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Loss of information of a statistic for a family of non-regular distributions, II: more general case

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Abstract In the paper of Akahira (Ann Inst Statist Math 48:349–364, 1996), it was shown that the second order asymptotic loss of information in reducing to a statistic consisting of extreme values and an asymptotically ancillary statistic vanished for a family of non-regular distributions whose densities have the same values and the sum of differential coefficients at the endpoints of the bounded support is equal to zero. In this paper, the result can be shown to be extended to the case of a family of non-regular distributions without the above restriction.

Keywords Rényi measure · Loss of information · Non-regular case · Extreme statistics · Asymptotically ancillary statistic · Truncated distributions

1 Introduction

Under regularity conditions, the amounts of information like Fisher, Kullback–Leibler etc are useful in statistical estimation. But, in the non-regular case when the regularity conditions do not hold, such informations do not necessarily work well. In the previous paper of Akahira (1996), the amount of information extended to as Rényi measure is introduced and the second order asymptotic loss of information associated with a statistic is defined. Further, it is shown that the second order asymptotic loss of information of a statistic consisting of extreme values and an asymptotically ancillary statistic vanishes for a family of non-regular distributions whose densities have the same values and the sum of differential coefficients at the endpoints of the bounded support is equal to zero. In a non-regular location shift family, the limiting behavior of relative Rényi entropy is discussed by Hayashi (2000, 2010). On the other hand, in non-regular cases, the asymptotic most accuracy of estimators with the asymptotic

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distributions is investigated by Akahira (1975b), and the two-sided asymptotic efficiency of asymptotically median unbiased estimators in sense of the concentration probability around the true parameter is also done by Akahira (1982, 1988, 1991a). Further, related results can be found in Papaioannou and Kempthorne (1971), Akahira (1995), Akahira and Takeuchi (1990, 1995), Haussler and Opper (1997) and Akahira et al. (2007).

In this paper, without the above conditions on the density, we consider the amounts of information of data and the statistic. It is shown that the second order asymptotic loss of information associated with the statistic still vanishes in such a situation.

2 The amounts of information

Suppose that $X_1, X_2, \dots, X_n, \dots$ is a sequence of independent and identically distributed (i.i.d.) real random variables with a density function $f(x,\theta)$ with respect to a σ-finite measure μ, where θ belongs to a parameter space Θ. Akahira and Takeuchi (1991) define as an amount of information on X_1 between $f(\cdot,\theta_1)$ and $f(\cdot,\theta_2)$ for any points θ_1 and θ_2 in Θ

$$I_{X_1}(\theta_1,\theta_2) = -8\log\int \{f(x,\theta_1)f(x,\theta_2)\}^{1/2}\,\mathrm{d}\mu(x). \tag{1}$$

It is remarked in (1) that the integral of the right-hand side is called affinity between $f(\cdot,\theta_1)$ and $f(\cdot,\theta_2)$ (see, e.g. Matusita (1955) and also LeCam (1990)). Further, Akahira (1996) extends the amount (1) of information to as Rényi measure:

$$I_{X_1}^{(\alpha)}(\theta_1,\theta_2) = -\frac{8}{1-\alpha^2}\log\int f(x,\theta_1)^{(1-\alpha)/2}f(x,\theta_2)^{(1+\alpha)/2}\mathrm{d}\mu(x) \tag{2}$$

for $|\alpha|<1$. If $\alpha=0$, then (1) follows from (2). Let $T_1 = T_1(\boldsymbol{X})$ and $T_2 = T_2(\boldsymbol{X})$ be statistics based on a sample $\boldsymbol{X} := (X_1,\dots,X_n)$ of size n. Let $f_\theta(t_1,t_2)$ be a joint density of T_1 and T_2 with respect to a direct product measure $\mu_{T_1}\otimes\mu_{T_2}$, $f_\theta(t_1|t_2)$ be a conditional density of T_1, given T_2, with respect to the measure μ_{T_1}, and $g_\theta(t_2)$ be a marginal density of T_2 with respect to μ_{T_2}. Here, the amount $I_{T_1|T_2}(\theta_1,\theta_2)$ of conditional information of T_1, given T_2, between $f_{\theta_1}(t_1|t_2)$ and $f_{\theta_2}(t_1|t_2)$ for any points θ_1 and θ_2 in Θ is defined by

$$I_{T_1|T_2}^{(\alpha)}(\theta_1,\theta_2) = -\frac{8}{1-\alpha^2}\log\int f_{\theta_1}(t_1|t_2)^{(1-\alpha)/2}f_{\theta_2}(t_1|t_2)^{(1+\alpha)/2}\mathrm{d}\mu_{T_1}(t_1). \tag{3}$$

Then we have the following.

Lemma 1 (Akahira (1996)) *For any θ_1, θ_2 in Θ and $|\alpha|<1$,*

$$I_{T_1,T_2}^{(\alpha)}(\theta_1,\theta_2) = -\frac{8}{1-\alpha^2}\log\int\left[\exp\left\{-\frac{1-\alpha^2}{8}I_{T_1|T_2}^{(\alpha)}(\theta_1,\theta_2)\right\}\right] \cdot g_{\theta_1}(t_2)^{(1-\alpha)/2}g_{\theta_2}(t_2)^{(1+\alpha)/2}d\mu_{T_2}(t_2)$$

$$= -\frac{8}{1-\alpha^2} \log E\left[\exp\left\{-\frac{1-\alpha^2}{8} I^{(\alpha)}_{T_1|T_2}(\theta_1, \theta_2)\right\}\right]$$
$$+ I^{(\alpha)}_{T_2}(\theta_1, \theta_2), \tag{4}$$

where the expectation $E[\,\cdot\,]$ is taken under the density

$$g_{\theta_1}(t_2)^{(1-\alpha)/2} g_{\theta_2}(t_2)^{(1+\alpha)/2} \Big/ \int g_{\theta_1}(t_2)^{(1-\alpha)/2} g_{\theta_2}(t_2)^{(1+\alpha)/2} \mathrm{d}\mu_{T_2}(t_2).$$

For the proof, see Akahira (1996). Since $X_1, \ldots, X_n$ are i.i.d., it follows that the amount (2) of information on $\boldsymbol{X}$ between $f(\cdot, \theta_1)$ and $f(\cdot.\theta_2)$ for any points θ_1, θ_2 in Θ is given by

$$I^{(\alpha)}_{\boldsymbol{X}}(\theta_1, \theta_2) = n I^{(\alpha)}_{X_1}(\theta_1, \theta_2) \tag{5}$$

for $|\alpha| < 1$. It can be also shown that

$$I^{(\alpha)}_{T_1, T_2}(\theta_1, \theta_2) \le I^{(\alpha)}_{\boldsymbol{X}}(\theta_1, \theta_2) \tag{6}$$

for $|\alpha| < 1$ (see Akahira and Takeuchi (1991) for $\alpha = 0$). Further, for each α with $|\alpha| < 1$ we can consider the loss of information of any statistic $T = T(\boldsymbol{X})$ as $I^{(\alpha)}_{\boldsymbol{X}}(\theta_1, \theta_2) - I^{(\alpha)}_{T}(\theta_1, \theta_2)$, and, in the next section, discuss the asymptotic loss of information up to the second order, i.e. the order $o(n^{-1})$ when $|\theta_1 - \theta_2| = O(n^{-1})$.

The relationship between the amount $I^{(\alpha)}_{X_1}$ of information and that of Fisher information is given as follows. Under suitable regularity conditions,

$$I^{(\alpha)}_{X_1}(\theta, \theta + \Delta) = I_{X_1}(\theta)\Delta^2 + o(\Delta^2),$$

as $\Delta \to 0$, for any fixed α, where $I_{X_1}(\theta) := E_\theta[\{(\partial/\partial\theta) \log f(X, \theta)\}^2]$ which is called the amount of Fisher information (see Akahira (1996)). Under regularity conditions, the loss of information associated with statistics is investigated by Fisher (1925), Rao (1961), Ghosh and Subramanyam (1974), and others. In the double exponential case as a typical example of non-regular situation, the loss of information associated with the order statistics and related estimators is discussed by Akahira and Takeuchi (1990).

3 The calculation of the amount of information

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of i.i.d. random variables with a density with respect to the Lebesgue measure and consider a location parameter family $f(x, \theta)$, $\theta \in \mathbf{R}^1$, defined by $f(x, \theta) = f_0(x - \theta)$ for $x \in \mathbf{R}^1$. We assume the following conditions:

(A.1) $$f_0(x) > 0 \quad \text{for } a < x < b$$
$$f_0(x) \le 0 \quad \text{for } x \le a,\ x \ge b,$$

where both a and b are finite.

(A.2) $f_0(\cdot)$ is twice continuously differentiable in the open interval (a, b), and there exist

$$c_1 := \lim_{x\to a+0} f_0(x) = f_0(a+0) > 0,\ \ c_2 := \lim_{x\to b-0} f_0(x) = f_0(b-0) > 0,$$
$$c_1' := \lim_{x\to a+0} f_0'(x) = f_0'(a+0),\ \ c_2' := \lim_{x\to b-0} f_0'(x) = f_0'(b-0).$$

Under the above conditions, it is known that the order of consistency is equal to n (see Akahira (1975a)).

Remark 1 In Akahira (1996), in addition to the above conditions, we assume

(A.3) $\quad c := c_1 = c_2, \quad h := c_2' = -c_1'.$

The condition is assumed to simplify the calculation on the amount of information, but it seems to be restrictive. Indeed, an extension of the result to the case when the condition (A.3) is excluded is suggested in Remark 3.2 in Akahira (1996). Since the conclusion without the condition is interesting, the condition (A.3) is removed mainly in this paper. Here, let $I_0 := \int_a^b \{f_0'(x)\}^2/f_0(x)\mathrm{d}x$.

Then we have the following.

Theorem 1 *Assume that the conditions* (A.1) *and* (A.2) *hold. Then, for* $|\alpha| < 1$ *and a small* $|\Delta|$,

$$I_{X_1}^{(\alpha)}(\theta, \theta+\Delta) = \begin{cases} 4\left(\frac{c_1}{1+\alpha} + \frac{c_2}{1-\alpha}\right)\Delta + \Big[\frac{4}{1-\alpha^2}(c_1' + c_1^2) - \frac{2}{1-\alpha}\{(c_1' + c_2') \\ \quad +(c_1^2 - c_2^2)\} + I_0 - (c_1 - c_2)^2\Big]\Delta^2 + o(\Delta^2) \quad for\ \Delta > 0, \\ -4\left(\frac{c_1}{1-\alpha} + \frac{c_2}{1+\alpha}\right)\Delta + \Big[-\frac{4}{1-\alpha^2}(c_2' - c_2^2) + \frac{2}{1-\alpha}\{(c_1' + c_2') \\ \quad +(c_1^2 - c_2^2)\} + I_0 - (c_1 - c_2)^2\Big]\Delta^2 + o(\Delta^2) \quad for\ \Delta < 0. \end{cases} \tag{7}$$

Proof Without loss of generality, we assume that $\theta = 0$. Let $\Delta > 0$. Put $l(x) = \log f_0(x)$ and let α be any fixed in $(-1, 1)$. In a similar way to the proof of Theorem 1 in Akahira (1996), we have for a small Δ

$$\begin{aligned} &\int_{a+\Delta}^b f_0(x)^{(1-\alpha)/2} f_0(x-\Delta)^{(1+\alpha)/2}\mathrm{d}x \\ &= \int_{a+\Delta}^b f_0(x)\mathrm{d}x - \frac{1+\alpha}{2}\Delta\int_{a+\Delta}^b l'(x)f_0(x)\mathrm{d}x + \frac{1+\alpha}{4}\Delta^2\int_{a+\Delta}^b l''(x)f_0(x)\mathrm{d}x \\ &\quad + \frac{(1+\alpha)^2}{8}\Delta^2\int_{a+\Delta}^b \{l'(x)\}^2 f_0(x)\mathrm{d}x + o(\Delta^2). \end{aligned} \tag{8}$$

From (A.2) we obtain

$$
\begin{aligned}
I_{X_1}^{(\alpha)}(0,\Delta) &= -\frac{8}{1-\alpha^2}\log\int_{a+\Delta}^{b} f(x)^{(1-\alpha)/2} f(x-\Delta)^{(1+\alpha)/2}\mathrm{d}x \\
&= -\frac{8}{1-\alpha^2}\log\left[1-\frac{1}{2}\{(1-\alpha)c_1+(1+\alpha)c_2\}\Delta\right. \\
&\quad \left. -\frac{1}{4}\left\{(1-\alpha)c_1'-(1+\alpha)c_2'\right\}\Delta^2-\frac{1-\alpha^2}{8}I_0\Delta^2+o(\Delta^2)\right] \\
&= -\frac{8}{1-\alpha^2}\left[-\frac{1}{2}\{(1-\alpha)c_1+(1+\alpha)c_2\}\Delta\right. \\
&\quad -\frac{1}{4}\left\{(1-\alpha)c_1'-(1+\alpha)c_2'\right\}\Delta^2-\frac{1-\alpha^2}{8}I_0\Delta^2 \\
&\quad \left. -\frac{1}{8}\{(1-\alpha)c_1+(1+\alpha)c_2\}^2\Delta^2+o(\Delta^2)\right] \\
&= 4\left(\frac{c_1}{1+\alpha}+\frac{c_2}{1-\alpha}\right)\Delta+\frac{1}{1-\alpha^2}\left[\{(1-\alpha)c_1+(1+\alpha)c_2\}^2\right. \\
&\quad \left.+2(c_1'-c_2')-2\alpha(c_1'+c_2')\right]\Delta^2+I_0\Delta^2+o(\Delta^2). \qquad (9)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\{(1-\alpha)c_1+(1+\alpha)c_2\}^2+2(c_1'-c_2')-2\alpha(c_1'+c_2') \\
&\quad = 4(c_1'+c_1^2)-2(1+\alpha)(c_1'+c_2')-2(1+\alpha)(c_1^2-c_2^2)-(1-\alpha^2)(c_1-c_2)^2,
\end{aligned}
$$

it follows from (9) that

$$
\begin{aligned}
I_{X_1}(0,\theta) &= 4\left(\frac{c_1}{1+\alpha}+\frac{c_2}{1-\alpha}\right)\Delta+\left[\frac{4}{1-\alpha^2}(c_1'+c_1^2)\right. \\
&\quad \left.-\frac{2}{1-\alpha}\left\{(c_1'+c_2')+(c_1^2-c_2^2)\right\}+I_0-(c_1-c_2)^2\right]\Delta^2+o(\Delta^2).
\end{aligned}
$$

Next, in the case when $\Delta < 0$, we have

$$
\begin{aligned}
&\int_a^{b+\Delta} f(x)^{(1-\alpha)/2} f(x-\Delta)^{(1+\alpha)/2}\mathrm{d}x \\
&= \int_a^b f(x)\mathrm{d}x-\frac{1+\alpha}{2}\Delta\int_a^{b+\Delta} l'(x)f(x)\mathrm{d}x+\frac{1+\alpha}{4}\Delta^2\int_a^{b+\Delta} l''(x)f(x)\mathrm{d}x \\
&\quad +\frac{(1+\alpha)^2}{8}\Delta^2\int_a^{b+\Delta}\{l'(x)\}^2 f(x)\mathrm{d}x+o(\Delta^2).
\end{aligned}
$$

In a similar way to the case $\Delta > 0$, we obtain (7). This completes the proof. □

Remark 2 The relative Rényi entropy is asymptotically obtained by Hayashi (2000) as the third form of (9) for $\Delta > 0$, and is also done by Hayashi (2010) asymptotically and uniformly in α up to the order $o(\Delta)$. The another expression on $I_{X_1}^{(\alpha)}(\theta, \theta+\Delta)$ of (7) is given by

$$I_{X_1}^{(\alpha)}(\theta,\theta+\Delta) = \begin{cases} \frac{4}{1-\alpha^2}\{(1-\alpha)c_1 + (1+\alpha)c_2\}\Delta \\ \quad +\frac{1}{1-\alpha^2}\Big[2\{(1-\alpha)c_1' - (1+\alpha)c_2'\} \\ \quad +\{(1-\alpha)c_1+(1+\alpha)c_2\}^2\Big]\Delta^2 + I_0\Delta^2 + o(\Delta^2) & \text{for } \Delta > 0, \\ -\frac{4}{1-\alpha^2}\{(1+\alpha)c_1 + (1-\alpha)c_2\}\Delta \\ \quad +\frac{1}{1-\alpha^2}\Big[2\{(1+\alpha)c_1' - (1-\alpha)c_2'\} \\ \quad +\{(1+\alpha)c_1+(1-\alpha)c_2\}^2\Big]\Delta^2 + I_0\Delta^2 + o(\Delta^2) & \text{for } \Delta < 0. \end{cases}$$

Corollary 1 (Akahira (1996)) *Assume that the conditions* (A.1) *to* (A.3) *hold. Then, for* $|\alpha| < 1$ *and a small* $|\Delta|$

$$I_{X_1}^{(\alpha)}(\theta,\theta+\Delta) = \frac{8c}{1-\alpha^2}|\Delta| + \left\{\frac{4}{1-\alpha^2}(c^2-h) + I_0\right\}\Delta^2 + o(\Delta^2).$$

The proof is straightforward from Theorem 1 and the condition (A.3). Now we consider the extreme statistics $\underline{\theta}$ and $\bar{\theta}$ defined by

$$\underline{\theta} := \max_{1\le i\le n} X_i - b, \qquad \bar{\theta} := \min_{1\le i\le n} X_i - a.$$

Let θ_0 be a true parameter and put $\hat{\theta}^* = (\underline{\theta}+\bar{\theta})/2$. Then it is seen that $\hat{\theta}^*$ is a $\{n\}$-consistent estimator (see Akahira (1975a, 1995)). Put $U := n(\bar{\theta}-\theta_0)$ and $V := n(\underline{\theta}-\theta_0)$. From the joint density function of $(\min_{1\le i\le n} X_i,\ \max_{1\le i\le n} X_i)$, we can derive the second order asymptotic joint density function

$$f_{U,V}(u,v) = \begin{cases} c_1c_2 e^{c_2v-c_1u}\Big[1+\frac{1}{n}\Big\{-1+2(c_1u-c_2v) \\ \quad +\frac{1}{2}(c_2'v^2 - c_1'u^2 - (c_1u-c_2v)^2) + \Big(\frac{c_1'}{c_1}u + \frac{c_2'}{c_2}v\Big)\Big\} + o\left(\frac{1}{n}\right)\Big] \\ \quad \text{for } v < 0 < u, \\ 0 \quad \text{otherwise,} \end{cases} \tag{10}$$

since $b_{1+} = c_1$, $b_{1-} = -c_2$, $b_{2+} = c_1'/2$ and $b_{2-} = -c_2'/2$ in Theorem 3.2 in Akahira (1993). In particular, under the condition (A.3), we have

$$f_{U,V}(u,v) = \begin{cases} c^2 e^{c(v-u)}\big[1+\frac{1}{n}\big\{-1+2c(u-v) \\ \quad +\frac{1}{2}(h(u^2+v^2) - c^2(u-v)^2) - \frac{h}{c}(u-v)\big\} + o(\frac{1}{n})\big] \\ \quad \text{for } v < 0 < u, \\ 0 \quad \text{otherwise,} \end{cases} \tag{11}$$

which coincides with (10) in Akahira (1996) (see Akahira (1991a, 1993)). It is seen from (10) and (11) that U and V are asymptotically independent but not up to the second order, i.e. the order $o(1/n)$.

Lemma 2 *Assume that the conditions* (A.1) *and* (A.2) *hold. Then, for* $|\alpha| < 1$ *and any sufficiently small* Δ^2/n

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{U,V}(u-\theta_0, v-\theta_0)^{(1-\alpha)/2} f_{U,V}(u-\theta_0-\Delta, v-\theta_0-\Delta)^{(1+\alpha)/2} \mathrm{d}u\mathrm{d}v$$

$$= \begin{cases} \left[\exp\left\{-\frac{1}{2}((1-\alpha)c_1+(1+\alpha)c_2)\Delta\right\}\right]\Big[1 - \frac{c_1'+c_1^2}{2n}\Delta^2 \\ \quad + \frac{1+\alpha}{4n}\left\{(c_1'+c_2')+(c_1^2-c_2^2)\right\}\Delta^2\Big] + o\left(\frac{\Delta^2}{n}\right) \quad for\ \Delta > 0, \\ \left[\exp\left\{\frac{1}{2}((1+\alpha)c_1+(1-\alpha)c_2)\Delta\right\}\right]\Big[1 + \frac{c_2'-c_2^2}{2n}\Delta^2 \\ \quad - \frac{1+\alpha}{4n}\left\{(c_1'+c_2')+(c_1^2-c_2^2)\right\}\Delta^2\Big] + o\left(\frac{\Delta^2}{n}\right) \quad for\ \Delta < 0, \end{cases} \tag{12}$$

and

$$I^{(\alpha)}_{n\bar{\theta},n\underline{\theta}}(\theta_0, \theta_0+\Delta) = \begin{cases} 4\left(\frac{c_1}{1+\alpha}+\frac{c_2}{1-\alpha}\right)\Delta + \frac{4(c_1'+c_1^2)}{(1-\alpha^2)n}\Delta^2 \\ \quad -\frac{2}{(1-\alpha)n}\left\{(c_1'+c_2')+(c_1^2-c_2^2)\right\}\Delta^2 + o\left(\frac{\Delta^2}{n}\right) \\ \quad for\ \Delta > 0, \\ -4\left(\frac{c_1}{1-\alpha}+\frac{c_2}{1+\alpha}\right)\Delta - \frac{4(c_2'-c_2^2)}{(1-\alpha^2)n}\Delta^2 \\ \quad +\frac{2}{(1-\alpha)n}\left\{(c_1'+c_2')+(c_1^2-c_2^2)\right\}\Delta^2 + o\left(\frac{\Delta^2}{n}\right) \\ \quad for\ \Delta < 0. \end{cases} \tag{13}$$

Proof Without loss of generality, we assume that $\theta_0 = 0$. Let α be any fixed in $(-1, 1)$. Let $\Delta > 0$. From (10), we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{U,V}(u, v)^{(1-\alpha)/2} f_{U,V}(u-\Delta, v-\Delta)^{(1+\alpha)/2}\mathrm{d}u\mathrm{d}v$$

$$= \int_{-\infty}^{0}\int_{\Delta}^{\infty} c_1c_2\left[\exp\left\{\frac{1+\alpha}{2}(c_1-c_2)\Delta\right\}\right]\Bigg[1+\frac{1}{n}\Bigg\{-1+2(c_1u-c_2v)$$

$$-\frac{1}{2}\left(c_1'u^2-c_2'v^2+(c_1u-c_2v)^2\right)+\left(\frac{c_1'}{c_1}u+\frac{c_2'}{c_2}v\right)\Bigg\}$$

$$-\frac{1+\alpha}{2n}\Bigg\{2(c_1-c_2)\Delta+\left(\frac{c_1'}{c_1}+\frac{c_2'}{c_2}\right)\Delta-(c_1'u-c_2'v)\Delta$$

$$-(c_1-c_2)(c_1u-c_2v)\Delta+\frac{1}{2}\left(c_1'-c_2'+(c_1-c_2)^2\right)\Delta^2\Big\}\Big]\mathrm{d}u\mathrm{d}v$$
$$+o\left(\frac{\Delta^2}{n}\right). \tag{14}$$

Since

$$\begin{aligned}
\int_{-\infty}^{0}\int_{\Delta}^{\infty}\mathrm{e}^{c_2v-c_1u}\mathrm{d}u\mathrm{d}v &= \frac{1}{c_1c_2}\mathrm{e}^{-c_1\Delta},\\
\int_{-\infty}^{0}\int_{\Delta}^{\infty}u\mathrm{e}^{c_2v-c_1u}\mathrm{d}u\mathrm{d}v &= \frac{1}{c_1c_2}\left(\Delta+\frac{1}{c_1}\right)\mathrm{e}^{-c_1\Delta},\\
\int_{-\infty}^{0}\int_{\Delta}^{\infty}v\mathrm{e}^{c_2v-c_1u}\mathrm{d}u\mathrm{d}v &= -\frac{1}{c_1c_2^2}\mathrm{e}^{-c_1\Delta},\\
\int_{-\infty}^{0}\int_{\Delta}^{\infty}u^2\mathrm{e}^{c_2v-c_1u}\mathrm{d}u\mathrm{d}v &= \frac{1}{c_1c_2}\left(\Delta^2+\frac{2}{c_1}\Delta+\frac{2}{c_1^2}\right)\mathrm{e}^{-c_1\Delta},\\
\int_{-\infty}^{0}\int_{\Delta}^{\infty}v^2\mathrm{e}^{c_2v-c_1u}\mathrm{d}u\mathrm{d}v &= \frac{2}{c_1c_2^3}\mathrm{e}^{-c_1\Delta},\\
\int_{-\infty}^{0}\int_{\Delta}^{\infty}uv\mathrm{e}^{c_2v-c_1u}\mathrm{d}u\mathrm{d}v &= -\frac{1}{c_1c_2^2}\left(\Delta+\frac{1}{c_1}\right)\mathrm{e}^{-c_1\Delta},
\end{aligned}$$

it follows from (14) that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{U,V}(u,v)^{(1-\alpha)/2}f_{U,V}(u-\Delta,v-\Delta)^{(1+\alpha)/2}\mathrm{d}u\mathrm{d}v\\
&=\left[\exp\left\{-\frac{1}{2}\left((1-\alpha)c_1+(1+\alpha)c_2\right)\right\}\Delta\right]\\
&\quad\times\left[1-\frac{c_1'+c_1^2}{2n}\Delta^2+\frac{1+\alpha}{4n}\left\{(c_1'+c_2')+(c_1^2-c_2^2)\right\}\Delta^2\right]+o\left(\frac{\Delta^2}{n}\right).
\end{aligned}\tag{15}$$

We also have from (15)

$$\begin{aligned}
&I^{(\alpha)}_{n\bar{\theta},n\underline{\theta}}(0,\Delta)\\
&=-\frac{8}{1-\alpha^2}\log\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{U,V}(u,v)^{(1-\alpha)/2}f_{U,V}(u-\Delta,v-\Delta)^{(1+\alpha)/2}\mathrm{d}u\,\mathrm{d}v\\
&=4\left(\frac{c_1}{1+\alpha}+\frac{c_2}{1-\alpha}\right)\Delta+\frac{4(c_1'+c_1^2)}{(1-\alpha^2)n}\Delta^2\\
&\quad-\frac{2}{(1-\alpha)n}\left\{(c_1'+c_2')+(c_1^2-c_2^2)\right\}\Delta^2+o\left(\frac{\Delta^2}{n}\right).
\end{aligned}$$

Next, in the case when $\Delta < 0$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{U,V}(u,v)^{(1-\alpha)/2} f_{U,V}(u-\Delta, v-\Delta)^{(1+\alpha)/2}\mathrm{d}u\mathrm{d}v$$

$$= \int_{-\infty}^{\Delta}\int_{0}^{\infty} c_1c_2\left[\exp\left\{\frac{1+\alpha}{2}(c_1-c_2)\Delta\right\}\right]\left[1+\frac{1}{n}\left\{-1+2(c_1u-c_2v)\right.\right.$$

$$\left. -\frac{1}{2}\left(c_1'u^2-c_2'v^2+(c_1u-c_2v)^2\right)+\left(\frac{c_1'}{c_1}u+\frac{c_2'}{c_2}v\right)\right\}$$

$$-\frac{1+\alpha}{2n}\left\{2(c_1-c_2)\Delta+\left(\frac{c_1'}{c_1}+\frac{c_2'}{c_2}\right)\Delta-(c_1'u-c_2'v)\Delta\right.$$

$$\left.\left. -(c_1-c_2)(c_1u-c_2v)\Delta+\frac{1}{2}(c_1'-c_2'+(c_1-c_2)^2)\Delta^2\right\}\right]\mathrm{d}u\mathrm{d}v$$

$$+o\left(\frac{\Delta^2}{n}\right).$$

In a similar way to the case $\Delta > 0$, we obtain (12) and (13). Thus we complete the proof. $\square$

Remark 3 In Lemma 2, we further assume that the condition (A.3) holds. Then

$$I_{n\bar{\theta},n\underline{\theta}}^{(\alpha)}(\theta_0,\theta_0+\Delta) = \frac{8}{1-\alpha^2}c|\Delta| + \frac{4}{(1-\alpha^2)n}(c^2-h)\Delta^2 + o\left(\frac{\Delta^2}{n}\right),$$

which coincides with the result in Akahira (1996).

4 Loss of information associated with the statistic

First, let

$$Z_1(\theta) = -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{f_0'(X_i-\theta)}{f_0(X_i-\theta)} - \sqrt{n}(c_1-c_2)$$

for $\underline{\theta} < \theta < \bar{\theta}$. Let $Z_1^* = Z_1(\hat{\theta}^*)$. Then it is noted that Z_1^* is an asymptotically ancillary statistic. Let $C := I_0 - (c_1-c_2)^2$. Since, by the Schwarz inequality

$$I_0 = \int_a^b \left(\frac{f_0'(x)}{f_0(x)}\right)^2 f_0(x)\mathrm{d}x \ge \left(\int_a^b f_0'(x)\mathrm{d}x\right)^2 = (c_2-c_1)^2, \tag{16}$$

it is seen that $C \ge 0$. Then we have the following.

Lemma 3 *Assume that the conditions* (A.1) *and* (A.2) *hold,* $C > 0$ *and* $\Delta = O(1/n)$. *Then the amount of conditional information of* $Z_1^*/(C\sqrt{n})$ *given* $\bar{\theta}$ *and* $\underline{\theta}$ *is obtained by*

$$I^{(\alpha)}_{Z_1^*/(C\sqrt{n})|\bar{\theta},\underline{\theta}}(\theta_0, \theta_0 + \Delta) = Cn\Delta^2 + o\left(\frac{1}{n}\right) \tag{17}$$

as $n \to \infty$ *for* $|\alpha| < 1$.

Proof Without loss of generality we assume that $\theta_0 = 0$. Put $Z_1^0 = Z_1(0)$. Then the asymptotic conditional cumulants $Z_1^0/\sqrt{C}$ given $U = u$ and $V = v$, under $\theta_0 = 0$, are obtained by

$$\begin{aligned}
E_0\left[\frac{Z_1^0}{\sqrt{C}}\middle| u, v\right] &= -\frac{1}{\sqrt{Cn}}\left(\frac{c_1'}{c_1} + \frac{c_2'}{c_2} - 2c_{12}\right) + \frac{1}{\sqrt{Cn}}(c_1c_{12} + c_1')u \\
&\quad - \frac{1}{\sqrt{Cn}}(c_2c_{12} + c_2')v + O_p\left(\frac{1}{n\sqrt{n}}\right), \\
V_0\left(\frac{Z_1^0}{\sqrt{C}}\middle| u, v\right) &= 1 - \frac{u}{Cn}\left(\frac{{c_1'}^2}{c_1} + 2c_{12}(c_1c_{12} + c_1') - c_1I_0\right) \\
&\quad + \frac{v}{Cn}\left(\frac{{c_2'}^2}{c_2} + 2c_{12}(c_2c_{12} + c_2') - c_2I_0\right) \\
&\quad - \frac{2}{n} + O_p\left(\frac{1}{n^2}\right), \\
k_{3,0}\left(\frac{Z_1^0}{\sqrt{C}}\middle| u, v\right) &= \frac{1}{C\sqrt{Cn}}\left\{-K + c_{12}(2c_{12}^2 - 3I_0)\right\} + O_p\left(\frac{1}{n\sqrt{n}}\right), \\
k_{4,0}\left(\frac{Z_1^0}{\sqrt{C}}\middle| u, v\right) &= \frac{1}{C^2n}\left\{H + 4c_{12}K + 6c_{12}^2(2I_0 - c_{12}^2)\right\} + O_p\left(\frac{1}{n^2}\right),
\end{aligned}$$

where $c_{12} := c_1 - c_2$,

$$K := \int_a^b \frac{(f_0'(x))^3}{f_0^2(x)}\mathrm{d}x, \quad H := \int_a^b \frac{(f_0'(x))^4}{f_0^3(x)}\mathrm{d}x - 3I_0^2.$$

Hence the Edgeworth expansion of the conditional distribution of $Z_1^0/\sqrt{C}$ given $U = u$ and $V = v$ is obtained by

$$\begin{aligned}
F_{Z_1^0/\sqrt{C}}(z|u, v) &= \Phi(z) + \frac{1}{\sqrt{Cn}}\left\{\frac{c_1'}{c_1} + \frac{c_2'}{c_2} - 2c_{12} - (c_1c_{12} + c_1')u + (c_2c_{12} + c_2')v\right\}\phi(z) \\
&\quad + \frac{1}{6C\sqrt{Cn}}\left\{K - c_{12}(2c_{12}^2 - 3I_0)\right\}(z^2 - 1)\phi(z)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{24C^2n}\left\{H+4c_{12}K+6C_{12}^2(2I_0-c_{12}^2)\right\}(z^3-3z)\phi(z)\\
&-\frac{1}{72C^3n}\left\{K-c_{12}(2c_{12}^2-3I_0)\right\}^2(z^5-10z^3+15z)\phi(z)\\
&+\frac{1}{2Cn}\left\{2C+\left(\frac{c_1'^{\,2}}{c_1}+2c_{12}(c_1c_{12}+c_1')-c_1I_0\right)u\right.\\
&-\left(\frac{c_2'^{\,2}}{c_2}+2c_{12}(c_2c_{12}+c_2')-c_2I_0\right)v\\
&\left.-\left(\frac{c_1'}{c_1}+\frac{c_2'}{c_2}-2c_{12}-(c_1c_{12}+c_1')u+(c_2c_{12}+c_2')v\right)^2\right\}z\phi(z)\\
&-\frac{1}{6C^2n}\left(K-c_{12}(2c_{12}^2-3I_0)\right)\left\{\frac{c_1'}{c_1}+\frac{c_2'}{c_2}-2c_{12}-(c_1c_{12}+c_1')u\right.\\
&\left.+(c_2c_{12}+c_2')v\right\}z(z^2-1)\phi(z)+o\left(\frac{1}{n}\right),
\end{aligned}
\tag{18}
$$

where $\Phi(z)=\int_{-\infty}^{z}\phi(u)du$ with $\phi(u)=(1/\sqrt{2\pi})\mathrm{e}^{-u^2/2}$. Then it follows from (18) that the asymptotic conditional density of $Z_1^0/\sqrt{C}$ given $U=u$ and $V=v$ is obtained by

$$
\begin{aligned}
f_{Z_1^0/\sqrt{C}}(z|u,v)=\phi(z)&-\frac{1}{\sqrt{Cn}}\left\{\frac{c_1'}{c_1}+\frac{c_2'}{c_2}-2c_{12}-(c_1c_{12}+c_1')u+(c_2c_{12}+c_2')v\right\}z\phi(z)\\
&-\frac{1}{6C\sqrt{Cn}}\left\{K-c_{12}(2c_{12}^2-3I_0)\right\}(z^3-3z)\phi(z)\\
&+\frac{1}{24C^2n}\left\{H+4c_{12}K+6c_{12}^2(2I_0-c_{12}^2)\right\}(z^4-6z^2+3)\phi(z)\\
&+\frac{1}{72C^3n}\left\{K-c_{12}(2c_{12}^2-3I_0)\right\}^2(z^6-15z^4+45z^2-15)\phi(z)\\
&+\frac{1}{2Cn}\left[-\left\{2C+\left(\frac{c_1'^{\,2}}{c_1}+2c_{12}(c_1c_{12}+c_1')-c_1I_0\right)u\right.\right.\\
&-\left(\frac{c_2'^{\,2}}{c_2}+2c_{12}(c_2c_{12}+c_2')-c_2I_0\right)v\\
&\left.\left.-\left(\frac{c_1'}{c_1}+\frac{c_2'}{c_2}-2c_{12}-(c_1c_{12}+c_1')u+(c_2c_{12}+c_2')v\right)^2\right\}\right](z^2-1)\phi(z)\\
&+\frac{1}{6C^2n}\left(K-c_{12}(2c_{12}^2-3I_0)\right)\left\{\frac{c_1'}{c_1}+\frac{c_2'}{c_2}-2c_{12}-(c_1c_{12}+c_1')u\right.\\
&\left.+(c_2c_{12}+c_2')v\right\}(z^4-4z^2+1)\phi(z)+o\left(\frac{1}{n}\right)\\
=:\ \phi(z)&-\frac{1}{\sqrt{Cn}}a_0z\phi(z)-\frac{1}{6C\sqrt{Cn}}a_1(z^3-3z)\phi(z)
\end{aligned}
$$

$$+\frac{1}{24C^2n}a_2(z^4-6z^2+3)\phi(z)+\frac{1}{72C^3n}a_3(z^6-15z^4+45z^2-15)\phi(z)$$
$$+\frac{1}{2Cn}a_4(z^2-1)\phi(z)+\frac{1}{6C^2n}a_5(z^4-4z^2+1)\phi(z)+o\left(\frac{1}{n}\right)\text{ (say).} \tag{19}$$

Since $Z_1^* = Z_1^0 + o_p(1)$, it follows from (18) that

$$\begin{aligned}
&f_{Z_1^*/\sqrt{C}}(z|u,v)^{(1-\alpha)/2} f_{Z_1^*/\sqrt{C}}(z-\sqrt{Cn}\Delta|u,v)^{(1+\alpha)/2}\\
&= \phi(z)\left\{1+\frac{1+\alpha}{2}\sqrt{Cn}\Delta z+\frac{1+\alpha}{4}Cn\Delta^2(z^2-1)-\frac{1-\alpha^2}{8}Cn\Delta^2 z^2\right.\\
&\quad -\frac{a_0 z}{\sqrt{Cn}}-\frac{(1+\alpha)a_0}{2}\Delta z^2-\frac{a_1}{6C^{3/2}\sqrt{n}}(z^3-3z)-\frac{(1+\alpha)a_1}{12C\sqrt{n}}\Delta z(z^3-3z)\\
&\quad +\frac{(1-\alpha^2)a_0 a_1}{12C^2 n}z(z^3-3z)+\frac{a_2}{24C^2n}(z^4-6z^2+3)\\
&\quad +\frac{a_3}{72C^3n}(z^6-15z^4+45z^2-15)+\frac{a_4}{2Cn}(z^2-1)\\
&\quad \left.+\frac{a_5}{6C^2n}(z^4-4z^2+1)+\frac{(1+\alpha)a_0}{2}\Delta+\frac{(1+\alpha)a_1}{4C}\Delta(z^2-1)+o_p\left(\frac{1}{n}\right)\right\},
\end{aligned}$$

hence

$$\begin{aligned}
&I^{(\alpha)}_{Z_1^*/\sqrt{C}|u,v}(0,\sqrt{Cn}\Delta)\\
&= -\frac{8}{1-\alpha}\log\int_{-\infty}^{\infty} f_{Z_1^*/\sqrt{C}}(z|u,v)^{(1-\alpha)/2} f_{Z_1^*/\sqrt{C}}(z-\sqrt{Cn}\Delta|u,v)^{(1+\alpha)/2}\mathrm{d}z\\
&= -\frac{8}{1-\alpha^2}\log\left(1-\frac{1-\alpha^2}{8}Cn\Delta^2+o\left(\frac{1}{n}\right)\right)\\
&= Cn\Delta^2+o\left(\frac{1}{n}\right).
\end{aligned}$$

Since

$$I^{(\alpha)}_{Z_1^*/\sqrt{C}|u,v}(0,\sqrt{Cn}\Delta) = I^{(\alpha)}_{Z_1^*/(C\sqrt{n})|\bar{\theta},\underline{\theta}}(0,\Delta),$$

we obtain (17). Thus we complete the proof. □

From Lemmas 2 and 3, we have the following.

Theorem 2 *Assume that the conditions* (A.1) *and* (A.2) *hold,* $C > 0$ *and* $\Delta = O(1/n)$. *Then the amount of information of the statistic* $(Z_1^*/(C\sqrt{n}), \bar{\theta}, \underline{\theta})$ *is given by*

$$
\begin{aligned}
&I^{(\alpha)}_{Z_1^*/(C\sqrt{n}),\bar{\theta},\underline{\theta}}(\theta_0, \theta_0+\Delta)\\
&=\begin{cases}
4\left(\frac{c_1}{1+\alpha}+\frac{c_2}{1-\alpha}\right)n\Delta+\Big[\frac{4}{1-\alpha^2}(c_1'+c_1^2)-\frac{2}{1-\alpha}\left\{(c_1'+c_2')+(c_1^2-c_2^2)\right\}\\
\quad +I_0-(c_1-c_2)^2\Big]n\Delta^2+o(n\Delta^2) \quad for\ \Delta>0,\\
-4\left(\frac{c_1}{1-\alpha}+\frac{c_2}{1+\alpha}\right)n\Delta+\Big[-\frac{4}{1-\alpha^2}(c_2'-c_2^2)+\frac{2}{1-\alpha}\left\{(c_1'+c_2')+(c_1^2-c_2^2)\right\}\\
\quad +I_0-(c_1-c_2)^2\Big]n\Delta^2+o(n\Delta^2) \quad for\ \Delta<0
\end{cases}
\end{aligned}
\tag{20}
$$

as $n\to\infty$ *for* $|\alpha|<1$.

Proof Without loss of generality we assume that $\theta_0=0$. Let α be any fixed in $(-1,1)$. From Lemma 2, we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{U,V}(u,v)^{(1-\alpha)/2} f_{U,V}(u-n\Delta, v-n\Delta)^{(1+\alpha)/2}\mathrm{d}u\,\mathrm{d}v\\
&\quad=\begin{cases}
\left[\exp\left\{-\frac{1}{2}\left((1-\alpha)c_1+(1+\alpha)c_2\right)n\Delta\right\}\right]\\
\quad\times\left[1-\frac{c_1'+c_1^2}{2}n\Delta^2+\frac{1+\alpha}{4}\left\{(c_1'+c_2')+(c_1^2-c_2^2)\right\}n\Delta^2\right]\\
\quad+o\left(n\Delta^2\right) \quad \text{for } \Delta>0,\\
\left[\exp\left\{\frac{1}{2}\left((1+\alpha)c_1+(1-\alpha)c_2\right)n\Delta\right\}\right]\\
\quad\times\left[1+\frac{c_2'-c_2^2}{2}n\Delta^2-\frac{1-\alpha}{4}\left\{(c_1'+c_2')+(c_1^2-c_2^2)\right\}n\Delta^2\right]\\
\quad+o(n\Delta^2) \quad \text{for } \Delta<0
\end{cases}\\
&\quad=: k(n\Delta)\ \text{(say)},
\end{aligned}
$$

and

$$
\begin{aligned}
I^{(\alpha)}_{\bar{\theta},\underline{\theta}}(0,\Delta)&=I^{(\alpha)}_{n\bar{\theta},n\underline{\theta}}(0,n\Delta)\\
&=\begin{cases}
4\left(\frac{c_1}{1+\alpha}+\frac{c_2}{1-\alpha}\right)n\Delta+\frac{4(c_1'+c_1^2)}{1-\alpha^2}n\Delta^2\\
\quad-\frac{2}{1-\alpha}\{(c_1'+c_2')+(c_1^2-c_2^2)\}n\Delta^2+o(n\Delta^2) \quad \text{for } \Delta>0,\\
-4\left(\frac{c_1}{1-\alpha}+\frac{c_2}{1+\alpha}\right)n\Delta-\frac{4(c_2'-c_2^2)}{1-\alpha^2}n\Delta^2\\
\quad+\frac{2}{1-\alpha}\{(c_1'+c_2')+(c_1^2-c_2^2)\}n\Delta^2+o(n\Delta^2) \quad \text{for } \Delta<0.
\end{cases}
\end{aligned}
\tag{21}
$$

From Lemma 3, we also obtain

$$
I^{(\alpha)}_{Z_1^*/(C\sqrt{n})|\bar{\theta},\underline{\theta}}(0,\Delta)=Cn\Delta^2+o(n\Delta^2),
$$

hence, by Lemma 1

$$
\begin{aligned}
& I^{(\alpha)}_{Z_1^*/(C\sqrt{n}),\bar{\theta},\underline{\theta}}(0,\Delta) \\
& \quad = -\frac{8}{1-\alpha^2}\log E\left[\exp\left\{-\frac{1-\alpha^2}{8}Cn\Delta^2+o(n\Delta^2)\right\}\right]+I^{(\alpha)}_{\bar{\theta},\underline{\theta}}(0,\Delta). \quad (22)
\end{aligned}
$$

Since, by Lemma 1,

$$
\begin{aligned}
& E\left[\exp\left\{-\frac{1-\alpha^2}{8}Cn\Delta^2+o(n\Delta^2)\right\}\right] \\
& \quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\exp\left\{-\frac{1-\alpha^2}{8}Cn\Delta^2+o(n\Delta^2)\right\}\right] \\
& \qquad \times \frac{1}{k(n\Delta)}f_{U,V}(u,v)^{(1-\alpha)/2}f_{U,V}(u-n\Delta,v-n\Delta)^{(1+\alpha)/2}\mathrm{d}u\,\mathrm{d}v \\
& \quad = \left[\exp\left\{-\frac{1-\alpha^2}{8}Cn\Delta^2\right\}\right](1+o(n\Delta^2)),
\end{aligned}
$$

and $C = I_0 - (c_1 - c_2)^2$, it follows from (21) and (22) that (20) holds. This completes the proof. □

Now, the second order asymptotic loss of information of any statistic $T = T(\boldsymbol{X})$ is defined as

$$
L_n^{(\alpha)}(T) = \frac{1}{n\Delta^2}\left\{I_{\boldsymbol{X}}^{(\alpha)}(\theta,\theta+\Delta) - I_T^{(\alpha)}(\theta,\theta+\Delta)\right\} + o(1) \tag{23}
$$

for $|\alpha| < 1$, where $\Delta = O(1/n)$ (see Akahira (1996)). Then we have the following.

Theorem 3 *Assume that the conditions* (A.1) *and* (A.2) *hold and* $\Delta = O(1/n)$. *Then the second order asymptotic loss of information of the statistic*

$$
T_n^* := (Z_1^*/(C\sqrt{n}),\bar{\theta},\underline{\theta})
$$

vanishes, that is,

$$
L_n^{(\alpha)}(T_n^*) = o(1)
$$

as $n \to \infty$ *for* $|\alpha| < 1$.

The proof is straightforward from Theorems 1 and 2, since

$$
\begin{aligned}
\frac{1}{n\Delta^2}\left\{I_{\boldsymbol{X}}^{(\alpha)}(0,\Delta) - I_{T_n^*}^{(\alpha)}(0,\Delta)\right\} &= \frac{1}{n\Delta^2}\left\{nI_{X_1}^{(\alpha)}(0,\Delta) - I_{T_n^*}^{(\alpha)}(0,\Delta)\right\} \\
&= o(1)
\end{aligned}
$$

as $n \to \infty$ for $|\alpha| < 1$. Here, it follows from (16) that $C = 0$, i.e. $I_0 = (c_1 - c_2)^2$ if only if f_0 is the uniform density or the truncated exponential density. From Corollary 1 and Remark 3, we easily see that $L_n^{(\alpha)}(\bar{\theta},\underline{\theta}) = o(1)$ as $n \to \infty$ for $|\alpha| < 1$. Hence, in the case when $C = 0$, the second order asymptotic loss of information of $(\bar{\theta},\underline{\theta})$

vanishes, which coincides with the fact that $(\bar{\theta}, \underline{\theta})$ is a sufficient statistic for θ for a fixed n in the uniform and truncated exponential cases.

Remark 4 From (7), (21) and (23) it is seen that the second order asymptotic loss of information of the extreme statistic $(\bar{\theta}, \underline{\theta})$ is given by

$$L_n^{(\alpha)}(\bar{\theta}, \underline{\theta}) = \frac{1}{n\Delta^2}\left\{nI_{X_1}^{(\alpha)}(\theta, \theta+\Delta) - I_{\bar{\theta},\underline{\theta}}^{(\alpha)}(\theta, \theta+\Delta)\right\}$$
$$= I_0 - (c_1 - c_2)^2 + o(1) = C + o(1) \tag{24}$$

as $n \to \infty$ for $|\alpha| < 1$. Further, it follows from (16) that $I_0 = (c_1 - c_2)^2$ if and only if $f_0(\cdot)$ is a density of truncated exponential distribution. In the case, it is seen from (24) that $L_n^{(\alpha)}(\bar{\theta}, \underline{\theta}) = o(1)$.

Remark 5 The result of Theorem 3 corresponds to the fact that the statistic T_n^* is second order asymptotically sufficient in Akahira (1991b). This means that, in T_n^*, the extreme statistic $(\bar{\theta}, \underline{\theta})$ has the information on the endpoints $a+\theta$ and $b+\theta$ and also the amount of conditional information of $Z_1^*/(C\sqrt{n})$ given $\bar{\theta}$ and $\underline{\theta}$ consists of the information on the inside of the interval $(a+\theta, b+\theta)$ and the difference between the values of the density at the endpoints. Further, imposing the condition (A.3), we see that $C = I_0$ since $c_1 = c_2$, hence, as is seen from (17), the information at the endpoints of the support of f_0 in the conditional information of $Z_1^*/(C\sqrt{n})$ given $\bar{\theta}$ and $\underline{\theta}$ disappears. This fact shows that we are blind to the real non-regular structure all the better for such a specialization, hence it is meaningful to consider the loss of information of the statistic without the condition (A.3).

5 Examples

In the previous setup, we now give some examples on similar truncated distributions to what are treated in Akahira (1996).

Example 1 (truncated normal distribution) Let $X_1, \ldots, X_n$ be i.i.d. random variables with a density function

$$f_0(x-\theta) = \begin{cases} ce^{-(x-\theta)^2/2} & \text{for } a < x-\theta < b, \\ 0 & \text{otherwise,} \end{cases}$$

where c is a normalizing constant depending on a and b, $a < b$ and $-\infty < \theta < \infty$. Then it is easily seen that

$$c_1 = \lim_{x\to a+0} f_0(x) = ce^{-a^2/2}, \quad c_2 = \lim_{x\to b-0} f_0(x) = ce^{-b^2/2},$$
$$c_1' = \lim_{x\to a+0} f_0'(x) = -ace^{-a^2/2}, \quad c_2' = \lim_{x\to b-0} f_0'(x) = -bce^{-b^2/2},$$
$$I_0 = 1 - c(be^{-b^2/2} - ae^{-a^2/2})$$

since $f_0(x) = ce^{-x^2/2}$ and $f_0'(x) = -cxe^{-x^2/2}$ for $a < x < b$. Since the conditions (A.1) and (A.2) are satisfied, it follows from Theorem 3 that the second order asymptotic loss of information of the statistic $(Z_1^*/(C\sqrt{n}), \bar{\theta}, \underline{\theta})$ vanishes, where $Z_1^* = \sqrt{n}\{\bar{X} - \hat{\theta}^* - c(e^{-a^2/2} - e^{-b^2/2})\}$, $\bar{X} = \sum_{i=1}^n X_i/n$, $\hat{\theta}^* = (\underline{\theta} + \bar{\theta})/2$, $C = 1 - c(be^{-b^2/2} - ae^{-a^2/2}) - c^2(e^{-a^2/2} - e^{-b^2/2})^2$ with $\underline{\theta} = \max_{1\le i\le n} X_i - b$ and $\bar{\theta} = \min_{1\le i\le n} X_i - a$. Further, it is seen from Remark 4 that the second order asymptotic loss of information of the statistic $(\bar{\theta}, \underline{\theta})$ is given by $L_n^{(\alpha)}(\bar{\theta}, \underline{\theta}) = C + o(1)$ as $n \to \infty$.

Example 2 (truncated Cauchy distribution) Let $X_1, \cdots, X_n$ be i.i.d. random variables with a density function

$$f_0(x-\theta) = \begin{cases} \frac{c}{1+(x-\theta)^2} & \text{for } a < x - \theta < b, \\ 0 & \text{otherwise,} \end{cases}$$

where c is a normalizing constant depending on a and b, $a < b$ and $-\infty < \theta < \infty$. Then it is easily seen that

$$c_1 = \lim_{x\to a+0} f_0(x) = c/(1+a^2), \quad c_2 = \lim_{x\to b-0} f_0(x) = c/(1+b^2),$$
$$c_1' = \lim_{x\to a+0} f_0'(x) = -2ac/(1+a^2)^2, \quad c_2' = \lim_{x\to b-0} f_0'(x) = -2bc/(1+b^2)^2,$$
$$I_0 = 4c\int_a^b \frac{x^2}{(1+x^2)^3}\mathrm{d}x,$$

since $f_0(x) = c/(1+x^2)$ and $f_0'(x) = -2cx/(1+x^2)^2$ for $a < x < b$. Since the conditions (A.1) and (A.2) are satisfied, it follows from Theorem 3 that the second order asymptotic loss of information of the statistic $(Z_1^*/(C\sqrt{n}), \bar{\theta}, \underline{\theta})$ vanishes, where

$$Z_1^* = \frac{2}{\sqrt{n}}\sum_{i=1}^n \frac{X_i - \hat{\theta}^*}{1+(X_i - \hat{\theta}^*)^2} - c\sqrt{n}\left(\frac{1}{1+a^2} - \frac{1}{1+b^2}\right), \quad \hat{\theta}^* = \frac{1}{2}(\underline{\theta} + \bar{\theta}),$$
$$C = I_0 - c^2\left(\frac{1}{1+a^2} - \frac{1}{1+b^2}\right)^2$$

with $\underline{\theta} = \max_{1\le i\le n} X_i - b$ and $\bar{\theta} = \min_{1\le i\le n} X_i - a$. Further, it is seen from Remark 4 that the second order asymptotic loss of information of the statistic $(\bar{\theta}, \underline{\theta})$ is given by $L_n^{(\alpha)}(\bar{\theta}, \underline{\theta}) = C + o(1)$ as $n \to \infty$.

Example 3 Let $X_1, \ldots, X_n$ be i.i.d. random variables with a density function

$$f_0(x-\theta) = \begin{cases} c\exp\left[\{1-(x-\theta)^2\}^p\right] & \text{for } a < x - \theta < b, \\ 0 & \text{otherwise ,} \end{cases}$$

where c is a normalizing constant depending on a and b, $a < b$ and $-\infty < \theta < \infty$. Then it is seen that

$$c_1 = \lim_{x \to a+0} f_0(x) = c \exp\{(1-a^2)^p\}, \quad c_2 = \lim_{x \to b-0} f_0(x) = c \exp\{(1-b^2)^p\},$$
$$c_1' = \lim_{x \to a+0} f_0'(x) = -2cpa(1-a^2)^{p-1} \mathrm{e}^{(1-a^2)^p},$$
$$c_2' = \lim_{x \to b-0} f_0'(x) = -2cpb(1-b^2)^{p-1} \mathrm{e}^{(1-b^2)^p},$$
$$I_0 = 4p^2 \int_a^b x^2 (1-x^2)^{2(p-1)} c \mathrm{e}^{(1-x^2)^p} \mathrm{d}x,$$

since $f_0(x) = c \exp\{(1-x^2)^p\}$ and $f_0'(x) = -2cpx(1-x^2)^{p-1} e^{(1-x^2)^p}$ for $a < x < b$. Since the conditions (A.1) and (A.2) are satisfied, it follows from Theorem 3 that the second order asymptotic loss of information of the statistic $(Z_1^*/(C\sqrt{n}), \bar{\theta}, \underline{\theta})$ vanishes, where

$$Z_1^* = \frac{1}{\sqrt{n}} \sum_{i=1}^n 2p(X_i - \hat{\theta}^*) \left\{1 - (X_i - \hat{\theta}^*)^2\right\}^{p-1}$$
$$-c\sqrt{n} \left[\exp\{(1-a^2)^p\} - \exp\left\{(1-b^2)^p\right\}\right],$$
$$\hat{\theta}^* = \frac{1}{2}(\underline{\theta} + \bar{\theta}), \quad C = I_0 - c^2 \left[\exp\left\{(1-a^2)^p\right\} - \exp\left\{(1-b^2)^p\right\}\right]^2$$

with $\underline{\theta} = \max_{1 \le i \le n} X_i - b$ and $\bar{\theta} = \min_{1 \le i \le n} X_i - a$. Further, it is seen from Remark 4 that the second order asymptotic loss of information of the statistic $(\bar{\theta}, \underline{\theta})$ is given by $L_n^{(\alpha)}(\bar{\theta}, \underline{\theta}) = C + o(1)$ as $n \to \infty$.

6 Conclusion

The condition (A.3) that the density has the same values and the same absolute ones of differential coefficient at the endpoints of the bounded support is assumed in Akahira (1996). But it may not be an essential condition. Indeed, in this paper, the result is extended to the case when the condition (A.3) does not necessarily hold. This is also an answer to Remark 3.2 in Akahira (1996), and brings a wider application to statistical inference.

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