A supplement to Qualitative inequalities for squared partial correlations of a Gaussian random vector

Sanjay Chaudhuri

June 25, 2013

1 Mixed ancestral graphs

In this supplement we briefly discuss mixed ancestral graphs. Our discussion closely follows Richardson and Spirtes (2002). We also refer to the same text for a more detailed treatment of the class of these graphs.

A graph G is an ordered pair (V, E) where V is a set of vertices and E is a set of edges.

A mixed graph is a graph containing three types of edges, undirected (—), directed ($\rightarrow$) and bidirected ($\leftrightarrow$). The following terminology is used to describe relations between variables in such a graph:

1. If $\alpha - \beta$ in G, then α is a neighbour of β and $\alpha \in ne(\beta)$.
2. If $\alpha \rightarrow \beta$ in G, then α is a parent of β and $\alpha \in pa(\beta)$.
3. If $\beta \rightarrow \alpha$ in G, then α is a child of β and $\alpha \in ch(\beta)$.
4. If $\alpha \leftrightarrow \beta$ in G, then α is a spouse of β and $\alpha \in sp(\beta)$.

Definition 1. A vertex α is said to be an ancestor of a vertex β if either there is a directed path $\alpha \rightarrow \cdots \rightarrow \beta$ from α to β, or $\alpha = \beta$. Further, for $X \subseteq V$ its ancestor set is defined as:

$$an(X) = \{\alpha \ : \ \alpha \text{ is an ancestor of } \beta \text{ for some } \beta \in X\}.$$

Definition 2. A vertex α is said to be anterior to a vertex β if there is a path ${}_{\alpha}\pi_{\beta}$ on which every edge is either of the form $\gamma - \delta$, or $\gamma \rightarrow \delta$ with δ between γ and β, or $\alpha = \beta$; that is, there are no edges $\gamma \leftrightarrow \delta$ and there are no edges $\delta \rightarrow \gamma$ pointing toward α. Further, for $X \subseteq V$ its anterior set is defined as:

$$ant(X) = \{\alpha \ : \ \alpha \text{ is an anterior to } \beta \text{ for some } \beta \in X\}.$$

Definition 3. An ancestral graph G is a mixed graph in which the following conditions hold for all vertices α in G:

This research was partially supported by Grant R-155-000-081-112 from National University of Singapore.

Department of Statistics and Applied probability
National University of Singapore
Singapore, 117546 E-mail: sanjay@stat.nus.edu.sg

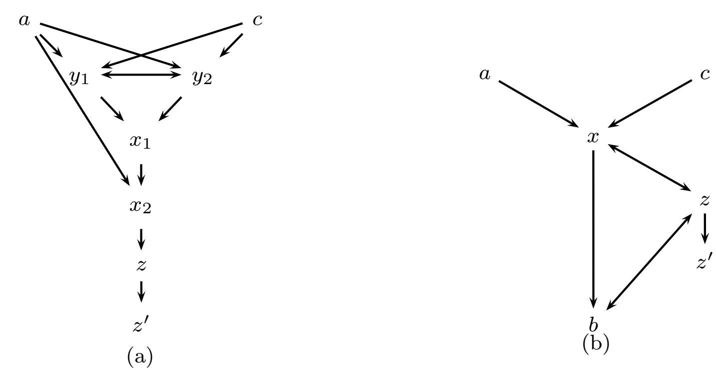

Fig. 1

1. $\alpha \notin ant\,(pa(\alpha) \cup sp(\alpha))$ and
2. if $ne(\alpha) \neq \emptyset$ then $pa(\alpha) \cup sp(\alpha) = \emptyset$.

The d-separation criterion for DAGs can be extended to m-separation criterion for mixed ancestral graphs.

A non-endpoint vertex ζ on a path is a collider on the path if the edges preceding and succeeding ζ on the path have an arrowhead at ζ, ie., $\rightarrow \zeta \leftarrow$, $\leftrightarrow \zeta \leftrightarrow$, $\leftrightarrow \zeta \leftarrow$, $\rightarrow \zeta \leftrightarrow$. A non-endpoint vertex ζ on a path which is not a collider is a noncollider on the path.

A path between vertices α and β in an ancestral graph G is said to be m-connecting given a set Z (possibly empty), with α, $\beta \notin Z$ if:

1. every noncollider on the path is not in Z, and
2. every collider on the path is in the $ant(Z)$.

If there is no path m-connecting α and β given Z, then α and β are said to be m-separated given Z. Non empty sets X and Y are m-separated given Z, if for every pair α, β with $\alpha \in X$ and $\beta \in Y$, α and β are m-separated given Z (X, Y and Z are disjoint sets).

A distribution F is said to satisfy the conditional independence relations represented by a mixed ancestral graph if for disjoint subsets X, Y and Z, $X \perp\!\!\!\perp Y|Z$ according to F whenever X is m-separated from Y given Z.

2 Examples of mixed ancestral graphs in the main text

Example 1 Consider the Mixed ancestral graph in Figure 1(a). There are more than one paths connecting a and c. Each of them has a collider on it. As for example, y_1 is a collider on the path $\{a, y_1, c\}$. So a is m-separated from c given $\emptyset$. Thus $a \perp\!\!\!\perp c$. Further note that, x_2 is a noncollider on each path connecting $\{a, c\}$ and z. Thus, $ac \perp\!\!\!\perp z|x_2$. Similarly, $ac \perp\!\!\!\perp z'|z$.

Example 2 Now we consider the graph in Figure 1(b). Clearly $a \perp\!\!\!\perp c$. x is a collider on the paths $\{a, x, z\}$ and $\{c, x, z\}$. Further, b is a collider on the paths $\{a, x, b, z\}$ and

$\{c, x, b, z\}$. So b and x m-separates a and c from z given $\emptyset$. So $ac \perp\!\!\!\perp z$. Now note that, x is a noncollider on the paths $\{a, x, b\}$ and $\{c, x, b\}$. Also z is a collider on the paths $\{a, x, z, b\}$ and $\{c, x, z, b\}$. This implies $\{a, c\}$ is m-separated from b given x, but not given zx.

References

Richardson T, Spirtes P (2002) Ancestral graph markov models. The Annals of Statistics 30(4):962–1030