Ann Inst Stat Math (2011) 63:347–373
DOI 10.1007/s10463-009-0257-x

Tests of serial independence for continuous multivariate time series based on a Möbius decomposition of the independence empirical copula process

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Received: 24 March 2008 / Revised: 16 December 2008 / Published online: 28 July 2009

Abstract Genest and Rémillard have recently studied tests of randomness based on a decomposition of the serial independence empirical copula process into a finite number of asymptotically independent sub-processes. A generalization of this decomposition that can be used to test serial independence in the continuous multivariate time series framework is investigated. The weak limits of the Cramér–von Mises statistics derived from the various processes under consideration are determined. As these statistics are not distribution-free, the consistency of the bootstrap methodology is investigated. Extensive simulations are used to study the finite-sample behavior of the tests for continuous time series of dimension one to three, and comparisons with the portmanteau test are provided, as well as, in the one-dimensional case, with the ranked-based version of the Brock, Dechert, and Scheinkman test. Finally, the studied tests are applied to a real trivariate financial time series.

Keywords Serial copula · Test of serial independence · Empirical process · Möbius decomposition · Cramér–von Mises statistic · Bootstrap · Permutation

1 Introduction

Testing for multivariate serial independence is a natural first step in multivariate time series modeling. Unlike their univariate counterparts which have received considerable

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attention in the literature, tests of multivariate serial independence are much less developed. A few exceptions are the multivariate portmanteau test (Hosking 1980) and the multivariate Wald-Wolfowitz rank test (Hallin and Puri 1995). In the univariate case, some powerful tests of serial independence are based on the empirical distribution function (Skag and Tjøstheim 1993; Delgado 1996). These approaches can be regarded as the serial analogs of the well-known test of independence proposed by Blum et al. (1961).

Our departure point is the work of Ghoudi et al. (2001) and Genest and Rémillard (2004). Inspired by the work, among others, of Deheuvels (1981), Ghoudi et al. (2001) investigated tests based on a *Möbius decomposition* of the Blum et al. (1961) statistic considered by Delgado (1996). In order to obtain margin-free test statistics, a version of this decomposition based on the empirical copula was studied by Genest and Rémillard (2004). The decomposition proposed by Ghoudi et al. (2001) was also recently extended to the multivariate time series setting by Beran et al. (2007) using a characterization of serial independence defined from probabilities of half-spaces. Because it uses the empirical probability distribution, the resulting procedure can be used to test for serial independence in both discrete and continuous multivariate time series. The price for this versatility is a high computational cost as each execution necessitates to numerically solve an optimization problem (Beran et al. 2007, Sect. 6). In the case of continuous multivariate time series, serial independence can also be naturally characterized through the underlying unique copula, which leads to a multivariate generalization of the margin-free rank-based procedures proposed in Genest and Rémillard (2004). This is the direction followed in this work.

Given a stationary and ergodic univariate sequence of continuous random variables $X_1, X_2, \ldots$ and an integer $p > 1$, Genest and Rémillard (2004) first form p-dimensional vectors of observations $Y_i = (X_i, \ldots, X_{i+p-1})$, $i \in \{1, \ldots, n\}$, where p is the *embedding dimension*. From the stationarity assumption, all the p-dimensional vectors Y_i have the same cumulative distribution function (c.d.f.) H. By Sklar (1959)'s theorem, H can be uniquely represented by a copula $C : [0, 1]^p \to [0, 1]$, such that

$$H(x_1, \ldots, x_p) = C[F(x_1), \ldots, F(x_p)], \qquad (x_1, \ldots, x_p) \in \mathbb{R}^p,$$

where F is the common c.d.f. of each X_i. Given that under serial independence of $X_1, X_2, \ldots$, the copula C coincides with the independence copula $\prod_{k=1}^p u_k$, $(u_1, \ldots, u_p) \in [0, 1]^p$, Genest and Rémillard considered test statistics derived from the process

$$\sqrt{n}\left\{C_n^s(u_1, \ldots, u_p) - \prod_{k=1}^p u_k\right\}, \qquad (u_1, \ldots, u_p) \in [0, 1]^p, \tag{1}$$

where C_n^s is the serial version of the empirical copula computed from $Y_1, \ldots, Y_n$. Under serial independence of $X_1, X_2, \ldots$, the empirical process (1) can be decomposed, using the *Möbius transform* (Rota 1964), into a collection of sub-processes $\sqrt{n}\mathcal{M}_A(C_n^s)$, $A \subseteq \{1, \ldots, p\}$, $A \ni 1$, $|A| > 1$, that converge jointly to tight centered mutually independent Gaussian processes (the map $\mathcal{M}_A$ will be precisely defined in Sect. 3.1). Instead of one Cramér–von Mises test statistic based on (1), this led

Genest and Rémillard (2004) to consider a collection of statistics of the form

$$\int_{[0,1]^p} \left[\sqrt{n}\mathcal{M}_A(C_n^s)(\mathbf{u})\right]^2 \mathrm{d}\mathbf{u}, \qquad A \subseteq \{1,\ldots,p\}, A \ni 1, |A| > 1,$$

that are asymptotically mutually independent under the null hypothesis of serial independence. Such a decomposition is motivated by the fact that each of these Cramér–von Mises statistics can be seen as focusing on a particular type of departure from serial independence. The resulting test of randomness may thus be more powerful than that based on a single Cramér–von Mises statistic derived from (1).

The contribution of this paper is three-fold. First, generalizing the work of Genest and Rémillard (2004), the asymptotic behavior of the serial independence empirical copula process and of its *Möbius decomposition* is obtained in the continuous multivariate time series setting. Second, the bootstrap approach, that can be used to practically carry out the tests under consideration, is shown to be valid. Third, extensive simulations, that could be conducted because of the good computational properties of the derived procedures, suggest that the proposed tests have substantial power. As an illustration, for the univariate version of the alternatives under consideration, the resulting tests generally seem to outperform the rank-based Brock, Dechert, and Scheinkman test recently studied by Genest et al. (2007a).

This paper is organized as follows. The second section focuses on a statistic for testing randomness based on the multivariate generalization of process (1), while the third section studies its Möbius decomposition. As the derived test statistics are margin-free but not distribution-free, the validity of the bootstrap is investigated in the fourth section. The fifth section gives important implementation details, while the sixth section presents extensive simulations. In the last section, the studied procedures are used to test serial independence in a trivariate time series whose marginal series are Apple's, Google's, and Microsoft's daily stock return data for 2007.

Note that all the tests of serial independence studied in this work are implemented in the R package `copula` (Yan and Kojadinovic 2008).

2 Statistic based on the serial independence copula process

2.1 Notation and setting

In the rest of the paper, for any integer $k > 0$, the set $\{1,\ldots,k\}$ will be denoted by $[k]$. Also, we adopt the convention $[0] = \emptyset$.

Consider a stationary ergodic sequence of q-dimensional continuous random vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots,$ the common c.d.f. of each $\mathbf{X}_i$ is denoted by F and the associated copula by C. The q components of $\mathbf{X}_i$ will be denoted by $X_i^{(1)}, \ldots, X_i^{(q)}$, respectively, and, as we continue, this notation will be used to access the components of any vector. Furthermore, let $p > 1$ be an integer, let $n' = n + p - 1$ and, for any $j \in [q]$, let $R_1^{(j)}$, $R_2^{(j)}, \ldots, R_{n'}^{(j)}$ be the ranks associated with the univariate sequence $X_1^{(j)}, X_2^{(j)}, \ldots,$ $X_{n'}^{(j)}$.

The empirical c.d.f. computed from the random vectors $\mathbf{X}_1, \ldots, \mathbf{X}_{n'}$ is denoted by F_n, i.e.,

$$F_n(\mathbf{x}) = \frac{1}{n'} \sum_{i=1}^{n'} \prod_{k=1}^{q} 1[X_i^{(k)} \leq x^{(k)}] = \frac{1}{n'} \sum_{i=1}^{n'} 1[\mathbf{X}_i \leq \mathbf{x}], \qquad \mathbf{x} \in \mathbb{R}^q,$$

where inequalities between vectors are to be understood component-wise. The corresponding marginal c.d.f.s are denoted by $F_n^{(1)}, \ldots, F_n^{(q)}$. The ranks $R_i^{(j)}$ are then related to the $X_i^{(j)}$ through the equalities $R_i^{(j)} = n' F_n^{(j)}(X_i^{(j)})$, $i \in [n']$, $j \in [q]$.

As we continue, the following notation will be needed. Given $B \subseteq [p]$ and $\mathbf{u} \in [0, 1]^{pq}$, the vector $\mathbf{u}_B \in [0, 1]^{pq}$ is defined by

$$u_B^{(i)} = \begin{cases} u^{(i)}, & \text{if } i \in \bigcup_{j \in B} \{(j-1)q + 1, \ldots, jq\}, \\ 1, & \text{otherwise.} \end{cases}$$

Moreover, given $\mathbf{u} \in [0, 1]^{pq}$ and $j \in [p]$, the vector $\mathbf{u}_{\langle j \rangle} \in [0, 1]^q$ is defined by $u_{\langle j \rangle}^{(i)} = u^{(i+(j-1)q)}$, $i \in [q]$. The vector $\mathbf{u}_{\langle j \rangle}$ is clearly a subvector of $\mathbf{u}$ whereas $\mathbf{u}_{\{j\}} \in [0, 1]^{pq}$.

Next, we form the pq-dimensional random vectors $\mathbf{Y}_i = (\mathbf{X}_i, \ldots, \mathbf{X}_{i+p-1})$, $i \in [n]$. The serial empirical copula computed from $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ is then given, for any $\mathbf{u} \in [0, 1]^{pq}$, by

$$C_n^s(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{p} \prod_{k=1}^{q} 1\left[F_n^{(k)}\left(X_{i+j-1}^{(k)}\right) \leq u_{\langle j \rangle}^{(k)}\right] = \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{p} \prod_{k=1}^{q} 1\left[R_{i+j-1}^{(k)} \leq n' u_{\langle j \rangle}^{(k)}\right].$$

A natural extension of the serial independence empirical copula process (1) to the multivariate time series setting is then

$$\sqrt{n}\left[C_n^s(\mathbf{u}) - \prod_{j=1}^{p} C_n^s(\mathbf{u}_{\{j\}})\right], \qquad \mathbf{u} \in [0, 1]^{pq}. \tag{2}$$

2.2 Weak convergence of the serial empirical copula under randomness

Let $\ell^\infty([0, 1]^{pq})$ be the space of all bounded real-valued functions on $[0, 1]^{pq}$. The following result, characterizing the behavior of the serial version of the empirical copula under serial independence, will be used to establish the asymptotic behavior of the empirical process (2) under the null hypothesis.

Theorem 1 *Suppose that C has continuous partial derivatives. Then, under serial independence of* $\mathbf{X}_1, \mathbf{X}_2, \ldots$, *the process* $\sqrt{n}\left[C_n^s(\mathbf{u}) - \prod_{j=1}^{p} C(\mathbf{u}_{\langle j \rangle})\right]$, $\mathbf{u} \in [0, 1]^{pq}$, *converges weakly in* $\ell^\infty([0, 1]^{pq})$ *to the tight centered Gaussian process*

$$\mathbb{C}(\mathbf{u}) = \mathbb{H}(\mathbf{u}) - \sum_{i=1}^{p} \prod_{\substack{k=1 \\ k \neq i}}^{p} C(\mathbf{u}_{\langle k \rangle}) \sum_{j=1}^{q} \partial_j C(\mathbf{u}_{\langle i \rangle}) \mathbb{H}(1, \dots, 1, u_{\langle i \rangle}^{(j)}, 1, \dots, 1), \quad \mathbf{u} \in [0, 1]^{pq}, \tag{3}$$

where $\partial_j C$ denotes the jth partial derivative of C and where $\mathbb{H}(\mathbf{u})$, $\mathbf{u} \in [0, 1]^{pq}$, is a tight centered Gaussian process with covariance function

$$E[\mathbb{H}(\mathbf{u})\mathbb{H}(\mathbf{v})] = \gamma_0(\mathbf{u}, \mathbf{v}) + \sum_{\delta=1}^{p-2} \left[\gamma_\delta(\mathbf{u}, \mathbf{v}) + \gamma_\delta(\mathbf{v}, \mathbf{u})\right], \quad \mathbf{u}, \mathbf{v} \in [0, 1]^{pq},$$

where

$$\begin{aligned} \gamma_\delta(\mathbf{u}, \mathbf{v}) = & \prod_{j \in [p] \setminus [p-\delta]} C(\mathbf{u}_{\langle j \rangle}) \prod_{j \in [\delta]} C(\mathbf{v}_{\langle j \rangle}) \Bigg[\prod_{j \in [p-\delta]} C(\mathbf{u}_{\langle j \rangle} \wedge \mathbf{v}_{\langle j+\delta \rangle}) \\ & - \sum_{i \in [p-\delta]} C(\mathbf{u}_{\langle i \rangle} \wedge \mathbf{v}_{\langle i+\delta \rangle}) \prod_{j \in [p-\delta] \setminus \{i\}} C(\mathbf{u}_{\langle j \rangle}) C(\mathbf{v}_{\langle j+\delta \rangle}) \\ & + (p - \delta - 1) \prod_{j \in [p-\delta]} C(\mathbf{u}_{\langle j \rangle}) C(\mathbf{v}_{\langle j+\delta \rangle}) \Bigg]. \end{aligned} \tag{4}$$

Hence, as expected, under serial independence, C_n^s is a consistent estimator of the copula $\otimes^p C$ of the $\mathbf{Y}_i$, where $\otimes$ denotes the tensor product.

2.3 Serial independence empirical copula process

In order to study the asymptotic behavior of the empirical process (2), we consider the map $\mathcal{I} : \ell^\infty([0, 1]^{pq}) \to \ell^\infty([0, 1]^{pq})$ defined by

$$\mathcal{I}(f)(\mathbf{x}) = f(\mathbf{x}) - \prod_{k=1}^{p} f(\mathbf{x}_{\{k\}}), \qquad \mathbf{x} \in [0, 1]^{pq}. \tag{5}$$

Clearly, $\sqrt{n}\mathcal{I}(C_n^s)$ is the empirical process defined by (2). Furthermore, it is easy to verify that the map $\mathcal{I}$ is Hadamard differentiable (van der Vaart and Wellner 1996, Chap. 3.9) tangentially to $\ell^\infty([0, 1]^{pq})$ and that its derivative at $f \in \ell^\infty([0, 1]^{pq})$ is

$$\mathcal{I}'_f(a)(x) = a(x) - \sum_{i=1}^{p} a(x_{\{i\}}) \prod_{\substack{j=1 \\ j \neq i}}^{p} f(x_{\{j\}}), \quad x \in [0, 1]^{pq}.$$

The next theorem establishes the asymptotic behavior of the serial independence empirical copula process (2).

Theorem 2 *Suppose that C has continuous partial derivatives. Then, under serial independence of $\mathbf{X}_1, \mathbf{X}_2, \ldots$, the empirical process $\sqrt{n}\mathcal{I}(C_n^s)(\mathbf{u})$, $\mathbf{u} \in [0, 1]^{pq}$, converges weakly in $\ell^\infty([0, 1]^{pq})$ to the tight centered Gaussian process*

$$\mathcal{I}'_{\otimes^p C}(\mathbb{C})(\mathbf{u}) = \mathbb{H}(\mathbf{u}) - \sum_{k=1}^{p} \mathbb{H}(\mathbf{u}_{\{k\}}) \prod_{\substack{j=1\\ j\neq k}}^{p} C(\mathbf{u}_{\langle j\rangle}), \qquad \mathbf{u} \in [0, 1]^{pq}.$$

For $\mathbf{u}, \mathbf{v} \in [0, 1]^{pq}$, the covariance function is given by

$$E[\mathcal{I}'_{\otimes^p C}(\mathbb{C})(\mathbf{u})\mathcal{I}'_{\otimes^p C}(\mathbb{C})(\mathbf{v})] = E[\mathbb{H}(\mathbf{u})\mathbb{H}(\mathbf{v})] = \gamma_0(\mathbf{u}, \mathbf{v}) + \sum_{\delta=1}^{p-2} \left[\gamma_\delta(\mathbf{u}, \mathbf{v}) + \gamma_\delta(\mathbf{v}, \mathbf{u})\right],$$

where γ_δ is defined by (4).

2.4 Resulting Cramér–von Mises statistic

A natural next step consists of considering as measure of departure from serial independence, the Cramér–von Mises statistic derived from the empirical process (2):

$$I_n = n \int_{[0,1]^{pq}} \left[C_n^s(\mathbf{u}) - \prod_{k=1}^{p} C_n^s(\mathbf{u}_{\{k\}}) \right]^2 \mathrm{d}\mathbf{u} = n \int_{[0,1]^{pq}} \mathcal{I}(C_n^s)(\mathbf{u})^2 \mathrm{d}\mathbf{u}. \qquad (6)$$

Not only is the above statistic clearly related to that proposed by Blum et al. (1961) but it can also be seen as an unnormalized extension of the sample version of the measure of dependence γ defined by Schweizer and Wolff (1981). To obtain a meaningful normalized version of I_n that could be used as a measure of dependence, it would be necessary to determine a sharp upper bound for I_n subject to the marginal empirical copulas $C_n^s(\mathbf{u}_{\{k\}})$, $k \in [p]$. This issue is related to the so-called *Fréchet problem* in the probability literature (see e.g. McNeil et al. 2005, Sect. 6.2.1), and would need to be investigated more in depth in the setting under consideration.

From Theorem 2 and the continuous mapping theorem (van der Vaart and Wellner 1996, Theorem 1.3.6), we obtain that, if C has continuous partial derivatives, then, under serial independence of $\mathbf{X}_1, \mathbf{X}_2, \ldots$, the random variable I_n converges in distribution to

$$\int_{[0,1]^{pq}} \mathcal{I}'_{\otimes^p C}(\mathbb{C})(\mathbf{u})^2 \mathrm{d}\mathbf{u}.$$

The expression of the statistic I_n in terms of the ranks can be obtained after a simple but tedious calculation. We have

$$
\begin{aligned}
I_n = & \frac{1}{n} \sum_{i=1}^{n} \sum_{l=1}^{n} \prod_{j=1}^{p} \prod_{k=1}^{q} \left[1 - \frac{R_{i+j-1}^{(k)} \vee R_{l+j-1}^{(k)}}{n'} \right] \\
& - \frac{2}{n^p} \sum_{i=1}^{n} \prod_{j=1}^{p} \sum_{l=1}^{n} \prod_{k=1}^{q} \left[1 - \frac{R_{i+j-1}^{(k)} \vee R_{l+j-1}^{(k)}}{n'} \right] \\
& + \frac{1}{n^{2p-1}} \prod_{j=1}^{p} \sum_{i=1}^{n} \sum_{l=1}^{n} \prod_{k=1}^{q} \left[1 - \frac{R_{i+j-1}^{(k)} \vee R_{l+j-1}^{(k)}}{n'} \right].
\end{aligned}
$$

3 Statistics derived from the Möbius decomposition of the serial independence copula process

In order to obtain potentially more powerful tests, following Genest and Rémillard (2004), we shall now derive $2^{p-1} - 1$ test statistics based on a *Möbius decomposition* of the process (2) that are asymptotically mutually independent under the hypothesis of serial independence of $\mathbf{X}_1, \mathbf{X}_2, \ldots$.

3.1 Möbius decomposition

The aim of this section is to generalize the decomposition obtained in Genest and Rémillard (2004, Sect. 2), itself extending that of Deheuvels (1981) to the serial setting.

Let $\mathcal{P} = \{B \subseteq [p] : |B| > 1\}$. The set $\mathcal{P}$ clearly contains $2^p - p - 1$ elements. Now, for any $A \subseteq [p]$, consider the map $\mathcal{M}_A : \ell^\infty([0, 1]^{pq}) \to \ell^\infty([0, 1]^{pq})$ defined by

$$
\mathcal{M}_A(f)(\mathbf{x}) = \sum_{B \subseteq A} (-1)^{|A|-|B|} f(\mathbf{x}_B) \prod_{k \in A \setminus B} f(\mathbf{x}_{\{k\}}), \qquad \mathbf{x} \in [0, 1]^{pq}. \tag{7}
$$

The $2^p - p - 1$ empirical processes $\{\sqrt{n}\mathcal{M}_A(C_n^s) : A \in \mathcal{P}\}$ are known as the *Möbius decomposition* of $\sqrt{n}\mathcal{I}(C_n^s)$. Furthermore, it is easy to verify that, for any $A \in \mathcal{P}$, the map $\mathcal{M}_A$ is Hadamard differentiable tangentially to $\ell^\infty([0, 1]^{pq})$ and that its derivative at $f \in \ell^\infty([0, 1]^{pq})$ is

$$
\begin{aligned}
\mathcal{M}'_{A,f}(a)(\mathbf{x}) \; = & \sum_{B \subseteq A} (-1)^{|A|-|B|} \Bigg[f(\mathbf{x}_B) \sum_{k \in A \setminus B} a(\mathbf{x}_{\{k\}}) \prod_{\substack{i \in A \setminus B \\ i \neq k}} f(\mathbf{x}_{\{i\}}) \\
& + a(\mathbf{x}_B) \prod_{k \in A \setminus B} f(\mathbf{x}_{\{k\}}) \Bigg], \qquad \mathbf{x} \in [0, 1]^{pq}.
\end{aligned}
$$

Given a subset $A = \{i_1, \dots, i_k\} \in \mathcal{P}$ and $\delta \in \{-\min A+1, \dots, p-\max A\}$, let the δ-translate of A, denoted by $A+\delta$, be the element of $\mathcal{P}$ defined by $\{i_1+\delta, \dots, i_k+\delta\}$. The following result then generalizes Proposition 2.1 in Genest and Rémillard (2004), itself generalizing Theorem 2.2 in Ghoudi et al. (2001).

Theorem 3 *Suppose that C has continuous partial derivatives. Then, under serial independence of $\mathbf{X}_1, \mathbf{X}_2, \dots$, the vector of 2^p-p-1 empirical processes $\{\sqrt{n}\mathcal{M}_A(C_n^s)(\mathbf{u}), \mathbf{u} \in [0,1]^{pq} : A \in \mathcal{P}\}$ converges weakly in $\ell^\infty([0,1]^{pq})$ to the corresponding vector of tight centered Gaussian processes $\{\mathcal{M}'_{A,\otimes^p C}(\mathbb{C})(\mathbf{u}), \mathbf{u} \in [0,1]^{pq} : A \in \mathcal{P}\}$, where*

$$\mathcal{M}'_{A,\otimes^p C}(\mathbb{C})(\mathbf{u}) = \sum_{B\subseteq A}(-1)^{|A|-|B|}\mathbb{H}(\mathbf{u}_B)\prod_{k\in A\setminus B} C(\mathbf{u}_{\langle k\rangle}), \qquad \mathbf{u} \in [0,1]^{pq}.$$

The cross-covariance function $E[\mathcal{M}'_{A,\otimes^p C}(\mathbb{C})(\mathbf{u})\mathcal{M}'_{B,\otimes^p C}(\mathbb{C})(\mathbf{v})]$ is given by

$$\begin{cases} \prod_{j\in A}\left[C(\mathbf{u}_{\langle j\rangle}\wedge \mathbf{v}_{\langle j+\delta\rangle}) - C(\mathbf{u}_{\langle j\rangle})C(\mathbf{v}_{\langle j+\delta\rangle})\right], & \text{if } B = A+\delta, \\ 0, & \text{otherwise,} \end{cases}$$

where $A, B \subseteq [p]$, $|A| > 1$, $|B| > 1$, and $\mathbf{u}, \mathbf{v} \in [0,1]^{pq}$.

As noticed by Ghoudi et al. (2001) and as can be concluded from the previous theorem, a process $\sqrt{n}\mathcal{M}_A(C_n^s)$ and a process $\sqrt{n}\mathcal{M}_{A+\delta}(C_n^s)$ are roughly the same. It follows that attention can be restricted to the $2^{p-1}-1$ processes $\sqrt{n}\mathcal{M}_A(C_n^s)$ for $A \in \mathcal{P}_1 = \{B \in \mathcal{P} : B \ni 1\}$.

3.2 Resulting Cramér–von Mises statistics

The $2^{p-1}-1$ Cramér–von Mises statistics obtained from the Möbius decomposition of the serial independence empirical copula process are given by

$$M_{A,n} = n\int_{[0,1]^{pq}} \mathcal{M}_A(C_n^s)(\mathbf{u})^2 \mathrm{d}\mathbf{u}, \qquad A \in \mathcal{P}_1.$$

From Theorem 3 and the continuous mapping theorem, we have that, if C has continuous partial derivatives, then, under serial independence of $\mathbf{X}_1, \mathbf{X}_2, \dots$, the random vector $\{M_{A,n} : A \in \mathcal{P}_1\}$ converges in distribution to the random vector

$$\left\{\int_{[0,1]^{pq}} \mathcal{M}'_{A,\otimes^p C}(\mathbb{C})(\mathbf{u})^2 \mathrm{d}\mathbf{u} : A \in \mathcal{P}_1\right\},$$

whose components are mutually independent. The expression of the statistics in terms of the ranks is given next. For any $A \in \mathcal{P}_1$, a simple but tedious calculation gives

$$M_{A,n} = \frac{1}{n}\sum_{i=1}^{n}\sum_{l=1}^{n}\prod_{j\in A}\left\{\prod_{k=1}^{q}\left[1-\frac{R_{i+j-1}^{(k)}\vee R_{l+j-1}^{(k)}}{n'}\right]\right.$$
$$-\frac{1}{n}\sum_{m=1}^{n}\prod_{k=1}^{q}\left[1-\frac{R_{i+j-1}^{(k)}\vee R_{m+j-1}^{(k)}}{n'}\right]-\frac{1}{n}\sum_{m=1}^{n}\prod_{k=1}^{q}\left[1-\frac{R_{l+j-1}^{(k)}\vee R_{m+j-1}^{(k)}}{n'}\right]$$
$$\left.+\frac{1}{n^2}\sum_{r=1}^{n}\sum_{s=1}^{n}\prod_{k=1}^{q}\left[1-\frac{R_{r+j-1}^{(k)}\vee R_{s+j-1}^{(k)}}{n'}\right]\right\}.$$

Note that, for $q = 1$, the above expression does not exactly coincide with that given in Genest and Rémillard (2004, p. 348) as the latter is based on the approximation $n \approx n'$.

4 Validity of the bootstrap

As can be seen from Theorems 2 and 3, as soon as $q > 1$, the derived statistics are not distribution-free anymore. In such a framework, a sensible way of obtaining critical values and p-values involves using the bootstrap or the permutation methodology. We focus in this section on the former, although we shall use the latter in Sects. 5 and 6 as it appears more natural in a ranked-based context and easier to implement.

Recall that the empirical c.d.f. computed from $\mathbf{X}_1, \ldots, \mathbf{X}_{n'}$ is denoted by F_n and that it is a consistent estimator of F from the Glivenko-Cantelli lemma for stationary ergodic sequences. Under the hypothesis of serial independence, a natural way of forming the bootstrap sample then consists of sampling with replacement from the empirical c.d.f. F_n. As we continue, the resulting sample will be denoted by $\mathbf{X}_1^*, \ldots, \mathbf{X}_{n'}^*$ and the serial version of the empirical copula computed from it by C_n^{s*}. The following result can then be stated.

Theorem 4 *Suppose that C has continuous partial derivatives. Under serial independence of $\mathbf{X}_1, \mathbf{X}_2, \ldots$, the conditional distribution of the process*

$$\sqrt{n}\left[C_n^{s*}(\mathbf{u}) - \prod_{j=1}^{p} C_n^{s}(\mathbf{u}_{\{j\}})\right], \qquad \mathbf{u} \in [0, 1]^{pq},$$

given the data, converges to the same limiting distribution as that of

$$\sqrt{n}\left[C_n^{s}(\mathbf{u}) - \prod_{j=1}^{p} C(\mathbf{u}_{\langle j\rangle})\right], \qquad \mathbf{u} \in [0, 1]^{pq},$$

in $\ell^\infty([0, 1]^{pq})$ in probability.

Let I_n^* denote the version of I_n computed from the bootstrap sample, and $M_{A,n}^*$, $A \in \mathcal{P}_1$, be the version of $M_{A,n}$, $A \in \mathcal{P}_1$, respectively, computed from the bootstrap sample.

Corollary 1 *Suppose that C has continuous partial derivatives. Then, under serial independence of $\mathbf{X}_1, \mathbf{X}_2, \ldots$, the conditional distribution of the random variable I_n^* given the data, converges to the same limiting distribution as that of I_n in probability, and, the conditional distribution of the random vector $\{M_{A,n}^* : A \in \mathcal{P}_1\}$ given the data, converges to the same limiting distribution as that of $\{M_{A,n} : A \in \mathcal{P}_1\}$ in probability.*

5 Implementation issues

The first section describes the practical computation of the test statistics. The second section, included for completeness, presents the practical implementation of the tests. It mainly consists of transposing the approach proposed in Genest and Rémillard (2004) and used in Beran et al. (2007) to the current context.

5.1 Practical computation of the statistics

In order to increase the speed of the calculations of the statistics I_n and $M_{A,n}$, $A \in \mathcal{P}_1$, it is convenient to first compute the quantity

$$J(R, i, l, j) = \prod_{k=1}^{q} \left[1 - \frac{R_{i+j-1}^{(k)} \vee R_{l+j-1}^{(k)}}{n'} \right], \qquad i, l \in [n], \ j \in [p],$$

that depend on the ranks, and then, for any $i \in [n]$ and any $j \in [p]$, the quantities

$$K(R, i, j) = \frac{1}{n} \sum_{l=1}^{n} \prod_{k=1}^{q} \left[1 - \frac{R_{i+j-1}^{(k)} \vee R_{l+j-1}^{(k)}}{n'} \right] = \frac{1}{n} \sum_{l=1}^{n} J(R, i, l, j),$$

and

$$L(R, j) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{l=1}^{n} \prod_{k=1}^{q} \left[1 - \frac{R_{i+j-1}^{(k)} \vee R_{l+j-1}^{(k)}}{n'} \right] = \frac{1}{n} \sum_{i=1}^{n} K(R, i, j).$$

The statistics are then given by

$$I_n = \frac{1}{n} \sum_{i=1}^{n} \sum_{l=1}^{n} \prod_{j=1}^{p} J(R, i, l, j) - 2 \sum_{i=1}^{n} \prod_{j=1}^{p} K(R, i, j) + n \prod_{j=1}^{p} L(R, j),$$

and

$$M_{A,n} = \frac{1}{n} \sum_{i=1}^{n} \sum_{l=1}^{n} \prod_{j \in A} \left[J(R, i, l, j) - K(R, i, j) - K(R, l, j) + L(R, j) \right], \ A \in \mathcal{P}_1.$$

5.2 Computation of the p-values

Let Q_n stand for I_n or $M_{A,n}$, $A \in \mathcal{P}_1$. An approximate p-value for Q_n can be obtained as follows:

1. Let $Q_{n,0}$ be the value of Q_n computed from the available sample.
2. Generate N random permutations $\sigma_1, \ldots, \sigma_N$ on $[n']$. For any $i \in [N]$, let $Q_{n,i}$ be the value of Q_n obtained from the sample $\mathbf{X}_{\sigma_i(1)}, \ldots, \mathbf{X}_{\sigma_i(n')}$.
3. An approximate p-value for the test statistic is then

$$\frac{1}{N+1}\left\{\frac{1}{2} + \sum_{i=1}^{N} 1[Q_{n,i} \geq Q_{n,0}]\right\}.$$

Note that, as frequently done, the definition of the approximate p-value has been modified in order to obtain estimated values in the open interval (0, 1) so that transformations by inverse c.d.f.s of continuous distributions are always well-defined.

Let us now focus on the statistics $M_{A,n}$, $A \in \mathcal{P}_1$. Under serial independence, the p-values obtained from these statistics are approximately independent uniform on [0, 1]. This led Genest and Rémillard (2004) to consider a global test of randomness based on Fisher's p-value combination method. Additional combination rules were studied in the non-serial case in Genest et al. (2007b). We restrict ourselves to the approaches proposed by Fisher (1932) and Tippett (1931). The corresponding global p-values are obtained as follows:

1. Let $M_{A,n,0}$, $A \in \mathcal{P}_1$, be the statistics computed from the original data.
2. Generate N random permutations on $[n']$ and let $M_{A,n,i}$, $A \in \mathcal{P}_1$, be the statistics computed from the ith randomized sample.
3. An approximate p-value for the statistic $M_{A,n,j}$, $A \in \mathcal{P}_1$, is then

$$\psi(M_{A,n,j}) = \frac{1}{N+1}\left\{\frac{1}{2} + \sum_{i=1}^{N} 1[M_{A,n,i} \geq M_{A,n,j}]\right\}, \quad j \in \{0, 1, \ldots, N\}.$$

   Next, for all $i \in \{0, 1, \ldots, N\}$, compute

$$W_{n,i} = -2 \sum_{A \in \mathcal{P}_1} \log\left[\psi(M_{A,n,i})\right] \quad \text{and} \quad T_{n,i} = \min_{A \in \mathcal{P}_1}\left[\psi(M_{A,n,i})\right]. \tag{8}$$

4. An approximate p-value for the global test *à la* Fisher (resp. *à la* Tippett) is then given by

$$\frac{1}{N}\sum_{i=1}^{N} 1[W_{n,i} \geq W_{n,0}] \qquad \left(\text{resp. } \frac{1}{N}\sum_{i=1}^{N} 1[T_{n,i} \leq T_{n,0}]\right).$$

Less computationally expensive tests *à la* Fisher and *à la* Tippett can be obtained by computing and combining only p-values of statistics $M_{A,n}$, $A \in \mathcal{P}_1$, for $|A| \leq h$, where h is to be fixed in $\{2, \ldots, p-1\}$. The case $h = 2$ appears of particular interest. The corresponding statistics will be denoted by $W_n^{(2)}$ and $T_n^{(2)}$ as we continue. They can be regarded as related to those considered in Skag and Tjøstheim (1993, Sect. 5).

6 Simulations

Extensive simulations were conducted in order to study the performance of the proposed test statistics for sample size $n = 100$ under a large collection of scenarios. The 9 models in Table 5 of Genest et al. (2007a) were generalized to the multivariate case and used to generate the data; see Table 1 for ease of reference. For each type of model, there are 4 factors in the experimental design of the simulation: the dimension q of the data, the embedding dimension p, a serial dependence parameter $\theta \in \mathbb{R}$, and the dependence parameter τ of the copula of the noise given in terms of Kendall's tau. There are three levels for q (1, 2, and 3), three levels for p (2, 4, and 6), three levels for θ (0, 1/8, and 1/4), and two levels for τ (0 and 0.5). Note that Kendall's τ is related to Pearson's correlation coefficient ρ in the bivariate normal case by $\tau = 2\arcsin(\rho)/\pi$. A Kendall's tau of 0.5 corresponds to a Pearson's rho of approximately 0.707.

Table 1 Time-series models used for data generation

Model	Equation
1. AR(1)-Gaussian	$\mathbf{X}_i = \theta\mathbf{X}_{i-1} + \boldsymbol{\varepsilon}_i$, where $\boldsymbol{\varepsilon}_i$ are marginally Gaussian with equicorrelated Gaussian copula
2. AR(1)-Laplace	$\mathbf{X}_i = \theta\mathbf{X}_{i-1} + \boldsymbol{\varepsilon}_i$, where $\boldsymbol{\varepsilon}_i$ are marginally Laplacian with equicorrelated Gaussian copula
3. AR(1)-Cauchy	$\mathbf{X}_i = \theta\mathbf{X}_{i-1} + \boldsymbol{\varepsilon}_i$, where $\boldsymbol{\varepsilon}_i$ are marginally Cauchy with equicorrelated Gaussian copula
4. MA(1)-Gaussian	$\mathbf{X}_i = \boldsymbol{\varepsilon}_i - \theta\boldsymbol{\varepsilon}_{i-1}$, where $\boldsymbol{\varepsilon}_i$ are marginally Gaussian with equicorrelated Gaussian copula
5. GARCH(1,1)-Gaussian	$X_i^{(j)} = \sqrt{h_i^{(j)}}\varepsilon_i^{(j)}$, with $h_i^{(j)} = 1 + \theta h_{i-1}^{(j)} + 2\theta(X_{i-1}^{(j)})^2$, where $\boldsymbol{\varepsilon}_i$ are marginally Gaussian with equicorrelated Gaussian copula
6. ARCH(1)-Gaussian	$X_i^{(j)} = \sqrt{h_i^{(j)}}\varepsilon_i^{(j)}$, with $h_i^{(j)} = 1 + \theta(X_{i-1}^{(j)})^2$, where $\boldsymbol{\varepsilon}_i$ are marginally Gaussian with equicorrelated Gaussian copula
7. TAR(1)-Uniform	$X_i^{(j)} = -\theta X_{i-1}^{(j)}\operatorname{sign}(X_{i-1}^{(j)} - 0.5) + \varepsilon_i^{(j)}$, where $\boldsymbol{\varepsilon}_i$ are from an equicorrelated Gaussian copula
8. (Randomized) Tent Map	$X_i^{(j)} = (1-\eta_i^{(j)})\varepsilon_i^{(j)} + \eta_i^{(j)}(1 - \lvert 2X_{i-1}^{(j)} - 1\rvert)$, where $\boldsymbol{\varepsilon}_i$ are from an equicorrelated Gaussian copula, and $\eta_i^{(j)}$ are i.i.d. Bernoulli(4θ)
9. Clayton Markovian	$X_i^{(j)} = \left(\left(X_{i-1}^{(j)}\right)^{-\theta}\left(\left(\varepsilon_i^{(j)}\right)^{-\theta/(\theta+1)} - 1\right) + 1\right)^{-1/\theta}$ if $\theta > 0$ and $X_i^{(j)} = \varepsilon_i^{(j)}$ if $\theta = 0$, where $\boldsymbol{\varepsilon}_i$ are from an equicorrelated Gaussian copula

As in Genest et al. (2007a), to ensure near-stationarity, time series of length 200 were generated and the final 100 observations were used for each replicate under each alternative model. For Model 8, the deterministic tent map is obtained when $\theta = 1/4$. For this value of θ, numerical rounding seems to result in a series vanishing to zero if the formula in Table 1 is followed. Instead, we use Equation (1) in Chatterjee and Yilmaz (1992): $X_i^{(j)} = X_i^{(j-1)}/w$ for $X_i^{(j-1)} <= w$ and $X_i^{(j)} = (1 - X_i^{(j-1)})/(1 - w)$ otherwise, where $w = 0.5 - eps^{0.75}$ with eps being the smallest positive double precision number in R. For Model 9, when $\theta = 0$, the data are i.i.d. from a Gaussian copula; when $\theta > 0$, the data are serial Markovian in a way such that the current observation and the last observation are from a Clayton copula, and, at the same time, the innovation used in the generation is from a Gaussian copula. To the best of our knowledge, this multivariate time series model has not been studied in the literature and may have applications in other contexts.

The tests under comparison are the global test based on the statistic I_n given in (6), Fisher's and Tippett's combined tests based on the statistics W_n and T_n defined in (8), and two similar tests involving only subsets of cardinality 2 based on the statistics $W_n^{(2)}$ and $T_n^{(2)}$ mentioned at the end of Sect. 5.2. The multivariate portmanteau test (Hosking 1980) as implemented in Johansen (1995, p. 22) is included for comparison. For one-dimensional time series, it coincides with the Ljung and Box (1978) test as implemented in the R function `Box.test`. It will be designated by P as we continue. Note that the powers of the tests based on I_n, W_n and T_n, and those of the tests based on $W_n^{(2)}$ and $T_n^{(2)}$ were computed from different replicates, which explains why they do not exactly coincide when $p = 2$ in the tables to be presented.

When $q = 1$, the studied test statistics essentially reduce to those studied by Genest and Rémillard (2004) and the permutation approach used here is equivalent to their procedure for simulating under the null hypothesis. The simulation results are summarized in Table 2. To facilitate the comparison with those presented in Tables 6 and 7 of Genest et al. (2007a), we included the results of the rank-based version of the Brock, Dechert, and Scheinkman test studied in Genest et al. (2007a) which was computed with the aid of the function `bds.test` provided in the R package `tseries`. The corresponding test statistic is denoted by $\tilde{S}_{n,.3}$, where 0.3 is the approximate standard deviation of a standard uniform variable. Its critical value at the 5% significance level was computed from 1 million observations for sample size $n = 100$. The results for $\tilde{S}_{n,.3}$ agree with those presented in Table 6 of Genest et al. (2007a) (in which Model 7 and 8 are permuted as pointed out by a referee).

As one can see from Table 2, all tests, except maybe the portmanteau test P for Model 3, appear to have maintained their nominal level. Tests based on the statistics I_n, W_n, T_n, $W_n^{(2)}$ and $T_n^{(2)}$ are more powerful than, or as powerful as, the portmanteau test except for Models 1 and 4, for which the latter is expected to be near-optimal. For all models except for $p = 2$ and 4 in Models 5 and 6, the studied tests tend to have higher power than the rank-based version of the Brock, Dechert, and Scheinkman test. It is interesting to restrict one's attention to the conditional heteroskedastic models (Models 5 and 6). Given the form of the models considered in the study, one would rather naturally expect the power of the tests based on W_n and T_n to decrease as the embedding dimension p increases. Indeed, the test statistics $M_{A,n}$, $A \in \mathcal{P}_1$, should

Table 2 Percentage of rejection of alternatives 1 to 9 at the 5% significance level as estimated from 1,000 replicates of series of length $n = 100$: univariate case ($q = 1$)

p	θ	Model 1							Model 2							Model 3						
		P	$\tilde{S}_{n,.3}$	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$	P	$\tilde{S}_{n,.3}$	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$	P	$\tilde{S}_{n,.3}$	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$
2	0	6.0	4.7	4.7	4.7	4.7	5.4	5.4	5.0	5.9	5.3	5.3	5.3	4.8	4.8	2.8	5.0	4.4	4.4	4.4	4.6	4.6
	1/8	20.8	9.2	17.3	17.3	17.3	17.6	17.6	19.8	16.6	23.5	23.5	23.5	24.6	24.6	6.3	53.6	59.5	59.5	59.5	63.0	63.0
	1/4	66.8	34.4	57.7	57.7	57.7	54.9	54.9	66.3	52.0	69.2	69.2	69.2	67.0	67.0	84.2	95.6	97.2	97.2	97.2	96.7	96.7
4	0	4.6	6.0	5.4	6.4	6.4	5.9	5.0	5.1	5.6	5.2	5.3	5.2	6.4	5.7	4.9	5.1	5.5	5.8	5.6	5.3	5.1
	1/8	13.9	8.1	15.2	8.5	8.6	11.5	11.6	12.6	11.7	20.9	12.4	11.2	15.2	15.6	5.6	38.8	50.2	37.9	36.9	40.8	44.7
	1/4	48.2	25.7	48.2	28.2	31.7	36.1	38.9	47.4	40.9	57.5	35.4	43.2	51.1	54.3	28.5	87.6	90.0	82.7	89.6	90.9	93.7
6	0	5.8	6.3	3.6	6.4	6.2	4.2	4.1	4.3	4.7	5.4	5.2	6.3	5.0	4.4	3.8	5.7	4.5	6.3	6.4	4.7	4.4
	1/8	11.1	7.1	16.4	7.4	7.4	10.9	9.2	10.9	8.9	18.2	7.7	8.2	13.0	12.4	6.2	26.8	43.1	30.0	27.8	36.0	39.7
	1/4	43.1	17.9	40.9	14.2	18.8	31.5	35.4	44.0	27.4	50.0	21.4	30.6	39.2	44.8	21.8	71.5	80.9	64.0	77.8	79.9	87.4

p	θ	Model 4							Model 5							Model 6						
		P	$\tilde{S}_{n,.3}$	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$	P	$\tilde{S}_{n,.3}$	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$	P	$\tilde{S}_{n,.3}$	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$
2	0	6.0	4.7	4.7	4.7	4.7	5.4	5.4	6.0	4.7	4.7	4.7	4.7	5.4	5.4	6.0	4.7	4.7	4.7	4.7	5.4	5.4
	1/8	23.1	12.3	18.2	18.2	18.2	22.2	22.2	11.4	20.4	6.1	6.1	6.1	6.6	6.6	7.4	12.1	5.5	5.5	5.5	5.7	5.7
	1/4	70.0	36.4	57.1	57.1	57.1	58.9	58.9	20.3	53.6	10.0	10.0	10.0	8.4	8.4	10.3	19.9	6.5	6.5	6.5	6.9	6.9
4	0	4.6	6.0	5.4	6.4	6.4	5.9	5.0	4.6	6.0	5.4	6.4	6.4	5.9	5.0	4.6	6.0	5.4	6.4	6.4	5.9	5.0
	1/8	15.5	9.9	11.1	11.3	10.1	13.2	12.7	8.4	16.8	5.1	11.5	10.3	6.3	6.5	7.1	9.4	6.9	8.2	6.7	6.4	5.8
	1/4	49.2	27.2	36.8	28.7	34.4	41.6	44.7	22.7	53.2	7.9	29.9	17.1	10.3	7.8	9.4	17.4	6.4	13.2	9.1	5.5	6.7
6	0	5.8	6.3	3.6	6.4	6.2	4.2	4.1	5.8	6.3	3.6	6.4	6.2	4.2	4.1	5.8	6.3	3.6	6.4	6.2	4.2	4.1
	1/8	16.1	8.2	4.2	8.2	8.1	12.6	13.6	7.7	13.3	6.2	28.1	14.8	5.9	4.7	7.4	6.9	6.8	12.8	8.3	6.5	5.9
	1/4	41.1	17.1	10.6	12.6	18.2	33.2	36.6	21.9	43.9	8.3	71.0	49.5	11.3	8.2	7.8	11.4	5.9	24.9	14.7	5.1	5.5

Table 2 continued

p	θ	Model 7							Model 8							Model 9						
		P	$\tilde{S}_{n,.3}$	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$	P	$\tilde{S}_{n,.3}$	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$	P	$\tilde{S}_{n,.3}$	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$
2	0	5.6	4.8	5.5	5.5	5.5	5.0	5.0	5.6	4.8	5.5	5.5	5.5	5.0	5.0	5.6	4.8	5.5	5.5	5.5	5.0	5.0
	1/8	41.8	19.1	36.8	36.8	36.8	35.4	35.4	6.4	89.9	92.1	92.1	92.1	92.5	92.5	11.3	9.3	9.9	9.9	9.9	10.6	10.6
	1/4	88.3	60.7	89.4	89.4	89.4	88.6	88.6	31.6	100.0	100.0	100.0	100.0	100.0	100.0	33.5	21.6	30.9	30.9	30.9	30.3	30.3
4	0	4.8	4.8	3.8	4.9	4.9	3.7	4.6	4.8	4.8	3.8	4.9	4.9	3.7	4.6	4.8	4.8	3.8	4.9	4.9	3.7	4.6
	1/8	29.2	14.3	19.6	18.5	19.0	23.6	26.1	5.4	73.0	39.9	54.8	52.7	66.2	72.7	8.5	8.3	11.7	8.3	7.4	9.2	9.6
	1/4	74.1	45.0	62.9	50.2	66.1	71.4	78.0	25.9	100.0	100.0	100.0	100.0	100.0	100.0	20.5	14.7	24.1	14.3	13.7	20.6	21.1
6	0	6.3	4.5	4.6	5.9	6.5	5.6	5.1	6.3	4.5	4.6	5.9	6.5	5.6	5.1	6.3	4.5	4.6	5.9	6.5	5.6	5.1
	1/8	24.8	10.7	6.5	8.1	10.8	17.2	19.6	6.1	53.1	21.1	28.6	26.9	46.4	59.9	8.5	5.3	12.2	7.7	7.1	9.3	7.9
	1/4	65.2	29.8	25.0	16.8	46.8	54.7	68.9	26.6	100.0	100.0	100.0	100.0	100.0	100.0	21.0	11.7	26.1	11.6	11.5	17.6	18.1

not highlight any strong dependence unless $A = \{1, 2\}$, and the number of subsets $A \neq \{1, 2\}$ increases exponentially with p. For a similar reason, as soon as $p > 2$, one would expect the power of $W_n^{(2)}$ and $T_n^{(2)}$ to be higher than that of W_n and T_n, respectively, for the models considered in Table 1. The results presented in Table 2 agree with these expectations for all but Models 5 and 6. This may be due to the fact that these conditional heteroskedastic models have a longer range of serial dependence than the other models.

Let us now turn to the multivariate case. The additional factor is then the parameter of the equicorrelated Gaussian copula of the innovations given in terms of Kendall's τ. For $q = 2$ and $q = 3$, the simulation results are summarized in Tables 3 and 4, respectively. Since there are two many combinations of the experimental factors, only results for Model 1, 3, 7, and 9 are reported.

From Tables 3 and 4, one can see that the nominal levels of all the tests except the portmanteau are maintained and not affected by τ. On the contrary, the power of the studied tests tend to increase as τ goes from 0 to 0.5, this phenomenon being particularly strong for Model 9. For a given dimension q, as the embedding dimension p increases, the power of the proposed tests appears to decrease. Except for Model 1, the tests based on the truncated statistics $W_n^{(2)}$ and $T_n^{(2)}$ have higher power for $p > 2$ than W_n and T_n, respectively. They perform particularly better for Model 7 with $p = 6$, for which tests based on W_n and I_n have nearly zero power. The portmanteau test performs really well except for Model 3, where its nominal level appears at least to double as soon as $p > 2$. However, as in the univariate case, it deals very badly with Model 8 (results not reported), unlike the studied tests as most of them have nearly 100% power.

As the dimension q goes from 2 to 3, the power of the studied tests does not change much for Models 1, 3, and 9, but drops drastically for Model 7. The benefit of the truncated statistics $W_n^{(2)}$ and $T_n^{(2)}$ is most visible in this case for $p > 2$. As already mentioned, this is probably due to the noise brought by the higher order statistics used in W_n and T_n since the models in Table 1 are all lag-1 dependent. This explains why $W_n^{(2)}$ and $T_n^{(2)}$ give robustly good results. In particular, $T_n^{(2)}$ is frequently among the best for the models under consideration.

Finally, by comparing Table 2 with Tables 3 and 4, it appears that the studied tests are globally more affected by the multivariate character of a time series than the portmanteau test for the models under consideration. The fact that the latter is based on pairwise statistics only suggests to study multivarite tests of randomness based only on pairwise ranked-based measures of dependence.

7 Application

The studied tests were applied to 2007 stock return series of three companies: Apple, Google, and Microsoft. As there were 251 trading days in 2007, each of the return series is of length 250. Table 5 contains the p-values of the portmanteau test (P), the global test (I_n), the combined test *à la* Fisher (W_n), and the combined test *à la* Tippett (T_n) computed from the non-transformed return series. The p-values of the combined

Table 3 Percentage of rejection of alternatives 1, 3, 7, and 9 at the 5% significance level as estimated from 1,000 replicates of series of length $n = 100$: bivariate case ($q = 2$)

p	θ	τ	Model 1						Model 3					
			P	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$	P	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$
2	0	0.0	4.8	4.1	4.1	4.1	5.1	5.1	6.5	4.6	4.6	4.6	5.8	5.8
		0.5	5.0	5.2	5.2	5.2	4.9	4.9	7.3	4.4	4.4	4.4	4.3	4.3
	1/8	0.0	20.7	16.0	16.0	16.0	13.5	13.5	10.8	56.4	56.4	56.4	57.8	57.8
		0.5	19.3	14.9	14.9	14.9	15.8	15.8	12.8	60.1	60.1	60.1	60.7	60.7
	1/4	0.0	77.5	51.1	51.1	51.1	46.0	46.0	80.8	96.1	96.1	96.1	95.5	95.5
		0.5	76.5	53.7	53.7	53.7	54.5	54.5	82.6	97.1	97.1	97.1	96.0	96.0
4	0	0.0	5.6	4.5	3.9	4.4	4.8	4.4	10.2	5.9	5.9	5.5	4.3	5.3
		0.5	5.5	5.9	6.0	4.8	5.8	6.2	9.5	5.6	5.1	5.1	5.6	5.6
	1/8	0.0	13.8	17.4	14.5	11.7	12.2	12.5	13.8	48.6	42.9	37.0	40.5	42.0
		0.5	14.4	18.9	13.1	9.2	10.4	10.9	13.2	50.3	44.6	37.9	42.2	46.7
	1/4	0.0	58.3	45.4	40.7	30.3	33.3	34.8	32.8	85.8	85.8	85.9	86.2	90.3
		0.5	56.2	52.8	37.2	32.4	38.9	40.0	33.8	91.5	87.9	87.6	89.0	91.9
6	0	0.0	5.3	4.9	4.8	5.6	6.4	6.0	12.4	4.9	5.6	7.0	5.1	4.2
		0.5	5.0	4.3	4.4	5.6	4.4	4.4	13.0	4.5	5.7	7.1	5.7	4.6
	1/8	0.0	12.8	14.5	13.7	11.7	9.7	8.7	15.3	34.0	31.0	29.5	32.3	34.6
		0.5	12.9	14.9	13.0	9.6	11.1	11.1	14.5	33.7	36.3	31.2	32.7	37.2
	1/4	0.0	48.6	31.3	28.5	25.2	30.0	27.8	27.4	69.1	64.9	79.1	78.4	88.0
		0.5	51.7	37.8	27.1	24.9	30.1	31.9	29.0	69.7	68.4	79.6	81.4	87.6

p	θ	τ	Model 7						Model 9					
			P	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$	P	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$
2	0	0.0	5.0	3.6	3.6	3.6	5.4	5.4	5.0	3.6	3.6	3.6	5.4	5.4
		0.5	4.5	4.3	4.3	4.3	4.0	4.0	4.5	4.3	4.3	4.3	4.0	4.0
	1/8	0.0	47.7	29.0	29.0	29.0	28.8	28.8	12.7	8.0	8.0	8.0	9.0	9.0
		0.5	51.2	28.6	28.6	28.6	30.3	30.3	10.7	13.9	13.9	13.9	12.9	12.9
	1/4	0.0	95.8	75.1	75.1	75.1	73.6	73.6	41.5	25.7	25.7	25.7	27.0	27.0
		0.5	96.1	74.2	74.2	74.2	73.1	73.1	37.9	38.7	38.7	38.7	39.7	39.7
4	0	0.0	5.2	4.9	5.4	4.6	5.2	5.6	5.2	4.9	5.4	4.6	5.2	5.6
		0.5	6.6	5.7	5.0	5.4	5.0	4.9	6.6	5.7	5.0	5.4	5.0	4.9
	1/8	0.0	34.0	0.7	6.2	13.7	18.6	18.6	11.5	13.9	14.6	9.5	8.0	7.0
		0.5	33.6	3.3	10.3	14.3	20.1	19.5	11.1	19.2	16.3	12.3	10.0	10.0
	1/4	0.0	84.9	2.4	16.0	48.3	50.7	58.3	26.9	26.5	26.1	17.8	17.8	16.9
		0.5	83.3	14.1	26.8	44.7	51.2	55.8	24.4	38.5	32.9	24.2	24.5	24.8
6	0	0.0	6.2	5.3	4.8	5.7	6.3	5.3	6.2	5.3	4.8	5.7	6.3	5.3
		0.5	7.1	4.5	4.4	4.5	5.4	5.4	7.1	4.5	4.4	4.5	5.4	5.4
	1/8	0.0	27.4	0.7	1.1	8.8	14.6	13.8	9.4	10.7	12.6	10.4	7.8	6.8
		0.5	28.5	0.8	3.1	8.9	16.9	15.9	8.8	13.9	16.2	11.6	8.8	8.6
	1/4	0.0	76.7	0.1	1.2	33.3	42.0	50.6	22.3	19.1	23.6	17.5	13.9	13.6
		0.5	75.3	0.2	3.9	28.4	37.5	48.1	21.3	30.8	31.7	23.3	20.8	20.6

Table 4 Percentage of rejection of alternatives 1, 3, 7, and 9 at the 5% significance level as estimated from 1,000 replicates of series of length $n = 100$: trivariate case ($q = 3$)

p	θ	τ	Model 1						Model 3					
			P	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$	P	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$
2	0	0.0	6.8	4.9	4.9	4.9	4.7	4.7	10.7	6.3	6.3	6.3	5.6	5.6
		0.5	4.4	5.3	5.3	5.3	5.1	5.1	10.4	5.0	5.0	5.0	5.5	5.5
	1/8	0.0	21.6	17.2	17.2	17.2	20.2	20.2	15.6	54.6	54.6	54.6	53.0	53.0
		0.5	18.9	17.1	17.1	17.1	19.2	19.2	15.5	60.7	60.7	60.7	59.9	59.9
	1/4	0.0	80.6	47.1	47.1	47.1	49.9	49.9	76.8	93.6	93.6	93.6	92.2	92.2
		0.5	80.3	49.1	49.1	49.1	52.0	52.0	73.2	95.0	95.0	95.0	96.0	96.0
4	0	0.0	6.5	5.4	5.9	6.6	6.1	6.2	14.9	6.0	5.6	4.9	4.8	4.5
		0.5	4.6	5.1	5.2	4.7	4.2	4.8	16.2	5.5	5.5	5.7	5.3	4.8
	1/8	0.0	16.2	14.3	14.6	11.4	10.0	11.4	17.8	32.8	36.2	35.8	38.5	38.0
		0.5	16.3	16.4	14.8	12.4	12.2	11.2	19.7	45.1	47.3	41.7	43.1	45.1
	1/4	0.0	62.6	34.7	38.4	33.3	33.7	34.0	38.1	71.6	79.0	82.3	82.8	87.2
		0.5	64.9	44.6	44.8	35.8	37.2	40.1	37.5	80.0	84.3	83.5	84.7	87.8
6	0	0.0	5.8	5.9	5.9	5.3	4.9	5.0	17.4	3.7	4.1	5.0	5.6	6.0
		0.5	5.3	5.7	4.1	5.2	5.2	5.0	19.4	4.4	6.3	6.7	5.6	4.8
	1/8	0.0	14.0	12.8	12.5	10.8	10.5	11.2	19.0	24.8	27.3	26.8	28.9	35.3
		0.5	14.4	13.9	14.3	12.2	11.4	11.0	21.7	31.0	35.1	33.2	34.1	38.6
	1/4	0.0	57.4	25.5	26.0	24.6	28.8	29.9	37.0	56.4	58.4	72.9	73.8	81.2
		0.5	55.4	30.3	30.0	28.8	28.9	34.3	33.2	58.8	64.7	77.4	79.3	85.1
p	θ	τ	Model 7						Model 9					
			P	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$	P	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$
2	0	0.0	4.1	6.5	6.5	6.5	4.6	4.6	4.1	6.5	6.5	6.5	4.6	4.6
		0.5	6.1	5.8	5.8	5.8	4.9	4.9	6.1	5.8	5.8	5.8	4.9	4.9
	1/8	0.0	49.1	14.7	14.7	14.7	18.5	18.5	12.9	12.9	12.9	12.9	12.3	12.3
		0.5	54.3	23.2	23.2	23.2	21.2	21.2	9.8	16.8	16.8	16.8	17.3	17.3
	1/4	0.0	97.6	55.0	55.0	55.0	49.7	49.7	40.8	24.9	24.9	24.9	27.1	27.1
		0.5	98.5	52.4	52.4	52.4	56.7	56.7	34.8	39.9	39.9	39.9	42.9	42.9
4	0	0.0	5.2	4.5	3.8	4.2	4.5	5.0	5.2	4.5	3.8	4.2	4.5	5.0
		0.5	6.2	5.3	5.7	5.7	4.9	5.3	6.2	5.3	5.7	5.7	4.9	5.3
	1/8	0.0	36.2	1.0	1.6	6.9	12.1	10.9	10.7	11.9	12.9	10.8	9.9	8.5
		0.5	36.1	0.8	2.8	9.0	15.8	14.0	10.6	19.8	22.0	16.3	11.6	12.8
	1/4	0.0	89.4	0.1	1.4	24.8	34.4	33.1	29.0	22.5	24.5	19.2	18.0	18.9
		0.5	87.8	0.0	7.0	30.0	37.4	39.4	28.8	36.0	38.0	28.6	27.1	27.1
6	0	0.0	7.5	4.4	5.0	6.1	3.3	3.5	7.5	4.4	5.0	6.1	3.3	3.5
		0.5	5.8	5.8	4.6	4.9	4.4	4.6	5.8	5.8	4.6	4.9	4.4	4.6
	1/8	0.0	32.3	1.2	1.2	3.9	11.3	9.9	11.1	10.1	11.1	9.2	7.1	7.1
		0.5	33.6	1.4	1.8	6.2	12.3	12.5	11.1	14.2	16.4	13.3	10.3	10.1
	1/4	0.0	84.4	0.3	0.7	15.4	26.8	26.3	26.9	17.2	19.6	16.1	14.8	15.7
		0.5	82.7	0.1	0.6	19.5	28.2	31.2	26.4	27.0	29.5	24.3	22.8	25.9

Table 5 p-values of the test statistics computed from the non-transformed return series

p	P	I_n	W_n	T_n	$W_n^{(2)}$	$T_n^{(2)}$
2	0.396	0.354	0.354	0.354	0.360	0.360
3	0.169	0.061	0.084	0.068	0.422	0.562
4	0.206	0.067	0.019	0.086	0.298	0.436
5	0.412	0.518	0.010	0.017	0.391	0.495
6	0.257	0.353	0.002	0.002	0.535	0.565
7	0.345	0.801	0.002	0.000	0.616	0.599

tests only involving subsets of cardinality 2 ($W_n^{(2)}$ and $T_n^{(2)}$) are also reported. The number of pseudo-random permutations was set to 10,000. The embedding dimension p takes each integer value from 2 to 7. Note that the issue of choosing p is common to all tests that use an embedding dimension and that there does not seem to be any statistical inference procedures for choosing p available in the literature.

It is a stylized fact of financial times series that series of absolute or squared returns exhibit serial correlation. For the data under consideration, the portmanteau test does indeed highlight strong evidence against serial independence for the transformed series whose components have been squared: the p-values corresponding to $p = 2, \ldots, 7$ are all below 0.001. As can be seen from Table 5, when applied to the non-transformed series, the portmanteau test does not detect any serial dependence for these values of p. In fact, the first embedding dimension that leads to a p-value of the portmanteau test below 5% is 11. As we have seen in the previous section, another inconvenience of the portmanteau test is that it may not maintain its nominal level for heavy-tailed distributions. The sample excess kurtoses for the three return series are 1.96, 1.50, and 5.71, respectively. At the 5% level, the global test I_n does not detect any serial dependence either, although weak evidence against independence seems to appear for $p = 3$ and 4. The combined tests based on the Möbius decomposition yield quite small p-values at embedding dimension 4 and higher. The combined tests truncated at cardinality 2, however, report no evidence of serial dependence for the three series.

In order to exploit the Möbius decomposition more in depth, Genest and Rémillard (2004) proposed a graphical representation of the values of the observed test statistics called a *dependogram*: for each subset $A \in \mathcal{P}_1$, a vertical bar is drawn whose height represents the value of $M_{A,n}$. The corresponding approximate critical values (at the corrected significance level $1-(1-\alpha)^{1/(2^{p-1}-1)}$) are represented on the bars by black bullets. Subsets such that the bar exceeds the critical value can be considered as being composed of dependent vectors.

Figure 1 shows the sample dependogram of asymptotic global level $\alpha = 5\%$ for embedding dimension $p = 6$. As the statistics for subsets $\{1, 2\}, \ldots, \{1, 6\}$ are not significant, for better illustration, a zoom was performed in the plot. The rejection of serial independence by W_n and T_n appears to be essentially due to subsets $\{1, 3, 6\}$, $\{1, 2, 3, 6\}$, and $\{1, 2, 3, 4, 6\}$. The fact that the statistics for subsets $\{1, 2\}, \ldots, \{1, 6\}$ are not significant, while, for instance, that for subset $\{1, 3, 6\}$ is significant, suggests the presence of complex serial dependence involving lags 2 and 5.

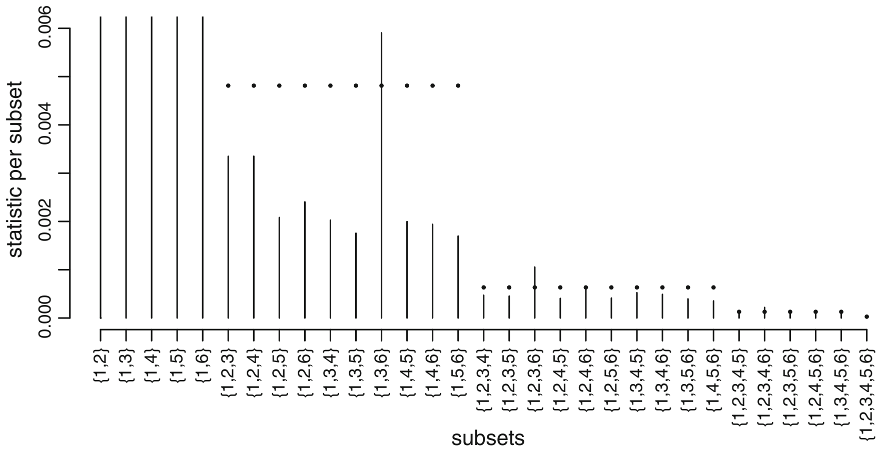

Fig. 1 Serial dependogram of asymptotic global level $\alpha = 5\%$ for the three stock return series with embedding dimension $p = 6$

A Proofs

A.1 Additional notation and definitions

Let $F^{(1)}, \dots, F^{(q)}$ denote the marginal c.d.f.s obtained from F and let $\mathbf{U}_i = (U_i^{(1)}, \dots, U_i^{(q)})$, $i \in [n']$, be the continuous random vectors obtained by applying the probability-integral transformation $U_i^{(j)} = F^{(j)}(X_i^{(j)})$ for all $i \in [n']$ and all $j \in [q]$. It follows that the vectors $\mathbf{U}_i$, $i \in [n']$, have standard uniform marginals and that each has as c.d.f. the copula C. Now, form the random vectors $\mathbf{V}_i = (\mathbf{U}_i, \dots, \mathbf{U}_{i+p-1})$, $i \in [n]$, and let H_n be the empirical c.d.f. computed from $\mathbf{V}_1, \dots, \mathbf{V}_n$, that is,

$$
\begin{aligned}
H_n(\mathbf{u}) &= \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{p} 1[\mathbf{U}_{i+j-1} \le \mathbf{u}_{\langle j \rangle}] \\
&= \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{p} \prod_{k=1}^{q} 1\Big[F^{(k)}(X_{i+j-1}^{(k)}) \le u_{\langle j \rangle}^{(k)}\Big], \quad \mathbf{u} \in [0,1]^{pq}. \qquad (9)
\end{aligned}
$$

Also, for any c.d.f. $G : \mathbb{R} \to [0,1]$, define its generalized inverse by $G^{-}(u) = \inf\{x \in \mathbb{R} : G(x) \ge u\}$. Finally, for any integer $d \ge 1$, let $D([0,1]^d)$ (resp. $\mathcal{C}([0,1]^d)$) be the space of *càdlàg* (resp. continuous) functions on $[0,1]^d$ equipped with the Skorohod (resp. uniform) topology.

A.2 Proof of Theorem 1

In the spirit of Genest and Rémillard (2004, Proposition 2.1), in order to prove Theorem 1, we study the empirical process

$$\mathbb{H}_n(\mathbf{u}) = \sqrt{n}\left[H_n(\mathbf{u}) - \prod_{j=1}^{p} C(\mathbf{u}_{\langle j\rangle})\right], \quad \mathbf{u} \in [0,1]^{pq}, \tag{10}$$

and its Möbius decomposition through the processes

$$\mathbb{M}_{A,n}(\mathbf{u}) = \sum_{B\subseteq A} (-1)^{|A|-|B|}\mathbb{H}_n(\mathbf{u}_B) \prod_{j\in A\setminus B} C(\mathbf{u}_{\langle j\rangle}), \quad \mathbf{u} \in [0,1]^{pq}, \ A \in \mathcal{P}. \tag{11}$$

Notice that, as shown for instance in Ghoudi et al. (2001), there holds

$$\sum_{A\in\mathcal{P}} \mathbb{M}_{A,n}(\mathbf{u}) \prod_{j\in[p]\setminus A} C(\mathbf{u}_{\langle j\rangle}) = \mathbb{H}_n(\mathbf{u}), \quad \mathbf{u} \in [0,1]^{pq}. \tag{12}$$

We can then state the following result, which is a straightforward extension of Theorem 2.2 in Ghoudi et al. (2001) (see also Theorem 3 in Beran et al. 2007) to the multivariate time series setting.

Lemma 1 *Under mutual independence of* $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_{n'}$*, the vector of processes* $\{\mathbb{M}_{A,n}(\mathbf{u}), \mathbf{u} \in [0,1]^{pq} : A \in \mathcal{P}\}$ *converges weakly in* $D([0,1]^{pq})$ *to a vector of tight centered Gaussian processes* $\{\mathbb{M}_A(\mathbf{u}), \mathbf{u} \in [0,1]^{pq} : A \in \mathcal{P}\}$ *having cross-covariance function*

$$E[\mathbb{M}_A(\mathbf{u}), \mathbb{M}_B(\mathbf{v})] = \begin{cases} \prod_{j\in A}\left[C(\mathbf{u}_{\langle j\rangle} \wedge \mathbf{v}_{\langle j+\delta\rangle}) - C(\mathbf{u}_{\langle j\rangle})C(\mathbf{v}_{\langle j+\delta\rangle})\right], & \text{if } B = A+\delta,\\ 0, & \text{otherwise,}\end{cases}$$

where $A, B \subseteq [p]$, $|A| > 1$, $|B| > 1$, *and* $\mathbf{u}, \mathbf{v} \in [0,1]^{pq}$.

The following lemma is the analogue of Corollary 2.1 given in Ghoudi et al. (2001) which seems to contain a minor error. It can also be regarded as an extension of Lemma 1 in Delgado (1996). We provide a proof as no proof was given in Ghoudi et al. (2001).

Lemma 2 *Under mutual independence of* $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_{n'}$*, the process* $\mathbb{H}_n(\mathbf{u})$, $\mathbf{u} \in [0,1]^{pq}$*, converges weakly in* $D([0,1]^{pq})$ *to the tight centered Gaussian process*

$$\mathbb{H}(\mathbf{u}) = \sum_{A\in\mathcal{P}} \mathbb{M}_A(\mathbf{u}) \prod_{j\in[p]\setminus A} C(\mathbf{u}_{\langle j\rangle}), \qquad \mathbf{u} \in [0,1]^{pq}, \tag{13}$$

with covariance function $E[\mathbb{H}(\mathbf{u})\mathbb{H}(\mathbf{v})] = \gamma_0(\mathbf{u},\mathbf{v}) + \sum_{\delta=1}^{p-2}\left[\gamma_\delta(\mathbf{u},\mathbf{v}) + \gamma_\delta(\mathbf{v},\mathbf{u})\right]$*, where* γ_δ *is defined by* (4).

Proof The convergence follows from Lemma 1 and the continuous mapping theorem since, as can be seen from (12), the map that transforms the Möbius decomposition $\{\mathbb{M}_{A,n}(\mathbf{u}), u \in [0,1]^{pq} : A \in \mathcal{P}\}$ into $\mathbb{H}_n(\mathbf{u})$, $\mathbf{u} \in [0,1]^{pq}$, is continuous. Let us now prove the second claim. Let $\mathbf{u}, \mathbf{v} \in [0,1]^{pq}$. We have

$$E[\mathbb{H}(\mathbf{u})\mathbb{H}(\mathbf{v})] = \sum_{A\in\mathcal{P}}\sum_{A'\in\mathcal{P}} E[\mathbb{M}_A(\mathbf{u})\mathbb{M}_{A'}(\mathbf{v})] \prod_{j\in[p]\setminus A} C(\mathbf{u}_{\langle j\rangle}) \prod_{j\in[p]\setminus A'} C(\mathbf{v}_{\langle j\rangle}).$$

From Lemma 1, we know that $E[\mathbb{M}_A(\mathbf{u})\mathbb{M}_{A'}(\mathbf{v})]$ is non-zero if and only if A' is a δ-translate of A. When $A \in \mathcal{P}$, possible values for δ are $-p+2, -p+3, \dots, p-2$. For $0 < \delta \leq p-2$, having $A+\delta \subseteq [p]$ is equivalent to having $A \subseteq [p-\delta]$, and, having $A-\delta \subseteq [p]$ is equivalent to having $A \subseteq [p]\setminus[\delta]$. Hence, the covariance function can be rewritten as

$$\begin{aligned} E[\mathbb{H}(\mathbf{u})\mathbb{H}(\mathbf{v})] = & \sum_{\delta=1}^{p-2} \sum_{\substack{A\subseteq[p-\delta]\\|A|>1}} E[\mathbb{M}_A(\mathbf{u})\mathbb{M}_{A+\delta}(\mathbf{v})] \prod_{j\in[p]\setminus A} C(\mathbf{u}_{\langle j\rangle}) \prod_{j\in[p]\setminus(A+\delta)} C(\mathbf{v}_{\langle j\rangle}) \\ & + \sum_{\substack{A\subseteq[p]\\|A|>1}} E[\mathbb{M}_A(\mathbf{u})\mathbb{M}_A(\mathbf{v})] \prod_{j\in[p]\setminus A} C(\mathbf{u}_{\langle j\rangle})C(\mathbf{v}_{\langle j\rangle}) \\ & + \sum_{\delta=1}^{p-2} \sum_{\substack{A\subseteq[p]\setminus[\delta]\\|A|>1}} E[\mathbb{M}_A(\mathbf{u})\mathbb{M}_{A-\delta}(\mathbf{v})] \prod_{j\in[p]\setminus A} C(\mathbf{u}_{\langle j\rangle}) \prod_{j\in[p]\setminus(A-\delta)} C(\mathbf{v}_{\langle j\rangle}). \end{aligned}$$

Now, let $\gamma_\delta(\mathbf{u},\mathbf{v}) = \sum_{\substack{A\subseteq[p-\delta]\\|A|>1}} E[\mathbb{M}_A(\mathbf{u})\mathbb{M}_{A+\delta}(\mathbf{v})] \prod_{j\in[p]\setminus A} C(\mathbf{u}_{\langle j\rangle}) \prod_{j\in[p]\setminus(A+\delta)} C(\mathbf{v}_{\langle j\rangle})$, and notice that the covariance can be simply rewritten as $E[\mathbb{H}(\mathbf{u})\mathbb{H}(\mathbf{v})] = \sum_{\delta=1}^{p-2}\gamma_\delta(\mathbf{u},\mathbf{v}) + \gamma_0(\mathbf{u},\mathbf{v}) + \sum_{\delta=1}^{p-2}\gamma_\delta(\mathbf{v},\mathbf{u})$. Next, from Lemma 1, we have that $\gamma_\delta(\mathbf{u},\mathbf{v})$ is equal to

$$\sum_{\substack{A\subseteq[p-\delta]\\|A|>1}} \prod_{j\in A} [C(\mathbf{u}_{\langle j\rangle} \wedge \mathbf{v}_{\langle j+\delta\rangle}) - C(\mathbf{u}_{\langle j\rangle})C(\mathbf{v}_{\langle j+\delta\rangle})] \prod_{j\in[p]\setminus A} C(\mathbf{u}_{\langle j\rangle}) \prod_{j\in[p]\setminus(A+\delta)} C(\mathbf{v}_{\langle j\rangle}).$$

Notice that $\prod_{j\in[p]\setminus A} C(\mathbf{u}_{\langle j\rangle}) = \prod_{j\in[p]\setminus[p-\delta]} C(\mathbf{u}_{\langle j\rangle}) \prod_{j\in[p-\delta]\setminus A} C(\mathbf{u}_{\langle j\rangle})$. Similarly, we have $\prod_{j\in[p]\setminus(A+\delta)} C(\mathbf{v}_{\langle j\rangle}) = \prod_{j\in[\delta]} C(\mathbf{v}_{\langle j\rangle}) \prod_{j\in([p]\setminus[\delta])\setminus(A+\delta)} C(\mathbf{v}_{\langle j\rangle})$, which is equal to $\prod_{j\in[\delta]} C(\mathbf{v}_{\langle j\rangle}) \prod_{j\in[p-\delta]\setminus A} C(\mathbf{v}_{\langle j+\delta\rangle})$. The result then follows after simplification. □

Before finally giving the proof of Theorem 1, we state a last lemma due to Fermanian et al. (2004, p 849) (see also Lemma 3.9.28 in van der Vaart and Wellner 1996).

Lemma 3 *Let G be a c.d.f. with compact support on $[0,1]^d$ and marginal c.d.f.s $G^{(1)},\dots,G^{(d)}$ that are continuously differentiable on $[0,1]$ with strictly positive densities. Furthermore, assume that G is continuously differentiable on $[0,1]^d$. Then, the map $\phi : D([0,1]^d) \to \ell^\infty([0,1]^d)$ defined by*

$$\phi(G)(\mathbf{u}) = G(G^{(1)-}(u^{(1)}),\dots,G^{(d)-}(u^{(d)})), \qquad \mathbf{u} \in [0,1]^d, \tag{14}$$

is Hadamard differentiable tangentially to $\mathcal{C}([0,1]^d)$, and its derivative at G is

$$\phi'_G(f)(\mathbf{u}) = f\left(G^{(1)-}(u^{(1)}), \ldots, G^{(d)-}(u^{(d)})\right) - \sum_{i=1}^{d} \partial_i G\left(G^{(1)-}(u^{(1)}), \ldots, G^{(d)-}(u^{(d)})\right) \times \frac{f(1, \ldots, 1, G^{(i)-}(u^{(i)}), 1 \ldots, 1)}{g^{(i)}(G^{(i)-}(u^{(i)}))},$$

where $g^{(1)}, \ldots, g^{(d)}$ *are the probability density functions corresponding to* $G^{(1)}, \ldots,$ $G^{(d)}$ *respectively.*

Proof of Theorem 1 Let $H_n^{(j,k)}$, $j \in [p]$, $k \in [q]$, be the marginal c.d.f.s obtained from H_n. First, notice that, under serial independence of $\mathbf{X}_1, \mathbf{X}_2, \ldots$, the $H_n^{(j,k)}$ are defined from i.i.d. samples. The empirical quantile processes obtained from the corresponding generalized inverses are then known to converge to well-behaved Gaussian processes (see e.g. van der Vaart and Wellner 1996, Sect. 3.9.4.2). Next, observe that the c.d.f. $\otimes^p C$ satisfies the conditions of Lemma 3. Then, invoke the functional delta method (van der Vaart and Wellner 1996, Theorems 3.9.4) with the Hadamard differentiable map (14) applied to Lemma 2. It follows that $\sqrt{n}\left[\phi(H_n)(\mathbf{u}) - \phi(\otimes^p C)(\mathbf{u})\right]$, $\mathbf{u} \in [0,1]^{pq}$, converges weakly in $\ell^\infty([0,1]^{pq})$ to the process $\mathbb{C}(\mathbf{u})$, $\mathbf{u} \in [0,1]^{pq}$, defined in (3). Moreover, it is easy to see from (14) that $\phi(\otimes^p C) = \otimes^p C$. The result then follows from the fact that, almost surely,

$$\sup_{\mathbf{u}\in[0,1]^{pq}} \left|\phi(H_n)(\mathbf{u}) - C_n^s(\mathbf{u})\right| \leq O(1/n). \tag{15}$$

Indeed, as shown in Fermanian et al. (2004, Lemma 1 and p 854), there holds

$$\sup_{\mathbf{u}\in[0,1]^{pq}} \left|\phi(H_n)(\mathbf{u}) - \frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{p}\prod_{k=1}^{q} 1\left[G_n^{(j,k)}(X_{i+j-1}^{(k)}) \leq u_{\langle j\rangle}^{(k)}\right]\right| \leq O(1/n),$$

where G_n is the empirical c.d.f. computed from the $\mathbf{Y}_i = (\mathbf{X}_i, \ldots, \mathbf{X}_{i+p-1})$, $i \in [n]$, and where $G_n^{(j,k)}$, $j \in [p]$, $k \in [q]$, are the corresponding marginals. Inequality (15) then follows from the fact that, clearly, for any $j \in [p]$ and any $k \in [q]$, $\sup_{\mathbf{u}\in[0,1]^{pq}} \left|G_n^{(j,k)}(\mathbf{u}) - F_n^{(k)}(\mathbf{u})\right| \leq O(1/n)$ almost surely.

A.3 Proof of Theorem 2

Proof From Theorem 1 and the application of the functional delta method with the Hadamard differentiable map $\mathcal{I}$, we have that the empirical process $\sqrt{n}\mathcal{I}(C_n^s)(\mathbf{u})$, $\mathbf{u} \in [0,1]^{pq}$ converges weakly in $\ell^\infty([0,1]^{pq})$ to the tight centered Gaussian process

$$\mathcal{I}'_{\otimes^p C}(\mathbb{C})(\mathbf{u}) = \mathbb{C}(\mathbf{u}) - \sum_{k=1}^{p} \mathbb{C}(\mathbf{u}_{\{k\}}) \prod_{\substack{j=1\\ j\neq k}}^{p} C(\mathbf{u}_{\langle j\rangle}), \qquad \mathbf{u} \in [0,1]^{pq}.$$

The first claim then follows from the expression of $\mathbb{C}$ given in (3). Let us now prove the second claim. Starting from the expression of the limiting process, for any $\mathbf{u}, \mathbf{v} \in [0,1]^{pq}$, $E[\mathcal{I}'_{\otimes^p C}(\mathbb{C})(\mathbf{u})\mathcal{I}'_{\otimes^p C}(\mathbb{C})(\mathbf{v})]$ is equal to

$$E[\mathbb{H}(\mathbf{u})\mathbb{H}(\mathbf{v})] - \sum_{l=1}^{p} E[\mathbb{H}(\mathbf{u})\mathbb{H}(\mathbf{v}_{\langle l\rangle})] \prod_{\substack{m=1\\ m\neq l}}^{p} C(\mathbf{v}_{\langle m\rangle}) - \sum_{k=1}^{p} E[\mathbb{H}(\mathbf{v})\mathbb{H}(\mathbf{u}_{\langle k\rangle})] \prod_{\substack{j=1\\ j\neq k}}^{p} C(\mathbf{u}_{\langle j\rangle})$$
$$+ \sum_{k=1}^{p}\sum_{l=1}^{p} E[\mathbb{H}(\mathbf{u}_{\langle k\rangle})\mathbb{H}(\mathbf{v}_{\langle l\rangle})] \prod_{\substack{j=1\\ j\neq k}}^{p} C(\mathbf{u}_{\langle j\rangle}) \prod_{\substack{m=1\\ m\neq l}}^{p} C(\mathbf{v}_{\langle m\rangle}).$$

The result then follows from the fact that only the first term is non-zero since, as can be checked from (4), for any $l \in [p]$, $\gamma_\delta(\mathbf{u}, \mathbf{v}_{\langle l\rangle}) = \gamma_\delta(\mathbf{u}_{\langle l\rangle}, \mathbf{v}) = 0$ for all $0 \le \delta \le p-2$ and all $\mathbf{u}, \mathbf{v} \in [0,1]^{pq}$. □

A.4 Proof of Theorem 3

Proof Let $\boldsymbol{\mathcal{M}} : \ell^\infty([0,1]^{pq}) \to (\ell^\infty([0,1]^{pq}))^{2^p-p-1}$ denote the (Hadamard differentiable) map whose $2^p - p - 1$ components are the maps $\mathcal{M}_A$, $A \in \mathcal{P}$. From Theorem 1 and the application of the functional delta method with the Hadamard differentiable map $\boldsymbol{\mathcal{M}}$, we obtain that $\sqrt{n}\boldsymbol{\mathcal{M}}(C_n^s)(\mathbf{u})$, $\mathbf{u} \in [0,1]^{pq}$ converges weakly in $(\ell^\infty([0,1]^{pq}))^{2^p-p-1}$ to $\boldsymbol{\mathcal{M}}'_{\otimes^p C}(\mathbb{C})(\mathbf{u})$, whose corresponding components are defined by

$$\mathcal{M}'_{A,\otimes^p C}(\mathbb{C})(\mathbf{u}) = \sum_{B\subseteq A} (-1)^{|A|-|B|} \left[\otimes^p C(\mathbf{u}_B) \sum_{k\in A\setminus B} \mathbb{C}(\mathbf{u}_{\{k\}}) \prod_{\substack{i\in A\setminus B\\ i\neq k}} C(\mathbf{u}_{\langle i\rangle}) \right.$$
$$\left. + \mathbb{C}(\mathbf{u}_B) \prod_{k\in A\setminus B} C(\mathbf{u}_{\langle k\rangle}) \right].$$

Then, from the expression of $\mathbb{C}$ given in (3), we get

$$\sum_{B\subseteq A} (-1)^{|A|-|B|} \left[\sum_{k\in A\setminus B} \left\{ \mathbb{H}(\mathbf{u}_{\{k\}}) - \sum_{i=1}^{q} \partial_i C(\mathbf{u}_{\langle k\rangle}) \mathbb{H}(1,\ldots,1,u_{\langle k\rangle}^{(i)},1,\ldots,1) \right\} \prod_{\substack{i\in A\\ i\neq k}} C(\mathbf{u}_{\langle i\rangle}) \right.$$
$$\left. + \mathbb{H}(\mathbf{u}_B) \prod_{k\in A\setminus B} C(\mathbf{u}_{\langle k\rangle}) - \sum_{k\in B}\sum_{i=1}^{q} \partial_i C(\mathbf{u}_{\langle k\rangle}) \mathbb{H}(1,\ldots,1,u_{\langle k\rangle}^{(i)},1,\ldots,1) \prod_{\substack{i\in A\\ i\neq k}} C(\mathbf{u}_{\langle i\rangle}) \right].$$

The first claim then follows after simplification. Let us now prove the second claim. From the expression of the limiting process and using (13) (and, in particular, the "inverse" relation, analog to (11)), we see that, for any $A \in \mathcal{P}$, the processes $\mathcal{M}'_{A,\otimes^p C}(\mathbb{C})$ and $\mathbb{M}_A$ have the same expression in terms of the process $\mathbb{H}$ and are therefore identical. It follows that they have the same cross-covariance function, which is given in Lemma 1. ◻

A.5 Proof of Theorem 4

Recall that $\mathbf{U}_i = (U_i^{(1)}, \ldots, U_i^{(q)})$, $i \in [n']$, are the continuous random vectors obtained from the available data by $U_i^{(j)} = F^{(j)}(X_i^{(j)})$ for all $i \in [n']$ and all $j \in [q]$. Similarly, let $\mathbf{U}_1^*, \ldots, \mathbf{U}_{n'}^*$ be the sample obtained from the bootstrap sample by the same probability-integral transformations. As previously, form the random vectors $\mathbf{V}_i = (\mathbf{U}_i, \ldots, \mathbf{U}_{i+p-1})$, $i \in [n]$, and the vectors $\mathbf{V}_i^* = (\mathbf{U}_i^*, \ldots, \mathbf{U}_{i+p-1}^*)$, $i \in [n]$. Furthermore, let H_n (resp. H_n^*) be the empirical c.d.f. computed from $\mathbf{V}_1, \ldots, \mathbf{V}_n$ (resp. $\mathbf{V}_1^*, \ldots, \mathbf{V}_n^*$). Now, by analogy with (10) and (11), define the empirical process $\mathbb{H}_n^*(\mathbf{u}) = \sqrt{n}\left[H_n^*(\mathbf{u}) - \prod_{j=1}^p H_n(\mathbf{u}_{\{j\}})\right]$, $\mathbf{u} \in [0,1]^{pq}$, and its Möbius decomposition through the processes

$$\mathbb{M}_{A,n}^*(\mathbf{u}) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \mathbb{H}_n^*(\mathbf{u}_B) \prod_{j \in A \setminus B} H_n(\mathbf{u}_{\{j\}}), \quad \mathbf{u} \in [0,1]^{pq}, \ \ A \in \mathcal{P}.$$

Lemma 4 *Under serial independence of* $\mathbf{X}_1, \mathbf{X}_2, \ldots$, *the conditional distribution of the vector of* $2^p - p - 1$ *processes* $\{\mathbb{M}_{A,n}^*(\mathbf{u}), \mathbf{u} \in [0,1]^{pq} : A \in \mathcal{P}\}$ *given the data, converges to the same limiting distribution as that of* $\{\mathbb{M}_{A,n}(\mathbf{u}), \mathbf{u} \in [0,1]^{pq} : A \in \mathcal{P}\}$ *in* $D([0,1]^{pq})$ *in probability.*

Proof First, define the slightly modified process $\tilde{\mathbb{H}}_n^*(\mathbf{u}) = \sqrt{n}\big[H_n^*(\mathbf{u}) - \prod_{j=1}^p G_n(\mathbf{u}_{\langle j \rangle})\big]$, $\mathbf{u} \in [0,1]^{pq}$, and its Möbius decomposition

$$\tilde{\mathbb{M}}_{A,n}^*(\mathbf{u}) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \mathbb{H}_n^*(\mathbf{u}_B) \prod_{j \in A \setminus B} G_n(\mathbf{u}_{\langle j \rangle}), \qquad \mathbf{u} \in [0,1]^{pq}, \ \ A \in \mathcal{P}, \tag{16}$$

where G_n is the empirical c.d.f. computed from $\mathbf{U}_1, \ldots, \mathbf{U}_{n'}$. Due to the fact that, clearly, for any $j \in [p]$, $\sup_{\mathbf{u} \in [0,1]^{pq}} |G_n(\mathbf{u}_{\langle j \rangle}) - H_n(\mathbf{u}_{\{j\}})| \le O(1/n)$ almost surely, the processes $\tilde{\mathbb{H}}_n^*$ and $\mathbb{H}_n^*$ are asymptotically equivalent. The same holds for the processes $\tilde{\mathbb{M}}_{A,n}^*$ and $\mathbb{M}_{A,n}^*$, $A \in \mathcal{P}$. Asymptotic normality of the marginals can then be proved by proceeding exactly as in the proof of Theorem 5 of Beran et al. (2007).

Asymptotic tightness is established by means of the representation used initially in Ghoudi et al. (2001, Theorems 2.1 and 2.2).As in Beran et al. (2007), assume with-out loss of generality that there exists an integer r such that $n = rp$. Then, starting from (16), for any $A \in \mathcal{P}$ and any $\mathbf{u} \in [0,1]^{pq}$, $\tilde{\mathbb{M}}^*_{A,n}(\mathbf{u})$ can be written as

$$\sum_{B \subseteq A} (-1)^{|A|-|B|} \sqrt{rp} \left[\frac{1}{rp} \sum_{m=0}^{r-1} \sum_{l=1}^{p} \prod_{j \in B} 1\left[\mathbf{U}^*_{mp+l+j-1} \right.\right.$$

$$\left.\left. \leq \mathbf{u}_{\langle j \rangle}\right] - \prod_{j \in B} G_n(\mathbf{u}_{\langle j \rangle}) \right] \prod_{j \in A \setminus B} G_n(\mathbf{u}_{\langle j \rangle})$$

$$= \frac{1}{\sqrt{p}} \sum_{l=1}^{p} \sum_{B \subseteq A} (-1)^{|A|-|B|} \prod_{j \in A \setminus B} G_n(\mathbf{u}_{\langle j \rangle}) \frac{1}{\sqrt{r}} \sum_{m=0}^{r-1}$$

$$\times \left[\prod_{j \in B} 1[\mathbf{U}^*_{mp+l+j-1} \leq \mathbf{u}_{\langle j \rangle}] - \prod_{j \in B} G_n(\mathbf{u}_{\langle j \rangle}) \right]. \tag{17}$$

Now, fix $l \in [p]$ and $B \subseteq A$, and let $B = \{j_1, \ldots, j_{|B|}\}$. First, notice that the $|B| \times q$-dimensional random vectors $(\mathbf{U}^*_{mp+l+j_1-1}, \ldots, \mathbf{U}^*_{mp+l+j_{|B|}-1})$, $m \in \{0, \ldots, r-1\}$, are not only independent (since $|B| \leq p$ necessarily) but can be regarded as a random sample from the empirical c.d.f. $\prod_{j \in B} G_n(\mathbf{u}_{\langle j \rangle})$, $\mathbf{u} \in [0,1]^{pq}$. Then, as in the proof of Theorem 5 of Beran et al. (2007), it follows from Lemma 2.8.8 in van der Vaart and Wellner (1996) that the last term between brackets in (17) converges weakly and is therefore asymptotically tight. The asymptotic tightness of $\tilde{\mathbb{M}}^*_{A,n}$ then follows from the fact that the term $\prod_{j \in A \setminus B} G_n(\mathbf{u}_{\langle j \rangle})$ converges and the fact that there is a finite number of l's and B's. $\square$

Corollary 2 *Under serial independence of $\mathbf{X}_1, \mathbf{X}_2, \ldots$, the conditional distribution of the process $\mathbb{H}^*_n(\mathbf{u})$, $\mathbf{u} \in [0,1]^{pq}$, given the data, converges to the same limiting distribution as that of $\mathbb{H}_n(\mathbf{u})$, $\mathbf{u} \in [0,1]^{pq}$ in $D([0,1]^{pq})$ in probability.*

Proof of Theorem 4 Observe that $\otimes^p C$ satisfies the conditions of Lemma 3. Next, invoke the functional delta method (van der Vaart and Wellner 1996, Theorem 3.9.4) and the functional delta method for the bootstrap (Kosorok 2008, Theorem 12.1) with the Hadamard differentiable map (14) applied to the previous corollary. Then, the conditional distribution of $\sqrt{n}\big[\phi(H^*_n)(\mathbf{u}) - \prod_{j=1}^{p} \phi(H_n)(\mathbf{u}_{\{j\}})\big]$, $\mathbf{u} \in [0,1]^{pq}$, given the data, converges to the same limiting distribution as that of the process $\sqrt{n}\big[\phi(H_n)(\mathbf{u}) - \prod_{j=1}^{p} C(\mathbf{u}_{\langle j \rangle})\big]$, $\mathbf{u} \in [0,1]^{pq}$ in $\ell^\infty([0,1]^{pq})$ in probability. The result follows from (15) and a similar relation linking $\phi(H^*_n)$ and C^{s*}_n. $\square$

Acknowledgments The authors are grateful to the Associate Editor and two referees whose comments helped in improving the quality of this work.

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