Ann Inst Stat Math (2011) 63:267–289
DOI 10.1007/s10463-009-0230-8

Asymptotic properties of conditional quantile estimator for censored dependent observations

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Received: 12 May 2008 / Revised: 21 January 2009 / Published online: 6 May 2009

Abstract In this paper, we establish strong uniform convergence and asymptotic normality of the conditional quantile estimator for the censorship model when the data exhibit some kind of dependence. It is assumed that the observations form a stationary α-mixing sequence. The strong uniform convergence in iid framework has recently been discussed by Ould-Saïd (Stat Probab Lett 76:579–586, 2006). As a by-product, we also obtain a uniform weak convergence rate for the product-limit estimator of the lifetime and censoring distributions under dependence, which is interesting independently.

Keywords Strong uniform convergence · Asymptotic normality · Censored data · α-mixing sequence · Conditional quantile estimator

1 Introduction

Censored dependent data appear in a number of applications. For example, as reported by Kang and Koehler (1997), in survival analysis a cohort may consist in a set of failure or censoring times corresponding to a repeated procedure performed on a single patient. The example in that paper refers to an angioplasty procedure for removing obstructions from blood vessels in arms and legs. A different example on the analysis of viral marker reaction times in repeated blood samples is described in

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Wei et al. (1989). In the context of censored time series analysis, Shumway et al. (1988) considered (hourly or daily) measurements of the concentration of a given substance subject to some detection limits, thus being potentially censored from the right. Other applications include toxicological and agricultural studies (Koehler and Symanowski 1995; Koehler 1995).

So with such type of applications in mind, let $T_1, T_2, \ldots$ and $C_1, C_2, \ldots$ be two independent sequences of nonnegative random variables corresponding respectively to survival (or failure) times having continuous distribution function (df) F, and to censoring times with continuous df G; and where both the T_i and the C_i are expected to exhibit some kind of dependence. Let $Y_i = \min(T_i, C_i) = T_i \wedge C_i$ and $\delta_i = I(T_i \le C_i)$, where $I(\cdot)$ denotes the indicator function. In the censored setup, one only observes $(Y_1, \delta_1), \ldots, (Y_n, \delta_n)$. The Kaplan–Meier estimators of the distribution functions F and G are defined by, respectively

$$F_n(x) = 1 - \prod_{i=1}^{n} \left[1 - \frac{\delta_{(i)}}{n-i+1}\right]^{I(Y_{(i)} \le x)} \quad \text{and}$$
$$G_n(x) = 1 - \prod_{i=1}^{n} \left[1 - \frac{1-\delta_{(i)}}{n-i+1}\right]^{I(Y_{(i)} \le x)},$$

where $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n)}$ denote the order statistics of $Y_1, Y_2, \ldots, Y_n$, and $\delta_{(i)}$ is the concomitant of $Y_{(i)}$. Clearly, the Y_i have common df $H(x) = 1 - (1 - F(x))(1 - G(x))$, and the uncensored model is the special case of the censored model with $G = 0$.

There is a vast literature devoted to the study of the Kaplan–Meier estimator $F_n(x)$ for censored independent observations. We refer to papers by Breslow and Crowley (1974) for the weak convergence of the Kaplan–Meier process (see Theorem 5 in this paper), Gu and Lai (1990) for the iterated logarithm, Stute and Wang (1993) for almost sure convergence, Chen and Lo (1997) for the rate of (both strong and weak) uniform convergence, and Lemdani and Ould-Saïd (2001) for the relative deficiency of $F_n(x)$ with respect to a smoothed estimator. Martingale methods for analyzing properties of $F_n(x)$ are described in the monography by Gill (1980). Ying and Wei (1994) explored the consistency and asymptotic normality of $F_n(x)$ in a ϕ-mixing context, Cai and Roussas (1992) and Cai (1998, 2001) studied the uniform strong consistency with rates and asymptotic normality of $F_n(x)$ for dependent data.

Let X be a real-valued random variable and let $F(t|x)$ be the conditional df of $T_1 := T$ given $X{=}x$. In the context of regression, it is of interest to estimate $F(y|x)$ and/or the pertaining quantile function $\xi_p(x) = \inf\{t : F(t|x) \ge p\}$ for given $p \in (0, 1)$. Indeed, it is well known that conditional quantile functions can give a good description of the data (see, e.g. Chaudhuri et al. 1997), such as robustness to heavy-tailed error distributions and outliers, and especially the conditional median functions. For independent data and without censoring, many authors considered this problem; see for example Mehra et al. (1991), Chaudhuri (1991a,b), Fan et al. (1994), and Xiang (1996). Under censoring, Dabrowska (1992) established a Bahadur-type representation of kernel quantile estimator; see also Van Keilegom and Veraverbeke (1998) for the fixed design regression framework or Iglesias-Pérez (2003) for the inclusion of left-truncation. Xiang (1995) obtained the deficiency of the sample quantile estimator with respect to a kernel estimator using coverage probability. Qin and Tsao (2003) studied empirical likelihood inference for median regression models and they showed that the limiting distribution of empirical ratio for the parameter vector estimate is weighted sum of χ^2 distributions. Recently, Ould-Saïd (2006) constructed a kernel estimator of the conditional quantile under iid censorship model and established its strong uniform convergence rate.

There is some literature devoted to conditional df and conditional quantile estimation under dependence. To mention some examples, Cai (2002) investigated the asymptotic normality and the weak convergence of a weighted Nadaraya–Watson conditional df and quantile estimator for α-mixing time series. Honda (2000) dealt with α-mixing processes and proved the uniform convergence and asymptotic normality of an estimate of $\xi_p(x)$ using the local polynomial fitting method. Ferraty et al. (2005) considered quantile regression under dependence when the conditioning variable is infinite dimensional. Nonparametric conditional median predictors for time series based on the double kernel method and the constant kernel method were proposed by Gannoun et al. (2003). A nice extension of the conditional quantile process theory to set-indexed processes under strong mixing was established in Polonik and Yao (2002). Also, Zhou and Liang (2000) reported asymptotic analysis of a kernel conditional median estimator for dependent data. However, for the best of our knowledge, there are few papers dealing with the estimation of a conditional df and its quantile for censored dependent data. As an exception, Lecoutre and Ould-Saïd (1995) provided the uniform strong consistency of a kernel type estimator of the conditional df under censoring and strong mixing condition.

In this paper we investigate the asymptotic properties of the conditional df and quantile estimators in Ould-Saïd (2006) for censored data under α-mixing. Specifically, we establish the strong uniform convergence (with rate) of the estimators. Besides, we obtain the asymptotic normal distribution of the conditional quantile estimator. As a by-product, we give a rate of uniform weak convergence for the product-limit estimator under dependence (see Lemma 6 (iii) below). In the sequel, $\{(X_k, T_k, C_k) =: \zeta_k,\ k \geq 1\}$ is assumed to be a stationary α-mixing sequence of random vectors from (X, T, C). In addition to the assumptions and notation for T and C we made at the beginning of the introduction, we assume throughout that C and (X, T) are independent. Recall that the sequence $\{\zeta_k, k \geq 1\}$ is said to be α-mixing if the α-mixing coefficient

$$\alpha(m) := \sup_{k\geq 1} \sup \left\{ |P(AB) - P(A)P(B)| : A \in \mathcal{F}_{m+k}^{\infty}, B \in \mathcal{F}_1^k \right\}$$

converges to zero as $m \to \infty$, where $\mathcal{F}_l^m$ denotes the σ-algebra generated by $\zeta_l, \zeta_{l+1}, \ldots, \zeta_m$ with $l \leq m$. Among various mixing conditions used in the literature, α-mixing is reasonably weak and has many practical applications; see, e.g., Doukhan (1994), page 99, for more details. In particular, the stationary autoregressive-moving average (ARMA) processes, which are widely applied in time series analysis, are α-mixing with exponential mixing coefficient, i.e., $\alpha(k) = O(\rho^k)$ for some $0 < \rho < 1$.

The rest of the paper is organized as follows. Section 2 introduces the kernel estimator of the conditional quantile, and formulates main results for the strong uniform convergence and asymptotic normality of the estimator. Section 3 gives proofs of the main results. Some preliminary lemmas, which are used in the proofs of the main results, are collected in Appendix.

In the sequel, $l(x)$ is the marginal density of X, and $F_1(x,t) = F(t|x)l(x)$, so we have $F(t|x) = F_1(x,t)/l(x)$; also, let $f_1(x,t)$ be the joint density function of (X,T). Let C and c denote generic finite positive constants, whose values are unimportant and may change from line to line. Let $C(l)$ represent the set of continuity points of function l and $\text{supp}(l) = \{x \in R | l(x) > 0\}$. Let Ω be a compact set of R which is included in $\text{supp}(l)$. For brevity we write $L_s(x)$ instead of $1 - L(x)$ for any function $L(x)$. All limits are taken as the sample size n tends to ∞, unless specified otherwise.

2 Estimators and main results

Throughout this paper, let τ be a strictly positive real number such that $\tau < \tau_H$, where $\tau_H = \sup\{y : H(y) < 1\}$. Following Ould-Saïd (2006), define the estimator of $F(t|x)$ by

$$F_n(t|x) = \frac{\frac{1}{nh_n}\sum_{i=1}^n \frac{\delta_i I(Y_i \le t)}{1-G_n(Y_i)} K\left(\frac{x-X_i}{h_n}\right)}{\frac{1}{nh_n}\sum_{i=1}^n K\left(\frac{x-X_i}{h_n}\right)} := \frac{F_{1n}(x,t)}{l_n(x)}$$

with the convention $0/0 = 0$, where K is a kernel function on R and the bandwidth $0 < h_n \to 0$. Then, a natural estimator of $\xi_p(x)$ is given by $\xi_{pn}(x) = \inf\{t : F_n(t|x) \ge p\}$.

2.1 Strong uniform convergence

We will first present the strong uniform convergence for the estimators $F_n(t|x)$ and $\xi_{pn}(x)$ of $F(t|x)$ and $\xi_p(x)$, respectively, under α-mixing assumptions. In order to formulate the results of the strong uniform convergence, we need to impose the following assumptions.

- (A1) For all integers $j \ge 1$, the joint density $l_j(\cdot,\cdot)$ of X_1 and X_{j+1} exists on $R \times R$. Furthermore, for some $\eta > 0$, $l_j(x,y) \le C_1$ for all $x, y \in R : |x-y| \le \eta$, $l(x) \le C_2$ for all $x \in R$ with positive constants C_1 and C_2.
- (A2) Let K be a Lipschitz-continuous function on R with compact support and $\int_R K(t)\text{d}t = 1$, $\int_R tK(t)\text{d}t = 0$.
- (A3) The bandwidth h_n satisfies $h_n = O(\ln\ln(n)/\ln(n))$ and $nh_n^2/\ln\ln(n) \to \infty$.
- (A4) $F_1(\cdot,\cdot)$ is differentiable to order 2 with respect to the first component and $\sup_{(x,t)\in R\times(0,\tau]} |\frac{\partial^2 F_1(x,t)}{\partial x^2}| < \infty$.
- (A5) The marginal density $l(\cdot)$ has a bounded derivative of order 2 and $l(x) \ge \gamma_0$ on Ω for some $\gamma_0 > 0$.

(A6) For each fixed $p \in (0, 1)$, the function $\xi_p(x)$ satisfies that, for any $\epsilon > 0$ and $\eta_p(x)$, there exists a $\beta > 0$ such that $\sup_{x\in\Omega} |\xi_p(x) - \eta_p(x)| \geq \epsilon$ implies that $\sup_{x\in\Omega} |F(\xi_p(x)|x) - F(\eta_p(x)|x)| \geq \beta$.

(A7) The conditional df $F(t|x)$ has bounded first derivative with respect to t, $f(t|x)$, and there exists $\gamma_1 > 0$ such that $f(t|x) \geq \gamma_1$ for all $(x, t) \in \Omega \times (0, \tau]$.

Remark 1 Assumptions (A1)–(A7) are very common in functional estimation. These assumptions were used by many authors; for example (A1) was assumed in Liebscher (2001) for complete samples, (A2)–(A7) were used by Ould-Saïd (2006).

Theorem 1 *Let* $\alpha(k) = O(k^{-\gamma})$ *for some* $\gamma > 5$*. Suppose that* (A1)–(A5) *are satisfied. If* $n^{\gamma-5}h_n^{2\gamma+2}(\ln(n))^{-2}(\ln\ln(n))^{-\gamma} \to \infty$*, then we have*

$$\sup_{x\in\Omega}\sup_{0<t\leq\tau} |F_n(t|x) - F(t|x)| = O\left(\max\left\{\left(\frac{\ln\ln(n)}{nh_n^2}\right)^{1/2}, h_n^2\right\}\right) \ a.s..$$

Theorem 2 *Under the assumptions of Theorem 1, if* (A6) *holds, and if* $p < F(\tau|x)$ *for each* x *in* Ω*, then we have* $\lim_{n\to\infty}\sup_{x\in\Omega} |\xi_{pn}(x) - \xi_p(x)| = 0 \ a.s.$ *In addition, if* (A7) *holds, we have*

$$\sup_{x\in\Omega} |\xi_{pn}(x) - \xi_p(x)| = O\left(\max\left\{\left(\frac{\ln\ln(n)}{nh_n^2}\right)^{1/2}, h_n^2\right\}\right) \ a.s.$$

Remark 2 (a) In the particular case of exponential decay, $\alpha(k) = O(\rho^k)$ for some $0 < \rho < 1$, we have that $\alpha(k) = O(k^{-\gamma})$ for sufficient large γ, and hence the condition $\alpha(k) = O(k^{-\gamma})$ for some $\gamma > 5$ is satisfied.

(b) In the censored iid case, Ould-Saïd (2006) obtained (see his Proposition 1 and Theorem 1).

$$\sup_{x\in\Omega}\sup_{0<t\leq\tau} |F_n(t|x) - F(t|x)| = O\left(\max\left\{\left(\frac{\ln\ln(n)}{nh_n}\right)^{1/2}, h_n^2\right\}\right) \ a.s. \quad \text{and}$$

$$\sup_{x\in\Omega} |\xi_{pn}(x) - \xi_p(x)| = O\left(\max\left\{\left(\frac{\ln\ln(n)}{nh_n}\right)^{1/2}, h_n^2\right\}\right) \ a.s.$$

under $\sqrt{\ln\ln(n)/n} = o(h_n^3)$. Actually, under this assumption, the rates above in Ould-Saïd (2006) and the rates we report in Theorems 1 and 2, respectively, reduce to the same $O(h_n^2)$ rate.

(c) Jabbari et al. (2007), under assumption that $\{(X_k, T_k), k \geq 1\}$ are stationary α-mixing random variables, $\{C_k, k \geq 1\}$ is a sequence of iid censoring random variables and $\{(X_k, T_k), k \geq 1\}$ and $\{C_k, k \geq 1\}$ are independent, proved Theorems 1 and 2, however their proofs have gap (see proof of their Lemma 4, and note that their $\overline{G}_n(t)$ relates to $\{Y_i\}$), since they use convergence rates for the censoring distribution in Deheuvels and Einmahl (2000) which are only valid for the iid setup.

2.2 Asymptotic normality

Now we give the asymptotic normality of the estimator $\xi_{pn}(x)$. Let $U(x)$ represent a neighborhood of x. We need the following assumptions for the asymptotic normality.

(B1) For all integers $j \geq 1$, the joint density $l_j(\cdot, \cdot)$ of X_1 and X_{j+1} exists on $R \times R$. Furthermore, $l_j(s, t) \leq C$ for $(s, t) \in U(x) \times U(x)$.
(B2) The kernel K is a bounded function on R with compact support and $\int_R K(t)\mathrm{d}t=1$, $\int_R tK(t)\mathrm{d}t = 0$.
(B3) The sequence $\alpha(n)$ satisfies
 (i) for every $q = q_n$ such that $q=o((nh_n)^{1/2})$, $\lim_{n\to\infty}(nh_n^{-1})^{1/2}\alpha(q)=0$;
 (ii) There exist $r > 2$ and $\delta > 1 - 2/r$ such that $\sum_{l=1}^{\infty} l^{\delta}[\alpha(l)]^{1-2/r} < \infty$.
(B4) $f_1(s, y)$ is continuous at $(x, \xi_p(x))$.

Remark 3 Similar assumption to (B3) was employed by Roussas and Tran (1992), Zhou and Liang (2000) and Masry (2005). In addition, note that assumption (B3)(ii) is equivalent to $\alpha(l) = O(1/l^{2+\theta})$ for some $\theta > 0$. Actually, the proof of Theorem 3 below shows that assumption (B3)(ii) can be weakened as $h_n^{-(1-2/r)} \sum_{l=[c_n]}^{\infty} \alpha(l)^{1-2/r} \to 0$ as $n \to \infty$ for $c_n \to \infty$ and $c_n h_n \to 0$ as $n \to \infty$.

Theorem 3 *Let $\alpha(k) = O(k^{-\gamma})$ for some $\gamma > 3$, and $x \in C(l) \cap \mathrm{supp}(l)$ with $p < F(\tau|x)$. Suppose that (B1)–(B4) are satisfied, and that $\sup_{x\in U(x)} |\frac{\partial^2 F_1(x,t)}{\partial x^2}| < \infty$ for $t \leq \tau$. If $n^{\gamma} h_n^{1+\gamma} \to \infty$ and $nh_n^5 \to 0$, then*

$$\sqrt{nh_n}(\xi_{pn}(x) - \xi_p(x)) \xrightarrow{\mathcal{D}} N(0, \sigma^2(x))$$

with $\sigma^2(x)=\frac{\Delta^2(x)}{f_1^2(x,\xi_p(x))}>0$ *and* $\Delta^2(x)=\int_R K^2(s)\mathrm{d}s \int_0^{\xi_p(x)} \frac{f_1(x,t)\mathrm{d}t}{1-G(t)}:=\kappa^2 \int_0^{\xi_p(x)} \frac{f_1(x,t)\mathrm{d}t}{1-G(t)}$.

Remark 4 In Theorem 3, we assume $n^{\gamma} h_n^{1+\gamma} \to \infty$, which can be satisfied easily. For example, $n^3 h_n^4 \to \infty$ implies $n^{\gamma} h_n^{1+\gamma} \to \infty$ since $\gamma > 3$.

3 Proof of main results

We are now ready to prove our main results.

Proof of Theorem 1 Note that

$$\sup_{x\in\Omega}\sup_{0<t\leq\tau} |F_n(t|x) - F(t|x)| \leq \frac{1}{\inf_{x\in\Omega} l_n(x)} \left\{ \sup_{x\in\Omega}\sup_{0<t\leq\tau} |F_{1n}(x, t) - F_1(x, t)| + \sup_{x\in\Omega}\sup_{0<t\leq\tau} |F_1(x, t)|\gamma_0^{-1} \sup_{x\in\Omega} |l_n(x) - l(x)| \right\}. \tag{1}$$

So, we need to investigate $\sup_{x\in\Omega}|l_n(x) - l(x)|$ and $\sup_{x\in\Omega}\sup_{0<t\le\tau}|F_{1n}(x,t) - F_1(x,t)|$.

We first consider $\sup_{x\in\Omega}\sup_{0<t\le\tau}|F_{1n}(x,t) - F_1(x,t)|$. It is easy to see

$$\begin{aligned}
F_{1n}(x,t) - F_1(x,t) &= \frac{1}{nh_n}\sum_{i=1}^{n}\delta_i I(Y_i \le t) K\left(\frac{x-X_i}{h_n}\right)\left[\frac{1}{G_{ns}(Y_i)} - \frac{1}{G_s(Y_i)}\right] \\
&\quad + \frac{1}{nh_n}\sum_{i=1}^{n}\left[\frac{\delta_i I(Y_i \le t)}{G_s(Y_i)} K\left(\frac{x-X_i}{h_n}\right)\right. \\
&\quad \left. - E\left(\frac{\delta_i I(Y_i \le t)}{G_s(Y_i)} K\left(\frac{x-X_i}{h_n}\right)\right)\right] \\
&\quad + \left\{\frac{1}{nh_n}\sum_{i=1}^{n} E\left(\frac{\delta_i I(Y_i \le t)}{G_s(Y_i)} K\left(\frac{x-X_i}{h_n}\right)\right) - F_1(x,t)\right\} \\
&:= D_{1n}(x,t) + D_{2n}(x,t) + D_{3n}(x,t). \qquad (2)
\end{aligned}$$

Since (A2) implies that K is bounded, according to Lemma 6(ii) in Appendix we have

$$\begin{aligned}
\sup_{x\in\Omega}\sup_{0<t<\tau}|D_{1n}(x,t)| &\le \frac{\|K\|_\infty \sup_{0<t\le\tau}|G_n(t) - G(t)|}{h_n G_s(\tau)[G_s(\tau) - \sup_{0<t\le\tau}|G_n(t) - G(t)|]} \\
&= O\left(\left(\frac{\ln\ln(n)}{nh_n^2}\right)^{1/2}\right) \quad a.s. \qquad (3)
\end{aligned}$$

Note that for $t \le \tau$

$$E\left[\frac{\delta_1 I(Y_1 \le t)}{G_s(Y_1)}\bigg| X_1\right] = E\left\{\frac{I(T_1 \le t)}{G_s(T_1)} E[I(T_1 \le C_1)|T_1]\bigg| X_1\right\} = E[I(T_1 \le t)|X_1].$$

So, from (A2) we find

$$\begin{aligned}
D_{3n}(x,t) &= \frac{1}{h_n} E\left\{K\left(\frac{x-X_1}{h_n}\right) E\left[\frac{\delta_1 I(Y_1 \le t)}{G_s(Y_1)}\bigg| X_1\right]\right\} - F_1(x,t) \\
&= \frac{1}{h_n} E\left\{K\left(\frac{x-X_1}{h_n}\right) I(T_1 \le t)\right\} - F_1(x,t) \\
&= \frac{1}{h_n}\int K\left(\frac{x-u}{h_n}\right) l(u) F(t|u)\mathrm{d}u - F_1(x,t) \\
&= \int K(s)\left[F_1(x - h_n s, t) - F_1(x,t)\right]\mathrm{d}s \\
&= \frac{h_n^2}{2!}\int s^2 K(s) \frac{\partial^2}{\partial x^2} F_1(x_n^*, t)\mathrm{d}s,
\end{aligned}$$

where x_n^* is between x and $x - h_n s$, which, by using (A4), yields that

$$\sup_{x\in\Omega}\sup_{0<t\le\tau}|D_{3n}(x,t)| = O(h_n^2). \tag{4}$$

Now we consider $D_{2n}(x,t)$. Let intervals $A(x_k)(k = 1,\ldots,s_n)$ with length $(h_n^2\ln\ln(n)/n)^{1/2}$, and centered at points x_k, cover the compact set Ω. Since Ω is bounded, $s_n = O((h_n^2\ln\ln(n)/n)^{-1/2})$; divide the interval $(0,\tau]$ into subintervals $(t_j;t_{j+1}]$, $j = 1,\ldots,m_n$, where $m_n = O((\ln\ln(n)/n)^{-1/2})$, and $0 = t_1 < t_2 < \cdots < t_{m_n+1} = \tau$ are such that $H(t_{j+1}) - H(t_j) = O((\ln\ln(n)/n)^{1/2})$. It can be checked that

$$\begin{aligned}\sup_{x\in\Omega}\sup_{0<t\le\tau}|D_{2n}(x,t)| \le & \max_{1\le k\le s_n}\sup_{x\in A(x_k)}\sup_{0<t\le\tau}|D_{2n}(x,t)-D_{2n}(x_k,t)| \\ &+\max_{1\le k\le s_n}\max_{1\le j\le m_n}\sup_{t_j\le t\le t_{j+1}}|D_{2n}(x_k,t)-D_{2n}(x_k,t_j)| \\ &+\max_{1\le k\le s_n}\max_{1\le j\le m_n}|D_{2n}(x_k,t_j)|. \end{aligned} \tag{5}$$

Because K is Lipschitz-continuous by (A2) and since the length of $A(x_k)$ is $(h_n^2\ln\ln(n)/n)^{1/2}$, we have

$$\max_{1\le k\le s_n}\sup_{x\in A(x_k)}\sup_{0<t\le\tau}|D_{2n}(x,t)-D_{2n}(x_k,t)| = O\left(\left(\frac{\ln\ln(n)}{nh_n^2}\right)^{1/2}\right)\ a.s. \tag{6}$$

Put $H_n(x) = \frac{1}{n}\sum_{i=1}^n I(Y_i\le x)$. By using Lemma 6 (i) one can obtain that

$$\begin{aligned}&\max_{1\le k\le s_n}\max_{1\le j\le m_n}\sup_{t_j\le t\le t_{j+1}}\left|D_{2n}(x_k,t)-D_{2n}(x_k,t_j)\right| \\ &\le \frac{C}{h_n}\max_{1\le j\le m_n}\left|H_n(t_{j+1})-H_n(t_j)+H(t_{j+1})-H(t_j)\right| \\ &= O\left(\left(\frac{\ln\ln(n)}{nh_n^2}\right)^{1/2}\right)\ a.s. \end{aligned} \tag{7}$$

Set $\beta_i(x_k,t_j) = \frac{\delta_i I(Y_i\le t_j)}{G_s(Y_i)}K\Big(\frac{x_k-X_i}{h_n}\Big) - E\Big(\frac{\delta_i I(Y_i\le t_j)}{G_s(Y_i)}K(\frac{x_k-X_i}{h_n})\Big)$. Then $D_{2n}(x_k,t_j) = \frac{1}{nh_n}\sum_{i=1}^n\beta_i(x_k,t_j)$, and for some $\epsilon_0 > 0$

$$\begin{aligned}&P\left(\max_{1\le k\le s_n}\max_{1\le j\le m_n}|D_{2n}(x_k,t_j)|\ge\epsilon_0\sqrt{\ln\ln(n)/(nh_n^2)}\right) \\ &\le s_n m_n\sup_{(x,t)\in\Omega\times(0,\tau]}P\left(\left|\sum_{i=1}^n\beta_i(x,t)\right|\ge\epsilon_0\sqrt{n\ln\ln(n)}\right).\end{aligned}$$

Note that $\sup_{(x,t)\in\Omega\times(0,\tau]} |\beta_i(x,t)| \le 2G_s^{-1}(\tau)\|K\|_\infty < \infty$, $\sup_{(x,t)\in\Omega\times(0,\tau]} E\beta_i^2(x,t) = O(h_n)$ and for $i \neq j$

$$\begin{aligned}
&\sup_{(x,t)\in\Omega\times(0,\tau]} |\mathrm{Cov}(\beta_i(x,t), \beta_j(x,t))| \\
&\le C \sup_{x\in\Omega} \left\{ E\left| K\!\left(\frac{x-X_i}{h_n}\right) K\!\left(\frac{x-X_j}{h_n}\right)\right| + E\left|K\!\left(\frac{x-X_i}{h_n}\right)\right| E\left|K\!\left(\frac{x-X_j}{h_n}\right)\right|\right\} \\
&= Ch_n^2 \sup_{x\in\Omega} \left\{ \int |K(s)K(t)| l_{j-i}(x-h_n s, x-h_n t)\mathrm{d}s\,\mathrm{d}t \right. \\
&\quad \left. + \left(\int |K(s)| l(x-h_n s)\mathrm{d}s\right)^2 \right\} \\
&= O(h_n^2)
\end{aligned}$$

by (A1). Hence, applying Lemma 4 for $X_i = \beta_i(x,t)$, $n = j$, $m = \infty$ and noting that $2(1-1/\gamma) > 1$, for $m \in \mathbb{N}$, $0 < m \le n/2$, we have

$$D_m := \max_{1\le j\le 2m} \mathrm{Var}\left(\sum_{i=1}^{j} \beta_i(x,t)\right) \le Cm\left((h_n^2)^{1-1/\gamma} + h_n\right) = O(mh_n).$$

Choosing $m = [(n^5(\ln(n))^2/h_n^2)^{1/(2\gamma)}]$, noticing (A3) implies $\ln\ln(n)/h_n \ge C\ln(n)$, and

$$n^{\gamma-5}h_n^{2\gamma+2}(\ln(n))^{-2}(\ln\ln(n))^{-\gamma} \to \infty \ \text{ implies } \ nh_n/(m(n\ln\ln(n))^{1/2}) \to \infty.$$

Therefore, from Lemma 3 we obtain that

$$\begin{aligned}
&P\left(\max_{1\le k\le s_n}\max_{1\le j\le m_n} |D_{2n}(x_k,t_j)| \ge \epsilon_0\sqrt{\ln\ln(n)/(nh_n^2)}\right) \\
&\le s_n m_n \left[4\exp\left\{-\frac{\epsilon_0^2 n\ln\ln(n)/16}{nm^{-1}D_m + 2\|K\|_\infty \epsilon_0 (n\ln\ln(n))^{1/2} m/3}\right\}\right. \\
&\quad \left. + \frac{32\cdot 2\|K\|_\infty}{\epsilon_0(n\ln\ln(n))^{1/2}} n\alpha(m)\right] \\
&\le O\left(\frac{n}{h_n\ln\ln(n)}\right)\left[4\exp\{-c\epsilon_0^2\ln(n)\} + \frac{ch_n}{n^2(\ln(n))(\ln\ln(n))^{1/2}}\right] \\
&\le O(1)\left(\frac{1}{n^2} + \frac{1}{n(\ln(n))(\ln\ln(n))^{3/2}}\right)
\end{aligned}$$

provided ϵ_0 large enough. Then, it follows from the Borel–Cantelli lemma that

$$\max_{1\le k\le s_n}\max_{1\le j\le m_n} |D_{2n}(x_k,t_j)| = O\left(\left(\frac{\ln\ln(n)}{nh_n^2}\right)^{1/2}\right) \ a.s. \tag{8}$$

From (5)–(8) we have

$$\sup_{x\in\Omega}\sup_{0<t\le\tau}|D_{2n}(x,t)| = O\left(\left(\frac{\ln\ln(n)}{nh_n^2}\right)^{1/2}\right) \ a.s. \tag{9}$$

Therefore, from (2)–(4) and (9) we obtain that

$$\sup_{x\in\Omega}\sup_{0<t\le\tau}|F_{1n}(x,t)-F_1(x,t)| = O\left(\left(\frac{\ln\ln(n)}{nh_n^2}\right)^{1/2}\right) \ a.s. \tag{10}$$

Next we prove

$$\sup_{x\in\Omega}|l_n(x)-l(x)| = O\left(\max\left\{\left(\frac{\ln\ln(n)}{nh_n^2}\right)^{1/2}, h_n^2\right\}\right) \ a.s. \tag{11}$$

We write

$$l_n(x)-l(x) = [l_n(x)-El_n(x)] + [El_n(x)-l(x)]\,. \tag{12}$$

Under assumptions (A2) and (A5), using Taylor expansion of $l(\cdot)$, we have

$$\sup_{x\in\Omega}|El_n(x)-l(x)| = \sup_{x\in\Omega}\left|\int_R K(u)[l(x-h_nu)-l(x)]\mathrm{d}u\right| = O(h_n^2). \tag{13}$$

Following the lines in the analysis of $D_{2n}(x,t)$ above, it is easy to verify that

$$\sup_{x\in\Omega}|l_n(x)-El_n(x)| = O\left(\left(\frac{\ln\ln(n)}{nh_n^2}\right)^{1/2}\right) \ a.s. \tag{14}$$

Therefore, (12)–(14) yield (11).

Finally, by (A5), the proof is completed from (1), (10) and (11).

Proof of Theorem 2 We observe that

$$\begin{aligned}|F(\xi_{pn}(x)|x)-F(\xi_p(x)|x)| \le{}& |F_n(\xi_{pn}(x)|x)-F(\xi_{pn}(x)|x)| \\ &+|F_n(\xi_{pn}(x)|x)-F(\xi_p(x)|x)|.\end{aligned}$$

Since $F(\cdot|x)$ is continuous, we have $F(\xi_p(x)|x)=p$ and

$$|F_n(\xi_{pn}(x)|x)-F(\xi_p(x)|x)| = F_n(\xi_{pn}|x)-p \le F_n(\xi_{pn}(x)|x)-F_n(\xi_{pn}(x)^-|x),$$

where $F_n(\xi_{pn}(x)^-|x)$ stands for the left-hand limit of $F_n(t|x)$ at $t = \xi_{pn}(x)^-$. Now, using again the continuity of $F(\cdot|x)$, we have

$$\begin{aligned} 0 &\le F_n(\xi_{pn}(x)|x) - F_n(\xi_{pn}(x)^-|x) \\ &\le |F_n(\xi_{pn}(x)|x) - F(\xi_{pn}|x)| + |F(\xi_{pn}^-|x) - F_n(\xi_{pn}^-|x)| \\ &\le 2 \sup_{0<t\le\tau} |F_n(t|x) - F(t|x)|, \end{aligned}$$

where for the last inequality we use the fact that $\xi_{pn}(x) \le \tau$ eventually (which follows from Theorem 2.1 and condition $p < F(\tau|x)$). Hence,

$$\begin{aligned} |F(\xi_{pn}(x)|x) - F(\xi_p(x)|x)| &\le \sup_{0<t\le\tau} |F_n(t|x) - F(t|x)| + 2 \sup_{0<t\le\tau} |F_n(t|x) - F(t|x)| \\ &\le 3 \sup_{x\in\Omega} \sup_{0<t\le\tau} |F_n(t|x) - F(t|x)|. \end{aligned} \tag{15}$$

Therefore, the first part of the theorem is deduced from Theorem 1 and (A6). Note that

$$F(\xi_{pn}(x)|x) - F(\xi_p(x)|x) = (\xi_{pn}(x) - \xi_p(x)) f(\xi_{pn}^*(x)|x),$$

where $\xi_{pn}^*(x)$ is between $\xi_p(x)$ and $\xi_{pn}(x)$. Then, by (15), we have

$$\sup_{x\in\Omega} |\xi_{pn}(x) - \xi_p(x)| f(\xi_{pn}^*(x)|x) \le 3 \sup_{x\in\Omega} \sup_{0<t\le\tau} |F_n(t|x) - F(t|x)|.$$

Then, the second part of the theorem follows from Theorem 1 and (A7).

Proof of Theorem 3 We observe that for $y \in R$

$$\begin{aligned} \Gamma_n(y) &= P\left(\frac{\sqrt{nh_n}(\xi_{pn}(x) - \xi_p(x))}{\sigma(x)} \le y \right) \\ &= P\left(p \le F_n\Big(\sigma(x) y (nh_n)^{-1/2} + \xi_p(x) \Big| x\Big)\right) \end{aligned}$$

and

$$\begin{aligned} &\Phi(y) - \Gamma_n(y) \\ &\quad = P\left(\frac{F_{1n}(x, \sigma(x) y (nh_n)^{-1/2} + \xi_p(x))}{l_n(x)} < p \right) - \Phi(-y) \\ &\quad = P\left(\frac{\sqrt{nh_n} l(x)}{\Delta(x) l_n(x)} \cdot F_{1n}(x, \sigma(x) y (nh_n)^{-1/2} + \xi_p(x)) < \frac{\sqrt{nh_n} l(x) p}{\Delta(x)} \right) - \Phi(-y). \end{aligned} \tag{16}$$

Put $\eta_{ni}(x,t) = \frac{\delta_i I(Y_i \le t)}{G_s(Y_i)} K(\frac{x - X_i}{h_n})$. Then

$$
\begin{aligned}
&\frac{\sqrt{nh_n}l(x)}{\Delta(x)l_n(x)}\cdot F_{1n}(x,\sigma(x)y(nh_n)^{-1/2}+\xi_p(x))-\frac{\sqrt{nh_n}l(x)p}{\Delta(x)}\\
&=\frac{l(x)}{l_n(x)}\cdot\frac{1}{\sqrt{nh_n}\Delta(x)}\left\{\sum_{i=1}^n\delta_i I\left(Y_i\le\sigma(x)y(nh_n)^{-1/2}+\xi_p(x)\right)\right.\\
&\quad\times K\left(\frac{x-X_i}{h_n}\right)\left[\frac{1}{G_{ns}(Y_i)}-\frac{1}{G_s(Y_i)}\right]+\sum_{i=1}^n\Big[\eta_{ni}(x,\sigma(x)y(nh_n)^{-1/2}+\xi_p(x))\\
&\quad\left.-E\eta_{ni}(x,\sigma(x)y(nh_n)^{-1/2}+\xi_p(x))\Big]\right\}\\
&\quad+\left\{\frac{l(x)}{l_n(x)}\cdot\frac{1}{\sqrt{nh_n}\Delta(x)}\sum_{i=1}^n E\eta_{ni}(x,\sigma(x)y(nh_n)^{-1/2}+\xi_p(x))-\frac{\sqrt{nh_n}l(x)p}{\Delta(x)}\right\}\\
&:=\frac{l(x)}{l_n(x)}\cdot\frac{1}{\sqrt{nh_n}\Delta(x)}\{I_{1n}(x)+I_{2n}(x)\}+I_{3n}(x).
\end{aligned}\tag{17}
$$

From (16) and (17), to prove Theorem 3 it suffices to show that

$$
\begin{aligned}
&l_n(x)\xrightarrow{P}l(x),\quad \frac{1}{\sqrt{nh_n}\Delta(x)}I_{1n}(x)\xrightarrow{P}0,\\
&I_{3n}(x)\xrightarrow{P}y,\quad \frac{1}{\sqrt{nh_n}\Delta(x)}I_{2n}(x)\xrightarrow{\mathcal{D}}N(0,1).
\end{aligned}
$$

Step 1. By $n^\gamma h_n^{1+\gamma}\to\infty$, the proof for $l_n(x)\xrightarrow{P}l(x)$ is standard (cf. Bosq 1998), so it is omitted here.

Step 2. We prove $\frac{1}{\sqrt{nh_n}\Delta(x)}I_{1n}(x)$. Note that, since $p<F(\tau|x)$, we have $\xi_p(x)<\tau$ and hence

$$
\begin{aligned}
\frac{1}{\sqrt{nh_n}\Delta(x)}|I_{1n}(x)|\le&\frac{\sqrt{nh_n}\sup_{0\le y\le\tau}|G_n(y)-G(y)|}{\Delta(x)(1-G(\tau))[(1-G(\tau))-\sup_{0\le y\le\tau}|G_n(y)-G(y)|]}\\
&\times\frac{1}{nh_n}\sum_{i=1}^n\left|K\left(\frac{x-X_i}{h_n}\right)\right|
\end{aligned}\tag{18}
$$

and

$$
\begin{aligned}
P\left(\frac{1}{nh_n}\sum_{i=1}^n\left|K\left(\frac{x-X_i}{h_n}\right)\right|>\epsilon\right)&\le\frac{1}{\epsilon nh_n}\sum_{i=1}^n E\left|K\left(\frac{x-X_i}{h_n}\right)\right|\\
&=\frac{1}{\epsilon}\int_R|K(t)|l(x-th_n)\mathrm{d}t\to\frac{l(x)}{\epsilon}\int_R|K(t)|\mathrm{d}t.
\end{aligned}\tag{19}
$$

Therefore, by using Lemma 6(iii), from (18) and (19) we obtain that

$$
\frac{1}{\sqrt{nh_n}\Delta(x)}|I_{1n}(x)|=O_P(h_n^{1/2})=o_P(1).
$$

Step 3. We verify $I_{3n}(x) \xrightarrow{P} y$. Note that

$$\begin{aligned}
&\frac{1}{\sqrt{nh_n}\Delta(x)}\sum_{i=1}^{n} E\eta_{ni}\left(x, \sigma(x)y(nh_n)^{-1/2} + \xi_p(x)\right) - \frac{\sqrt{nh_n}l(x)p}{\Delta(x)} \\
&= \frac{n}{\sqrt{nh_n}\Delta(x)} E\left(\frac{\delta_1 I(Y_1 \le \sigma(x)y(nh_n)^{-1/2} + \xi_p(x))}{G_s(Y_1)} K\left(\frac{x - X_1}{h_n}\right)\right) \\
&\quad - \frac{\sqrt{nh_n}l(x)F(\xi_p(x)|x)}{\Delta(x)} \\
&= \frac{n}{\sqrt{nh_n}\Delta(x)} E\left\{K\left(\frac{x - X_1}{h_n}\right)\right. \\
&\quad \times \left. E\left(\frac{I(T_1 \le C_1)I(T_1 \le \sigma(x)y(nh_n)^{-1/2} + \xi_p(x))}{G_s(T_1)}\middle| X_1\right)\right\} - \frac{\sqrt{nh_n}F_1(x, \xi_p(x))}{\Delta(x)} \\
&= \frac{n}{\sqrt{nh_n}\Delta(x)} E\left\{K\left(\frac{x - X_1}{h_n}\right) I\left(T_1 \le \sigma(x)y(nh_n)^{-1/2} + \xi_p(x)\right)\right\} \\
&\quad - \frac{\sqrt{nh_n}F_1(x, \xi_p(x))}{\Delta(x)} \\
&= \frac{n}{\sqrt{nh_n}\Delta(x)} \int_R K\left(\frac{x - u}{h_n}\right) l(u) \int_0^{\sigma(x)y(nh_n)^{-1/2} + \xi_p(x)} F(dy|u)du \\
&\quad - \frac{\sqrt{nh_n}F_1(x, \xi_p(x))}{\Delta(x)} \\
&= \frac{\sqrt{nh_n}}{\Delta(x)} \int_R K(s)l(x - h_n s)F\left(\sigma(x)y(nh_n)^{-1/2} + \xi_p(x)|x - h_n s\right) \mathrm{d}s \\
&\quad - \frac{\sqrt{nh_n}F_1(x, \xi_p(x))}{\Delta(x)} \\
&= \frac{\sqrt{nh_n}}{\Delta(x)} \int_R K(s)\left[F_1(x - h_n s, \sigma(x)y(nh_n)^{-1/2} + \xi_p(x))\right. \\
&\quad \left. - F_1\left(x, \sigma(x)y(nh_n)^{-1/2} + \xi_p(x)\right)\right] \mathrm{d}s \\
&\quad + \frac{\sqrt{nh_n}}{\Delta(x)}\left[F_1(x, \sigma(x)y(nh_n)^{-1/2} + \xi_p(x)) - F_1(x, \xi_p(x))\right] \\
&:= I_{31n}(x) + I_{32n}(x).
\end{aligned}$$

From the assumption (B2) we have

$$\begin{aligned}
I_{31n}(x) &= \frac{\sqrt{nh_n}}{\Delta(x)}\left|\int_R K(s)\left[(-h_n s)\frac{\partial F_1\left(x, \sigma(x)y(nh_n)^{-1/2} + \xi_p(x)\right)}{\partial x}\right.\right. \\
&\quad \left.\left. + \frac{h_n^2 s^2}{2!}\frac{\partial^2 F_1\left(x_n^*, \sigma(x)y(nh_n)^{-1/2} + \xi_p(x)\right)}{\partial x^2}\right] ds\right| \\
&= O\left(\sqrt{nh_n^5}\right) = o(1),
\end{aligned}$$

where x_n^* is between x and $x - h_n s$.

$$I_{32n}(x) = \frac{\sqrt{nh_n}}{\Delta(x)} f_1(x, \theta_n(x)) \cdot \sigma(x) y (nh_n)^{-1/2} \to \frac{\sigma(x)}{\Delta(x)} f_1(x, \xi_p(x)) y = y,$$

where $\theta_n(x)$ is between $\xi_p(x)$ and $\sigma(x)y(nh_n)^{-1/2} + \xi_p(x)$. Therefore $I_{3n}(x) \stackrel{P}{\to} y$ from *Step 1*.

Step 4. Now, we prove $\frac{1}{\sqrt{nh_n}\Delta(x)} I_{2n}(x) \stackrel{\mathcal{D}}{\to} N(0,1)$. Here, we will employ Bernstein's big-block and small-block procedure. Partition the set $\{1, 2, \ldots, n\}$ into $2w_n+1$ subsets with large blocks of size $p = p_n$ and small blocks of size $q = q_n$ and set $w = w_n = [\frac{n}{p+q}]$.

(B3) implies that there exists a sequence of positive integers $\delta_n \to \infty$ such that $\delta_n q = o(\sqrt{nh_n})$, $\delta_n (nh_n^{-1})^{1/2} \alpha(q) \to 0$. Define the large block size $p = p_n = [(nh_n)^{1/2}/\delta_n]$. Then a simple calculation shows

$$q/p \to 0, \quad w\alpha(q) \to 0, \quad wq/n \to 0, \quad p/n \to 0, \quad p/(nh_n)^{1/2} \to 0. \tag{20}$$

Put $Z_i = \frac{1}{\sqrt{h_n}\Delta(x)} \left\{ \eta_{ni}(x, \sigma(x)y(nh_n)^{-1/2} + \xi_p(x)) - E\eta_{ni}(x, \sigma(x)y(nh_n)^{-1/2} + \xi_p(x)) \right\}$. Let y_{mn}, y'_{mn}, y''_{wn} be defined as follows:

$$y_{mn} = \sum_{i=k_m}^{k_m+p-1} Z_i, \quad y'_{mn} = \sum_{j=l_m}^{l_m+q-1} Z_j, \quad y''_{wn} = \sum_{k=w(p+q)+1}^{n} Z_k,$$

where $k_m = (m-1)(p+q)+1$, $l_m = (m-1)(p+q)+p+1$. Then

$$\frac{1}{\sqrt{nh_n}\Delta(x)} I_{2n}(x) = n^{-1/2} \sum_{i=1}^{n} Z_i = n^{-1/2} \left\{ \sum_{m=1}^{w} y_{mn} + \sum_{m=1}^{w} y'_{mn} + y''_{wn} \right\}$$
$$:= n^{-1/2} \left\{ S'_n + S''_n + S'''_n \right\}.$$

It suffices to show that

$$\frac{1}{n} E(S''_n)^2 \to 0, \quad \frac{1}{n} E(S'''_n)^2 \to 0, \tag{21}$$

$$\mathrm{Var}(n^{-1/2} S'_n) \to 1, \tag{22}$$

$$\left| E\exp\left(it \sum_{m=1}^{w} n^{-1/2} y_{mn} \right) - \prod_{m=1}^{w} E\exp\left(itn^{-1/2} y_{mn} \right) \right| \to 0, \tag{23}$$

$$g_n(\epsilon) = \frac{1}{n} \sum_{m=1}^{w} E y_{mn}^2 I(|y_{mn}| > \epsilon\sqrt{n}) \to 0 \ \forall \epsilon > 0. \tag{24}$$

Relation (21) implies that S''_n and S'''_n are asymptotically negligible, (23) shows that the summands y_{mn} in S'_n are asymptotically independent, and (22) and (24) are the standard Lindeberg–Feller conditions for asymptotic normality of S'_n under independence.

We first establish (21). Obviously

$$\begin{aligned}\frac{1}{n}E(S''_n)^2 &= \frac{1}{n}\sum_{i=1}^{w}\sum_{i=l_m}^{l_m+q-1} EZ_i^2 + \frac{2}{n}\sum_{i=1}^{w}\sum_{l_m\le i<j\le l_m+q-1}\mathrm{Cov}(Z_i, Z_j)\\ &\quad +\frac{2}{n}\sum_{1\le i<j\le w}\mathrm{Cov}(y'_{in}, y'_{jn})\\ &:= J_{1n}(x) + J_{2n}(x) + J_{3n}(x).\end{aligned}$$

Note that

$$\begin{aligned} EZ_i^2 &= \frac{1}{h_n\Delta^2(x)}\Big[E\eta_{ni}^2\Big(x, \sigma(x)y(nh_n)^{-1/2} + \xi_p(x)\Big)\\ &\quad -\Big(E\eta_{ni}(x, \sigma(x)y(nh_n)^{-1/2} + \xi_p(x))\Big)^2\Big]\\ &= \frac{1}{h_n\Delta^2(x)}\Bigg\{E\Bigg[K^2\left(\frac{x-X_1}{h_n}\right)\\ &\quad \times E\left(\frac{I(T_1\le C_1)I(T_1\le\sigma(x)y(nh_n)^{-1/2}+\xi_p(x))}{G_s^2(T_1)}\Big|X_1\right)\Bigg]\\ &\quad -\Bigg(E\Bigg[K\left(\frac{x-X_1}{h_n}\right)\\ &\quad \times E\left(\frac{I(T_1\le C_1)I(T_1\le\sigma(x)y(nh_n)^{-1/2}+\xi_p(x))}{G_s(T_1)}\Big|X_1\right)\Bigg]\Bigg)^2\Bigg\}\\ &= \frac{1}{h_n\Delta^2(x)}\Bigg\{E\left[K^2\left(\frac{x-X_1}{h_n}\right)\frac{I(T_1\le\sigma(x)y(nh_n)^{-1/2}+\xi_p(x))}{G_s(T_1)}\right]\\ &\quad -\left(E\left[K\left(\frac{x-X_1}{h_n}\right)I(T_1\le\sigma(x)y(nh_n)^{-1/2}+\xi_p(x))\right]\right)^2\Bigg\}\\ &= \frac{1}{\Delta^2(x)}\int_R K^2(s)\int_0^{\sigma(x)y(nh_n)^{-1/2}+\xi_p(x)}\frac{f_1(x-h_ns,t)}{G_s(t)}\mathrm{d}s\,\mathrm{d}t\\ &\quad -\frac{h_n}{\Delta^2(x)}\left(\int_R K(s)\int_0^{\sigma(x)y(nh_n)^{-1/2}+\xi_p(x)} f_1(x-h_ns,t)\mathrm{d}s\,\mathrm{d}t\right)^2\\ &\to \frac{1}{\Delta^2(x)}\int_R K^2(s)\mathrm{d}s\int_0^{\xi_p(x)}\frac{f_1(x,t)\mathrm{d}t}{G_s(t)} = 1, \end{aligned}\tag{25}$$

which yields that $J_{1n}(x) = O(wq/n) = o(1)$ from (20). Since

$$|J_{2n}(x)| \le \frac{2}{n}\sum_{1\le i<j\le n}|\mathrm{Cov}(Z_i, Z_j)| \quad\text{and}\quad |J_{3n}(x)| \le \frac{2}{n}\sum_{1\le i<j\le n}|\mathrm{Cov}(Z_i, Z_j)|,$$

to prove $|J_{2n}(x)| = o(1)$ and $|J_{3n}(x)| = o(1)$, it suffices to show that

$$\frac{1}{n}\sum_{1\le i<j\le n}|\mathrm{Cov}(Z_i, Z_j)| \to 0. \tag{26}$$

Next, let c_n (specified below) is a sequence of integers such that $c_n \to \infty$ and $c_n h_n \to 0$. Put

$$S_1 = \{(i, j)|i, j \in \{1, 2, \ldots, n\}, 1 \le j - i \le c_n\},$$
$$S_2 = \{(i, j)|i, j \in \{1, 2, \ldots, n\}, c_n + 1 \le j - i \le n - 1\}.$$

We write

$$\frac{1}{n}\sum_{1\le i<j\le n}|\mathrm{Cov}(Z_i, Z_j)| = \frac{1}{n}\sum_{S_1}|\mathrm{Cov}(Z_i, Z_j)| + \frac{1}{n}\sum_{S_2}|\mathrm{Cov}(Z_i, Z_j)|. \tag{27}$$

From (B1) we find

$$\begin{aligned}|\mathrm{Cov}(Z_i, Z_j)| &\le \frac{1}{h_n\Delta^2(x)G^2(\tau)}\left\{E\left|K\left(\frac{x-X_i}{h_n}\right)K\left(\frac{x-X_j}{h_n}\right)\right|\right.\\ &\quad \left. +E\left|K\left(\frac{x-X_i}{h_n}\right)\right|E\left|K\left(\frac{x-X_j}{h_n}\right)\right|\right\}\\ &= \frac{h_n}{\Delta^2(x)G^2(\tau)}\int_R\int_R |K(s)K(t)|\\ &\quad \times\left\{l_{j-i}(x - sh_n, x - th_n) + l(x - sh_n)l(x - th_n)\right\}\mathrm{d}s\,\mathrm{d}t\\ &= O(h_n).\end{aligned}$$

Hence

$$\frac{1}{n}\sum_{S_1}|\mathrm{Cov}(Z_i, Z_j)| = O(c_n h_n) \to 0. \tag{28}$$

On the other hand, it follows from Lemma 2 that $|\mathrm{Cov}(Z_i, Z_j)| \le C[\alpha(j - i)]^{1-2/r} (E|Z_i|^r)^{2/r}$ and

$$\begin{aligned}E|Z_i|^r &\le Ch_n^{-r/2}G_s^{-r}(\tau)E\left|K\left(\frac{x-X_i}{h_n}\right)\right|^r\\ &= Ch_n^{-(r/2-1)}G_s^{-r}(\tau)\int_R |K(s)|^r l(x - h_n s)\mathrm{d}s\\ &= O\left(h_n^{-(r/2-1)}\right),\end{aligned}$$

which implies that

$$\begin{aligned}\frac{1}{n}\sum_{S_2}|\mathrm{Cov}(Z_i,Z_j)| &\le \frac{C}{n}\sum_{j=1}^{n}\sum_{j-i=c_n+1}^{n-1}[\alpha(j-i)]^{1-2/r}h_n^{-(1-2/r)}\\ &\le Ch_n^{-(1-2/r)}\sum_{l=c_n}^{\infty}\alpha(l)^{1-2/r}\\ &\le Cc_n^{-\delta}h_n^{-(1-2/r)}\sum_{l=c_n}^{\infty}l^{\delta}\alpha(l)^{1-2/r}.\end{aligned}$$

Therefore, by choosing $c_n = h_n^{-(1-2/\gamma)/\delta}$ and in view of (B3) we obtain that

$$\frac{1}{n}\sum_{S_2}|\mathrm{Cov}(Z_i,Z_j)| \to 0. \tag{29}$$

Thus, (26) is verified from (27)–(29).

As to $n^{-1}E(S_n''')^2$, from (25) and (26) we have

$$\begin{aligned}\frac{1}{n}E(S_n''')^2 &= \frac{1}{n}\sum_{i=w(p+q)+1}^{n}\mathrm{Var}(Z_i)+\frac{2}{n}\sum_{w(p+q)+1\le i<j\le n}\mathrm{Cov}(Z_i,Z_j)\\ &\le C\cdot\frac{n-w(p+q)}{n}+\frac{2}{n}\sum_{1\le i<j\le n}|\mathrm{Cov}(Z_i,Z_j)|\to 0.\end{aligned}$$

Next we prove (22). Since $wp/n\to 1$, from (25) and (26) one can get $\mathrm{Var}(n^{-1/2}S_n')\to 1$.

As to (23), according to Lemma 1 we have

$$\left|E\exp\left(it\sum_{m=1}^{w}n^{-1/2}y_{mn}\right)-\prod_{m=1}^{w}E\exp\left(itn^{-1/2}y_{mn}\right)\right|\le 16w\alpha(q+1),$$

which tends to zero by (20).

Finally, we establish (24). Since $\max_{1\le m\le w}|y_{mn}| = O(p/\sqrt{h_n})$, we have that $\{|y_{mn}| > \epsilon\sqrt{n}\}$ is an empty set by (20), and (24) is shown.

Acknowledgments The authors are grateful to the Editor, an associate Editor, and two anonymous referees for providing a detailed list of comments which greatly improved the presentation of the paper. Authors were supported by the Grants MTM2008-03129 of the Spanish Ministry of Science and Innovation, and PGIDIT07PXIB300191PR of the Xunta de Galicia, Spain, and also by the National Natural Science Foundation of China (10871146).

Appendix

This appendix states some lemmas, which are used in the proof of the main result in Sect. 3.

Lemma 1 (Volkonskii and Rozanov 1959) *Let $V_1, \ldots, V_n$ be α-mixing random variables measurable with respect to the σ-algebra $\mathcal{F}_{i_1}^{j_1}, \ldots, \mathcal{F}_{i_m}^{j_m}$, respectively, with $1 \le i_1 < j_1 < \cdots < j_m \le n$, $i_{l+1} - j_l \ge w \ge 1$ and $|V_j| \le 1$ for $l, j = 1, 2, \ldots, m$. Then*

$$\left| E\left(\prod_{j=1}^{m} V_j\right) - \prod_{j=1}^{m} EV_j \right| \le 16(m-1)\alpha_w,$$

where $\mathcal{F}_a^b = \sigma\{V_i, a < i \le b\}$ denotes σ-field generated by $V_{a+1}, V_{a+2}, \ldots, V_b$, α_n is the mixing coefficient.

Lemma 2 (Hall and Heyde 1980, Corollary A.2, p. 278) *Suppose that X and Y are random variables such that $E|X|^p < \infty$, $E|X|^q < \infty$, where $p, q > 1$, $p^{-1} + q^{-1} < 1$. Then*

$$|EXY - EXEY| \le 8\|X\|_p \|Y\|_q \left\{ \sup_{A \in \sigma(X), B \in \sigma(Y)} |P(AB) - P(A)P(B)| \right\}^{1-p^{-1}-q^{-1}}.$$

Lemma 3 (Liebscher 2001, Proposition 5.1) *Assume that $EX_i = 0$ and $|X_i| \le S < \infty$ a.s. $(i = 1, 2, \ldots, n)$. Then, for $n, m \in \mathbb{N}$, $0 < m \le n/2$, $\varepsilon > 0$,*

$$P\left(\left|\sum_{i=1}^{n} X_i\right| > \varepsilon\right) \le 4\exp\left\{-\frac{\varepsilon^2}{16}\left(nm^{-1}D_m + \frac{1}{3}\varepsilon Sm\right)^{-1}\right\} + 32\frac{S}{\varepsilon} n\alpha(m),$$

where $D_m = \max_{1 \le j \le 2m} \mathrm{Var}(\sum_{i=1}^{j} X_i)$.

Lemma 4 (Liebscher 1996, Lemma 2.3) *Assume $\alpha(k) \le C_1 k^{-\gamma}$, for some $\gamma > 1$. Let $\sup_{1 \le i, j \le n, i \ne j} |\mathrm{Cov}(X_i, X_j)| =: R^*(n) < \infty$ be satisfied. Moreover, let $R_m(n) < \infty$ for some m, $2\gamma/(\gamma - 1) < m \le \infty$, where $R_m(n) = \sup_{1 \le i \le n}(E|X_i|^m)^{1/m}$, for $1 \le m < \infty$, and $R_\infty(n) = \sup_{1 \le i \le n} \mathrm{ess}_{w \in \Omega}|X_i|$. Then*

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) \le n\left\{C_2(\gamma, m)(R_m(n))^{2m/(\gamma(m-2))}(R^*(n))^{1-m/(\gamma(m-2))} + R_2^2(n)\right\}$$

holds with $C_2(\gamma, m) := \frac{20\gamma - 40\gamma/m}{\gamma - 1 - 2\gamma/m} C_1^{1/\gamma}$.

Lemma 5 (Cai and Roussas 1992, Theorem 3.1) *Let $\{\xi_n, n \ge 1\}$ be a stationary α-mixing sequence of random variables with arbitrary distribution function Q and* *mixing coefficient* $\alpha(k) = O(k^{-\gamma})$, *for some* $\gamma > 3$. *Define* $R(s,t) = \sum_{k\le t} g_k(s)$, $s \in R$, $t \ge 0$, *where* $g_k(s) = I(\xi_k \le s) - Q(s)$. *Then there exists a Kiefer process* $\{K(s,t), s \in R, t \ge 0\}$ *with covariance function* $E[K(s,t)K(s',t')] = \Gamma(s,s')$ $\min(t,t')$ *and* $\Gamma(s,s')$ *is defined by*

$$\Gamma(s,s') = \mathrm{Cov}(g_1(s), g_1(s')) + \sum_{k=2}^{\infty} \left\{\mathrm{Cov}(g_1(s), g_k(s')) + \mathrm{Cov}(g_1(s'), g_k(s))\right\},$$

such that, for some $\lambda > 0$ *depending only on* γ,

$$\sup_{(0\le)t\le T} \sup_{s\in R} |R(s,t) - K(s,t)| = O\left(T^{1/2}(\ln(T))^{-\lambda}\right) \ a.s.,\ T > 0.$$

Lemma 6 *Suppose that* $\alpha(k) = O(k^{-\gamma})$, *for some* $\gamma > 3$. *Then, for any* $\tau \in (0, \tau_H)$, *we have*

- (i) (Cai and Roussas 1992) $\limsup_{n\to\infty}\{(\frac{n}{2\ln\ln(n)})^{1/2} \sup_{x\in R} |H_n(x) - H(x)|\} = 1$ *a.s., where* $H_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(Y_i \le x)$,
- (ii) $\sup_{x\in[0,\tau]} |G_n(x) - G(x)| = O((\ln\ln(n)/n)^{1/2})$ *a.s.,*
- (iii) $\sup_{x\in[0,\tau]} |F_n(x) - F(x)| = O_P(n^{-1/2})$, $\sup_{x\in[0,\tau]} |G_n(x) - G(x)| = O_P$ $(n^{-1/2})$.

Remark 5 The conclusion of Lemma 6(iii) is very useful, and interesting independently. In the censored iid framework, Breslow and Crowley (1974) in their Theorem 5 proved that empirical process of $F_n(x)$ converges weakly to a zero mean Gaussian process, from which the first conclusion in Lemma 6(iii) can be easily obtained. For the best of our knowledge, no similar result is available under dependence.

Proof of Lemma 6 (ii) Theorem 2 of Cai (2001) shows that

$$\sup_{x\in[0,\tau]} |F_n(x) - F(x)| = O\left((\ln\ln(n)/n)^{1/2}\right) \quad a.s. \tag{30}$$

Note that $1 - H_n(x) = (1 - F_n(x))(1 - G_n(x))$. Therefore, from (i) and (30) it is easy to verify

$$\sup_{x\in[0,\tau]} |G_n(x) - G(x)| = O\left((\ln\ln(n)/n)^{1/2}\right) \ a.s.$$

(iii) As it is known (see, e.g., Gill 1980), for a df F on $[0, \tau_H)$, the cumulative hazard function Λ is defined by

$$\Lambda(t) = \int_0^t \frac{\mathrm{d}F(z)}{1 - F(z-)}.$$

and $\Lambda(t) = -\ln(1-F(t))$ for the case that F is continuous. The empirical cumulative hazard function $\Lambda_n(t)$ is given by

$$\Lambda_n(t) = \int_0^t \frac{\mathrm{d}N_n(z)}{Y_n(z)},$$

where $N_n(x) = \frac{1}{n}\sum_{i=1}^n I(Y_i \leq x, \delta_i = 1)$, $Y_n(x) = \frac{1}{n}\sum_{i=1}^n I(Y_i \geq x)$. Since $F^*(x) = P(Y_1 \leq x, \delta_1 = 1) = \int_0^x [1 - G(z)]\mathrm{d}F(z)$,

$$\Lambda(t) = \int_0^t \frac{\mathrm{d}F^*(z)}{H_s(z)}.$$

From Lemma 5, there exist two Kiefer processes $\{K^{(i)}(s,t) : s, t \geq 0\}$ $(i = 1, 2)$ such that for some $\lambda > 0$ depending only on γ

$$\sup_{0\leq x<\infty} \left|N_n(x) - F^*(x) - K^{(1)}(x,n)/n\right| = O\left(n^{-1/2}(\ln(n))^{-\lambda}\right) \ a.s. \quad (31)$$

$$\sup_{0\leq x<\infty} \left|Y_n(x) - H_s(x) - K^{(2)}(x,n)/n\right| = O\left(n^{-1/2}(\ln(n))^{-\lambda}\right) \ a.s. \quad (32)$$

Put $U_i = F^*(Y_i)$, $V_i = H_s(Y_i)$. Then the continuity of $F^*(x)$ and $H_s(x)$ implies that both of U_i and V_i are uniformly distributed random variables and

$$N_n(x) = E_n(F^*(x)), \quad Y_n(x) = E_n^*(H_s(x)),$$

where $E_n(x) = n^{-1}\sum_{i=1}^n I(U_i \leq x, \delta_i = 1)$, $E_n^*(x) = n^{-1}\sum_{i=1}^n I(V_i \leq x)$. Therefore we have

$$\sup_{0\leq x<\infty} |N_n(x) - F^*(x)| \leq \sup_{0\leq x<\infty} |E_n(F^*(x)) - F^*(x) - K^{(1)}(F^*(x), n)/n| + \sup_{0\leq x<\infty} |K^{(1)}(F^*(x), n)/n|, \quad (33)$$

$$\sup_{0\leq x<\infty} |Y_n(x) - H_s(x)| \leq \sup_{0\leq x<\infty} |E_n^*(H_s(x)) - H_s(x) - K^{(2)}(H_s(x), n)/n| + \sup_{0\leq x<\infty} |K^{(2)}(H_s(x), n)/n|. \quad (34)$$

Since $B^{(i)}(y) = K^{(i)}(y,n)/\sqrt{n}$, $0 \leq y \leq 1$ is a Brownian bridge (see Csörgö and Révész 1981, page 80), and

$$P\left(\sup_{0\leq y\leq 1} |B^{(i)}(y)| > u\right) = \sum_{k\neq 0} (-1)^{k+1}\mathrm{e}^{-2k^2u^2} \to 0 \quad \text{as } u \to \infty$$

for $i = 1, 2$ from Theorem 1.5.1 of Csörgö and Révész (1981, page 43), we have

$$\sup_{0 \le x < \infty} |K^{(1)}(F^*(x), n)/n| = O_P(n^{-1/2}) \text{ and}$$
$$\sup_{0 \le x < \infty} |K^{(2)}(H_s(x), n)/n| = O_P(n^{-1/2}),$$

which, together with (31)–(34), imply

$$\sup_{0 \le x < \infty} |N_n(x) - F^*(x)| = O_P(n^{-1/2}), \quad \sup_{0 \le x < \infty} |Y_n(x) - H_s(x)| = O_P(n^{-1/2}). \tag{35}$$

Similarly, one can obtain that

$$\sup_{0 \le x < \infty} |H_n(x) - H(x)| = O_P(n^{-1/2}). \tag{36}$$

Note that,

$$\Lambda_n(t) - \Lambda(t) = \int_0^t \left[\frac{1}{Y_n(z)} - \frac{1}{H_s(z)}\right] \mathrm{d}F^*(z) + \int_0^t \frac{1}{H_s(z)} \mathrm{d}[N_n(z) - F^*(z)]$$
$$+ \int_0^t \left[\frac{1}{Y_n(z)} - \frac{1}{H_s(z)}\right] \mathrm{d}[N_n(z) - F^*(z)].$$

Hence, following the line in Cai (1998, page 386), according to the continuity of $F^*(x)$ and $H_s(x)$ and (35) one can obtain that $\sup_{x \in [0,\tau]} |\Lambda_n(t) - \Lambda(t)| = O_P(n^{-1/2})$. Further, in view of

$$F_n(t) - F(t) = (1 - F(t))[\Lambda_n(t) - \Lambda(t)] + O(\ln \ln(n)/n) \;\; a.s.$$

[cf. the proof of Theorem 2 of Cai (1998, page 388)] we have

$$\sup_{x \in [0,\tau]} |F_n(x) - F(x)| = O_P(n^{-1/2}),$$

which, together with $1 - H_n(x) = (1 - F_n(x))(1 - G_n(x))$ and (36), yields that $\sup_{x \in [0,\tau]} |G_n(x) - G(x)| = O_P(n^{-1/2})$. □

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