Supplementary Material to
Prediction in Ewens-Pitman Sampling Formula and
random ssampling from number partitions
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1 The proofs omitted in the paper

1.1 §2.2 The forward equation (19)

The induction step to confirm (19). Write $S_{n,kl} = S_{n,kl}(-1, -\alpha, 0)$ for short, and assume $n^* + s_0 = n$.

$$
\begin{aligned}
f_{n+1}(\ell, s_0) &= \frac{1}{\theta+m+n}\binom{n}{s_0}\frac{(m-k\alpha|-1)_{s_0}}{(\theta+m|-1)_n}((n^*-\ell\alpha)(\theta+k\alpha|-\alpha)_\ell S_{n^*,\ell} \\
&\quad +(\theta+(k+\ell-1)\alpha)(\theta+k\alpha|-\alpha)_{\ell-1}S_{n^*,\ell-1}) \\
&\quad +\frac{m-k\alpha+s_0-1}{\theta+m+n}\frac{(m-k\alpha|-1)_{s_0-1}}{(\theta+m|-1)_n}\binom{n}{s_0-1}(\theta+k\alpha|-1)_\ell S_{n^*+1,\ell} \\
&= \binom{n}{s_0}\frac{(m-k\alpha|-1)_{s_0}}{(\theta+m|-1)_{n+1}}(\theta+k\alpha|-1)_\ell((n^*-\ell\alpha)S_{n^*,\ell}+S_{n^*,\ell-1}) \\
&\quad +\binom{n}{s_0-1}\frac{(m-k\alpha|-1)_{s_0}}{(\theta+m|-1)_{n+1}}(\theta+k\alpha|-1)_\ell S_{n^*+1,\ell} \\
&= \left(\binom{n}{s_0}+\binom{n}{s_0-1}\right)\frac{(m-k\alpha|-1)_{s_0}}{(\theta+m|-1)_{n+1}}(\theta+k\alpha|-1)_\ell S_{n^*+1,\ell}\,.
\end{aligned}
$$

1.2 §2.2 The last paragraph *A restriction on* s_0

If $m_1 + m_2 = m, k_1 + k_2 = k$ and $s_{01} + s_{02} = s_0$, the joint p.m.f. of (s_{01}, s_{02}) is

$$
\sum_{s_0=0}^{n}\binom{n}{s_{01}, s_{02}, n^*}\frac{(m_1-k_1\alpha|-1)_{s_{01}}(m_2-k_2\alpha|-1)_{s_{02}}(\theta+k\alpha|-1)_{n^*}}{(\theta+m|-1)_n}
$$

Hence, going back the derivation in (17)

$$
\begin{aligned}
&\mathbb{P}\{(s_{02}, K_n(m,k)) = (0,\ell) \mid m_1+m_2=m \;\&\; k_1+k_2=k\} \\
&= \sum_{s_0=0}^{n}\binom{n}{s_0}\frac{(m_1-k_1\alpha|-1)_{s_0}(\theta+k\alpha|-1)_{n^*}}{(\theta+m|-1)_n}\times\frac{(\theta+k\alpha|-\alpha)_\ell}{(\theta+k\alpha|-1)_{n^*}}S_{n^*,\ell}(-1,-\alpha,0), \\
&= \frac{(\theta+k\alpha|-\alpha)_\ell}{(\theta+m|-1)_n}\sum_{s_0=0}^{n}\binom{n}{s_0}(m_1-k_1\alpha|-1)_{s_0}S_{n^*,\ell}(-1,-\alpha,0), \\
&= \frac{(\theta+k\alpha|-\alpha)_\ell}{(\theta+m|-1)_n}S_{n,\ell}(-1,-\alpha,m_1-k_1\alpha), \quad n^*=n-s_0,\ 0\le s_0\le n,
\end{aligned}
$$

Further,

$$\mathbb{P}\{s_{02}=0 \mid m_1+m_2=m \ \&\ k_1+k_2=k\}$$

$$= \frac{1}{(\theta+m|-1)_n}\sum_{\ell=0}^{n}(\theta+k\alpha|-\alpha)_\ell S_{n,\ell}(-1,-\alpha,m_1-k_1\alpha)$$

$$= \frac{(\theta+k\alpha+m_1-k_1\alpha|-1)_n}{(\theta+m|-1)_n} = \frac{(\theta+k_2\alpha+m_1|-1)_n}{(\theta+m|-1)_n}.$$

1.3 An alternative proof of Proposition 7, §4.1.

Here, $E(S_{kj}),\ 0\le j\le k$ in general, are obtained. The conditional expectation of U_k given $Y_k=m$ are known (23), and Y_k have the marginal Hg $(n;kt_k,\nu-kt_k)$ of $\mathsf{SymMvHg}$ $(n;t)$. Hence

$$E(S_{kj}) = \left[\sum_{m=0}^{n} E(U_k)\binom{n}{m}\binom{\nu-n}{rc-m}\right]\Big/\binom{\nu}{rc}\ ,\ r=k, c=t_k.$$

Now using

$$\binom{n}{m}=\frac{(n)_j}{(m)_j}\binom{n-j}{m-j},\quad \binom{\nu-n}{rc-m}=\frac{(\nu-n)_{r-j}}{(rc-m)_{r-j}}\binom{\nu-n-r-j}{rc-m-r-j},$$

$$E(S_{kj}) = \frac{c}{(rc)_r}\binom{r}{j}(n)_j(\nu-n)_{r-j}\left[\sum_{m=j}^{n}\binom{n-j}{m-j}\binom{\nu-n-r-j}{rc-m-r-j}\right]\Big/\binom{\nu}{rc}$$

$$= \frac{c}{(rc)_r}\binom{r}{j}\binom{\nu-r}{rc-r}\Big/\binom{\nu}{rc} = c\binom{r}{j}\frac{(n)_j(\nu-n)_{r-j}}{(\nu)_r},\quad 0\le j\le k.$$

□

Check that $\sum_{j=0}^{k} E(S_{kj}) = t_k$.

1.4 Calculation of $Var(K_n(\tau))$ in Proposition 7, §4.1.

$$K_n(\tau)=\kappa-\sum_{k=1}^{\kappa}S_{k0},\quad E(K_n(\tau))=\kappa-\mu(\tau),\quad \mu(\tau):=\sum_{k=1}^{\kappa}E(S_{k0}).$$

$$(K_n(\tau))^2=\kappa^2-2\kappa(\sum_{k=1}^{\kappa}S_{k0})+(\sum_{k=1}^{\kappa}S_{k0})^2$$

$$=\kappa^2-2\kappa(\sum_{k=1}^{\kappa}S_{k0})+\sum_{k=1}^{\kappa}(S_{k0})_2+\sum_{k=1}^{\kappa}S_{k0}+2\sum_{1\le j<k\le\nu}S_{j0}S_{k0}.$$

$$Var(K_n(\tau))=\kappa^2-2\kappa\mu(\tau)+\sum_{k=1}^{\kappa}E((S_{k0})_2)+\mu(\tau)+2\sum_{1\le j<k\le\nu}E(S_{j0}S_{k0})-(\kappa-\mu(\tau))^2$$

From (23),

$$E((U_0)_\ell)=\frac{(c)_\ell(\nu-m)_{r\ell}}{(\nu)_{r\ell}},\quad E((S_{k0})_2)=\frac{(\tau_k)_2(\nu-n)_{2k}}{(\nu)_{2k}}.$$

Since (Y_j, Y_k), as a marginal of $\mathsf{MvHg}\ (n;\tau)$, follows $\mathsf{MvHg}\ (n;\tau_j,\tau_k,\nu-\tau_j-\tau_k)$,

$$
\begin{aligned}
E(S_{j0}S_{k0}) &= \sum_{(y_j,y_k)} E[S_{j0}S_{k0}|(S_{j0},S_{k0})=(y_j,y_k)]\mathbb{P}\{(S_{j0},S_{k0})=(y_j,y_k)\} \\
&= \sum_{(y_j,y_k)} \frac{\tau_j(j\tau_j-y_j)_j}{(j\tau_j)_j}\frac{\tau_k(k\tau_k-y_k)_k}{(k\tau_k)_k}\binom{j\tau_j}{y_j}\binom{k\tau_k}{y_k}\binom{\nu-j\tau_j-k\tau_k}{n-y_j-y_k}\Big/\binom{\nu}{n} \\
&= \tau_j\tau_k\sum_{(y_j,y_k)}\binom{j\tau_j-j}{y_j}\binom{k\tau_k-k}{y_k}\binom{\nu-j\tau_j-k\tau_k}{n-y_j-y_k}\Big/\binom{\nu}{n} \\
&= \tau_j\tau_k\binom{\nu-j-k}{n}\Big/\binom{\nu}{n} = \tau_j\tau_k\frac{(\nu-j-k)_{j+k}}{(\nu)_{j+k}}.
\end{aligned}
$$

The second equality is due to

$$
(j\tau_j-y_j)_j\binom{j\tau_j}{y_j} = (j\tau_j)_j\binom{j\tau_j-j}{y_j}.
$$

1.5 Examples of Proposition 11, §4.2.

The generalized Stirling numbers $S_{n,k}(-1,-\alpha,0)$ are as follows:

$n\backslash k$	1	2	3	4	5
1	1				
2	$1-\alpha$	1			
3	$(1-\alpha\vert-1)_2$	$3(1-\alpha)$	1		
4	$(1-\alpha\vert-1)_3$	$(1-\alpha)(11-7\alpha)$	$6(1-\alpha)$	1	
5	$(1-\alpha\vert-1)_4$	$5(1-\alpha\vert-1)_2(5-3\alpha)$	$5(1-\alpha)(7-5\alpha)$	$10(1-\alpha)$	1

$S_{n,n-1}=(1-\alpha|-1)_{n-1}$, $\qquad S_{n,n-1}=\binom{n}{2}(1-\alpha)$

The case $\nu=5,\kappa=3$.

$$g_4(2) = \frac{S_{4,2}}{S_{5,3}} = \frac{(1-\alpha)(11-7\alpha)}{S_{5,3}}; \quad g_4(3) = (4-3\alpha)\frac{S_{4,3}}{S_{5,3}} = \frac{6(1-\alpha)(4-3\alpha)}{S_{5,3}}$$

$$g_3(1) = \frac{S_{3,1}}{S_{4,2}} g_4(2) = \frac{(1-\alpha|-1)_2}{S_{5,3}}$$

$$g_3(2) = (3-2\alpha)\frac{S_{3,2}}{S_{4,2}} g_4(2) + \frac{S_{3,2}}{S_{4,3}} g_4(3)$$

$$= (3-2\alpha)\frac{S_{3,2}}{S_{5,2}} + \frac{S_{3,2}}{S_{5,3}}(4-3\alpha)$$

$$= \frac{3(1-\alpha)}{S_{5,2}}(3-2\alpha+4-3\alpha)$$

$$g_3(3) = (3-3\alpha)\frac{S_{3,3}}{S_{4,3}} g_4(3) = \frac{3(1-\alpha)(4-3\alpha)}{S_{5,2}}$$

$$g_2(1) = \frac{S_{2,1}}{S_{3,2}} g_4(2) = \frac{(1-\alpha|-1)_2 + (1-\alpha)(7-5\alpha)}{S_{5,3}}$$

$$g_2(2) = (2-2\alpha)\frac{S_{2,2}}{S_{3,2}} g_3(2) + \frac{S_{2,2}}{S_{3,3}} g_3(3)$$

$$= \frac{2(1-\alpha)(3-5\alpha)}{S_{5,3}} + \frac{3(1-\alpha)(4-3\alpha)}{S_{5,3}} = \frac{1-\alpha}{S_{5,3}}(26-19\alpha)$$

The case $\nu = 5, \kappa = 2$.

$$g_4(1) = \frac{S_{4,1}}{S_{5,2}} = \frac{(1-\alpha|-1)_3}{S_{5,2}}; \quad g_4(2) = (4-2\alpha)\frac{S_{4,2}}{S_{5,2}} = \frac{2(1-\alpha|-1)_2(11-7\alpha)}{S_{5,2}}$$

$$g_3(1) = g_4(1) + \frac{S_{3,1}}{S_{4,2}} g_4(2) = \frac{S_{4,1}}{S_{5,2}} + \frac{S_{3,1}}{S_{5,2}}(4-2\alpha) = \frac{(1-\alpha|-1)_2}{S_{5,2}}(3-\alpha+4-2\alpha)$$

$$g_3(2) = (3-2\alpha)\frac{S_{3,2}}{S_{4,2}} g_4(2) = \frac{S_{3,2}}{S_{5,2}}(3-2\alpha)(4-2\alpha)$$

$$g_2(1) = g_3(1) + \frac{S_{2,1}}{S_{3,2}} g_3(2) = \frac{S_{4,1} + (4-2\alpha)S_{3,1}}{S_{5,2}} + \frac{S_{2,1}}{S_{5,2}}(3-2\alpha)(4-2\alpha)$$

$$= \frac{(1-\alpha|-1)_2}{S_{5,2}}(7-3\alpha+2(3-2\alpha))$$

$$g_2(2) = (2-2\alpha)\frac{S_{2,2}}{S_{3,2}} g_3(2) = \frac{2(1-\alpha)}{S_{5,2}}(3-2\alpha)(4-2\alpha)$$

1.6 The second paragraph after Examples of Proposition 11

The transition probabilities of the conditional upward random walks.

The case $(\nu, \kappa) = (5, 4)$

$$\begin{bmatrix} g_2(1) \\ g_2(2) \end{bmatrix} = \begin{bmatrix} 6/10 \\ 4/10 \end{bmatrix} g_1(1)$$

$$\begin{bmatrix} g_3(2) \\ g_3(3) \end{bmatrix} = \begin{bmatrix} 1 & 2/9 \\ 0 & 7/9 \end{bmatrix} \begin{bmatrix} g_2(1) \\ g_2(2) \end{bmatrix} = \begin{bmatrix} 3/10 \\ 7/10 \end{bmatrix}$$

$$\begin{bmatrix} g_4(3) \\ g_4(4) \end{bmatrix} = \begin{bmatrix} 1 & 3/7 \\ 0 & 4/7 \end{bmatrix} \begin{bmatrix} g_3(2) \\ g_3(3) \end{bmatrix} = \begin{bmatrix} 6/10 \\ 4/10 \end{bmatrix}$$

The case $(\nu, \kappa) = (5, 3)$

$$\begin{bmatrix} g_2(1) \\ g_2(2) \end{bmatrix} = \begin{bmatrix} (9-6\alpha)/5(7-5\alpha) \\ (26-19\alpha)/5(7-5\alpha) \end{bmatrix} g_1(1)$$

$$\begin{bmatrix} g_3(1) \\ g_3(2) \\ g_3(3) \end{bmatrix} = \begin{bmatrix} \frac{2-\alpha}{9-6\alpha} & 0 \\ \frac{7-5\alpha}{9-6\alpha} & \frac{14-10\alpha}{26-19\alpha} \\ 0 & \frac{12-9\alpha}{26-19\alpha} \end{bmatrix} \begin{bmatrix} g_2(1) \\ g_2(2) \end{bmatrix} = \begin{bmatrix} \frac{2-\alpha}{5(7-5\alpha)} \\ \frac{21-15\alpha}{5(7-5\alpha)} \\ \frac{12-9\alpha}{5(7-5\alpha)} \end{bmatrix}$$

$$\begin{bmatrix} g_4(2) \\ g_4(3) \end{bmatrix} = \begin{bmatrix} 1 & \frac{9-6\alpha}{21-15\alpha} & 0 \\ 0 & \frac{12-9\alpha}{21-15\alpha} & 1 \end{bmatrix} \begin{bmatrix} g_3(1) \\ g_3(2) \\ g_3(3) \end{bmatrix} = \begin{bmatrix} \frac{11-7\alpha}{5(7-5\alpha)} \\ \frac{24-18\alpha}{5(7-5\alpha)} \end{bmatrix}$$

The case $(\nu, \kappa) = (5, 2)$

$$\begin{bmatrix} g_2(1) \\ g_2(2) \end{bmatrix} = \begin{bmatrix} (13-7\alpha)/5(5-3\alpha) \\ (12-8\alpha)/5(5-3\alpha) \end{bmatrix} g_1(1)$$

$$\begin{bmatrix} g_3(2) \\ g_3(3) \end{bmatrix} = \begin{bmatrix} \frac{7-3\alpha}{13-7\alpha} & 0 \\ \frac{6-4\alpha}{13-7\alpha} & 1 \end{bmatrix} \begin{bmatrix} g_2(1) \\ g_2(2) \end{bmatrix} = \begin{bmatrix} \frac{7-3\alpha}{5(5-3\alpha)} \\ \frac{18-12\alpha}{5(5-3\alpha)} \end{bmatrix}$$

$$\begin{bmatrix} g_4(3) \\ g_4(4) \end{bmatrix} = \begin{bmatrix} \frac{3-\alpha}{7-3\alpha} & 0 \\ \frac{4-2\alpha}{7-3\alpha} & 1 \end{bmatrix} \begin{bmatrix} g_3(2) \\ g_3(3) \end{bmatrix} = \begin{bmatrix} \frac{3-\alpha}{5(5-3\alpha)} \\ \frac{22-14\alpha}{5(5-3\alpha)} \end{bmatrix}$$

THE END