_{i=1}^{n} Y_{ij}(u)\exp\{\alpha_0^T V_{ij}(u) + \widehat{\beta}_p^T(W_{ij},\alpha_0)Z_{ij} + \widehat{g}_p(W_{ij},\alpha_0)\} \\
& [V_{ij}(u) + \chi_2^T(W_{ij})Z_{ij} + \chi_1^T(W_{ij})]^{\otimes k},
\end{aligned}$$

$$\begin{aligned}
\tilde{R}_{nk}^{**}(u,\alpha_0) &= \frac{1}{n}\sum_{i=1}^{n} Y_{ij}(u)\exp\{\alpha_0^T V_{ij}(u) + \widehat{\beta}_p^T(W_{ij},\alpha_0)Z_{ij} + \widehat{g}_p(W_{ij},\alpha_0)\} \\
& \left[(\chi_{n2}^T(W_{ij}) - \chi_{22}^T(W_{ij}))Z_{ij} + (\chi_{n1}^T(W_{ij}) - \chi_{12}^T(W_{ij}))\right]^{\otimes k},
\end{aligned}$$

and

$$\begin{aligned}
L_{n2}^* &= \frac{1}{\sqrt{n}}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^\tau \left[\frac{R_{nj1}^*(u,\alpha_0)}{R_{nj0}^*(u,\alpha_0)} - \frac{\tilde{R}_{nj1}^*(u,\alpha_0)}{\tilde{R}_{nj0}^*(u,\alpha_0)}\right] dN_{ij}(u) \\
& \qquad - \frac{1}{\sqrt{n}}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^\tau \frac{\tilde{R}_{nj1}^{**}(u,\alpha_0)}{\tilde{R}_{nj0}^{**}(u,\alpha_0)} dN_{ij}(u) \\
&\equiv K_{n1} + K_{n2}. \qquad (20)
\end{aligned}$$

Then

$$
\begin{aligned}
L_{n3}^{*}+K_{n2} &= \frac{1}{\sqrt{n}}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^{\tau}\Bigg[(\chi_{n2}^{T}(W_{ij})-\chi_{2}^{T}(W_{ij}))Z_{ij}+(\chi_{n1}^{T}(W_{ij})-\chi_{1}^{T}(W_{ij})) \\
&\qquad -\frac{\tilde{R}_{nj1}^{**}(u,\alpha_0)}{\tilde{R}_{nj0}^{**}(u,\alpha_0)}\Bigg]dN_{ij}(u).
\end{aligned} \tag{21}
$$

It can be shown that

$$
L_{n3}^{*}+K_{n2}=o_p(1). \tag{22}
$$

These results lead to

$$
\begin{aligned}
\sqrt{n}\frac{\partial \ell_p(\alpha)}{\partial \alpha}|_{\alpha=\alpha_0} &= \frac{1}{\sqrt{n}}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^{\tau}\left\{V_{ij}(u)+\chi_{2}^{T}(W_{ij})Z_{ij}+\chi_{1}^{T}(W_{ij})-\frac{r_{j1}(u,\alpha_0)}{r_{j0}(u,\alpha_0)}\right\}dM_{ij}(u) \\
&\quad +K_{n1}+o_p(1).
\end{aligned}
$$

In the following, we show that K_{n1} may be written as

$$
K_{n1}=-\frac{1}{\sqrt{n}}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^{\tau}\sigma(W_{ij})Q(u,W_{ij})s(W_{ij})dM_{ij}(u)+o_p(1).
$$

(i) First, let us derive the asymptotic expression for $\widehat{g}_p(w,\alpha_0)$. We know that $\widehat{\varphi}_p^{*}\equiv\widehat{\varphi}_p^{*}(w_0,\alpha_0)$ satisfies

$$
\widehat{U}(\varphi_0^{*},w_0)+\frac{\partial \widehat{U}(\tilde{\varphi}^{*},w_0)}{\partial \varphi^{*}}(\widehat{\varphi}_p^{*}-\varphi_0^{*})=0, \tag{23}
$$

where $\tilde{\varphi}^{*}$ lies between $\widehat{\varphi}_p^{*}$ and φ_0^{*}. Hence $\tilde{\varphi}^{*}\to\varphi_0^{*}$ in probability. This, when combined with (A.15)-(A.17), yields

$$
\begin{aligned}
&\widehat{\varphi}_p^{*}(w_0,\alpha_0)-\varphi_0^{*} \\
=\ &\mathbb{A}^{*^{-1}}(w_0)n^{-1}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^{\tau}K_{h_1}(W_{ij}-w_0)\left[\tilde{U}_{ij}(w_0)-\frac{\Phi_{nj1}(u,\alpha_0,\varphi_0^{*})}{\Phi_{nj0}(u,\alpha_0,\varphi_0^{*})}\right]dM_{ij}(u) \\
&+\mathbb{A}^{*^{-1}}(w_0)\mathbb{B}_n^{*}(\tau,w_0)+O_p(h_1^4).
\end{aligned}
$$

Since $nh_1^4 \to 0$, we have

$$
\begin{aligned}
& \widehat{\beta}_p^*(w_0,\alpha_0) - \beta_0^*(w_0) \\
= \;& \mathbf{e}_p {\mathbb{A}^*}^{-1}(w_0) n^{-1} \sum_{j=1}^m \sum_{i=1}^n \int_0^\tau K_{h_1}(W_{ij}-w_0) \left[\tilde{U}_{ij}(w_0) - \frac{\Phi_{nj1}(u,\alpha_0,\varphi_0^*)}{\Phi_{nj0}(u,\alpha_0,\varphi_0^*)} \right] dM_{ij}(u) + o_p(\frac{1}{\sqrt{n}}) \\
= \;& \mathbf{e}_p {\mathbb{A}^*}^{-1}(w_0) n^{-1} \sum_{j=1}^m \sum_{i=1}^n \int_0^\tau K_{h_1}(W_{ij}-w_0) \left[\tilde{U}_{ij}(w_0) - C_{j1}(u,w_0) \right] dM_{ij}(u) + o_p(\frac{1}{\sqrt{n}})
\end{aligned}
$$

and

$$
\begin{aligned}
& \widehat{g}_p'(w_0,\alpha_0) - g_0'(w_0) \\
= \;& e_1^T {\mathbb{A}^*}^{-1}(w_0)(nh_1)^{-1} \sum_{j=1}^m \sum_{i=1}^n \int_0^\tau K_{h_1}(W_{ij}-w_0) \left[\tilde{U}_{ij}(w_0) - \frac{\Phi_{nj1}(u,\alpha_0,\varphi_0^*)}{\Phi_{nj0}(u,\alpha_0,\varphi_0^*)} \right] dM_{ij}(u) \\
& + e_1^T {\mathbb{A}^*}^{-1}(w_0) h_1^{-1} \mathbb{B}_n^*(\tau,w_0) + o_p(\frac{1}{\sqrt{n}})
\end{aligned}
$$

uniformly for $w_0 \in \cup_{j=1}^m supp(f_j)$. Then

$$
\begin{aligned}
& \widehat{g}_p(w,\alpha_0) - g_0(w) \\
& = \frac{e_1^T}{(nh_1)} \sum_{j=1}^m \sum_{i=1}^n \int_0^\tau \int_0^w {\mathbb{A}^*}^{-1}(w_0) K_{h_1}(W_{ij}-w_0) [\tilde{U}_{ij}(w_0) - C_{j1}(u,w_0)] dw_0 dM_{ij}(u) \\
& + \frac{e_1^T}{(nh_1)} \sum_{j=1}^m \sum_{i=1}^n \int_0^\tau \int_0^w {\mathbb{A}^*}^{-1}(w_0) K_{h_1}(W_{ij}-w_0) [C_{j1}(u,w_0) - \frac{\Phi_{nj1}(u,\alpha_0,\varphi_0^*)}{\Phi_{nj0}(u,\alpha_0,\varphi_0^*)}] dw_0 dM_{ij}(u) \\
& + e_1^T h_1^{-1} \int_0^w {\mathbb{A}^*}^{-1}(w_0) \mathbb{B}_n^*(\tau,w_0) dw_0 + o_p(\frac{1}{\sqrt{n}})
\end{aligned}
\tag{24}
$$

uniformly for $w \in \cup_{j=1}^m supp(f_j)$. By variable transformation and Taylor series expansion, we have, for a density kernel K with compact support,

$$
\begin{aligned}
& e_1^T h_1^{-1} \int_0^w \mathbb{A}^{*-1}(w_0) K_{h_1}(W_{ij}-w_0) \left[\tilde{U}_{ij}(w_0) - C_{j1}(u,w_0) \right] dw_0 \\
= \;& -e_1^T h_1^{-1} \int_{W_{ij}/h_1}^{(W_{ij}-w)/h_1} \left[\mathbb{A}^{*-1}(W_i) - [\mathbb{A}^{*-1}(W_i)]' t h_1 + o(h_1) \right] K(t) \\
& \qquad [\begin{pmatrix} Z_{ij} \\ Z_{ij}t \\ t \end{pmatrix} - C_{j1}(u,W_{ij}) - t h_1 C_{j1}'(u,W_{ij}) - o(h_1)] dt \\
= \;& -h_1^{-1} c_{n0}(W_{ij},w) + c_{n1}(W_{ij},w) + c_{n2}(W_{ij},w) + o_p(1)
\end{aligned}
\tag{25}
$$

uniformly for $i = 1, 2, \cdots, n$, and $w \in J_W$, where

$$
\begin{aligned}
& c_{n0}(W_{ij}, w) \\
= \;& e_1^T \mathbb{A}^{*-1}(W_{ij}) \Big[\int_{W_{ij}/h_1}^{(W_{ij}-w)/h_1} K(t) \begin{pmatrix} Z_{ij} \\ Z_{ij}t \\ t \end{pmatrix} dt - C_{j1}(u, W_{ij}) \int_{W_{ij}/h_1}^{(W_{ij}-w)/h_1} K(t)dt \Big],
\end{aligned}
$$

$$c_{n1}(W_{ij}, w) = e_1^T \mathbb{A}^{*-1}(W_{ij}) C'_{j1}(u, W_{ij}) \int_{W_i/h_1}^{(W_i-w)/h_1} tK(t)dt,$$

and

$$
\begin{aligned}
& c_{n2}(W_{ij}, w) \\
= \;& e_1^T \mathbb{A}^{*-1}(W_{ij}) \Big[\int_{W_{ij}/h_1}^{(W_{ij}-w)/h_1} tK(t) \begin{pmatrix} Z_{ij} \\ Z_{ij}t \\ t \end{pmatrix} dt - C_{j1}(u, W_{ij}) \int_{W_{ij}/h_1}^{(W_{ij}-w)/h_1} tK(t)dt \Big].
\end{aligned}
$$

By some algebraic manipulations, we obtain the following $\mathcal{F}_{t,j}$-predictable process approximation:

$$C_{j1}(u, w_0) - \frac{\Phi_{nj1}(u, \alpha_0, \varphi_0^*)}{\Phi_{nj0}(u, \alpha_0, \varphi_0^*)} = -h_1 \mu_2 \begin{pmatrix} 0_p \\ \frac{\mathcal{D}_w[t_{j1}(u,w_0)]}{t_{j0}(u,w_0)} \\ \mathcal{D}_w[\log(t_{j0}(u, w_0))] \end{pmatrix} (1 + o_p(1)) + r_{nj}(u),$$

where $r_{nj}(t)$ is $\mathcal{F}_{t,j}$-predictable, and of mean zero and variance $O(\frac{1}{nh_1})$. Then by an argument similar to that for (25), it can be shown that the $\mathcal{F}_{t,j}$-predicable process has the following property:

$$
\begin{aligned}
& e_1^T h_1^{-1} \int_0^w \mathbb{A}^{*-1}(w_0) K_{h_1}(W_{ij} - w_0) \left[C_{j1}(u, w_0) - \frac{\Phi_{nj1}(u, \alpha_0, \varphi_0^*)}{\Phi_{nj0}(u, \alpha_0, \varphi_0^*)} \right] dw_0 \\
= \;& e_1^T \mathbb{A}^{*-1}(W_{ij}) \left[\mu_2 \begin{pmatrix} 0_p \\ \frac{\mathcal{D}_w[t_{j1}(u,W_{ij})]}{t_{j0}(u,W_{ij})} \\ \mathcal{D}_w[\log(t_{j0}(u, W_{ij}))] \end{pmatrix} + r_{nj}(u) \right] \int_{\frac{W_i}{h_1}}^{\frac{W_i - w}{h_1}} K(t)dt + o_p(1) \quad (26)
\end{aligned}
$$

uniformly for $u \in [0, \tau]$, $i = 1, 2, \cdots, n$. Note that when $|W_{ij}| \geq h_1$ and $|W_{ij} - w| \geq h_1$,

$$c_{n0}(W_{ij}, w) \;=\; e_1^T \mathbb{A}^{*-1}(W_{ij}) \begin{pmatrix} Z_{ij} - \frac{t_{j1}(u,W_{ij})}{t_{j0}(u,W_{ij})} \\ 0_p \\ 0 \end{pmatrix}$$

$$
\begin{aligned}
& [I\{W_{ij} \le -h_1\} I\{W_{ij} - w \ge h_1\} - I\{W_{ij} \ge h_1\} I\{W_{ij} - w \le -h_1\}] \\
\triangleq\ & e_1^T \mathbb{A}^{*-1}(W_{ij}) \begin{pmatrix} Z_{ij} - \frac{t_{j1}(u,W_{ij})}{t_{j0}(u,W_{ij})} \\ 0_p \\ 0 \end{pmatrix} Id(W_{ij}, w; h_1), \\
c_{n1}(W_{ij}, w) \ =\ & 0,
\end{aligned}
$$

and

$$
c_{n2}(W_{ij}, w) = e_1^T [\mathbb{A}^{*-1}(W_{ij})]' \begin{pmatrix} 0_p \\ Z_{ij}\mu_2 \\ \mu_2 \end{pmatrix} Id(W_{ij}, w; h_1).
$$

Write $Id(w, w^*) = I\{0 > w\} I\{w > w^*\} - I\{0 < w\} I\{w < w^*\}$. Then

$$
E|Id(W_{ij}, w; h_1) - Id(W_{ij}, w)| = O(h_1).
$$

It follows that

$$
\begin{aligned}
c_{n0}(W_{ij}, w) &= e_1^T \mathbb{A}^{*-1}(W_{ij}) \begin{pmatrix} Z_{ij} - \frac{t_{j1}(u,W_{ij})}{t_{j0}(u,W_{ij})} \\ 0_p \\ 0 \end{pmatrix} Id(W_{ij}, w) + o_p(1) \\
&\triangleq C_0(K) Id(W_{ij}, w) + o_p(1)
\end{aligned} \tag{27}
$$

uniformly for $w \in \cup_{j=1}^m supp(f_j)$, $i = 1, 2, \cdots, n$. By the same argument, we have

$$
\begin{aligned}
c_{n2}(W_{ij}, w) &= e_1^T [\mathbb{A}^{*-1}(W_{ij})]' \begin{pmatrix} 0_p \\ Z_{ij}\mu_2 \\ \mu_2 \end{pmatrix} Id(W_{ij}, w) + o_p(1) \\
&\triangleq C_1(K) Id(W_{ij}, w) + o_p(1),
\end{aligned} \tag{28}
$$

uniformly for $w \in \cup_{j=1}^m supp(f_j)$, $i = 1, 2, \cdots, n$. Therefore, by (24) through (28),

$$
\begin{aligned}
& \widehat{g}_p(w, \alpha_0) - g_0(w) \\
=\ & n^{-1} \sum_{j=1}^m \sum_{i=1}^n \int_0^\tau e_1^T \mu_2 \Big\{ \mathbb{A}^{*-1}(W_{ij}) \begin{pmatrix} -h_1^{-1}(Z_{ij} - \frac{t_{j1}(u,W_{ij})}{t_{j0}(u,W_{ij})}) \\ \frac{\mathcal{D}_w[t_{j1}(u,W_{ij})]}{t_{j0}(u,W_{ij})} \\ \mathcal{D}_w[\log(t_{j0}(u, W_{ij}))] \end{pmatrix} \\
& + [\mathbb{A}^{*-1}(W_{ij})]' \begin{pmatrix} 0_p \\ Z_{ij} \\ 1 \end{pmatrix} \Big\} Id(W_{ij}, w) dM_{ij}(u) \\
& + e_1^T h_1^{-1} \int_0^w \mathbb{A}^{*-1}(w_0) \mathbb{B}_n^*(\tau, w_0) dw_0 + o_p(\frac{1}{\sqrt{n}}).
\end{aligned} \tag{29}
$$

(ii) Next, we derive the asymptotic expression of K_{n1} in terms of $\widehat{g}_p(\cdot,\alpha_0)-g_0(\cdot)$. Following techniques used in Cai et al. (2007a), it can be shown that

$$E\left\{[c_{n0}(W_i,w)]^{\otimes 2}\right\}=O(h_1) \tag{30}$$

uniformly for $w\in\cup_{j=1}^m supp(f_j)$. Since $\mathbb{B}_n(\tau,w_0)=O_p(h_1^2)$, by (29), we have

$$\widehat{g}_p(w,\alpha_0)-g_0(w)=O_p(h_1) \tag{31}$$

uniformly for $w\in\cup_{j=1}^m supp(f_j)$. Similar to arguments used in the proof of Theorem 4, we can show that

$$\widehat{\beta}_p^T(w,\alpha_0)-\beta_0^T(w)=O_p(\frac{1}{\sqrt{nh_1}}) \tag{32}$$

uniformly for $w\in J_W$. Applying the first order approximation $x/y=x_0/y_0+(x-x_0)/y_0-(y-y_0)x_0/y_0^2+O\{(x-x_0)^2+(y-y_0)^2\}$ to $R^*_{nj1}(u,\alpha_0)/R^*_{nj0}(u,\alpha_0)$ and $\tilde{R}^*_{nj1}(u,\alpha_0)/\tilde{R}^*_{nj0}(u,\alpha_0)$ yields

$$\begin{aligned}&\frac{\tilde{R}^*_{nj1}(u,\alpha_0)}{\tilde{R}^*_{nj0}(u,\alpha_0)}-\frac{R^*_{nj1}(u,\alpha_0)}{R^*_{nj0}(u,\alpha_0)}\\=&\frac{\tilde{R}^*_{nj1}(u,\alpha_0)-R^*_{nj1}(u,\alpha_0)}{R^*_{nj0}(u,\alpha_0)}-\frac{\tilde{R}^*_{nj0}(u,\alpha_0)-R^*_{nj0}(u,\alpha_0)}{R^*_{nj0}(u,\alpha_0)}\frac{R^*_{nj1}(u,\alpha_0)}{R^*_{nj0}(u,\alpha_0)}+r^*_{nj},\end{aligned} \tag{33}$$

where $r^*_{nj}=O\{[\tilde{R}^*_{nj1}(u,\alpha_0)-R^*_{nj0}(u,\alpha_0)]^2+[\tilde{R}^*_{nj0}(u,\alpha_0)-R^*_{nj1}(u,\alpha_0)]^2\}$. By (31), (32), and the definitions of $\tilde{R}^*_{njk}$ and R^*_{njk}, we have

$$\begin{aligned}&\sqrt{n}[\tilde{R}^*_{njk}(u,\alpha_0)-R^*_{njk}(u,\alpha_0)]\\=&\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_{ij}(u)\exp\{\alpha_0^TV_{ij}(u)+\beta_0^T(W_{ij})Z_{ij}+g_0(W_{ij})\}[V_{ij}(u)+\chi_2^T(W_{ij})Z_i+\chi_1^T(W_{ij})]^{\otimes k}\\&\Big[\exp\{\widehat{\beta}_p^T(W_i,\alpha_0)Z_i-\beta_0^T(W_i)Z_i+\widehat{g}_p(W_i,\alpha_0)-g_0(W_i)\}-1\Big]\\=&\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i(u)\exp\{\alpha_0^TV_i(u)+\beta_0^T(W_{ij})Z_{ij}+g_0(W_{ij})\}[V_{ij}(u)+\chi_2^T(W_{ij})Z_{ij}+\chi_1^T(W_{ij})]^{\otimes k}\\&\Big[\widehat{\beta}_p^T(W_{ij},\alpha_0)Z_{ij}-\beta_0^T(W_{ij})Z_{ij}+\widehat{g}_p(W_{ij},\alpha_0)-g_0(W_{ij})\Big]+o_p(1),\end{aligned} \tag{34}$$

which is bounded by $O_p(h_1)$. Together with (33), this leads to

$$
\begin{aligned}
& K_{n1} \\
= \;& \frac{1}{\sqrt{n}} \sum_{j=1}^{m} \sum_{i=1}^{n} \int_0^{\tau} \left[\frac{R^*_{nj1}(u,\alpha_0)}{R^*_{nj0}(u,\alpha_0)} - \frac{\tilde{R}^*_{nj1}(u,\alpha_0)}{\tilde{R}^*_{nj0}(u,\alpha_0)} \right] dN_{ij}(u) \\
= \;& -\sum_{j=1}^{m} \int_0^{\tau} \sqrt{n} \left[\frac{\tilde{R}^*_{nj1}(u,\alpha_0) - R^*_{nj1}(u,\alpha_0)}{R^*_{nj0}(u,\alpha_0)} - \frac{\tilde{R}^*_{nj0}(u,\alpha_0) - R^*_{nj0}(u,\alpha_0)}{R^*_{nj0}(u,\alpha_0)} \frac{R^*_{nj1}(u,\alpha_0)}{R^*_{nj0}(u,\alpha_0)} \right] d\bar{N}_{\cdot j}(u) \\
& + o_p(1),
\end{aligned}
$$

where $\bar{N}_j(u) = \frac{1}{n} N_{ij}(u)$. Note that the empirical process $\sqrt{n}[\bar{N}_j(u) - EN_{\cdot j}(u)]$ converges to a Gaussian process with mean zero, $\tilde{R}^*_{nj0}(u,\alpha_0) - R^*_{nj0}(u,\alpha_0) = o_p(1)$ uniformly for $u \in [0,\tau]$, and the empirical process satisfies $\sqrt{n}[R^*_{njk}(u,\alpha_0) - r_{jk}(u,\alpha_0)] = O_p(1), k = 0, 1$. Given these properties, we can show that

$$
\begin{aligned}
K_{n1} \;=\; & -\sum_{j=1}^{m} \int_0^{\tau} \sqrt{n} \Big[\frac{\tilde{R}^*_{nj1}(u,\alpha_0) - R^*_{nj1}(u,\alpha_0)}{r_{j0}(u,\alpha_0)} \\
& \qquad - \frac{\tilde{R}^*_{nj0}(u,\alpha_0) - R^*_{nj0}(u,\alpha_0)}{r_{j0}(u,\alpha_0)} \frac{r_{j1}(u,\alpha_0)}{r_{j0}(u,\alpha_0)} \Big] dEN_{\cdot j}(u)(1 + o_p(1)) + o_p(1).
\end{aligned}
$$

Then by (34),

$$
\begin{aligned}
& K_{n1} \\
= \;& -\sum_{j=1}^{m} \int_0^{\tau} r_{j0}^{-1}(u,\alpha_0) \Bigg\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i(u) \exp\{\alpha_0^T V_i(u) + \beta_0^T(W_{ij}) Z_i + g_0(W_{ij})\} \\
& [V_{ij}(u) + \chi_2^T(W_{ij}) Z_{ij} + \chi_1^T(W_{ij}) - \frac{r_{j1}(u,\alpha_0)}{r_{j0}(u,\alpha_0)}] \\
& \Big[\widehat{\beta}_p^T(W_{ij},\alpha_0) Z_{ij} - \beta_0^T(W_{ij}) Z_{ij} + \widehat{g}_p(W_{ij},\alpha_0) - g_0(W_{ij}) \Big] \Bigg\} dEN_{\cdot j}(u) \\
& (1 + o_p(1)) + o_p(1). \qquad (35)
\end{aligned}
$$

(iii) Now, we substitute the result of (i) in (ii) to obtain the asymptotic expression of K_{n1}. Let $s^*_{ij}(u,\alpha_0) = Y_{ij}(u) \exp\{\alpha_0^T V_{ij}(u) + \beta_0^T(W_{ij}) Z_{ij} + g_0(W_{ij})\}[V_{ij}(u) + \chi_2^T(W_{ij}) Z_{ij} + \chi_1^T(W_{ij}) - \frac{r_{j1}(u,\alpha_0)}{r_{j0}(u,\alpha_0)}]$, $i = 1, 2, \cdots, n$; $j = 1, \cdots, m$. Write

$$K_{n11}$$

$$
\begin{aligned}
&= \sum_{s=1}^{m}\int_0^\tau r_{s0}^{-1}(u,\alpha_0)\Big\{\frac{1}{nh_1\sqrt{n}}\sum_{i=1}^{n}s_{is}^*(u,\alpha_0)\sum_{j=1}^{m}\sum_{l=1}^{n}\int_0^\tau c_{n0}(W_{lj},W_{is})dM_{lj}(u)\Big\}dEN_{\cdot s}(u)\\
&= \frac{1}{\sqrt{n}}\sum_{j=1}^{m}\sum_{l=1}^{n}\int_0^\tau\int_0^\tau \frac{1}{nh_1}\sum_{s=1}^{m}\sum_{i=1}^{n} r_{s0}^{-1}(u,\alpha_0)s_{is}^*(u,\alpha_0)c_{no}(W_{lj},W_{is})dEN_{\cdot s}(u)dM_{lj}(u)\\
&\triangleq \frac{1}{\sqrt{n}}\sum_{j=1}^{m}\sum_{l=1}^{n}\int_0^\tau \xi_s(W_{lj})dM_{lj}(u). \qquad (36)
\end{aligned}
$$

By (30), $Var(\xi_s(\cdot)|W_{lj}) = O(\frac{1}{nh_1})$ uniformly for $l = 1, 2, \cdots, n$, and $i = 1, \cdots, m$. Taking iterative expectations, and exchanging the orders of integration, we obtain

$$
\begin{aligned}
E[\xi_s(W_{lj})|W_{lj}] &= E\left[\frac{1}{nh_1}\sum_{s=1}^{m}\sum_{i=1}^{n}\int_0^\tau r_{s0}^{-1}(u,\alpha_0)s_{is}^*(u,\alpha_0)c_{no}(W_{lj},W_{is})dEN_{\cdot s}(u)|W_{lj}\right]\\
&= \sum_{s=1}^{m}\int_0^\tau\int \frac{1}{h_1}r_{s0}^{-1}(u,\alpha_0)s_{is}^*(u,\alpha_0)c_{no}(W_{lj},w)f_s(w)dwdEN_{\cdot s}(u)\\
&= \frac{1}{h_1}\sum_{s=1}^{m}\int a_s(w)f_s(w)c_{no}(W_{lj},w)dw + O_p(\frac{1}{nh_1})\\
&\triangleq \xi_n(W_{lj}) + O_p(\frac{1}{nh_1}), \qquad (37)
\end{aligned}
$$

where

$$
a_s(w) = \int_0^\tau r_{s0}^{-1}(u,\alpha_0)\psi_s(u,w,\alpha_0)dEN_{\cdot s}(u),
$$

and

$$
\psi_s(u,w,\alpha_0) = E\left[\rho(u,V_s,Z_s,w)[V_s(u)+\chi_2^T(w)Z_s+\chi_1^T(w)-\frac{r_{s1}(u,\alpha_0)}{r_{s0}(u,\alpha_0)}]|w\right].
$$

Then, by (36) and (37),

$$
K_{n11} = \frac{1}{\sqrt{n}}\sum_{j=1}^{m}\sum_{l=1}^{n}\int_0^\tau \xi_n(W_{lj})dM_{lj}(u) + o_p(1).
$$

We will demonstrate that the first term on the r.h.s. above is of mean zero and variance o(1). This in turn shows that $K_{n11} = o_p(1)$. Actually, because $\xi_n(W_{lj})$ is $\mathcal{F}_{t,lj}$-predictable, and $M_{lj}(u)$ is the corresponding martingale, the mean is obviously zero. Now, note that the corresponding variance is of the same order as

$$\begin{aligned} E[\xi_n^{\otimes 2}(W_{lj})] &= \frac{1}{h_1^2}E[\sum_{s=1}^m \int a_s(w)f_s(w)c_{no}(W_{lj},w)dw]^{\otimes 2} \\ &\leqslant O(h_1^{-2})\sum_{s=1}^m \int f_s(w^*)[\int a_s(w)f_s(w)c_{no}(w^*,w)dw]^{\otimes 2}dw^*. \end{aligned} \tag{38}$$

Also, note that $b_{j1}(u,w_0) - b_{j0}(u,w_0)E(Z_{ij}|w_0) = O_p(h_1)$ for each $w_0 \in \cup_{j=1}^m supp(f_j)$, $i=1,\cdots,n$, $j=1,\cdots,m$. Similar to the proof of Cai, Fan, Jiang and Zhou (2007), and recognizing that $\int a_s(w)f_s(w)dw = 0$, $j=1,\cdots,m$, we can show that $E[\xi_n^{\otimes 2}(W_{lj})] = O(h_1)$. Therefore, $K_{n11} = o_p(1)$.

Write

$$\begin{aligned} & K_{n12} \\ = & -\sum_{j=1}^m \int_0^\tau r_{j0}^{-1}(u,\alpha_0)\Big\{\frac{1}{\sqrt{n}}\sum_{i=1}^n s_{ij}^*(u,\alpha_0)n^{-1}\sum_{s=1}^m\sum_{r=1}^n \cdot \\ & \int_0^\tau e_1^T\mu_2\Big\{\mathbb{A}^{*-1}(W_{rs})\begin{pmatrix} 0_p \\ \frac{\mathcal{D}_w[t_{s1}(u,W_{rs})]}{t_{s0}(u,W_{rs})} \\ \mathcal{D}_w[\log(t_{s0}(u,W_{rs}))] \end{pmatrix} \\ & +[\mathbb{A}^{*-1}(W_{rs})]'\begin{pmatrix} 0_p \\ Z_{rs} \\ 1 \end{pmatrix}\Big\}Id(W_{rs},W_{ij})dM_{rs}(u)\Big\}dEN_{\cdot j}(u). \end{aligned} \tag{39}$$

Let

$$\alpha_{nj}(u,w) = \frac{1}{n}\sum_{i=1}^n s_{ij}^*(u,\alpha_0)Id(w,W_{ij}).$$

Taking expectations iteratively, we have

$$E[\alpha_{nj}(u,w)] = \int \psi_j(u,w^*,\alpha_0)Id(w,w^*)f_j(w^*)dw^*.$$

It can be shown that

$$\alpha_{nj}(u,w) = \int \psi_j(u,w^*,\alpha_0)Id(w,w^*)f_j(w^*)dw^* + o_p(1)$$

uniformly for $u \in [0,\tau]$ and $w \in \cup_{j=1}^m supp(f_j)$. This, together with (39), leads to

$$\begin{aligned}
& K_{n12} \\
= \quad & -\frac{1}{\sqrt{n}}\sum_{s=1}^m\sum_{r=1}^n\int_0^\tau\Big[\sum_{j=1}^m\int_0^\tau r_{j0}^{-1}(u,\alpha_0)a_{nj}(u,W_{rs})dE._j(u)\Big]Q(u,W_{rs})dM_{rs}(u)+o_p(1) \\
= \quad & -\frac{1}{\sqrt{n}}\sum_{s=1}^m\sum_{r=1}^n\int_0^\tau\Big[\sum_{j=1}^m\int a_j(w)Id(W_{rs},w)f_j(w)dw\Big]Q(u,W_{rs})dM_{rs}(u). \qquad (40)
\end{aligned}$$

Recalling that $\int a_j(w)f_j(w)dw = 0$, we obtain

$$\sum_{j=1}^m\int a_j(w)Id(w^*,w)f_j(w)dw = \sum_{j=1}^m\int_{-\infty}^{w^*}a_j(w)f_j(w)dw = s(w^*).$$

Combining with (40), this yields

$$K_{n12} = -\frac{1}{\sqrt{n}}\sum_{s=1}^m\sum_{r=1}^n\int_0^\tau Q(u,W_{rs})s(W_{rs})dM_{rs}(u)+o_p(1).$$

Furthermore, because $b_{j1}(u,w_0)-b_{j0}(u,w_0)E(Z_{ij}|w_0) = O_p(h_1)$ for every $w_0 \in \cup_{j=1}^m supp(f_j)$, $i=1,\cdots,n$, $j=1,\cdot,m$, we can prove that

$$\sum_{j=1}^m\int_0^\tau r_{j0}^{-1}(u,\alpha_0)\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^n s_{ij}^*(u,\alpha_0)[\widehat{\beta}_p^T(W_{ij},\alpha_0)Z_{ij}-\beta_0^T(W_{ij})Z_{ij}]\right\}dEN._j(u) = o_p(1).$$

Combining the results for K_{n11} and K_{n12} yields the lemma. □

2 Proof of Lemma 3

By (A.2), we can show that

$$\begin{aligned}
\frac{\partial^2\ell_p(\alpha)}{\partial\alpha\partial\alpha^T}\mid_{\alpha=\alpha_0} \quad = \quad & -\frac{1}{n}\sum_{j=1}^m\sum_{i=1}^n\int_0^\tau\left[\frac{R_{nj2}(u,\alpha_0)}{R_{nj0}(u,\alpha_0)}-\frac{R_{nj1}^{\otimes 2}(u,\alpha_0)}{R_{nj0}^2(u,\alpha_0)}\right]dN_{ij}(u) \\
& +\sum_{i=1}^n\int_0^\tau\left[\kappa_{nj1}(W_{ij})+\kappa_{nj2}(W_{ij})Z_{ij}(u)-\frac{K_{nj1}(u,\alpha_0)}{K_{nj0}(u,\alpha_0)}\right]dN_{ij}(u),
\end{aligned}$$

where $K_{njk}(u,\alpha_0)=\frac{1}{n}\sum_{i=1}^{n}Y_{ij}(u)\exp\{\alpha_0^TV_{ij}(u)+\widehat{\beta}_p^T(W_{ij},\alpha_0)Z_{ij}(u)+\widehat{g}_p(W_{ij},\alpha_0)\}[\kappa_{nj1}(W_{ij})+\kappa_{nj2}(W_{ij})Z_{ij}(u)]^{\otimes k}$, for $k=0,1$. By Lemma 2, and recognizing that $\widehat{\beta}_p(w,\alpha_0)=\beta_0(w)+o_p(1)$, $\widehat{g}_p(w,\alpha_0)=g_0(w)+o_p(1)$ uniformly for $w\in\cup_{j=1}^m supp(f_j)$, we have

$$\begin{aligned}
&\frac{\partial^2\ell_p(\alpha)}{\partial\alpha\partial\alpha^T}\mid_{\alpha=\alpha_0}\\
=\;& -\frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^\tau\left[\frac{r_{j2}(u,\alpha_0)}{r_{j0}(u,\alpha_0)}-\frac{r_{j1}^{\otimes 2}(u,\alpha_0)}{r_{j0}^2(u,\alpha_0)}\right]dN_{ij}(u)\\
&+\sum_{j=1}^{m}\frac{1}{n}\sum_{i=1}^{n}\int_0^\tau\left[\kappa_1(W_{ij})+\kappa_2(W_{ij})Z_{ij}(u)-\frac{K^*_{nj1}(u,\alpha_0)}{K^*_{nj0}(u,\alpha_0)}\right]dN_{ij}(u)+o_p(1), \quad (41)
\end{aligned}$$

where $K^*_{njk}(u,\alpha_0)$ has a similar definition to $K_{njk}(u,\alpha_0)$, except that $\kappa_{nj1}(W_{ij})$, $\kappa_{nj2}(W_{ij})$, $\widehat{\beta}_p(W_{ij},\alpha_0)$ and $\widehat{g}_p(W_{ij},\alpha_0)$ are replaced by $\kappa_1(W_{ij})$, $\kappa_2(W_{ij})$, $\beta_0(W_{ij})$ and $g_0(W_{ij})$ respectively. Since

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^\tau&\left[\kappa_1(W_{ij})+\kappa_2(W_{ij})Z_{ij}(u)-\frac{K^*_{nj1}(u,\alpha_0)}{K^*_{nj0}(u,\alpha_0)}\right]Y_{ij}(u)\lambda_{0j}(u)\\
&\exp\{\alpha_0^TV_{ij}(u)+\beta_0^T(W_{ij})Z_{ij}(u)+g_0(W_{ij})\}du=0,
\end{aligned}$$

the second term on the right-hand side of (41) is

$$\frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^\tau\left[\kappa_1(W_{ij})+\kappa_2(W_{ij})Z_{ij}(u)-\frac{K^*_{nj1}(u,\alpha_0)}{K^*_{nj0}(u,\alpha_0)}\right]dM_{ij}(u)=o_p(1).$$

Therefore,

$$\begin{aligned}
\frac{\partial^2\ell_p(\alpha)}{\partial\alpha\partial\alpha^T}\mid_{\alpha=\alpha_0} &= -\frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^\tau\left[\frac{r_{j2}(u,\alpha_0)}{r_{j0}(u,\alpha_0)}-\frac{r_{j1}^{\otimes 2}(u,\alpha_0)}{r_{j0}^2(u,\alpha_0)}\right]dN_{ij}(u)+o_p(1)\\
&= -I(\alpha_0)+o_p(1).
\end{aligned}$$

□

3 Proof of Theorem 5

Let $\theta = (\beta^T(\cdot), \alpha^T, g(\cdot))^T$, $\theta_0 = (\beta_0^T(\cdot), \alpha_0^T, g_0(\cdot))^T$ and $\widehat{\theta}_p = (\widehat{\beta}_p^T(\cdot), \widehat{\alpha}_p^T, \widehat{g}_p(\cdot))^T$. By the same argument of Lemma 1, we have, for any $j = 1, \cdots, m$,

$$\sup_{0\le t\le\tau} \sup_{\|\theta-\theta_0\|\le\|\widehat{\theta}_p-\theta_0\|} n^{-1}|\Delta_{nj}(t,\theta) - \Delta_{nj}(t,\theta_0)| \xrightarrow{P} 0, \tag{42}$$

where

$$\Delta_{nj}(t,\theta) = \sum_{i=1}^{n} I(W_{ij} \in J_W)Y_{ij}(t)\exp(\beta^T(W_{ij})Z_{ij}(u) + \alpha^T V_{ij}(u) + g(W_{ij}))$$

and

$$\Delta_{nj}(t,\theta_0) = \sum_{i=1}^{n} I(W_{ij} \in J_W)Y_{ij}(t)\exp(\beta_0^T(W_{ij})Z_{ij}(u) + \alpha_0^T V_{ij}(u) + g_0(W_{ij})).$$

By the definition of $\widehat{\Lambda}_{0jp}$, we have

$$\begin{aligned}
&\widehat{\Lambda}_{0jp}(t) - \Lambda_{0j}(t)\\
= &\int_0^\tau \left\{\frac{1}{\Delta_{nj}(u,\widehat{\theta}_p)} - \frac{1}{\Delta_{nj}(u,\theta_0)}\right\} d\bar{N}_{nj}(u) + \int_0^\tau \left\{\frac{d\bar{N}_n(u)}{\Delta_{nj}(u,\theta_0)} - d\Lambda_{0j}(u)\right\}\\
= &-\int_0^\tau \frac{\Delta_{nj}(u,\widehat{\theta}_p) - \Delta_{nj}(u,\theta_0)}{\Delta_{nj}(u,\widehat{\theta}_p)} d\Lambda_{0j}(u) - \int_0^\tau \frac{\Delta_{nj}(u,\widehat{\theta}_p) - \Delta_{nj}(u,\theta_0)}{\Delta_{nj}(u,\widehat{\theta}_p)\Delta_{nj}(u,\theta_0)} d\bar{M}_{nj}(u)\\
&+\int_0^\tau \frac{1}{\Delta_{nj}(u,\theta_0)} d\bar{M}_{nj}(u),
\end{aligned}$$

where $\bar{N}_{nj}(u) = \sum_{i=1}^{n} N_{ij}(u)$, and $\bar{M}_{nj}(u) = \sum_{i=1}^{n} M_{ij}(u)$. From (42) it is easy to see that the first term on the right-hand side converges to zero in probability uniformly on $(0,\tau]$ as $n \to \infty$. The last two terms of the above expression are square integrable local martingales with variation processes

$$\int_0^\tau \frac{\Delta_{nj}(u,\widehat{\theta}_p) - \Delta_{nj}(u,\theta_0)}{(\Delta_{nj}(u,\widehat{\theta}_p))^2\Delta_{nj}(u,\theta_0)} d\Lambda_{0j}(u)$$

and

$$\int_0^\tau \frac{1}{\Delta_{nj}(u,\theta_0)} d\Lambda_{0j}(u),$$

respectively. Since $\Delta_{nj}(t, \theta_0) = O_p(n)$, the above variance processes converge to zero in probability uniformly on $(0, \tau]$ as $n \to \infty$. The terms converge to zero in probability uniformly on $(0, \tau]$ by an argument similar to that of Andersen and Gill (1982) via the Lenglart inequality. Therefore

$$\widehat{\Lambda}_{0jp}(t) \xrightarrow{P} \Lambda_{0j}(t)$$

uniformly on $(0, \tau]$. Thus, we can prove by the standard argument of kernel estimation that

$$\widehat{\lambda}_{0jp}(t) \xrightarrow{P} \lambda_{0j}(t)$$

uniformly on $(0, \tau]$.

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