Supplementory material

The Harmonic Moment Tail Index Estimator: asymptotic distribution and robustness

Jan Beran[1], Dieter Schell[1] and Milan Stehlík[2]

[1]Department of Mathematics and Statistics, University of Konstanz, Germany
[2]Institute of Applied Statistics, Johannes Keppler University, Austria

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Proof of theorem 1. Following Resnick (2007), we divide the proof in four steps. The first two steps are stated without proof, because they correspond to Resnick (2007), pp. 81-84. Steps three and four are stated in more detail.

Step 1. $X_{n-k,n}$ is a consistent estimator of $U(n/k)$ as $n \to \infty$, $k \to \infty$ and $k/n \to 0$. Therefore we can replace $U(n/k)$ in the tail empirical measure by $X_{n-k,n}$.

Step 2. Replacing $U(n/k)$ by $X_{n-k,n}$ we define an estimator of $\nu_{n,k}$,

$$\hat{\nu}_{n,k}(\cdot) := \frac{1}{k}\sum_{i=1}^{n} 1\{X_i/X_{n-k,n} \in \ \cdot\ \}.$$

Then, defining a scaling operator of Radon measures in $M_+(0,\infty]$ by

$$T : M_+(0,\infty] \times (0,\infty) \to M_+(0,\infty]$$

where $(\mu, x)(A) \mapsto T(\mu, x)(A) := \mu(xA)$ for any $A \in \mathcal{E}$, and showing its continuity in $(\nu_\gamma, 1)$, we can show consistency of $\hat{\nu}_{n,k}$, i.e.

$$\hat{\nu}_{n,k} \xrightarrow{P} \nu_\gamma,$$

(as $n \to \infty$, $k \to \infty$ and $k/n \to 0$) by the continuous mapping theorem using

$$T(\nu_\gamma, 1)(\cdot) = \nu_\gamma(\cdot).$$

Step 3. We are now ready to prove consistency of the Harmonic Moment Tail Index estimator. Therefore, we will use the functional

$$T^{(\beta)}(\mu) = \int_1^\infty \mu(x, \infty] \frac{dx}{x^\beta},$$

defined on $M_+(0, \infty]$, where $\beta > 1 - \frac{1}{\gamma}$. For $\beta = 1$ and $T^{(1)}(\hat{\nu}_{n,k})$ this step can also be found in Resnick (2007). We modify this step in order to obtain the desired consistency result. Unfortunately the continuous mapping theorem is not directly applicable to $T^{(\beta)}(\hat{\nu}_{n,k})$. Therefore we define

$$X_{M_n} := \int_1^M \hat{\nu}_{n,k}(x, \infty] x^{-\beta} dx \quad \text{and} \quad X_M := \int_1^M \nu_\gamma(x, \infty] x^{-\beta} dx$$

Since the integration is over a finite region, the continuous mapping theorem yields

$$\begin{aligned} \int_1^M \hat{\nu}_{n,k}(x, \infty] x^{-\beta} dx \Rightarrow \int_1^M \nu_\gamma(x, \infty] x^{-\beta} dx &= \int_1^M x^{-\frac{1}{\gamma} - \beta} dx \\ &= \gamma(\gamma(\beta - 1) + 1)^{-1}(1 - M^{1 - \frac{1}{\gamma} - \beta}), \end{aligned}$$

due to $\hat{\nu}_{n,k} \xrightarrow{P} \nu_\gamma$. Moreover,

$$X_M \Rightarrow \int_1^\infty \nu_\gamma(x, \infty] x^{-\beta} dx = \gamma/(\gamma(\beta - 1) + 1) =: X,$$

as $M \to \infty$ provided that $\beta > 1 - \frac{1}{\gamma}$. Now, with $Y_n = Y_n^{(\beta)} := \int_1^\infty \hat{\nu}_{n,k}(x, \infty] x^{-\beta} dx$, we obtain

$$\begin{aligned} d(X_{M_n}, Y_n) :&= \int_1^M \hat{\nu}_{n,k}(x, \infty] x^{-\beta} dx - \int_1^\infty \hat{\nu}_{n,k}(x, \infty] x^{-\beta} dx \\ &= \int_M^\infty \hat{\nu}_{n,k}(x, \infty] x^{-\beta} dx. \end{aligned}$$

Hence, according to Theorem 4.2 in Billingsley (1968) we have to show that for all $\varepsilon > 0$,

$$\lim_{M\to\infty} \limsup_{n\to\infty} P(d(X_{M_n}, Y_n) > \varepsilon) \\ = \lim_{M\to\infty} \limsup_{n\to\infty} P\left(\int_M^\infty \hat{\nu}_{n,k}(x,\infty] x^{-\beta} dx > \varepsilon\right) \to 0,$$

in order to obtain $Y_n \xrightarrow{P} \gamma/(\gamma(\beta-1)+1)$. To achieve this, we use an analogous decomposition as in Resnick (2007):

$$P\left(\int_M^\infty \hat{\nu}_{n,k}(x,\infty] x^{-\beta} dx > \delta\right) \leq \\ P\left(\int_M^\infty \hat{\nu}_{n,k}(x,\infty] x^{-\beta} dx > \delta, \frac{X_{n-k,n}}{U(n/k)} \in (1-\eta, 1+\eta)\right) \\ + P\left(\int_M^\infty \hat{\nu}_{n,k}(x,\infty] x^{-\beta} dx > \delta, \frac{X_{n-k,n}}{U(n/k)} \notin (1-\eta, 1+\eta)\right) := P_1 + P_2$$

The second probability is negligible, since

$$P_2 \leq P\left(\left|\frac{X_{n-k,n}}{U(n/k)} - 1\right| \geq \eta\right) \to 0,$$

as $n \to \infty$, $k \to \infty$ and $n/k \to \infty$. This is a consequence of the result in the first step of the proof. Moreover, P_1 is bounded from above by

$$P\left(\int_M^\infty \nu_{n,k}((1-\eta)x, \infty] x^{-\beta} dx > \delta\right) := P_3,$$

since, under the condition $X_{n-k,n} \geq (1-\eta)U(n/k)$ we have

$$\hat{\nu}_{n,k}(x,\infty] = \frac{1}{k}\sum_{i=1}^n 1\{X_i/X_{n-k,k} \in (x,\infty]\} \\ \leq \frac{1}{k}\sum_{i=1}^n 1\{X_i/\left((1-\eta)U(k/n)\right) \in (x,\infty]\} = \nu_{n,k}((1-\eta)x,\infty]$$

Using Markov's inequality we have

$$
\begin{aligned}
P_3 &= P\left((1-\eta)^{\beta-1}\int_{(1-\eta)M}^{\infty} \nu_{n,k}(x,\infty]x^{-\beta}dx > \delta\right) \\
&\leq \frac{(1-\eta)^{\beta-1}}{\delta} E\left(\int_{(1-\eta)M}^{\infty} \nu_{n,k}(x,\infty]x^{-\beta}dx\right) \\
&= \frac{(1-\eta)^{\beta-1}}{\delta}\int_{M(1-\eta)}^{\infty} \frac{n}{k}P(X_1 > U(n/k)x)x^{-\beta}dx.
\end{aligned}
$$

Note, that regularly varying tails imply

$$
\frac{n}{k}P(X_1 > U(n/k)x) = \frac{n}{k}P\left(X_1/U(n/k) \in (x,\infty]\right) \to \nu_\gamma((x,\infty]) = x^{-\frac{1}{\gamma}},
$$

provided $n\to\infty$, $k\to\infty$ but $k/n\to 0$. Thus, by Karamata's theorem, we obtain

$$
\begin{aligned}
P\left(\int_{(1-\eta)M}^{\infty} \nu_{n,k}(x,\infty]x^{-\beta}dx > \delta\right) &\overset{n\to\infty}{\to} \frac{(1-\eta)^{\beta-1}}{\delta}\int_{(1-\eta)M}^{\infty} x^{-\frac{1}{\gamma}-\beta}dx \\
&= O(M^{1-\frac{1}{\gamma}-\beta}),
\end{aligned}
$$

so that $P_1 + P_2 \to 0$ as $M\to\infty$.

Step 4. So far, we have shown

$$
Y_n^{(\beta)} \overset{P}{\to} \int_1^{\infty} \nu_\gamma(x,\infty]x^{-\beta}dx = \gamma/(1+\gamma(\beta-1)), \tag{1}
$$

Rewriting $Y_n^{(\beta)}$ we obtain

$$
Y_n^{(\beta)} = (X_{n-k,n})^{\beta-1}\int_{X_{n-k,n}}^{\infty} \frac{n}{k}\left(1-F_n(s)\right)s^{-\beta}ds.
$$

Since $\beta\neq 1$, partial integration yields

$$
\int_t^{\infty}(1-F(s))\frac{ds}{s^{\beta}} = -\frac{1}{1-\beta}(1-F(t))t^{1-\beta} + \frac{1}{1-\beta}\int_t^{\infty} s^{1-\beta}dF(s),
$$

so that

$$
\begin{aligned}
Y_n^{(\beta)} &= \frac{1}{1-\beta}\left((X_{n-k,n})^{\beta-1}\int_{X_{n-k,n}}^{\infty}\frac{n}{k}s^{1-\beta}dF_n(s) - 1\right) \\
&= \frac{1}{1-\beta}\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{X_{n-k,n}}{X_{n-i+1,n}}\right)^{\beta-1} - 1\right).
\end{aligned} \tag{2}
$$

Combining (1) and (2) we obtain

$$\frac{1}{k}\sum_{i=1}^{k}\left(\frac{X_{n-k,n}}{X_{n-i+1,n}}\right)^{\beta-1} \xrightarrow{P} \frac{1}{\gamma(\beta-1)+1}.$$

Hence,

$$H_{k,n}^{(\beta)} = \frac{1}{\beta-1}\left(\frac{1}{\frac{1}{k}\sum_{i=1}^{k}\left(\frac{X_{n-k,n}}{X_{n-i+1,n}}\right)^{\beta-1}} - 1\right) \xrightarrow{P} \gamma.$$

$\square$