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Modelling conflicting information using subexponential distributions and related classes

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Abstract In the Bayesian modelling the data and the prior information concerning a certain parameter of interest may conflict, in the sense that the information carried by them disagree. The most common form of conflict is the presence of outlying information in the data, which may potentially lead to wrong posterior conclusions. To prevent this problem we use robust models which aim to control the influence of the atypical information in the posterior distribution. Roughly speaking, we conveniently use heavy-tailed distributions in the model in order to resolve conflicts in favour of those sources of information which we believe is more credible. The class of heavy-tailed distributions is quite wide and the literature have been concerned in establishing conditions on the data and prior distributions in order to reject the outlying information. In this work we focus on the subexponential and $\mathfrak{L}$ classes of heavy-tailed distributions, in which we establish sufficient conditions under which the posterior distribution automatically rejects the conflicting information.

Keywords Bayesian robustness · Conflicting information · Regularly varying distributions · Subexponential distributions

1 Introduction

In Bayesian context the idea of *surprising events* such as outliers, gross errors (such as copying), is associated with the presence of conflicting information. Broadly speaking,

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we say that two sources of information *conflict* if they disagree; that is, the functions (likelihood/densities) concerning the parameter of interest are concentrated far away from each other. As extensively discussed in the literature (e.g. Finetti 1961; Lindley 1968; O'Hagan and Forster 2004), problems of conflicts are directly related with the tails thickness. A surprising event occurs on the tails of the distributions and if we model it with a light-tailed distribution it will yield very low posterior probabilities, which may disturb the posterior estimates. Clarke and Gustafson (1998) provides a way of quantifying the sensitivity of the posterior estimates under perturbations in the prior information, data model and data themselves. A detailed discussion about conflicts and their potential effects on the posterior distribution can be found in O'Hagan and Forster (2004, Sect. 3.35).

To the purpose of resolving problems of conflicts, a long literature has been developed aiming to establish *sufficient* conditions on the distributions in the model in order to make the posterior distribution unaffected by conflicts (surprising events). In the pure location-parameter case, Dawid (1973) and O'Hagan (1979) proposed sufficient conditions on the data and prior distribution which allows to resolve conflict by rejecting the conflicting information in favour of the other source. Some further development of the ideas of Dawid and O'Hagan can be found in O'Hagan (1988, 1990), Pericchi et al. (1993) and Pericchi and Sansó (1995), O'Hagan and Le (1994) and Le and O'Hagan (1998) and finally Haro-Lòpez and Smith (1999), who proposed some conditions on multivariate υ-spherical family (Fernandez et al. 1995) involving location and scale parameters in order to bound the influence of the likelihood over the posterior distribution. However, their approach establishes conditions which are quite difficult to verify, and does not provide explicitly the limiting posterior distribution.

Andrade and O'Hagan (2006) used the theory of *regular variation* in order to resolve conflicts in Bayesian modelling of location and scale parameters structures; Andrade and O'Hagan (2011) generalised their idea to location-scale structures. The advantage of regarding heavy tails as regularly varying distributions is that regular variation provides a much easier interpretation of tails decay, since any distribution with regularly varying tails can be represented simply as a power function. The concept Credence (O'Hagan 1988), as a form of expressing the credibility of a source of information, has an equivalent in the regular variation theory.

The regularly varying class is quite restrictive, since it embraces only distributions whose tails behave like a power law function, which may lead to some well-known problems such as moments indeterminacy and convergence of MCMC algorithms. In this work we consider the subexponential and the $\mathfrak{L}$ classes of heavy-tailed distributions, which embraces the regular variation class; however, they also comprise distribution whose tails are lighter than those of the regularly varying class, but heavier than the exponential function. The use of these classes change completely the approaches based on regular variation and those proposed by Dawid (1973) and O'Hagan (1979), due to the tails thickness of the subexponential and $\mathfrak{L}$ distribution. We consider a general problem of many observations which involves a group of outliers and propose *sufficient* conditions on the location and the scale parameter structures to resolve posterior conflicts of information. More precisely, consider an i.i.d. random sample $X = (x_1, x_2, \ldots, x_n)$, let $X_L = (x_1, x_2, \ldots, x_k)$ and $X^U = (x_{k+1}, \ldots, x_n)$, where X_L is a group of outliers, that is X_L is large in relation to X^U. Considering a general model $x_i|y \overset{D}{\sim} f(x|y)$ $(i = 1, 2, \ldots, n)$ and $y \overset{D}{\sim} p(y)$, where y is a location or a scale parameter, the idea is to find sufficient conditions on f and on p so that the posterior distribution becomes unaffected by the outlying group. Using the Subexponential and $\mathfrak{L}$ classes, we propose conditions in which the posterior distribution tends to some quantity which does not involve X_L, i.e. $p(y|X) \to p(y)f(X^U|y)/K$ as $\min\{X_L\} \to \infty$, where K is the normalising constant.

In Sect. 2 we provide the definitions and some properties of the classes of distributions which we use throughout the paper. In Sect. 3 we find sufficient conditions on the location parameter structure in order to reject observations in the sample which are far away from the other sources of information. In Sect. 4 we consider the scale parameter case, in which we propose alternative conditions to those proposed by Andrade and O'Hagan (2006). We illustrate the theory in Sect. 5, where we provides examples involving distributions belonging to the classes studied. Finally, we make some general comments in Sect. 6.

2 Some classes of distribution functions

In this section we recall some important classes of functions (and some properties) that play an important role in the models that we will consider. The basic reference for these classes is Bingham et al. (1987), which will be cited as BGT from now on.

Definition 1 A measurable function f is *regularly varying* at ∞ and with index $\rho \in \Re$, written $f \in RV(\rho)$, if it satisfies:

$$\lim_{x\to\infty} \frac{f(xy)}{f(x)} = y^{\rho}, \quad \forall y > 0. \tag{1}$$

In particular, if $\rho = 0$, f is said to be slowly varying. The class RV generalises the Pareto distribution in which $P(X > x) = x^{-\alpha}$, a power function. When $0 < \alpha < 2$, this Pareto distribution has a "fat" tail. It means that X can exceed a high level x with a rather high probability. In order to include also heavy tail distributions such as $P(X > x) = x^{-\alpha}\log(x)$ or $P(X > x) = x^{-\alpha}\log(\log(x))$, the class RV was introduced. Equivalently to (1), we have

$$\frac{P(X > xt)}{P(X > x)} \to t^{-\alpha} \quad \text{as } x \to \infty. \tag{2}$$

Sometimes a heavy tail is not RV but it possesses properties similar to those of distributions that are in RV. As an example we consider $P(X > x) = \exp(-\beta[\log x])$ (where [.] is the integer part). This tail is not a regularly varying tail. In this case (2) is replaced by: for all $t > 0$,

$$\frac{P(X > xt)}{P(X > x)} \text{ is bounded as } x \to \infty.$$

Definition 2 A measurable function f is *O-regularly varying* at ∞, written $f \in ORV$, if it satisfies:

$$\lim_{x\to\infty} \sup \frac{f(xy)}{f(x)} < \infty, \quad \forall y > 0. \tag{3}$$

If $f \in ORV$, the upper index of f is given by:

$$\alpha(f) = \lim_{y\to\infty} \frac{\log \limsup_{x\to\infty} f(xy)/f(x)}{\log(y)}, \tag{4}$$

and the lower index of f is given by $\beta(f) = \alpha(1/f)$. It can be proved (see BGT, Sect. 2.0.1) that if $f \in ORV$, then for any $\beta < \beta(f)$ and $\alpha > \alpha(f)$, there exist constants C, D and x° so that

$$Cy^\beta \leq \frac{f(xy)}{f(x)} \leq Dy^\alpha, \quad \forall y \geq 1, \forall x \geq x^\circ. \tag{5}$$

Definition 3 A measurable function f is in the class $\mathfrak{L}$ if it satisfies:

$$\lim_{x\to\infty} \frac{f(x+y)}{f(x)} = 1, \quad \forall y \in \Re. \tag{6}$$

Equivalently, $f \in \mathfrak{L}$ if and only if $f \circ \log \in RV(0)$. It is well known that $f \in RV(\rho)\ \forall \rho \in \mathbb{R}$ implies that $f \in \mathfrak{L}$. The converse statement is false in general. Another definition of $\mathfrak{L}$ uses the $F(x) = P(X \leq x)$, that is $F(x)$ is said to have a long tail if for all $t > 0$, we have

$$\frac{P(X > x + t)}{P(X > x)} \to 1, \text{ as } x \to \infty$$

If the long-tailed quantity X exceeds some high level x, the probability approaches 1 that it will exceed any other higher level approaches 1.

Definition 4 A density function f is a subexponential density, written $f \in SD$, if $f \in \mathfrak{L}$ and if

$$\lim_{x\to\infty} \frac{f^{\otimes 2}(x)}{f(x)} = 2, \tag{7}$$

where $f^{\otimes 2}(x) = f \otimes f(x)$ is the 2-fold convolution of f.

Let $X_1, X_2, \ldots, X_n$ denote i.i.d. random variables with common distribution function F. Also consider the partial sums $S_n = \sum_1^n X_i$ and the partial maxima $M_n = \max(X_1, X_2, \ldots, X_n)$.

On many cases the sequence obeys the principle of the single jump. This means that the tail of the partial sum S_n is determined completely by the tail of the largest value M_n. More precisely, as $x \to \infty$, we have

$$P(S_n > x) \sim P(M_n > x). \tag{8}$$

Distributions that satisfy (8) are called subexponential distributions. It can be proved that subexponential distributions are long tailed. Moreover it can be proved that (8) with $n = 2$ implies (8) for all $n \geq 2$.

It can be proved for density functions that if $f \in RV$ or $f \in \mathfrak{L} \cap ORV$ imply that $f \in SD$. See Chover et al. (1973).

3 Location parameter models

3.1 Notation

Consider a random sample $X = (x_1, x_2, \ldots, x_n)$ of independent and identically distributed (iid) random variables with fixed sample size n. A general location parameter Bayesian model is of the form

$$\begin{aligned} x_i|y &\overset{D}{\sim} f(x_i \mid y) = f(x_i - y), 1 \leq i \leq n; \\ y &\overset{D}{\sim} p(y), \end{aligned}$$

where f is a fixed p.d.f. and $p(y)$ is the prior p.d.f. of y which is the parameter of interest.

Let $X_L = (x_1, x_2, \ldots, x_k)$ and $X^U = (x_{k+1}, \ldots, x_n)$. Suppose that X_L represents those pieces of information which are large in relation to X^U and the prior information. We clearly have

$$\begin{aligned} f(X|y) &= \Pi_{i=1}^{k} f(x_i - y) \times \Pi_{j=k+1}^{n} f(x_j - y) \\ &= f(X_L \mid y) \times f(X^U \mid y) \\ &= L \times U. \end{aligned}$$

The posterior p.d.f. of y is given by:

$$p(y \mid X) = \frac{f(X \mid y)p(y)}{\int_{\Re} f(X \mid y)p(y)\mathrm{d}y}. \tag{9}$$

We want to investigate what happens to $p(y \mid X)$ as $x = \min(x_1, x_2, \ldots, x_k) \to \infty$. This is a situation in which the sample fractions X_L and X^U conflict, in the sense that they carry very diverse information, that is the likelihood of y based on X_L is settled far away from the likelihood of y based on X^U and the prior distribution $p(y)$. This kind of conflict may disturb the posterior distribution and potentially lead to wrong conclusions. In order to avoid this behaviour, the idea is to establish conditions under which

$$p(y \mid X) \to p(y \mid X^U) \propto p(y) f(X^U|y), \quad \text{as } x \to \infty. \tag{10}$$

In this case we say that the influence of the data over the posterior distribution vanishes, leaving the posterior distribution depending only on the prior distribution and the likelihood of y based on X^U. The model rejects the data X_L in favour of the prior distribution and the rest of the data. As a matter of fact, the influence of the outliers diminishes because the information provided by the them appears in both the numerator and the denominator of the posterior distribution, hence they cancel out under the conditions established to obtain (10) This behaviour implies that the posterior distribution is robust to atypical data, that is if x becomes too far away from the prior mode and the data X^U, X_L is rejected.

3.2 Preliminary results

If $f \in \mathfrak{L}$, then as $z \to \infty$ we have $f(z-y)/f(z) \to 1$. If $f \in \mathfrak{L}$, it follows that

$$L = \Pi_{i=1}^{k} f(x_i - y) \sim \Pi_{i=1}^{k} f(x_i) (\text{as} x \to \infty).$$

Now we consider (cf. (9)) the integral $\int_{\Re} f(X \mid y) p(y) \mathrm{d}y$ and write

$$\int_{\Re} f(X \mid y) p(y) \mathrm{d}y = \int_{\Re} f(X_L \mid y) f(X^U \mid y) p(y) \mathrm{d}y.$$

Using Fatou's lemma, we get that for $f \in \mathfrak{L}$,

$$\lim \inf_{x \to \infty} \frac{\int_{\Re} f(X \mid y) p(y) \mathrm{d}y}{\Pi_{i=1}^{k} f(x_i)} \geq \int_{\Re} f(X^U \mid y) p(y) \mathrm{d}y$$

and then also that

$$\lim \sup_{x \to \infty} p(y \mid X) \leq \frac{f(X^U \mid y) p(y)}{\int_{\Re} f(X^U \mid y) p(y) \mathrm{d}y} = p(y \mid X^U).$$

3.3 Main results

The theorems in this section will establish sufficient conditions on the data and prior distributions in order to achieve (10). Since we are dealing with the limit of (9) as $x \to \infty$, we need to apply the dominated (bounded) convergence theorems to solve the limit of the integral in the denominator. Thus, the general strategy to prove the forthcoming theorems is to bound the integrands using the proposed conditions.

Theorem 5 (Densities in $\mathfrak{L} \cap ORV$) *Suppose that f is a bounded density and that $f \in \mathfrak{L} \cap ORV$ with $\alpha(f) < 0$. Also assume that*

$$\int_{x}^{\infty} p(y) \mathrm{d}y = o(\Pi_{i=1}^{k} f(x_i)), \quad as \quad x \to \infty. \tag{11}$$

Then we have

$$p(y \mid X) \to p(y|X^U), \quad as \quad x \to \infty.$$

Proof For the integral in (9), we write

$$\int_{\Re} f(X|y)p(y)\mathrm{d}y = \left(\int_{-\infty}^{0} + \int_{0}^{x/2} + \int_{x/2}^{\infty}\right) L \times U \times p(y)\mathrm{d}y$$
$$= I + II + III.$$

First consider II. In II we have $0 \leq y \leq x/2$ and it follows that

$$x_i - x/2 \leq x_i - y \leq x_i,\ 1 \leq i \leq k$$

and then also that

$$x_i/2 \leq x_i - y \leq x_i,\ 1 \leq i \leq k.$$

First note that for $f \in \mathfrak{L}$ we have $L \sim \Pi_{i=1}^{k} f(x_i)$, as $x \to \infty$. Since $f \in ORV$, it follows that in II, $L/\Pi_{i=1}^{k} f(x_i)$ is bounded. Since f is bounded (by assumption) we have that U is bounded, then there exists a constant C such that

$$\int_{\Re} U \times p(y)\mathrm{d}y < C \int_{\Re} p(y)\mathrm{d}y = C,$$

hence

$$\frac{II}{\Pi_{i=1}^{k} f(x_i)} \to \int_{0}^{\infty} U \times p(y)\mathrm{d}y.$$

Next we consider I. In I we have $x_i \leq x_i - y$. Using (5) we obtain that

$$\frac{f(x_i - y)}{f(x_i)} = \frac{f(x_i(x_i - y)/x_i)}{f(x_i)} \leq D\left(\frac{x_i - y}{x_i}\right)^{\alpha},$$

where $\alpha(f) < \alpha$. Since $\alpha(f) < 0$, we can choose $\alpha < 0$ and then we see that in I, $L/\Pi_{i=1}^{k} f(x_i)$ is bounded. Again Lebesgue's theorem can be applied to obtain that

$$\frac{I}{\Pi_{i=1}^{k} f(x_i)} \to \int_{-\infty}^{0} U \times p(y)\mathrm{d}y.$$

For the third term III we use the assumption that f is a bounded density. In this case we obtain that

$$III \leq (\sup f(z))^{n} \times \int_{x/2}^{\infty} p(y)\mathrm{d}y.$$

By our assumption on p we obtain that

$$III = o(\Pi_{i=1}^{k} f(x_i/2)).$$

Using $f \in ORV$, we conclude that $III = o(\Pi_{i=1}^{k} f(x_i))$. This proves the theorem. $\square$

Theorem 6 (Densities in $\mathfrak{L}$) *Suppose that $f \in \mathfrak{L}$ and that f is bounded. Also suppose that there exists $s > 0$ such that*

$$\int_{-\infty}^{0} e^{-sy} U \times p(y)\mathrm{d}y + \int_{0}^{\infty} e^{sy} U \times p(y)\mathrm{d}y < \infty.$$

If $\int_x^\infty p(y)\mathrm{d}y = o(\Pi_{i=1}^k f(x_i))$, as $x \to \infty$, we have that

$$p(y \mid X) \to p(y \mid X^U), \quad \text{as } x \to \infty.$$

Proof Since $f \in \mathfrak{L}$, we still have $L/\Pi_{i=1}^k f(x_i) \to 1$ as $x \to \infty$. Also we have $F := f \circ \log \in RV(0)$. Using (5), for each $\varepsilon > 0$, we can find constants A, B, z° so that

$$\begin{aligned} \frac{f(z-y)}{f(z)} &\leq Ae^{-\varepsilon y}, \ y \leq 0, z \geq z^\circ, \\ \frac{f(z-y)}{f(z)} &\leq Be^{\varepsilon y}, \ y \geq 0, z-y \geq z^\circ, z \geq z^\circ. \end{aligned} \tag{12}$$

Now we write the integral as follows:

$$\begin{aligned} \int_{\Re} f(X|y)p(y)\mathrm{d}y &= \left(\int_{-\infty}^{0} + \int_{0}^{x-z^\circ} + \int_{x-z^\circ}^{\infty}\right) L \times U \times p(y)\mathrm{d}y \\ &= I + II + III. \end{aligned}$$

First consider I. For $\mathfrak{L}$ we have $y \leq 0$ and $x_i \geq x$. It follows from (12) that for $x \geq z^\circ$ we have

$$L = \Pi_{i=1}^k f(x_i - y) \leq \Pi_{i=1}^k Af(x_i)e^{-\varepsilon y}.$$

Taking $s = \varepsilon k$, we can use Lebesgue's theorem to see that

$$\frac{I}{\Pi_{i=1}^k f(x_i)} \to \int_{-\infty}^{0} U \times p(y)\mathrm{d}y.$$

Now consider II. In II we have $0 \leq y \leq x - z^\circ$ so that

$$z^\circ \leq x_i - x + z^\circ \leq x_i - y \leq x_i.$$

It follows from (12) that for $x \geq z^\circ$ we have

$$L = \Pi_{i=1}^k f(x_i - y) \leq \Pi_{i=1}^k Bf(x_i)e^{\varepsilon y}.$$

Taking $s = \varepsilon k$, we can use Lebesgue's theorem to see that

$$\frac{II}{\Pi_{i=1}^{k} f(x_i)} \to \int_0^\infty U \times p(y)\mathrm{d}y.$$

Finally consider III. Since f has a bounded density, we find that

$$III \leq (\sup f(z))^n \int_{x-z^\circ}^\infty p(y)\mathrm{d}y.$$

Our assumption about p shows that

$$III = o(\Pi_{i=1}^{k} f(x_i - z^\circ)).$$

Using $f \in \mathfrak{L}$, we obtain that $III = o(\Pi_{i=1}^{k} f(x_i))$. This proves the result. □

Now, assume $k = 1$ in Sect. 3.1, we have only one outlier and then we have $L = f(x_1 - y)$. In this subsection we assume $f \in SD$, that is $f \in \mathfrak{L}$ and f satisfies $f \otimes f(x)/f(x) \to 2$, as $x \to \infty$.

Theorem 7 (Densities in SD) *Suppose that $f \in SD \subset \mathfrak{L}$ and that $p(|x|) = o(f(|x|))$, then $p \otimes f(x)/f(x) \to 1$.*

Proof We write

$$\begin{aligned} p \otimes f(x) &= \left(\int_{-\infty}^{-x^\circ} + \int_{-x^\circ}^{x^\circ} + \int_{x^\circ}^{\infty} \right) p(y) f(x-y)\mathrm{d}y \\ &= I + II + III. \end{aligned}$$

Since $p(-x) = o(1) f(-x)$ as $x \to \infty$, for $\varepsilon > 0$ we can find x° so that we have

$$I \leq \varepsilon \int_{-\infty}^{-x^\circ} f(y) f(x-y)\mathrm{d}y \leq \varepsilon f \otimes f(x),$$

It follows that

$$\limsup \frac{I}{f(x)} \leq 2\varepsilon.$$

For III, in a similar way we find that

$$\limsup \frac{III}{f(x)} \leq 2\varepsilon.$$

Now consider II. Using $f \in \mathfrak{L}$, we get that

$$\frac{II}{f(x)} \to \int_{-x^\circ}^{x^\circ} p(y)\mathrm{d}y.$$

By choosing x° sufficiently large, we obtain that

$$1-\int_{-x^\circ}^{x^\circ} p(y)\mathrm{d}y \leq \varepsilon,$$

We conclude that

$$\limsup_{x\to\infty}\left|\frac{p\otimes f(x)}{f(x)}-1\right| \leq 5\varepsilon.$$

Now let $\varepsilon \to 0$, to get the desired result. □

Depending on the distributions the verification of the conditions above may become difficult. Thus the following theorem provides the same result, but with conditions slightly different from those of Theorem 7.

Theorem 8 (Densities in ***SD***) *Assume that $f \in SD$, (this is: $f \in \mathfrak{L}$ and $f \otimes f(x)/f(x) \to 2$, as $x \to \infty$). Also assume that $p(|x|) \sim \alpha f(|x|)$ where $\alpha > 0$. Then $p \otimes f(x) \sim (\alpha+1)f(x)$.*

Proof We choose a in such a way that

$$(\alpha-\varepsilon)f(|x|) \leq p(|x|) \leq (\alpha+\varepsilon)f(|x|), \quad \forall x \text{ with } |x| \geq a.$$

Now choose $x^\circ \geq a$. We reconsider I and III from the proof of Theorem 7 and get that

$$(\alpha-\varepsilon)\int_{-\infty}^{-x^\circ} f(y)f(x-y)\mathrm{d}y \ \leq I \leq \ (\alpha+\varepsilon)\int_{-\infty}^{-x^\circ} f(y)f(x-y)\mathrm{d}y$$
$$(\alpha-\varepsilon)\int_{x^\circ}^{\infty} f(y)f(x-y)\mathrm{d}y \leq III \leq (\alpha+\varepsilon)\int_{x^\circ}^{\infty} f(y)f(x-y)\mathrm{d}y$$

It follows that

$$(\alpha+\varepsilon)\left\{f\otimes f(x)-\int_{-x^\circ}^{x^\circ} f(y)f(x-y)\mathrm{d}y\right\} \leq I+III$$
$$\leq (\alpha+\varepsilon)\left\{f\otimes f(x)-\int_{-x^\circ}^{x^\circ} f(y)f(x-y)\mathrm{d}y\right\},$$

and using $f \in \mathfrak{L}$ we obtain that

$$(\alpha-\varepsilon)\left(2-\int_{-x^\circ}^{x^\circ} f(y)\mathrm{d}y\right) \leq \lim_{x\to\infty}\binom{\sup}{\inf}\frac{I+III}{f(x)} \leq (\alpha+\varepsilon)\left(2-\int_{-x^\circ}^{x^\circ} f(y)\mathrm{d}y\right).$$

For II we obtain that (use $f \in \mathfrak{L}$), we get that

$$\frac{II}{f(x)} \to \int_{-x^\circ}^{x^\circ} p(y)\mathrm{d}y.$$

We can find x° sufficiently large such that

$$1-\varepsilon \le \int_{-x^\circ}^{x^\circ} f(y)\mathrm{d}y \le 1,\, 1-\varepsilon \le \int_{-x^\circ}^{x^\circ} p(y)\mathrm{d}y \le 1.$$

We get that

$$(\alpha-\varepsilon)+1-\varepsilon \le \lim_{x\to\infty}\begin{pmatrix}\sup\\ \inf\end{pmatrix}\frac{p\otimes f(x)}{f(x)} \le (\alpha+\varepsilon)(1+\varepsilon)+1.$$

Now let $\varepsilon \to 0$ to get the desired result. □

3.4 Many observations

Theorems 7 and 8 can be extended to many observations with some of them possibly being outliers. In order to show this, consider a random sample $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ iid with a p.d.f. $f(x_i|y) = f(x_i - y)\,\forall i$, where y is a location parameter. Let also $y \overset{D}{\sim} p(y)$ (prior distribution).

In practical problems we might have a few observations very large, that is $(x_1, \ldots, x_k)$ $(k \le n)$ tending to infinity. This is equivalent to think of the k observations close to each other and tending to infinity, that is $x_i = x + \xi_i$ for ξ_i fixed $(i = 1, \ldots, k)$. Thus we can write the joint distribution of $(x_1, \ldots, x_k)$ as:

$$f(x_1, \ldots, x_k|y) = \prod_{i=1}^{k} f(x+\xi_i - y) = g(x-y),$$

which clearly keeps the location structure. In fact, the outliers behave like a single observation as $x \to \infty$.

The joint distribution of the rest of the observations is given by:

$$f(x_{k+1}, \ldots, x_n|y) = \prod_{i=k+1}^{n} f(x_i - y) = U.$$

The posterior can be written as

$$p(y|\boldsymbol{x}) = \frac{U \times g(x-y) \times p(y)}{\int_{\mathbb{R}} U \times g(x-y) \times p(y)\mathrm{d}y}.$$

Let $p^*(y) = U \times p(y)$, it follows that we have the same structure of Theorem 8, holds if $p^*(|x|) \sim \alpha g(|x|)$ as $x \to \infty$. In this case the posterior distribution

$$p(y|\boldsymbol{x}) \to \frac{U \times p(y)}{\int_{\mathbb{R}} U \times p(y)\mathrm{d}y} \ (x \to \infty).$$

4 Scale parameter models

4.1 Notation

Due to (1), notice that regular variation provides a natural way to deal with scale parameters since the scale structure of a scale parameter model is the same as in the definition of regular variation. Consider a sample $X = (x_1, x_2, \dots, x_n)$ of independent and identically distributed (iid) with fixed sample size n. A typical scale parameter model is of the form

$$\begin{aligned} x_i|y &\overset{D}{\sim} f(x_i \mid y) = y^{-1}h(x_i/y), 1 \le i \le n; \\ y &\overset{D}{\sim} p(y), \end{aligned}$$

where f is the data p.d.f. and $p(y)$ is the prior p.d.f. of y which is the parameter of interest. For convenience, we assume that all random variables involved are concentrated on the positive halfline.

Suppose that $x_i, 1 \le i \le k < n$ are large. As before we define $X_L = (x_1, x_2, \dots, x_k)$, $x = \min(x_1, \dots, x_k)$ and $X^U = (x_{k+1}, \dots, x_n)$. We clearly have

$$\begin{aligned} f(X \mid y) &= y^{-n}\Pi_{i=1}^{k}h(x_i/y) \times \Pi_{j=k+1}^{n}h(x_j/y) \\ &= y^{-n} \times L \times U. \end{aligned}$$

The posterior p.d.f. of y is given by:

$$p(y \mid X) = \frac{f(X \mid y)p(y)}{\int_{\Re} y^{-n}L \times U \times p(y)\mathrm{d}y}. \tag{13}$$

4.2 Main result

Now we suppose that $h \in RV(\alpha)$. In this case it is easy to see that

$$\frac{L}{\Pi_{i=1}^{k}h(x_i)} \to y^{-\alpha k}, \text{ as } x \to \infty.$$

We need extra conditions to see what happens in (13) if $x \to \infty$. An alternative to the conditions proposed in Andrade and O'Hagan (2006) is provided by the next result.

Theorem 9 (Densities in ***RV***) *Suppose that $h \in RV(\alpha)$ with $\alpha < 0$ and suppose that h is bounded on bounded intervals. Assume that for $\varepsilon > 0$ we have*

$$\int_0^1 y^{-(\alpha+\varepsilon)k} U \times p(y)\mathrm{d}y + \int_1^\infty y^{-(\alpha-\varepsilon)k} U \times p(y)\mathrm{d}y < \infty, \tag{14}$$

Then

$$\frac{1}{\Pi_{i=1}^k h(x_i)} \int_{\Re} L \times U \times p(y)\mathrm{d}y \to \int_0^\infty y^{-\alpha k} U \times p(y)\mathrm{d}y < \infty \tag{15}$$

Proof We have

$$\int_{\Re} L \times U \times p(y)\mathrm{d}y = \left(\int_0^1 + \int_1^\infty\right) L \times U \times p(y)\mathrm{d}y = I + II.$$

First consider I and write

$$\frac{I}{\Pi_{i=1}^k h(x_i)} = \int_0^1 \Pi_{i=1}^k \frac{h(x_i/y)}{h(x_i)} U \times p(y)\mathrm{d}y.$$

Since $1 \leq 1/y$ and $h \in RV(\alpha)$, for each $\varepsilon > 0$ we can find constants C and z° such that

$$\frac{h(z/y)}{h(z)} \leq Cy^{-\alpha-\varepsilon}, \quad \forall z \geq z^\circ, \quad \forall y \leq 1.$$

It follows that

$$\Pi_{i=1}^k \frac{h(x_i/y)}{h(x_i)} \leq C^k y^{-\alpha k-\varepsilon k}, \quad \forall x \geq z^\circ, \quad \forall y \leq 1.$$

Using (14) it follows that we can apply dominated convergence and we find that

$$\frac{I}{\Pi_{i=1}^k h(x_i)} \to \int_0^1 y^{-\alpha k} U \times p(y)\mathrm{d}y.$$

Now consider II. Using $h \in RV(\alpha)$, for each $\varepsilon > 0$, we can find constants D and z° such that

$$\frac{h(x_i/y)}{h(x_i)} \leq Dy^{-\alpha+\varepsilon}, \quad \forall y \geq 1 \text{ and } x_i/y \geq z^\circ.$$

Now consider the case where $y \geq 1$, $x_i/y < z^\circ$ and $x_i \geq z^\circ$. Since we assume that h is bounded on bounded intervals, we get that

$$\begin{aligned}\frac{h(x_i/y)}{h(x_i)} &\leq \sup_{u \leq z^\circ} h(u) \frac{1}{h(x_i)} \\ &= \sup_{u \leq z^\circ} h(u) \frac{1}{x_i^{-\alpha+\varepsilon} h(x_i)} x_i^{-\alpha+\varepsilon}.\end{aligned}$$

Since $z^{-\alpha+\varepsilon}h(z) \to \infty$ as $z \to \infty$, we can find a constant D° so that

$$\frac{h(x_i/y)}{h(x_i)} \leq D^\circ x_i^{-\alpha+\varepsilon}, \forall y \geq 1, x_i/y < z^\circ, x_i \geq z^\circ.$$

Now it follows that

$$\frac{h(x_i/y)}{h(x_i)} \leq D^\circ y^{\alpha-\varepsilon} x_i^{-\alpha+\varepsilon} y^{-\alpha+\varepsilon}.$$

Since $\alpha < 0$ and $x_i/y < z^\circ$, we get that

$$\frac{h(x_i/y)}{h(x_i)} \leq D^\circ (z^\circ)^{-\alpha+\varepsilon} y^{-\alpha+\varepsilon} = F y^{-\alpha+\varepsilon}.$$

Combining these estimates, we have proved that we can find a constant G such that

$$\frac{h(x_i/y)}{h(x_i)} \leq G y^{-\alpha+\varepsilon}, \quad \forall y \geq 1, \quad \forall x_i \geq z^\circ.$$

Assumption (14) can be used and applying dominated convergence, we get that

$$\frac{II}{\Pi_{i=1}^k h(x_i)} \to \int_1^\infty y^{-\alpha k} U \times p(y)\mathrm{d}y.$$

Combining the results for I and II, we obtain (15). □

4.3 Remark

We briefly discuss conditions under which condition (14) holds. We prove the following result:

Proposition 10 *Suppose that $h \in RV(\alpha)$ is bounded on bounded intervals. Also assume that*

$$\int_0^1 y^{\theta-(n-k)(\alpha+\varepsilon)} p(y)\mathrm{d}y < \infty \; \textit{resp.} \int_1^\infty y^\theta p(y)\mathrm{d}y < \infty.$$

Then for fixed X^U,

$$\int_0^1 y^\theta U \times p(y)\mathrm{d}y < \infty, \textit{resp.} \int_1^\infty y^\theta U \times p(y)\mathrm{d}y < \infty.$$

Proof First consider an integral of form

$$\int_0^1 y^\theta U \times p(y)\mathrm{d}y.$$

Clearly we have

$$\int_0^1 y^\theta U \times p(y)\mathrm{d}y = \left(\int_0^a + \int_a^1\right) y^\theta U \times p(y)\mathrm{d}y = A + B,$$

where $0 < a < 1$. First consider B. Since we assume that h is bounded on bounded intervals, we have

$$B \le \int_a^1 y^\theta p(y)\mathrm{d}y \le \max(1, a^\theta) \int_a^1 p(y)\mathrm{d}y < \infty.$$

Next consider A. Using $z^{-\alpha-\varepsilon} h(z) \to 0$, we can find z° such that $h(z) \le \varepsilon z^{\alpha+\varepsilon}$, $z \ge z^\circ$. It follows that

$$h(x_i/y) \le \varepsilon (x_i/y)^{\alpha+\varepsilon}, x_i/y \ge z^\circ.$$

Since $y \le a$, we should choose a such that $x_i/a \ge z^\circ$, or $a \le x_i/z^\circ$. Having done this, we find

$$\begin{aligned} A &\le \int_a^1 y^\theta \Pi_{i=k+1}^n \varepsilon (x_i/y)^{\alpha+\varepsilon} p(y)\mathrm{d}y \\ &= \varepsilon^{n-k} \Pi_{i=k+1}^n x_i^{\alpha+\varepsilon} \int_a^1 y^{\theta-(n-k)(\alpha+\varepsilon)} p(y)\mathrm{d}y < \infty. \end{aligned}$$

Next consider an integral of the form

$$\int_1^\infty y^\theta U \times p(y)\mathrm{d}y.$$

Since we assume that h is bounded on bounded intervals, we have

$$h(x_i/y) \le \sup_{z \le x_i} h(z) = s(x_i)$$

and then also

$$\int_1^\infty y^\theta U \times p(y)\mathrm{d}y \le \Pi_{i=k+1}^n s(x_i) \int_1^\infty y^\theta p(y)\mathrm{d}y < \infty.$$

This proves the result. $\square$

5 Examples

We illustrate the theory with a general problem of estimating the location and the scale parameters of the random sample $\boldsymbol{x} = (2, 3, 3, 4, x_5)$, where we take x_5 arbitrarily large in order to observe the behaviour of the posterior distribution of the location and the scale parameters. The general model is

$$\begin{cases} x_i|y,\sigma \overset{D}{\sim} f(x_i|y,\sigma) = \sigma^{-1}h\left(\frac{x_i-y}{\sigma}\right) \ iid, \ i=1,\ldots,5 \\ y \qquad \overset{D}{\sim} p(y) \\ \sigma \qquad \overset{D}{\sim} \pi(\sigma) \end{cases}. \tag{16}$$

We use the `OpenBugs` (MCMC methods) for sampling from the posterior distributions, in all the cases the algorithm was run until its convergence, then the posterior estimates of the location and of the scale parameters were computed.

In order to achieve rejection of the outlying observation we need to model accordingly to the theorems above. This basically means to choose suitably heavy-tailed distributions for the data and for the prior distributions and lighter tails for the prior distributions of the location and the scale parameters. As our purpose is to illustrate the theory, we opt for quite strong prior information (small variances), which will make the MCMC algorithm to achieve convergence more quickly. Thus we expect to base the posterior estimates on the prior information and on the non-outlying observations $X^U = (2, 3, 3, 4)$. As for the data distribution, we propose four different choices for f, namely, models: (I) f light-tailed, (II) $f \in RV$, (III) $f \in \mathfrak{L}$ and (IV) $f \in SD$. Thus we assess the behaviour of the posterior estimates as we disturb the data by increasing x_5.

We need to verify if the distributions of Models (I)–(IV) satisfy the conditions of the Sects. 3 and 4.

Model I The traditional light-tailed choice for f is a normal distributions with mean (location) y and standard deviation (scale) σ. It is easy to verify that the normal distributions does not belong to any of the families above. In fact,

$$\lim_{x\to\infty} f(x-y)/f(x) = \begin{cases} 0, & y<0; \\ 1, & y=0; \\ \infty, & y>0. \end{cases},$$

hence $f \notin \mathfrak{L}$, therefore $f \notin RV$ and $f \notin SD$. Notice that $f \notin ORV$, since we the limit (3) is infinity as $y > 0$. As for the prior information, we assign $y \overset{D}{\sim} N(0, 0.05)$ and $\sigma \overset{D}{\sim} \mathrm{G}(3, 10)$.

Model II Besides being bounded, by (1), the Student's t distribution with d degrees of freedom and is regularly varying with index $-(d+1)$. Thus we assign to $f(x_i|y,\sigma^2)$ a t distribution with $d = 4$ degrees of freedom, mean y and variance σ^2. In addition we assign $y \overset{D}{\sim} N(0, 0.05)$ for the prior distribution of y and $\sigma \overset{D}{\sim} \mathrm{G}(3, 10)$. Now we need to verify the conditions of Theorems 5 (location parameter) and 9 (scale parameter).
Location parameter: We have to show that

$$[1-\Phi(x)]/\prod_{i=1}^{n} f(x_i) \to 0, \quad \text{as } x \to \infty,$$

where Φ is the cumulative distribution of the standard normal distribution. In fact, $1 - \Phi(x) = \mathrm{erfc}(x/\sqrt{2})/2$, where erfc is the complementary error function

$$erfc(x) = 2(2\pi)^{-1/2} \int_x^{\infty} e^{-x^2/2} \mathrm{d}x,$$

which has the asymptotic expansion

$$erfc\left(\frac{x}{\sqrt{2}}\right) = \frac{2e^{-x^2/2}}{\sqrt{2\pi}} \sum_{n=0}^{n} \frac{(-1)^n 1 \cdot 2 \cdot 3 \cdots (2n-1)}{x^{2n+1}}. \tag{17}$$

For any $k\,(\leq n) \prod_{i=1}^{k} f(x_i) \propto \prod_{i=1}^{k} \left(1 + x_i^2/(d-2)\right)^{-(d+1)/2}$, it follows that $[1 - \Phi(x)]/\prod_{i=1}^{k} f(x_i) \to 0$ as $x \to \infty$. Hence the conditions of Theorem 5 as satisfied. Here $k = 1$ which makes the condition even easier to verify.

Scale parameter: It's straightforward by application of Proposition 10, since the prior of the scale parameter is a Gamma distribution, it follows that

$$\int_1^{\infty} \sigma^{\theta} p(\sigma) d\sigma < \infty, \text{ for all } \theta > -\alpha = -3.$$

Model III The Exponential Power Distribution (EPD) (or generalised normal distribution) (Box and Tiao 1973) is of the form

$$f(x_i|y, \sigma^2) \propto \frac{1}{\sigma} e^{-\left|\frac{x_i - y}{\sigma}\right|^q}.$$

This structure generalises several well known distributions. For instance, if $q = 2$ f is a normal distribution and if $q = 1$ f is the double exponential. For $0 < q \leq 1$ we have $EPD \in \mathfrak{L}$, thus we choose $q = 1/3$. From (1), the EPD is not in the RV class, hence we cannot guarantee robustness of the posterior estimate of the scale parameter, here we assign $\sigma \overset{D}{\sim} \mathrm{G}(3, 0.01)$. As for the location parameter, let $y \overset{D}{\sim} N(0, 0.05)$, again we have to satisfy the conditions of Theorem 5. In fact, similarly to the strategy used in Model II above, $\prod_{i=1}^{k} f(x_i) \propto \sigma^{-k} \exp\{-\sum_{k=1}^{k} |x_i|^q\}$, which can be compared with (17), hence Condition (11) is verified.

Model IV The LogNormal distribution is a well known subexponential distribution (see Goldie and Kluppelberg 1998). The LogNormal distribution is also in $\mathfrak{L}$, but not in ORV. We consider the model

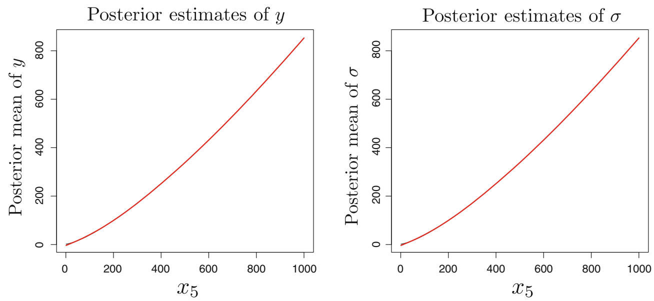
$$f(x_i|y, \sigma^2) \propto (x_i\sigma)^{-1} e^{-\frac{(\log x_i - y)^2}{2\sigma^2}}, \ i = 1, \ldots, 5$$

that is x_i is lognormally distributed with location parameter y and scale parameter σ. In addition we choose a LogNormal distribution for the prior distribution of y, that is $y \overset{D}{\sim} \text{LogN}(0, 0.05)$, hence we satisfy the condition that $p(x) \sim f(x)$ $(x \to \infty)$ (Theorem 8). Again, as in the $\mathfrak{L}$ the SD class will not produce a robust posterior distribution for the scale parameter, thus we arbitrarily choose $\sigma \overset{D}{\sim} \text{IG}(3, 10)$.

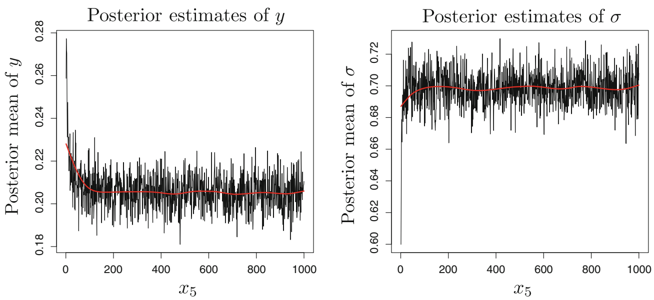
Note that we have different models, thus we cannot compare the models estimates. In fact, we compare the behaviour of the posterior estimates in the different models.

In order to verify robustness of each model, we considered the data $\boldsymbol{x}$ and we made the outlier x_5 to vary from 1 to 1,000. Thus we ran the MCMC algorithm 1,000 times for each model in order to generate a sample from the posterior distribution, then we computed and plot the posterior estimates of y and of σ corresponding the each value of x_5. Due to natural oscillation of MCMC outputs we added to each plot a trend line, which basically averages the outputs by some polynomial regression (commonly used in times series analysis), this helps to see the variation of the posterior estimates. Figure 1a (Model I) shows a quite common model used in Bayesian analysis, in which both the data and the prior distributions are light-tailed. In this case the model is quite sensitive to atypical information, that is note that, as $x_5 \uparrow \infty$ the posterior estimates follow the outlying information faithfully to the infinity. This was the behaviour identified by de Finetti (1961) and described in more details by Lindley (1968). In practice this basically means that if outliers are in the data, the posterior distribution may be disturbed by them, and potentially lead to wrong conclusions. As an alternative, in Model II the data distribution is in $\mathfrak{L} \cap ORV$, which is verified by the fact that a Student t distribution is regularly varying. The prior distributions are light-tailed, thus Theorem 5 and 9 assures that the posterior estimates will be robust to atypical data. In fact, Fig. 1b shows that the posterior estimates for y and for σ becomes unaffected by the outlying information when it becomes too large. As pointed out by Andrade and O'Hagan (2006), the posterior estimates reject the outlying data in favour of the rest of the data and the prior information. Model III assigns to the data a distribution which is in $\mathfrak{L}$, but not in ORV, thus as shows Fig. 1c we achieve robustness only on the location parameter, whose posterior estimates tend to a constant, the posterior estimates of the scale parameter tend to infinity as $x_5 \uparrow \infty$. Similarly, in Model IV (Fig. 1d) we cannot control the influence of the outlier in the posterior distribution of σ, which produces estimates very sensitive to changes of x_5, in contrast the posterior estimates of the location parameter y tends to a constant, rejecting the outlier.

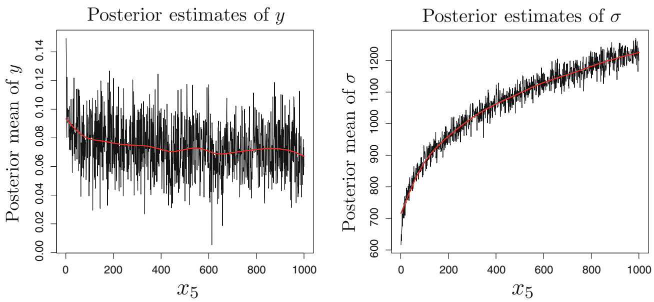
The classes of heavy-tailed distributions defined in Sect. 2, when combined with light-tailed prior distributions, produces quite diverse behaviours of the posterior distribution in the presence of conflicting information. Ultimately the posterior distribution will be based on that source of information whose distribution (data or prior) has lighter tails. Thus we assign a heavy-tailed distribution to the source of information which we believe might carry conflicting information and a distribution with lighter tails to the other source of information. In this way the conflicting information will be automatically discarded by the model. This happens due the fact that heavy-tailed distributions are more prepared to deal with events which occur far away from the mode of the distribution, due to the weight of its tails. See Andrade and O'Hagan



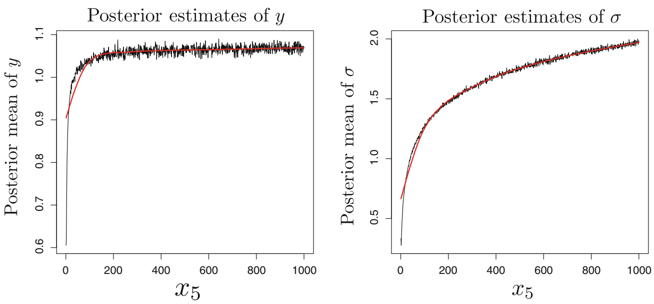
(a) Model I: Posterior estimates of y and σ.



(b) Model II: Posterior estimates of y and σ.



(c) Model III: Posterior estimates of y and σ.



(d) Model IV: Posterior estimates of y and σ.

Fig. 1 Posterior estimates for the location and the scale parameters

(2006) for further details. In the examples above the posterior estimates of the location and of the scale parameters are robust to conflicting information due the $\mathfrak{L}$ and ORV distributions assigned to the data, this behaviour is also achieved when we use regularly varying distributions (Andrade and O'Hagan 2011). With respect to models involving either an $\mathfrak{L}$ or a subexponential distribution, only the estimate of the location parameter will be unaffected by conflicts. This suggests that scale parameters should be modelled with distribution with approximately power law tails decay, which is provided by the RV and ORV classes.

6 Discussion

The results presented involve quite wide classes of heavy-tailed distribution, in particular the $\mathfrak{L}$ and SD embraces most of the distributions whose tails decay like e^{-x^q} $(q < 1)$ and those with regularly varying tails, which behave like a polynomial. Distributions like the EPD, Laplace and LogNormal have been used as heavy tails in practical applications (see Pericchi and Sansó 1995; Pericchi et al. 1993). By working in wider classes of heavy-tailed distribution, we expand the scope distributions, which can be used in order to achieve posterior robustness. As shown in the examples, we cannot achieve posterior robustness on the scale parameters within the classes $\mathfrak{L}$ and SD. As Andrade and O'Hagan (2006) point out, differently from the location case in which conflicts disturb only the location of the posterior distribution, in a scale parameter structure the posterior distribution is affected both on the location and on the scale as some observation increases, thus we need quite heavy tails to resolve those conflicts.

The results above concern the cases where we want to reject some observations or the whole sample in favour of the prior information or the prior information and the non-outlying data, although this is the most common form of conflict, in some situations one may wish to weaken the prior information in the model, perhaps for finding the prior information not so credible. In this case, the theory presented provides tools for making the model to behave in the way the modeller wishes. In order to reject some prior information we basically need to model accordingly to the Theorems above, but how focusing in the prior information, that is assign some heavy-tailed distribution to the prior information and some distribution with lighter tails to the data. For instance, in Theorem 5, in order to reject the prior distribution in favour of the data, we need to choose $p \in \mathfrak{L} \cap ORV$, with p bounded, $\alpha(p) < 0$ and $\int_x^\infty f(y)\mathrm{d}y = o(p(x))$ as $x \to \infty$. In general we need just to exchange the prior distribution with the data distribution in the theorems presented above. Andrade and O'Hagan (2006, 2011) provide some further description of how to reject prior information of the location and the scale parameter.

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