Ann Inst Stat Math (2013) 65:941–958
DOI 10.1007/s10463-013-0398-9

Mittag-Leffler vector random fields with Mittag-Leffler direct and cross covariance functions

Chunsheng Ma

Received: 30 March 2012 / Revised: 29 August 2012 / Published online: 30 March 2013

Abstract In terms of the two-parameter Mittag-Leffler function with specified parameters, this paper introduces the Mittag-Leffler vector random field through its finite-dimensional characteristic functions, which is essentially an elliptically contoured one and reduces to a Gaussian one when the two parameters of the Mittag-Leffler function equal 1. Having second-order moments, a Mittag-Leffler vector random field is characterized by its mean function and its covariance matrix function, just like a Gaussian one. In particular, we construct direct and cross covariances of Mittag-Leffler type for such vector random fields.

Keywords Covariance matrix function · Cross covariance · Direct covariance · Elliptically contoured random field · Gaussian random field · Mittag-Leffler function · Variogram

1 Introduction

A Mittag-Leffler function $E_{\alpha,\beta}(z)$, named after its originator, the Swedish mathematician Gö Mittag-Leffler (1846–1927), is defined by a series expansion

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C},$$

C. Ma (✉)
Department of Mathematics, Statistics, and Physics, Wichita State University, Wichita, KS 67260-0033, USA
e-mail: chunsheng.ma@wichita.edu

C. Ma
School of Economics, Wuhan University of Technology, Wuhan, Hubei 430070, China

where α and β are positive constants, and $\mathbb{C}$ is the set of complex numbers. For $\alpha = 0$ and $\beta = 1$, one may define $E_{0,1}(z)$ by $E_{0,1}(z) = \frac{1}{1-z}$, $|z| < 1$. This is a generalization of the exponential function, to which it reduces for $\alpha = \beta = 1$, $E_{1,1}(z) = \mathrm{e}^z$, $z \in \mathbb{C}$. It is easy to see that

$$E_{1,2}(z) = \frac{\mathrm{e}^z - 1}{z}, \quad E_{1,3}(z) = \frac{\mathrm{e}^z - 1 - z}{z^2},$$

and, for an integer $n \geq 2$,

$$E_{1,n}(z) = \frac{1}{z^{n-1}} \left(\mathrm{e}^z - \sum_{k=0}^{n-2} \frac{z^k}{k!} \right), \quad z \in \mathbb{C}.$$

Also, it is known that

$$E_{2,1}(z) = \cosh(\sqrt{z}), \quad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}, \quad E_{\frac{1}{2},1}(z) = \mathrm{e}^{z^2} \mathrm{erfc}(-z),$$

where $\mathrm{erfc}(z)$ is the error complement function defined by

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \mathrm{e}^{-t^2}\, \mathrm{d}t, \quad z \in \mathbb{C}.$$

For properties of the Mittag-Leffler function, we refer the reader to Erdélyi (1995), Djrbashian (1993), and Paris and Kaminski (2001). This and related functions have been widely adopted in the literature of statistics, probability, and various sciences. For example, Mittag-Leffler and related distributions have been studied by Pillai (1990), Fujita (1993), Lin (1998), Jose and Uma (2009), Jayakumar et al. (2010), Haubold et al. (2011), among others. The Mittag-Leffler and related functions appear naturally in connection with the description of relaxation phenomena in various complex physical, biophysical and chemical systems (Glockle and Nonnenmacher 1995; Blumenfeld and Mandelbrot 1997; Berberan-Santos 2005; Weron and Klauzer 2010), within the framework of fractional (non-integer) kinetic equations. Recently, there is an increasing interest in time series or stochastic processes involving Mittag-Leffler or related functions. One type of those time series or stochastic processes have Mittag-Leffler or related marginals, such as a Mittag-Leffler process of Jayakumar (2003), and first-order autoregressive processes considered by Jayakumar and Pillai (1993), Jose and Uma (2009), Jayakumar et al. (2010) and Jose et al. (2010). Another type is the so-called Mittag-Leffler noise investigated by Glockle and Nonnenmacher (1995), Kneller and Hinsen (2004), Viñales and Despósito (2007), and Uma et al. (2011). It is a univariate Gaussian process with the Mittag-Leffler covariance function $E_{1,\beta}(-\theta|x|^\beta)$, $x \in \mathbb{R}$, where $\theta > 0$ and $0 < \beta \leq 1$, and includes an Ornstein–Uhlenbeck process as a special case with $\beta = 1$. Barndorff-Nielsen and Leonenko (2005) constructed stationary processes with prescribed one-dimensional marginal laws and Mittag-Leffler covariance functions $E_{\alpha,\beta}(-|x|)$ and $E_{\alpha,1}(-|x|^\gamma)$, $x \in \mathbb{R}$, where $\alpha > 0$, $\beta > 0$, and $0 < \gamma < 1$. This paper adopts a different approach, and our objective is to introduce Mittag-Leffler vector random fields, which have second-order moments and form a subclass of elliptically contoured (spherically invariant) vector random fields (Du and Ma 2011; Ma 2011a), and to construct a particular type of covariance matrix functions for such random fields with Mittag-Leffler direct and cross covariances.

By an m-variate random field, we mean a family of m-variate real random vectors on the same probability space, $\{\mathbf{Z}(x) = (Z_1(x), \ldots, Z_m(x))', x \in \mathbb{D}\}$, where the index set $\mathbb{D}$ could be a temporal, spatial, or spatio-temporal domain. This paper is concerned with second-order random fields, each of whose components has second-order moments. For an m-variate second-order random field $\{\mathbf{Z}(x), x \in \mathbb{D}\}$, its mean or expectation (function) is defined by

$$E\mathbf{Z}(x) = (EZ_1(x), \ldots, EZ_m(x))', \quad x \in \mathbb{D}.$$

Its covariance matrix (function), $\mathbf{C}(x_1, x_2)$, is a two-point function with $m \times m$ entries,

$$\begin{aligned} C_{ij}(x_1, x_2) = E\{[Z_i(x_1) - EZ_i(x_1)][Z_j(x_2) - EZ_j(x_2)]\}, \quad x_1, x_2 \in \mathbb{D}, \\ i, j = 1, \ldots, m, \end{aligned}$$

where each diagonal entry is called a direct covariance (function) and each off-diagonal entry is called a cross covariance (function). In particular, a second-order vector random field $\{\mathbf{Z}(x), x \in \mathbb{D}\}$ is said to be (second-order, or weakly) stationary or homogeneous, if its mean function $E\mathbf{Z}(x)$ does not depend on $x \in \mathbb{D}$, and its covariance matrix function $\mathrm{cov}(\mathbf{Z}(x_1), \mathbf{Z}(x_2))$ depends on the lag $x_1 - x_2$ only. In such a case, we write $\mathrm{cov}(\mathbf{Z}(x_1), \mathbf{Z}(x_2))$ as $\mathbf{C}(x_1 - x_2)$ for simplicity. For properties of second-order vector random fields, we refer the reader to Cramér and Leadbetter (1967), Gikhman and Skorokhod (1969), Yaglom (1987) and Ma (2011a), among others. It is known that linear combinations with positive coefficients, Hadamard (or Schur) products, and convergent products of covariance matrix functions are again covariance matrix functions, under the Gaussian or second-order elliptically contoured setting (see, e.g., Ma 2011a,b,c,d).

The rest of this paper is organized as follows. Section 2 defines a Mittag-Leffler vector random field through its finite-dimensional characteristic functions, and Sect. 3 presents some covariance matrix structures for Mittag-Leffler vector random fields, with particular attention to Mittag-Leffler direct and cross covariances. Some concluding remarks are given in Sect. 4, while our theorems are proved in Sect. 5.

2 Mittag-Leffler vector random fields

This section provides the definition for a Mittag-Leffler vector random field through its finite-dimensional characteristic functions after Theorem 1, which guarantees the existence of such a vector random field with any given covariance matrix structure and with finite-dimensional characteristic functions of Mittag-Leffler type.

Theorem 1 *Let α and β be positive constants,* $0 < \alpha \le 1$*, and* $\beta \ge \alpha$*. If an $m \times m$ matrix function* $\mathbf{C}(x_1, x_2)$, $x_1, x_2 \in \mathbb{D}$*, possesses the following two properties:*

(i) *the transpose of* $\mathbf{C}(x_1, x_2)$ *equals* $\mathbf{C}(x_2, x_1)$, *i.e.,*

$$\{\mathbf{C}(x_1, x_2)\}' = \mathbf{C}(x_2, x_1), \quad x_1, x_2 \in \mathbb{D},$$

and

(ii) *the inequality*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{a}_i' \mathbf{C}(x_i, x_j) \mathbf{a}_j \geq 0 \tag{1}$$

holds for every natural number n, *any* $x_i \in \mathbb{D}$ *and any* $\mathbf{a}_i \in \mathbb{R}^m$, $i = 1, 2, \ldots, n$,

then there exists an m*-variate random field* $\{\mathbf{Z}(x), x \in \mathbb{D}\}$ *with mean* $\mathbf{0}$, *covariance matrix function* $\mathbf{C}(x_1, x_2)$, *and the finite-dimensional characteristic functions*

$$E \exp\left\{ \imath \sum_{k=1}^{n} \mathbf{Z}'(x_k) \boldsymbol{\omega}_k \right\} = \Gamma(\beta) E_{\alpha,\beta} \left(-\frac{\Gamma(\alpha+\beta)}{2\Gamma(\beta)} \sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j) \boldsymbol{\omega}_j \right), \quad \boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_n \in \mathbb{R}^m, \tag{2}$$

for every natural number n *and any* $x_i \in \mathbb{D}$, $i = 1, 2, \ldots, n$.

Conversely, the covariance matrix function $\mathbf{C}(x_1, x_2)$ *of an* m*-variate random field* $\{\mathbf{Z}(x), x \in \mathbb{D}\}$ *with the finite-dimensional characteristic functions* (2) *satisfies the above properties* (i) *and* (ii).

We call an m-variate random field $\{\mathbf{Z}(x) + \boldsymbol{\mu}(x), x \in \mathbb{D}\}$ a Mittag-Leffler random field, where $\{\mathbf{Z}(x), x \in \mathbb{D}\}$ is described in Theorem 1 and $\boldsymbol{\mu}(x), x \in \mathbb{D}$, is an m-valued (non-random) function. The term is so called because its finite-dimensional characteristic functions are of the Mittag-Leffler form. In the particular case where $\alpha = \beta = 1$, it reduces to a Gaussian vector random field, since $E_{1,1}(x) = \mathrm{e}^x$. From Theorem 1, one may see that a Mittag-Leffler vector random field is characterized by its mean function and its covariance matrix function, just like a Gaussian or a second-order elliptically contoured one. For the construction of such a random field, it suffices to construct its covariance matrix function.

It should be remarked that the two parameters α and β are restricted to $0 < \alpha \leq \min(1, \beta)$ of the Mittag-Leffler function employed in Theorem 1 or in the above definition of a Mittag-Leffler vector random field. Although it is not clear how to deal with other cases generally, the above procedure does not work for, for example, $E_{2,1}(-x) = \cosh(\sqrt{-x}) = \cos(\sqrt{x}), x \geq 0$. Nevertheless, $E_{2,1}(x^2) = \cosh(x)$ can be used to construct a covariance matrix function, as the following example shows.

Example 1 For given real functions $g_1(x), \ldots, g_m(x), x \in \mathbb{D}$, there exists an m-variate Mittag-Leffler random field with direct and cross covariances

$$C_{ij}(x_1, x_2) = \cosh(g_i(x_1) + g_j(x_2)), \quad x_1, x_2 \in \mathbb{D}, i, j = 1, \ldots, m.$$

To see this, it suffices to verify inequality (1), which follows from

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{a}_i'\mathbf{C}(x_i,x_j)\mathbf{a}_j=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m}a_{ik}\cosh(g_k(x_i)+g_l(x_j))a_{jl}$$
$$=\left\{\sum_{i=1}^{n}\sum_{k=1}^{m}a_{ik}\cosh(g_k(x_i))\right\}^2+\left\{\sum_{i=1}^{n}\sum_{k=1}^{m}a_{ik}\sinh(g_k(x_i))\right\}^2\geq 0.$$

A reason for we restrict $0<\alpha\leq\min(1,\beta)$ is due to the following Schneider's (1996) theorem on the completely monotone property of a Mittag-Leffler function with $0<\alpha\leq 1$ and $\beta\geq\alpha$, which contains Pollard's (1948) theorem as a special case with $\beta=1$. We cite Schneider's theorem here for convenience in the proofs of our theorems in Sect. 5.

Theorem 2 (Schneider's theorem) *The Mittag-Leffler function* $E_{\alpha,\beta}(-x)$, $x\geq 0$, *is completely monotone on* $[0,\infty)$ *if and only if*

$$0<\alpha\leq 1,\quad \beta\geq\alpha. \tag{3}$$

Moreover, in the Laplace–Stieltjes representation

$$E_{\alpha,\beta}(-x)=\frac{1}{\Gamma(\beta)}\int_0^\infty \mathrm{e}^{-xt}\,\mathrm{d}\mu_{\alpha,\beta}(t),\quad x\geq 0, \tag{4}$$

the underlying probability measure $\mu_{\alpha,\beta}(x)$ *on* $[0,\infty)$ *has the following properties involving* α *and* β*:*

(i) *for* $0<\alpha<1$ *and* $\beta=1$, $\mu_{\alpha,1}(x)$ *is absolutely continuous with respect to the Lebesgue measure and its density* $f_{\alpha,1}(x)$ *is given by*

$$f_{\alpha,1}(x)=\alpha^{-1}x^{-1-1/\alpha}p_\alpha(x^{-1/\alpha}),\quad x\geq 0,$$

where $p_\alpha(x)$ *is the density of the one-sided stable distribution whose Laplace transform is* $\exp(-t^\alpha)$*;*

(ii) *for* $\alpha=\beta=1$, $\mu_{1,1}(x)$ *is the Dirac measure at the point* 1*;*

(iii) *for* $\alpha=1$ *and* $\beta>1$, $\mu_{1,\beta}(x)$ *is a beta distribution function with density*

$$f_{1,\beta}(x)=\begin{cases}(\beta-1)(1-x)^{\beta-2}, & 0<x<1,\\ 0, & \text{elsewhere};\end{cases}$$

(iv) *for* $\alpha<1$ *and* $\beta\geq\alpha$, $\mu_{\alpha,\beta}(x)$ *is absolutely continuous with density*

$$f_{\alpha,\beta}(x)=\Gamma(\beta)\sum_{k=0}^{\infty}\frac{(-x)^k}{k!\Gamma(\beta-\alpha-\alpha k)},\quad x\geq 0.$$

3 Mittag-Leffler direct and cross covariances

As we have seen in the last section, a Mittag-Leffler vector random field is characterized by its mean function and its covariance matrix function. For the development of such a random field, we just need to construct its covariance matrix function. In this section, we concentrate on a particular subclass of Mittag-Leffler vector random fields whose direct and cross covariance functions are of the Mittag-Leffler form. Our construction method involves two ingredients or building blocks, a scalar (or univariate) variogram and a conditionally negative definite matrix, which are briefly reviewed below.

A symmetric, real $m \times m$ matrix $\boldsymbol{\Theta} = (\theta_{ij})$ is said to be conditionally negative definite (Bapat and Raghavan 1997), if the inequality

$$\sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j \theta_{ij} \leq 0$$

holds for any real numbers $a_1, \ldots, a_m$ subject to $\sum_{k=1}^{m} a_k = 0$. Without the restrict condition $\sum_{k=1}^{m} a_k = 0$, $\boldsymbol{\Theta} = (\theta_{ij})$ is often said to be negative definite. In general, a necessary condition for the above inequality is

$$\theta_{ii} + \theta_{jj} \leq 2\theta_{ij}, \quad i, j = 1, \ldots, m,$$

which implies that all entries of a conditionally negative definite matrix are non-negative whenever its diagonal entries are non-negative. If all its diagonal entries vanish, a conditionally negative definite matrix is also named a Euclidean distance matrix (Dattorro 2005). It is known that $\boldsymbol{\Theta} = (\theta_{ij})$ is conditionally negative definite if and only if an $m \times m$ matrix with entries $\exp(-\theta_{ij} u)$ is positive definite, for every fixed $u \geq 0$ (cf. Theorem 4.1.3 of Bapat and Raghavan 1997). Some simple examples of conditionally negative definite matrices are

(i) $\theta_{ij} = \theta_i + \theta_j$,
(ii) $\theta_{ij} = \theta_i - \theta_j$,
(iii) $\theta_{ij} = |\theta_i - \theta_j|$,
(iv) $\theta_{ij} = (\theta_i - \theta_j)^2$,
(v) $\theta_{ij} = -\theta_i \theta_j$,
(vi) $\theta_{ij} = \max(\theta_i, \theta_j)$, $i, j = 1, \ldots, m$, where θ_is are real numbers.

In what follows, $\|\mathbf{x}\|$ and $|\mathbf{x}|$ denotes the Euclidean norm and the ℓ_1 norm of $\mathbf{x} \in \mathbb{R}^d$, respectively. An $m \times m$ matrix with entries $\|\mathbf{x}_i - \mathbf{x}_j\|$ is a Euclidean distance matrix, for m given points $\mathbf{x}_i \in \mathbb{R}^d$ $(i = 1, \ldots, m)$. So is a matrix with entries $|\mathbf{x}_i - \mathbf{x}_j|$, $i, j = 1, \ldots, m$. The Euclidean norm and the ℓ_1 norm involves in the covariance matrix structures of Theorems 3 and 4, respectively, besides two conditionally negative definite matrices, one having entries β_{ij}, and the other having entries θ_{ij}.

Theorem 3 *Let λ be a positive constant with $0 < \lambda \leq 1$. If $\boldsymbol{\beta} = (\beta_{ij})$ is an $m \times m$ conditionally negative definite matrix with all entries greater than 1, and $\boldsymbol{\Theta} = (\theta_{ij})$ is* *an $m \times m$ conditionally negative definite matrix with positive entries, then there exists an m-variate stationary Mittag-Leffler random field with direct and cross covariances*

$$C_{ij}(\mathbf{x}) = \frac{\Gamma(\beta_{ij})}{\beta_{ij}-1}\theta_{ij}^{-\frac{1}{2}} E_{1,\beta_{ij}}\left(-\theta_{ij}^{\frac{\lambda}{2}}\|\mathbf{x}\|^{\lambda}\right), \quad \mathbf{x} \in \mathbb{R}^d, \quad i, j = 1, \ldots, m. \tag{5}$$

Theorem 4 *Let λ be a positive constant with $0 < \lambda \leq 1$. If $\boldsymbol{\beta} = (\beta_{ij})$ is an $m \times m$ conditionally negative definite matrix with all entries greater than 1, and $\boldsymbol{\Theta} = (\theta_{ij})$ is an $m \times m$ conditionally negative definite matrix with positive entries, then there exists an m-variate stationary Mittag-Leffler random field with direct and cross covariances*

$$C_{ij}(\mathbf{x}) = \frac{\Gamma(\beta_{ij})}{\beta_{ij}-1}\theta_{ij}^{-\frac{d}{2}} E_{1,\beta_{ij}}\left(-\theta_{ij}^{\frac{\lambda}{2}}|\mathbf{x}|^{\lambda}\right), \quad \mathbf{x} \in \mathbb{R}^d, \quad i, j = 1, \ldots, m. \tag{6}$$

Obviously, a major difference between (5) and (6) appears in the θ_{ij}'s exponent, associated with the different norm employed in the difference construction.

It is known that the Euclidean norm and the ℓ_1 norm are scalar (or univariate) variograms. A scalar variogram or structure function $\gamma(x_1, x_2)$, $x_1, x_2 \in \mathbb{D}$, associated with a scalar random field with second-order increments (see, e.g., Cressie 1993), is a non-negative function with $\gamma(x, x) \equiv 0$, $x \in \mathbb{D}$, and satisfies the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \gamma(x_i, x_j) \leq 0,$$

for every integer $n \geq 2$, any $x_k \in \mathbb{D}$, and any $a_k \in \mathbb{R}$ ($k = 1, \ldots, n$) subject to $\sum_{k=1}^{n} a_k = 0$. The covariance matrix structures in the following theorems involve a scalar variogram, besides a conditionally negative definite matrix.

Theorem 5 *If $\gamma(x_1, x_2)$, $x_1, x_2 \in \mathbb{D}$, is a scalar variogram, and $\boldsymbol{\beta} = (\beta_{ij})$ is an $m \times m$ conditionally negative definite matrix with all entries greater than 1, then there exists an m-variate Mittag-Leffler random field with direct and cross covariances*

$$C_{ij}(x_1, x_2) = \frac{\Gamma(\beta_{ij})}{\beta_{ij}-1} E_{1,\beta_{ij}}(-\gamma(x_1, x_2)), \quad x_1, x_2 \in \mathbb{D}, \quad i, j = 1, \ldots, m. \tag{7}$$

In the particular case where $\beta_{ij} \equiv 1$, $E_{1,1}(-\gamma(x_1, x_2)) = \exp(-\gamma(x_1, x_2))$, $x_1, x_2 \in \mathbb{D}$, is a scalar covariance function whenever $\gamma(x_1, x_2)$ is a scalar variogram, according to Schoenberg's theorem (see, e.g., Ma 2005).

The parameter α of the Mittag-Leffler function is assumed to be equal to 1 in each of covariance matrix structures in Theorems 3–5, in contrast to the parameter β that varies from entry to entry. It would be of interest to consider a case with distinct α's. Nevertheless, the parameter α is not limited to be 1 in Theorem 6 or 7 below.

Theorem 6 *Let α and β_k be positive constants, $0 < \alpha < 1$, and $\beta_k > \frac{\alpha}{2}$ ($k = 1, \ldots, m$). If $\gamma(x_1, x_2)$, $x_1, x_2 \in \mathbb{D}$, is a scalar variogram, then there exists an m-variate Mittag-Leffler random field with direct and cross covariances*

$$C_{ij}(x_1, x_2) = \Gamma(\beta_i + \beta_j - \alpha) E_{\alpha, \beta_i + \beta_j}(-\gamma(x_1, x_2)), \quad x_1, x_2 \in \mathbb{D}, \\ i, j = 1, \ldots, m. \tag{8}$$

Theorem 7 *Let α and β be positive constants, $0 < \alpha < 1$, and $\beta \geq \alpha$. If $\gamma(x_1, x_2)$ is a scalar variogram on $\mathbb{D}$ and $\boldsymbol{\Theta} = (\theta_{ij})$ is an $m \times m$ conditionally negative definite matrix with non-negative diagonal entries, then there is an m-variate Mittag-Leffler random field with direct and cross covariances*

$$C_{ij}(x_1, x_2) = E_{\alpha, \beta}(-\gamma(x_1, x_2) - \theta_{ij}), \quad x_1, x_2 \in \mathbb{D}, \quad i, j = 1, \ldots, m. \tag{9}$$

Example 2 For $\alpha \in (0, 1)$ and $\nu \in (0, 1]$, in (9) taking $\beta = 1$ and $\gamma(\mathbf{x}, \mathbf{0}) = \|\mathbf{x}\|^\nu$, $\mathbf{x} \in \mathbb{R}^d$, yields a stationary covariance matrix function with entries

$$C_{ij}(\mathbf{x}) = E_{\alpha, 1}(-\|\mathbf{x}\|^\nu - \theta_{ij}), \quad \mathbf{x} \in \mathbb{R}^d, \quad i, j = 1, \ldots, m.$$

In the particular case where $d = 1$ and $\theta_{ij} = 0$, $C_{ij}(x)$ reduces to the univariate correlation model proposed by Barndorff-Nielsen and Leonenko (2005), and by Viñales and Despósito (2007).

Although $E_{2,1}(x) = \cosh(\sqrt{x})$ and $E_{2,2}(x) = \frac{\sinh(\sqrt{x})}{\sqrt{x}}$, $x \geq 0$, do not belong to any of the families in Theorems 3–7, a quotient of $E_{2,1}(x)$ and the reciprocal of $E_{2,2}(x)$ can be used to construct covariance matrix structures as follows.

Theorem 8 *Let λ be a positive constant and $0 < \lambda < 1$. If $\gamma(x_1, x_2)$ is a scalar variogram on $\mathbb{D}$ and $\boldsymbol{\Theta} = (\theta_{ij})$ is an $m \times m$ conditionally negative definite matrix with non-negative diagonal entries, then there exist three m-variate Mittag-Leffler random fields, the first one has direct and cross covariances*

$$C_{ij}(x_1, x_2) = \frac{\cosh\{\lambda(\gamma(x_1, x_2) + \theta_{ij})^{\frac{1}{2}}\}}{\cosh\{(\gamma(x_1, x_2) + \theta_{ij})^{\frac{1}{2}}\}}, \quad x_1, x_2 \in \mathbb{D}, \quad i, j = 1, \ldots, m, \tag{10}$$

the second one has direct and cross covariances

$$C_{ij}(x_1, x_2) = \frac{\sinh\{\lambda(\gamma(x_1, x_2) + \theta_{ij})^{\frac{1}{2}}\}}{\sinh\{(\gamma(x_1, x_2) + \theta_{ij})^{\frac{1}{2}}\}}, \quad x_1, x_2 \in \mathbb{D}, i, j = 1, \ldots, m, \tag{11}$$

and the third one has direct and cross covariances

$$C_{ij}(x_1, x_2) = \frac{(\gamma(x_1, x_2) + \theta_{ij})^{\frac{1}{2}}}{\sinh\{(\gamma(x_1, x_2) + \theta_{ij})^{\frac{1}{2}}\}}, \quad x_1, x_2 \in \mathbb{D}, i, j = 1, \ldots, m. \tag{12}$$

In particular, letting $\lambda \to 0_+$ in (10) yields direct and cross covariances

$$C_{ij}(x_1, x_2) = \frac{1}{\cosh\{(\gamma(x_1, x_2) + \theta_{ij})^{\frac{1}{2}}\}}, \quad x_1, x_2 \in \mathbb{D}, \quad i, j = 1, \ldots, m.$$

4 Concluding remarks

The Mittag-Leffler vector random field introduced in this paper is formulated through its finite-dimensional characteristic functions, in terms of the two-parameter Mittag-Leffler function with $0 < \alpha \leq \min(1, \beta)$. This type of vector random fields allows every possible covariance matrix structure that satisfies the conditions of Theorem 1, just as a Gaussian one does. A Gaussian vector random field belongs to the family of Mittag-Leffler ones, which in turn belong to that of elliptically contoured ones (Ma 2011a). Examples of elliptically contoured vector random fields include Gaussian, Student's t (Ma 2013a), stable, logistic, hyperbolic (Du et al. 2012), Mittag-Leffler, Linnik, and Laplace ones. It should be remarked that the finite-dimensional characteristic functions of a Mittag-Leffler vector random field are of Mittag-Leffler type, but its finite-dimensional distribution functions are not of Mittag-Leffler type. In contrast, time series or stochastic processes whose finite-dimensional densities are of Mittag-Leffler type are studied in Pillai (1990), Jayakumar and Pillai (1993), Lin (1998), Jayakumar (2003), and Jose et al. (2010).

A Mittag-Leffler vector random field is characterized by its mean and covariance matrix functions, just like a Gaussian one. An important feature of such a random field is that there is not a restriction or a tight connection between its mean and covariance matrix functions, unlike a log-Gaussian (Matheron 1989), Chi-square (Ma 2011c), or K-distributed case (Ma 2013b), so that the Mittag-Leffler vector random field may be relatively more flexible for applications, just like a Gaussian one.

As an important ingredient or building block, we also employ the Mittag-Leffler function to construct covariance matrix functions, besides two other building blocks (see also Du and Ma 2011; Ma 2011b,c,d, 2013a), a scalar variogram and a conditionally negative definite matrix. Of course, other covariance matrix structures, which satisfy the conditions of Theorem 1, can be adopted for the Mittag-Leffler vector random field as well. An immediate application of these covariance matrix functions in geostatistics is for the co-kriging or linear predication (Cressie 1993). A recent example may be found in Kleiber and Nychka (2012), where non-stationary covariance matrix models are highlighted, while covariance matrix functions in our Theorems 5–8 may be stationary or non-stationary relevant to the choice of $\gamma(x_1, x_2)$. Covariance matrix structures constructed in this paper are, of course, admissible for other second-order elliptically contoured vector random fields, besides the Mittag-Leffler one, but may not be available for another non-Gaussian vector random field, such as a log-Gaussian, skew-Gaussian (Minozzo and Ferracuti 2012), Chi-square, or K-distributed one.

5 Proofs

5.1 Proof of Theorem 1

For a given matrix function $\mathbf{C}(x_1, x_2)$ that possesses the two properties (i) and (ii), it is known (e.g., Gikhman and Skorokhod 1969) that there is an m-variate Gaussian random field, $\{\mathbf{Y}(x), x \in \mathbb{D}\}$, say, with mean $\mathbf{0}$ and covariance matrix function $\mathbf{C}(x_1, x_2)$. Let U be a non-negative random variable with distribution $\mu_{\alpha,\beta}(x)$ defined in Theorem 2, and let U and $\{\mathbf{Y}(x), x \in \mathbb{D}\}$ be independent. The expectation of U is obtained from (4),

$$EU = -\frac{\mathrm{d}}{\mathrm{d}x} \int_0^\infty \mathrm{e}^{-xu}\, \mathrm{d}\mu_{\alpha,\beta}(u)\Big|_{x=0} = -\Gamma(\beta) E'_{\alpha,\beta}(0) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

where $E'_{\alpha,\beta}(0) = \frac{1}{\Gamma(\alpha+\beta)}$ follows from the Taylor series expansion of $E_{\alpha,\beta}(x)$.

Based on U and $\{\mathbf{Y}(x), x \in \mathbb{D}\}$, we define an m-variate random field

$$\mathbf{Z}(x) = (EU)^{-\frac{1}{2}} U^{\frac{1}{2}} \mathbf{Y}(x), \quad x \in \mathbb{D}.$$

Clearly, this is an m-variate random field with second-order moments, its mean function is $E\mathbf{Z}(x) = \mathbf{0}$, and its covariance matrix function is

$$\operatorname{cov}(\mathbf{Z}(x_1), \mathbf{Z}(x_2)) = \operatorname{cov}(\mathbf{Y}(x_1), \mathbf{Y}(x_2)) = \mathbf{C}(x_1, x_2), \quad x_1, x_2 \in \mathbb{D}.$$

For every natural number n and any $x_k \in \mathbb{D}$ $(k = 1, \ldots, n)$, the characteristic function of $(\mathbf{Z}'(x_1), \ldots, \mathbf{Z}'(x_n))'$ is

$$\begin{aligned}
E \exp\left\{\imath \sum_{k=1}^n \mathbf{Z}'(x_k)\boldsymbol{\omega}_k\right\} &= E \exp\left\{\imath (EU)^{-\frac{1}{2}} U^{\frac{1}{2}} \sum_{k=1}^n \mathbf{Y}'(x_k)\boldsymbol{\omega}_k\right\} \\
&= \int_0^\infty E \exp\left\{\imath (EU)^{-\frac{1}{2}} u^{\frac{1}{2}} \sum_{k=1}^n \mathbf{Y}'(x_k)\boldsymbol{\omega}_k\right\} \mathrm{d}\mu_{\alpha,\beta}(u) \\
&= \int_0^\infty \exp\left\{-\frac{1}{2EU} u \sum_{i=1}^n \sum_{j=1}^n \boldsymbol{\omega}'_i \mathbf{C}(x_i, x_j)\boldsymbol{\omega}_j\right\} \mathrm{d}\mu_{\alpha,\beta}(u) \\
&= \Gamma(\beta) E_{\alpha,\beta}\left(-\frac{1}{2EU} \sum_{i=1}^n \sum_{j=1}^n \boldsymbol{\omega}'_i \mathbf{C}(x_i, x_j)\boldsymbol{\omega}_j\right), \quad \boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_n \in \mathbb{R}^m,
\end{aligned}$$

where the third equality follows from the characteristic function of the normal random vector $(\mathbf{Y}'(x_1), \ldots, \mathbf{Y}'(x_n))'$, and the last equality follows from identity (4).

Conversely, it is easy to check that the covariance matrix function of an m-variate random field $\{\mathbf{Z}(x), x \in \mathbb{D}\}$ with the finite-dimensional characteristic functions (2) satisfies the properties (i) and (ii) in Theorem 1. □

5.2 Proof of Theorem 3

We start by showing that the matrix function

$$H(\mathbf{x}) = \begin{pmatrix} \theta_{11}^{-\frac{1}{2}} \exp(-\theta_{11}^{\frac{1}{2}}\|\mathbf{x}\|) & \theta_{12}^{-\frac{1}{2}} \exp(-\theta_{12}^{\frac{1}{2}}\|\mathbf{x}\|) & \cdots & \theta_{1m}^{-\frac{1}{2}} \exp(-\theta_{1m}^{\frac{1}{2}}\|\mathbf{x}\|) \\ \theta_{21}^{-\frac{1}{2}} \exp(-\theta_{21}^{\frac{1}{2}}\|\mathbf{x}\|) & \theta_{22}^{-\frac{1}{2}} \exp(-\theta_{22}^{\frac{1}{2}}\|\mathbf{x}\|) & \cdots & \theta_{2m}^{-\frac{1}{2}} \exp(-\theta_{2m}^{\frac{1}{2}}\|\mathbf{x}\|) \\ \cdots & \cdots & \cdots & \cdots \\ \theta_{m1}^{-\frac{1}{2}} \exp(-\theta_{m1}^{\frac{1}{2}}\|\mathbf{x}\|) & \theta_{m2}^{-\frac{1}{2}} \exp(-\theta_{m2}^{\frac{1}{2}}\|\mathbf{x}\|) & \cdots & \theta_{mm}^{-\frac{1}{2}} \exp(-\theta_{mm}^{\frac{1}{2}}\|\mathbf{x}\|) \end{pmatrix}, \quad \mathbf{x} \in \mathbb{R}^d, \tag{13}$$

is a stationary covariance matrix function. To do so, we evaluate its Fourier transform matrix, which is positively proportional to

$$\begin{pmatrix} (\theta_{11} + \|\boldsymbol{\omega}\|^2)^{-\frac{d+1}{2}} & (\theta_{12} + \|\boldsymbol{\omega}\|^2)^{-\frac{d+1}{2}} & \cdots & (\theta_{1m} + \|\boldsymbol{\omega}\|^2)^{-\frac{d+1}{2}} \\ (\theta_{21} + \|\boldsymbol{\omega}\|^2)^{-\frac{d+1}{2}} & (\theta_{22} + \|\boldsymbol{\omega}\|^2)^{-\frac{d+1}{2}} & \cdots & (\theta_{2m} + \|\boldsymbol{\omega}\|^2)^{-\frac{d+1}{2}} \\ \cdots & \cdots & \cdots & \cdots \\ (\theta_{m1} + \|\boldsymbol{\omega}\|^2)^{-\frac{d+1}{2}} & (\theta_{m2} + \|\boldsymbol{\omega}\|^2)^{-\frac{d+1}{2}} & \cdots & (\theta_{mm} + \|\boldsymbol{\omega}\|^2)^{-\frac{d+1}{2}} \end{pmatrix}, \quad \boldsymbol{\omega} \in \mathbb{R}^d,$$

by Theorem 1.14 of Stein and Weiss (1971, page 6). This matrix is positive definite for each fixed $\boldsymbol{\omega} \in \mathbb{R}^d$, since its entries can be rewritten as

$$\begin{aligned}(\theta_{ij} + \|\boldsymbol{\omega}\|^2)^{-\frac{d+1}{2}} &= \frac{1}{\Gamma(\frac{d+1}{2})} \int_0^\infty u^{\frac{d+1}{2}-1} \exp(-\|\boldsymbol{\omega}\|^2 u) \exp(-\theta_{ij} u)\, \mathrm{d}u, \quad i, j \\ &= 1, \ldots, m,\end{aligned}$$

and the matrix with entries $\exp(-\theta_{ij} u)$ is positive definite due to the assumption that $\boldsymbol{\Theta}$ is conditionally negative definite. It follows from the Cramér–Kolmogorov theorem (see Cramér 1940) that $H(\mathbf{x})$ is a covariance matrix function.

According to Pollard (1946), the function $\exp(-x^\lambda)$, $x \geq 0$, is the Laplace transform of a distribution function, $G(\omega)$, say, with support on the interval $[0, \infty)$; that is,

$$\exp(-x^\lambda) = \int_0^\infty \exp(-x\omega)\, \mathrm{d}G(\omega), \quad x \geq 0. \tag{14}$$

From (13) and Theorem 4 of Ma (2011b), we obtain that an $m \times m$ matrix function with entries

$$\theta_{ij}^{-\frac{1}{2}} \exp(-\theta_{ij}^{\frac{\lambda}{2}} \|\mathbf{x}\|^\lambda) = \theta_{ij}^{-\frac{1}{2}} \int_0^\infty \exp(-\theta_{ij}^{\frac{1}{2}} \|\mathbf{x}\|\omega)\, \mathrm{d}G(\omega), \quad \mathbf{x} \in \mathbb{R}^d, \quad i, j = 1, \ldots, m,$$

is a covariance matrix function. So is the matrix function with entries

$$\theta_{ij}^{-\frac{1}{2}} \exp(-\theta_{ij}^{\frac{\lambda}{2}} \|\mathbf{x}\|^{\lambda} u), \quad \mathbf{x} \in \mathbb{R}^d, \quad i, j = 1, \ldots, m, \tag{15}$$

where u is an arbitrary non-negative number.

For $\alpha = 1$, $\beta > 1$, according to Schneider's theorem, we can express $E_{1,\beta}(-x)$ as

$$E_{1,\beta}(-x) = \frac{\beta - 1}{\Gamma(\beta)} \int_0^1 (1-u)^{\beta-2} \mathrm{e}^{-xu} \, \mathrm{d}u, \quad x \geq 0. \tag{16}$$

Consequently, (5) can be rewritten as

$$C_{ij}(\mathbf{x}) = \theta_{ij}^{-\frac{1}{2}} \int_0^1 \exp\left(-\beta_{ij} \ln \frac{1}{1-u}\right) \exp(-\theta_{ij}^{\frac{\lambda}{2}} \|\mathbf{x}\|^{\lambda} u) \frac{du}{(1-u)^2}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d,$$
$$i, j = 1, \ldots, m.$$

For each fixed $u \in [0, 1)$, an $m \times m$ matrix with entries $\exp\left(-\beta_{ij} \ln \frac{1}{1-u}\right)$ is positive definite, since $\boldsymbol{\beta} = (\beta_{ij})$ is conditionally negative definite, and an $m \times m$ matrix with entries (15) is a covariance matrix function. Therefore, it follows from Theorem 4 of Ma (2011b) that an $m \times m$ matrix function with entries $C_{ij}(\mathbf{x})$ is a covariance matrix function. □

5.3 Proof of Theorem 4

As in the proof of Theorem 3 we start by showing that the matrix function

$$H(\mathbf{x}) = \begin{pmatrix} \theta_{11}^{-\frac{d}{2}} \exp(-\theta_{11}^{\frac{1}{2}}|\mathbf{x}|) & \theta_{12}^{-\frac{d}{2}} \exp(-\theta_{12}^{\frac{1}{2}}|\mathbf{x}|) & \cdots & \theta_{1m}^{-\frac{d}{2}} \exp(-\theta_{1m}^{\frac{1}{2}}|\mathbf{x}|) \\ \theta_{21}^{-\frac{d}{2}} \exp(-\theta_{21}^{\frac{1}{2}}|\mathbf{x}|) & \theta_{22}^{-\frac{d}{2}} \exp(-\theta_{22}^{\frac{1}{2}}|\mathbf{x}|) & \cdots & \theta_{2m}^{-\frac{d}{2}} \exp(-\theta_{2m}^{\frac{1}{2}}|\mathbf{x}|) \\ \cdots & \cdots & \cdots & \cdots \\ \theta_{m1}^{-\frac{d}{2}} \exp(-\theta_{m1}^{\frac{1}{2}}|\mathbf{x}|) & \theta_{m2}^{-\frac{d}{2}} \exp(-\theta_{m2}^{\frac{1}{2}}|\mathbf{x}|) & \cdots & \theta_{mm}^{-\frac{d}{2}} \exp(-\theta_{mm}^{\frac{1}{2}}|\mathbf{x}|) \end{pmatrix}, \quad \mathbf{x} \in \mathbb{R}^d, \tag{17}$$

is a stationary covariance matrix function. Its Fourier transform matrix is positively proportional to

$$\begin{pmatrix} \prod_{k=1}^{d} (\theta_{11} + \omega_k^2)^{-1} & \prod_{k=1}^{d} (\theta_{12} + \omega_k^2)^{-1} & \cdots & \prod_{k=1}^{d} (\theta_{1m} + \omega_k^2)^{-1} \\ \prod_{k=1}^{d} (\theta_{21} + \omega_k^2)^{-1} & \prod_{k=1}^{d} (\theta_{22} + \omega_k^2)^{-1} & \cdots & \prod_{k=1}^{d} (\theta_{2m} + \omega_k^2)^{-1} \\ \cdots & \cdots & \cdots & \cdots \\ \prod_{k=1}^{d} (\theta_{m1} + \omega_k^2)^{-1} & \prod_{k=1}^{d} (\theta_{m2} + \omega_k^2)^{-1} & \cdots & \prod_{k=1}^{d} (\theta_{mm} + \omega_k^2)^{-1} \end{pmatrix}, \quad \boldsymbol{\omega} \in \mathbb{R}^d,$$

and is positive definite for each fixed $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)' \in \mathbb{R}^d$, since its entries can be rewritten as

$$\prod_{k=1}^{d} (\theta_{ij} + \omega_k^2)^{-1} = \prod_{k=1}^{d} \int_0^\infty \exp(-\omega_k^2 u) \exp(-\theta_{ij} u) \, \mathrm{d}u, \quad i, j = 1, \ldots, m,$$

and, by assumption, $\boldsymbol{\Theta}$ is conditionally negative definite. According to the Cramér–Kolmogorov theorem, $H(\mathbf{x})$ is a covariance matrix function.

It then follows from (14), (17), and Theorem 4 of Ma (2011b) that an $m \times m$ matrix with entries

$$\theta_{ij}^{-\frac{d}{2}} \exp(-\theta_{ij}^{\frac{\lambda}{2}} |\mathbf{x}|^\lambda) = \theta_{ij}^{-\frac{d}{2}} \int_0^\infty \exp(-\theta_{ij}^{\frac{1}{2}} |\mathbf{x}| \omega) \, \mathrm{d}G(\omega), \quad \mathbf{x} \in \mathbb{R}^d, \quad i, j = 1, \ldots, m,$$

is a covariance matrix function. So is the matrix function with entries

$$\theta_{ij}^{-\frac{d}{2}} \exp(-\theta_{ij}^{\frac{\lambda}{2}} |\mathbf{x}|^\lambda u), \quad \mathbf{x} \in \mathbb{R}^d, \quad i, j = 1, \ldots, m, \tag{18}$$

where u is an arbitrary nonnegative number. In terms of identity (16), (6) can be rewritten as

$$C_{ij}(\mathbf{x}) = \theta_{ij}^{-\frac{d}{2}} \int_0^1 \exp\left(-\beta_{ij} \ln \frac{1}{1-u}\right) \exp(-\theta_{ij}^{\lambda} |\mathbf{x}|^\lambda u) \frac{du}{(1-u)^2}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d,$$
$$i, j = 1, \ldots, m,$$

which form a covariance matrix function by Theorem 4 of Ma (2011b). □

5.4 Proof of Theorem 5

By using identity (16), (7) can be rewritten as

$$C_{ij}(x_1, x_2) = \int_0^1 \mathrm{e}^{-\beta_{ij} \ln \frac{1}{1-u}} \mathrm{e}^{-\gamma(x_1, x_2) u} \frac{du}{(1-u)^2}, \quad x_1, x_2 \in \mathbb{D}, \quad i, j = 1, \ldots, m.$$

Since $\gamma(x_1, x_2)$ is a scalar variogram, $\mathrm{e}^{-\gamma(x_1, x_2) u}$, $x_1, x_2 \in \mathbb{D}$, is a scalar covariance function for each fixed $u \in [0, 1]$, by Schoenberg's theorem. Also, for each fixed $u \in (0, 1)$, an $m \times m$ matrix with entries $\mathrm{e}^{-\beta_{ij} \ln \frac{1}{1-u}}$ is positive definite. Thus, the matrix function with entries $\mathrm{e}^{-\beta_{ij} \ln \frac{1}{1-u}} \mathrm{e}^{-\gamma(x_1, x_2) u}$ is a covariance matrix function. So is the matrix function with entries (7) by Theorem 4 of Ma (2011b). □

5.5 Proof of Theorem 6

For $0 < \alpha \leq 1$ and $\beta > \alpha$, it follows from (2.15) of Schneider (1996) that the function $E_{\alpha,\beta}(-x)$ can be expressed as

$$E_{\alpha,\beta}(-x) = \int_0^\infty \mathrm{e}^{-xu} f_{\alpha,\beta}(u)\,\mathrm{d}u, \quad x \geq 0,$$

where

$$f_{\alpha,\beta}(x^{-\alpha}) = \frac{1}{\Gamma(\beta-\alpha)} x^{1+\alpha-\beta} h_{\alpha,\beta}(x), \quad x > 0,$$

and

$$h_{\alpha,\beta}(x) = \int_0^x y^{\beta-\alpha-1} g_\alpha(x-y)\,\mathrm{d}y, \quad x > 0,$$

where $g_\alpha(x)$ is a density of the one-sided stable distribution with the Laplace transform $\exp(-\omega^\alpha)$, $\omega \geq 0$. Alternatively, the function $E_{\alpha,\beta}(-x)$ can be expressed as

$$E_{\alpha,\beta}(-x) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^\infty \mathrm{e}^{-xu} u^{\frac{\beta-\alpha-1}{\alpha}} h_{\alpha,\beta}(u^{-\frac{1}{\alpha}})\,\mathrm{d}u, \quad x \geq 0.$$

Based on such an expression, we rewrite (8) as

$$C_{ij}(x_1,x_2) = \int_0^\infty \mathrm{e}^{-\gamma(x_1,x_2)u} u^{\frac{\beta_i+\beta_j-\alpha-1}{\alpha}} h_{\alpha,\beta_i+\beta_j}(u^{-\frac{1}{\alpha}})\,\mathrm{d}u,$$
$$x_1, x_2 \in \mathbb{D}, \quad i, j = 1, \ldots, m.$$

For each fixed $u > 0$, an $m \times m$ matrix with entries $u^{\frac{\beta_i+\beta_j-\alpha-1}{\alpha}}$ is clearly positive definite. Also, an $m \times m$ matrix with entries $h_{\alpha,\beta_i+\beta_j}(u^{-\frac{1}{\alpha}})$ is positive definite, since the following inequality holds for any $a_k \in \mathbb{R}$ ($k = 1, \ldots, m$),

$$\begin{aligned}
&\sum_{i=1}^m \sum_{j=1}^m a_i a_j h_{\alpha,\beta_i+\beta_j}(u^{-\frac{1}{\alpha}}) \\
&= \sum_{i=1}^m \sum_{j=1}^m a_i a_j \int_0^{u^{-\frac{1}{\alpha}}} y^{\beta_i+\beta_j-\alpha-1} g_\alpha(u^{-\frac{1}{\alpha}} - y)\,\mathrm{d}y \\
&= \int_0^{u^{-\frac{1}{\alpha}}} \left(\sum_{i=1}^m a_i y^{\beta_i} \right)^2 y^{-\alpha-1} g_\alpha(u^{-\frac{1}{\alpha}} - y)\,\mathrm{d}y \geq 0.
\end{aligned}$$

Thus, the matrix function with entries $\mathrm{e}^{-\gamma(x_1,x_2)u} u^{\frac{\beta_i+\beta_j-\alpha-1}{\alpha}} h_{\alpha,\beta_i+\beta_j}(u^{-\frac{1}{\alpha}})$ is a covariance matrix function, due to the fact that $\mathrm{e}^{-\gamma(x_1,x_2)u}$ is a scalar covariance function by Schoenberg's theorem. So is the matrix function with entries (8) by Theorem 4 of Ma (2011b). □

5.6 Proof of Theorem 7

For $0 < \alpha \leq 1$ and $\beta \geq \alpha$, according to Schneider's theorem, the function $E_{\alpha,\beta}(-x)$ can be expressed as

$$E_{\alpha,\beta}(-x) = \int_0^\infty \mathrm{e}^{-xu} f_{\alpha,\beta}(u)\,\mathrm{d}u, \quad x \geq 0,$$

where $f_{\alpha,\beta}(x)$ is a density function and

$$f_{\alpha,\beta}(x) = \Gamma(\beta) \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma(\beta - \alpha - \alpha n)}, \quad x \geq 0.$$

As a result, (9) can be rewritten as

$$C_{ij}(x_1, x_2) = \int_0^\infty \mathrm{e}^{-\theta_{ij}u} \mathrm{e}^{-\gamma(x_1,x_2)u} f_{\alpha,\beta}(u)du, \quad x_1, x_2 \in \mathbb{D}, \quad i, j = 1, \ldots, m.$$

Since $\gamma(x_1, x_2)$ is a scalar variogram, $\mathrm{e}^{-\gamma(x_1,x_2)u}$, $x_1, x_2 \in \mathbb{D}$, is a scalar covariance function for each fixed $u \geq 0$, by Schoenberg's theorem. Since $\boldsymbol{\Theta}$ is conditionally negative definite, an $m \times m$ matrix with entries $\mathrm{e}^{-\theta_{ij}u}$ is positive definite. Thus, the matrix function with entries $\mathrm{e}^{-\theta_{ij}u}\mathrm{e}^{-\gamma(x_1,x_2)u}$ is a covariance matrix function. So is the matrix function with entries (9) by Theorem 4 of Ma (2011b). □

5.7 Proof of Theorem 8

Notice that $\cosh(x)$ can be decomposed into an infinite product as

$$\cosh(x) = \prod_{n=1}^{\infty} \left(1 + \frac{4x^2}{(2n-1)^2\pi^2}\right), \quad x \in \mathbb{R}.$$

As a result, (10) can be rewritten as

$$C_{ij}(x_1, x_2) = \prod_{n=1}^{\infty} \left(\lambda^2 + \frac{(1-\lambda^2)(2n-1)^2\pi^2}{4(\gamma(x_1, x_2) + \theta_{ij}) + (2n-1)^2\pi^2}\right), \quad x_1, x_2 \in \mathbb{D}, \quad i, j = 1, \ldots, m.$$

Similarly, using the product expansion of $\sinh(x)$, $\sinh(x) = x\prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2\pi^2}\right)$, we rewrite (11) as

$$C_{ij}(x_1, x_2) = \lambda \prod_{n=1}^{\infty}\left(\lambda^2 + \frac{(1-\lambda^2)n^2\pi^2}{\gamma(x_1, x_2) + \theta_{ij} + n^2\pi^2}\right), \quad x_1, x_2 \in \mathbb{D}, \quad i, j = 1, \ldots, m,$$

and rewrite (12) as

$$C_{ij}(x_1, x_2) = \prod_{n=1}^{\infty}\left(1 + \frac{\gamma(x_1, x_2) + \theta_{ij}}{n^2\pi^2}\right)^{-1}, \quad x_1, x_2 \in \mathbb{D}, \quad i, j = 1, \ldots, m.$$

By Theorems 1–3 of Ma (2011b), the function (10), (11), or (12) forms a covariance matrix function once we are able to show that an $m \times m$ matrix function with entries

$$\{1 + (\gamma(x_1, x_2) + \theta_{ij})\tau\}^{-1}, \quad x_1, x_2 \in \mathbb{D}, \quad i, j = 1, \ldots, m,$$

is a covariance matrix function, for each fixed $\tau > 0$. In fact, this is true since these entries can be re-expressed as

$$\{1 + (\gamma(x_1, x_2) + \theta_{ij})\tau\}^{-1} = \int_0^{\infty} \mathrm{e}^{-\gamma(x_1,x_2)\tau u}\mathrm{e}^{-\theta_{ij}\tau u}\,\mathrm{d}u \quad x_1, x_2 \in \mathbb{D}, \quad i, j = 1, \ldots, m,$$

and the functions

$$\mathrm{e}^{-\gamma(x_1,x_2)\tau u}\mathrm{e}^{-\theta_{ij}\tau u} \quad x_1, x_2 \in \mathbb{D}, \quad i, j = 1, \ldots, m,$$

form a covariance matrix function, under the assumptions. □

Acknowledgments This work is partly supported by U.S. Department of Energy under Grant DE-SC0005359. The author is grateful to Professor N.N. Leonenko for helpful discussions regarding the Mittag-Leffler function, and to an anonymous referee for valuable comments and suggestions that led to a considerably improved presentation of this paper.

References

Bapat, R. B., Raghavan, T. E. S. (1997). *Nonnegative matrices and applications*. Cambridge: Cambridge University Press.

Barndorff-Nielsen, O. E., Leonenko, N. N. (2005). Spectral properties of superpositions of Ornstein–Uhlenbeck type processes. *Methodology and Computing in Applied Probability*, *7*, 335–352.

Berberan-Santos, M. N. (2005). Properties of the Mittag-Leffler relaxation function. *Journal of Mathematical Chemistry*, *38*, 629–635.

Blumenfeld, R., Mandelbrot, B. B. (1997). Lévy dusts, Mittag-Leffler statistics, mass fractal lacunarity, and perceived dimension. *Physical Review E*, *56*, 112–118.

Cramér, H. (1940). On the theory of stationary random processes. *Annals of Mathematics*, *41*, 215–230.

Cramér, H., Leadbetter, M. R. (1967). *Stationary and related stochastic processes: sample function properties and their applications*. New York: Wiley.
Cressie, N. (1993). *Statistics for spatial data* (revised ed.). New York: Wiley.
Dattorro, J. (2005). *Convex optimization and Euclidean distance geometry*. California: Meboo Publishing.
Djrbashian, M. M. (1993). *Harmonic analysis and boundary value problems in the complex domain*. Basel: Birkhäuser Verlag.
Du, J., Leonenko, N., Ma, C., Shu, H. (2012). Hyperbolic vector random fields with hyperbolic direct and cross covariance functions. *Stochastic Analysis and Applications*, *30*, 662–674.
Du, J., Ma, C. (2011). Spherically invariant vector random fields in space and time. *IEEE Transactions on Signal Processing*, *59*, 5921–5929.
Erdélyi, A. (Ed.). (1955). *Higher transcendental functions. Bateman project* (Vol. 3). New York: McGraw-Hill.
Fujita, Y. (1993). A generalization of the results of Pillai. *Annals of the Institute of Statistical Mathematics*, *45*, 361–365.
Gikhman, I. I., Skorokhod, A. V. (1969). *Introduction to the theory of random processes*. Philadelphia: W. B. Saunders Co.
Glockle, W. G., Nonnenmacher, T. F. (1995). A fractional calculus approach to self-similar protein dynamics. *Biophysical Journal*, *68*, 46–53.
Haubold, H. J., Mathai, A. M., Saxena, R. K. (2011). Mittag-Leffler functions and their applications. *Journal of Applied Mathematics*, *2011*, Article ID 298628.
Jayakumar, K. (2003). Mittag-Leffler process. *Mathematical and Computer Modelling*, *37*, 1427–1434.
Jayakumar, K., Pillai, R. N. (1993). The first-order autoregressive Mittag-Leffler process. *Journal of Applied Probability*, *30*, 462–466.
Jayakumar, K., Ristic, M. M., Mundassery, D. A. (2010). A generalization to bivariate Mittag-Leffler and bivariate discrete Mittag-Leffler autoregressive processes. *Communication in Statistics—Theory and Methods*, *39*, 942–955.
Jose, K. K., Uma, P. (2009). On Marshall-Olkin Mittag-Leffler distributions and processes. *Far East Journal of Theoretical Statistics*, *28*, 189–199.
Jose, K. K., Uma, P., Lekshmi, V. S., Haubold, H. J. (2010). Generalized Mittag-Leffler distributions and processes for applications in astrophysics and time series modeling. *Proceedings of the third UN/ESA/NASA workshop on the international heliophysical year 2007 and basic space science* (pp. 79–92). New York: Springer.
Kleiber, W., Nychka, D. (2012). Nonstationary modeling for multivariate spatial processes. *Journal of Multivariate Analysis*, *112*, 76–91.
Kneller, G. R., Hinsen, K. (2004). Fractional Brownian dynamics in proteins. *The Journal of Chemical Physics*, *12*, Article ID 10278.
Lin, G. D. (1998). On the Mittag-Leffler distributions. *Journal of Statistical Planning and Inference*, *74*, 1–9.
Ma, C. (2005). Semiparametric spatio-temporal covariance models with the autoregressive temporal margin. *Annals of the Institute of Statistical Mathematics*, *57*, 221–233.
Ma, C. (2011a). Vector random fields with second-order moments or second-order increments. *Stochastic Analysis and Applications*, *29*, 197–215.
Ma, C. (2011b). Covariance matrices for second-order vector random fields in space and time. *IEEE Transactions on Signal Processing*, *59*, 2160–2168.
Ma, C. (2011c). Covariance matrix functions of vector χ^2 random fields in space and time. *IEEE Transactions on Communications*, *59*, 2554–2561.
Ma, C. (2011d). Vector random fields with long-range dependence. *Fractals*, *19*, 249–258.
Ma, C. (2013a). Student's t vector random fields with power-law and log-law decaying direct and cross covariances. *Stochastic Analysis and Applications*, *31*, 167–182.
Ma, C. (2013b). K-distributed vector random fields in space and time. *Statistics and Probability Letters*, *83*, 1143–1150.
Matheron, G. (1989). The internal consistency of models in geostatistics. In M. Armstrong (Ed.), *Geostatistics* (Vol. 1, pp. 21–38). Netherlands: Kluwer.
Minozzo, M., Ferracuti, L. (2012). On the existence of some skew-normal stationary processes. *Chilean Journal of Statistics*, *3*, 159–172.
Paris, R. B., Kaminski, D. (2001). *Asymptotics and Mellin-Barnes integrals*. Cambridge: Cambridge University Press.

Pillai, R. N. (1990). Mittag-Leffler functions and related distributions. *Annals of the Institute of Statistical Mathematics*, *42*, 157–161.

Pollard, H. (1946). The representation of $e^{-x^{\lambda}}$ as a Laplace integral. *Bulletin of the American Mathematical Society*, *52*, 908.

Pollard, H. (1948). The completely monotonic character of the Mittag-Leffler function $E_\alpha(-x)$. *Bulletin of the American Mathematical Society*, *54*, 1115–1116.

Schneider, W. R. (1996). Completely monotone generalized Mittag-Leffler functions. *Expositiones Matematicae*, *14*, 3–16.

Stein, E. M., Weiss, G. (1971). *Introduction to Fourier analysis on Euclidean spaces*. Princeton: Princeton University Press.

Uma, B., Swaminathan, T. N., Ayyaswamy, P. S., Eckmann, D. M., Radhakrishnan, R. (2011). Generalized Langevin dynamics of nanoparticle using a finite element approach: Thermostating with correlated noise. *The Journal of Chemical Physics*, *135*, Article ID 114104.

Viñales, A. D., Despósito, M. A. (2007). Anomalous diffusion induced by a Mittag-Leffler correlated noise. *Physical Review E*, *75*, Article ID 042102.

Weron, K., Klauzer, A. (2010). Generalization of the Khinchin theorem to Lévy flights. *Physics Review Letters*, *105*, Article ID 260603.

Yaglom, A. M. (1987). *Correlation theory of stationary and related random functions*. New York: Springer.