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Weighted least-squares estimators of parametric functions of the regression coefficients under a general linear model

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Abstract The weighted least-squares estimator of parametric functions $\mathbf{K}\boldsymbol{\beta}$ under a general linear regression model $\{\mathbf{y},\ \mathbf{X}\boldsymbol{\beta},\ \sigma^2\boldsymbol{\Sigma}\}$ is defined to be $\mathbf{K}\hat{\boldsymbol{\beta}}$, where $\hat{\boldsymbol{\beta}}$ is a vector that minimizes $(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})'\mathbf{V}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})$ for a given nonnegative definite weight matrix $\mathbf{V}$. In this paper, we study some algebraic and statistical properties of $\mathbf{K}\hat{\boldsymbol{\beta}}$ and the projection matrix associated with the estimator, such as, their ranks, unbiasedness, uniqueness, as well as equalities satisfied by the projection matrices.

Keywords General linear regression model · Parametric functions · WLSE · Projection matrix · Unbiasedness of estimator · Uniqueness of estimator

1 Introduction

Throughout this paper, $\mathbb{R}^{m\times n}$ denotes the set of all $m\times n$ real matrices. The symbols $\mathbf{A}'$, $r(\mathbf{A})$ and $\mathscr{R}(\mathbf{A})$ stand for the transpose, the rank and the range (column space) of a matrix $\mathbf{A}\in\mathbb{R}^{m\times n}$, respectively. The Moore–Penrose inverse of $\mathbf{A}\in\mathbb{R}^{m\times n}$, denoted by $\mathbf{A}^+$, is defined to be the unique solution $\mathbf{G}$ to the four matrix equations

$$\text{(i) } \mathbf{AGA}=\mathbf{A},\quad \text{(ii) } \mathbf{GAG}=\mathbf{G},\quad \text{(iii) } (\mathbf{AG})'=\mathbf{AG},\quad \text{(iv) } (\mathbf{GA})'=\mathbf{GA}.$$

A matrix $\mathbf{G}$ is called a generalized inverse of $\mathbf{A}$, denoted by $\mathbf{A}^-$, if it satisfies (i). Further, let $\mathbf{P_A}$, $\mathbf{E_A}$ and $\mathbf{F_A}$ stand for the three orthogonal projectors $\mathbf{P_A}=\mathbf{AA}^+$, $\mathbf{E_A}=\mathbf{I}_m-\mathbf{AA}^+$ and $\mathbf{F_A}=\mathbf{I}_n-\mathbf{A}^+\mathbf{A}$.

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Consider a general linear regression model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad E(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad \mathrm{Cov}(\boldsymbol{\varepsilon}) = \sigma^2\boldsymbol{\Sigma}, \tag{1}$$

or in the compact form

$$\{\mathbf{y},\ \mathbf{X}\boldsymbol{\beta},\ \sigma^2\boldsymbol{\Sigma}\}, \tag{2}$$

where $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is an observable random vector, $\mathbf{X} \in \mathbb{R}^{n\times p}$ is a known matrix of arbitrary rank, $\boldsymbol{\beta} \in \mathbb{R}^{p\times 1}$ is a vector of unknown parameters, $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is a known or unknown nonnegative definite matrix, and σ^2 is an unknown positive parameter.

Let $\mathbf{K} \in \mathbb{R}^{k\times p}$ be a given matrix. Then the product $\mathbf{K}\boldsymbol{\beta}$ is called a vector of parametric functions of $\boldsymbol{\beta}$ in (2). The vector $\mathbf{K}\boldsymbol{\beta}$ is said to be estimable under (2) if there exists a matrix $\mathbf{L} \in \mathbb{R}^{k\times n}$ such that $E(\mathbf{L}\mathbf{y}) = \mathbf{K}\boldsymbol{\beta}$ holds. It is well known that $\mathbf{K}\boldsymbol{\beta}$ is estimable under (2) if and only if

$$\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'), \tag{3}$$

see, e.g., Alalouf and Styan (1979) and Tian et al. (2008).

The most popular method for approaching estimations of parametric functions under a linear model is to apply various criteria of minimizing the (weighted) sum of squared deviations under the model. Let $\mathbf{V} \in \mathbb{R}^{n\times n}$ be a given nonnegative definite (nnd) matrix, i.e., $\mathbf{V}$ can be written as $\mathbf{V} = \mathbf{Z}\mathbf{Z}'$ for some matrix $\mathbf{Z}$. Then the weighted least-squares estimator (WLSE) of $\boldsymbol{\beta}$ in (2), denoted by $\mathrm{WLSE}(\boldsymbol{\beta})$, is defined to be

$$\mathrm{WLSE}(\boldsymbol{\beta}) = \underset{\boldsymbol{\beta}}{\mathrm{argmin}}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}); \tag{4}$$

the WLSE of the parametric functions $\mathbf{K}\boldsymbol{\beta}$ under (2) is defined to be $\mathrm{WLSE}(\mathbf{K}\boldsymbol{\beta}) = \mathbf{K}\mathrm{WLSE}(\boldsymbol{\beta})$. As is well known, the normal matrix equation corresponding to (4) is given by

$$\mathbf{X}'\mathbf{V}\mathbf{X}\boldsymbol{\beta} = \mathbf{X}'\mathbf{V}\mathbf{y}. \tag{5}$$

This equation is always consistent. Solving the equation gives the following result.

Lemma 1 *Let $\mathbf{K} \in \mathbb{R}^{k\times p}$. Then the general expression of the WLSE of $\mathbf{K}\boldsymbol{\beta}$ under (2) can be written as*

$$\mathrm{WLSE}(\mathbf{K}\boldsymbol{\beta}) = \mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{y}, \tag{6}$$

where

$$\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}} = \mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V} + \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U} \tag{7}$$

is called the projection matrix associated with WLSE($\mathbf{K}\boldsymbol{\beta}$), *in which* $\mathbf{U} \in \mathbb{R}^{p\times n}$ *is arbitrary. The expectation and covariance matrix of* WLSE($\mathbf{K}\boldsymbol{\beta}$) *are given by*

$$E[\mathrm{WLSE}(\mathbf{K}\boldsymbol{\beta})] = \mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}\boldsymbol{\beta}, \tag{8}$$

$$\mathrm{Cov}[\mathrm{WLSE}(\mathbf{K}\boldsymbol{\beta})] = \sigma^2\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}\mathbf{P}'_{\mathbf{K};\mathbf{X};\mathbf{V}}. \tag{9}$$

In particular, the WLSE of the mean vector $\mathbf{X}\boldsymbol{\beta}$ *in* (2) *is given by*

$$\mathrm{WLSE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{P}_{\mathbf{X};\mathbf{V}}\mathbf{y}, \tag{10}$$

where

$$\mathbf{P}_{\mathbf{X};\mathbf{V}} = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V} + \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U}. \tag{11}$$

In this case, the matrix $\mathbf{P}_{\mathbf{X};\mathbf{V}}$ *in* (11) *is square.*

The WLSEs given in Lemma 1 are a most useful class of estimators in regression analysis. Many estimators under a general linear model can be regarded as special cases of the WLSEs with respect to some given weight matrices. If the covariance matrix $\boldsymbol{\Sigma}$ in (2) is known, the two commonly used choices of the weight matrix $\mathbf{V}$ in (4) are given by

(i) $\mathbf{V} = (\boldsymbol{\Sigma} + \mathbf{X}\mathbf{T}\mathbf{X}')^{-}$, where $\mathbf{T}$ is a symmetric matrix such that $r(\mathbf{V}) = r[\,\mathbf{X},\ \boldsymbol{\Sigma}\,]$,
(ii) $\mathbf{V} = \boldsymbol{\Sigma}^{-1}$ if $\boldsymbol{\Sigma}$ is nonsingular.

In these cases, the WLSEs of $\mathbf{K}\boldsymbol{\beta}$ are the best linear unbiased estimators (BLUEs) of $\mathbf{K}\boldsymbol{\beta}$ under (2) for the estimable $\mathbf{K}\boldsymbol{\beta}$. If the covariance matrix $\boldsymbol{\Sigma}$ is unknown, the weight matrix $\mathbf{V}$ in (4) may be chosen according to some estimations of $\boldsymbol{\Sigma}$ in (2), or some assumptions on the pattern of $\boldsymbol{\Sigma}$. In what follows, we assume that the weight matrix $\mathbf{V}$ in (4) is given.

It can be seen from (6), (8), (9) and (10) that algebraic and statistical properties of WLSE($\mathbf{K}\boldsymbol{\beta}$) and WLSE($\mathbf{X}\boldsymbol{\beta}$) are mainly determined by the projection matrices $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$, $\mathbf{P}_{\mathbf{X};\mathbf{V}}$ and the matrix products $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}$, $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}$, $\mathbf{P}_{\mathbf{X};\mathbf{V}}\mathbf{X}$ and $\mathbf{P}_{\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}$. These facts prompt us to give a comprehensive approach to the projection matrices $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$, $\mathbf{P}_{\mathbf{X};\mathbf{V}}$ and the products $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}$, $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}$, $\mathbf{P}_{\mathbf{X};\mathbf{V}}\mathbf{X}$ and $\mathbf{P}_{\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}$ from mathematical point of view. The projection matrix $\mathbf{P}_{\mathbf{X};\mathbf{V}}$ was studied by some authors, see, e.g., Baksalary and Puntanen (1989), Mitra and Rao (1974), and Rao (1974) among others. Takane et al. (2007) and Tian and Takane (2008a,b) recently reconsidered the projection matrix $\mathbf{P}_{\mathbf{X};\mathbf{V}}$ and the corresponding estimator WLSE($\mathbf{X}\boldsymbol{\beta}$) $= \mathbf{P}_{\mathbf{X};\mathbf{V}}\mathbf{y}$ by making use of the matrix rank method, and obtained some new results on $\mathbf{P}_{\mathbf{X};\mathbf{V}}$ and WLSE($\mathbf{X}\boldsymbol{\beta}$). In this paper, we consider the following problems on the projection matrix $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ in (6) and the corresponding WLSE($\mathbf{K}\boldsymbol{\beta}$) in (6):

(I) The maximal and minimal possible ranks of $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$, $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}$ and $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}$, and their rank invariance.
(II) Uniqueness of $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$, $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}$ and $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}$.
(III) Identifying conditions for $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X} = \mathbf{K}$ to hold.

(IV) Equalities satisfied by $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$.
(V) Uniqueness of WLSE($\mathbf{K}\boldsymbol{\beta}$).
(VI) Unbiasedness of WLSE($\mathbf{K}\boldsymbol{\beta}$).
(VII) Rank of Cov[WLSE($\mathbf{K}\boldsymbol{\beta}$)].

The proofs of the main results in the paper are given in Appendix.

In what follows, we give some well-known results on ranks of matrices and a linear matrix equation. These results will be used in Sects. 2 and 3 for deriving properties of $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ and WLSE($\mathbf{K}\boldsymbol{\beta}$) $= \mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{y}$. The results in the following lemma were shown by Marsaglia and Styan (1974).

Lemma 2 *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$ *and* $\mathbf{C} \in \mathbb{R}^{l\times n}$. *Then*

$$r[\,\mathbf{A},\ \mathbf{B}] = r(\mathbf{A}) + r(\mathbf{E_A B}) = r(\mathbf{B}) + r(\mathbf{E_B A}), \tag{12}$$

$$r\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix} = r(\mathbf{A}) + r(\mathbf{CF_A}) = r(\mathbf{C}) + r(\mathbf{AF_C}). \tag{13}$$

The following result was given by Tian and Styan (2001).

Lemma 3 *Any pair of idempotent matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *of the same size satisfy*

$$r(\,\mathbf{A} - \mathbf{B}\,) = r\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix} + r[\,\mathbf{A},\ \mathbf{B}\,] - r(\mathbf{A}) - r(\mathbf{B}). \tag{14}$$

Hence $\mathbf{A} = \mathbf{B}$ *if and only if* $\mathscr{R}(\mathbf{A}) = \mathscr{R}(\mathbf{B})$ *and* $\mathscr{R}(\mathbf{A}') = \mathscr{R}(\mathbf{B}')$.

We also need two formulas shown by De Moor and Golub (1991) on the extremal ranks of the matrix expression $\mathbf{A} - \mathbf{BZC}$; see also Tian (2002) and Tian and Cheng (2003).

Lemma 4 *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$ *and* $\mathbf{C} \in \mathbb{R}^{l\times n}$ *be given. Then*

$$\max_{\mathbf{Z}\in\mathbb{R}^{k\times l}} r(\,\mathbf{A} - \mathbf{BZC}\,) = \min\left\{ r[\,\mathbf{A},\ \mathbf{B}\,],\ \ r\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix}\right\}, \tag{15}$$

$$\min_{\mathbf{Z}\in\mathbb{R}^{k\times l}} r(\,\mathbf{A} - \mathbf{BZC}\,) = r[\,\mathbf{A},\ \mathbf{B}\,] + r\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix} - r\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{0}\end{bmatrix}. \tag{16}$$

The following result is well known; see Penrose (1955).

Lemma 5 *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$ *and* $\mathbf{C} \in \mathbb{R}^{l\times n}$ *be given. Then the matrix equation* $\mathbf{BZC} = \mathbf{A}$ *is solvable for* $\mathbf{Z}$ *if and only if* $\mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{B})$ *and* $\mathscr{R}(\mathbf{A}') \subseteq \mathscr{R}(\mathbf{C}')$, *or equivalently,* $\mathbf{BB}^+\mathbf{AC}^+\mathbf{C} = \mathbf{A}$. *In this case, the general solution can be written as* $\mathbf{Z} = \mathbf{B}^+\mathbf{AC}^+ + \mathbf{F_B U}_1 + \mathbf{U}_2\mathbf{E_C}$, *where* $\mathbf{U}_1,\ \mathbf{U}_2 \in \mathbb{R}^{k\times l}$ *are arbitrary.*

To simplify some matrix operations in Sects. 2 and 3, we use the following simple results on the Moore–Penrose inverses:

$$\mathbf{A}^+ = (\mathbf{A}'\mathbf{A})^+\mathbf{A}' = \mathbf{A}'(\mathbf{AA}')^+, \tag{17}$$

$$(\mathbf{X}'\mathbf{VX})(\mathbf{X}'\mathbf{VX})^+ = (\mathbf{X}'\mathbf{V})(\mathbf{X}'\mathbf{V})^+,\ \ (\mathbf{X}'\mathbf{VX})^+(\mathbf{X}'\mathbf{VX}) = (\mathbf{VX})^+(\mathbf{VX}). \tag{18}$$

2 Properties of the projection matrix $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$

Notice that $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$, $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}$ and $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}$ in (7), (8) and (9) are special cases of a general product $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{Q}$. We first show some algebraic properties of the product $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{Q}$.

Theorem 1 *Let* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ *be as given in* (7) *and let* $\mathbf{Q} \in \mathbb{R}^{n\times q}$. *Then*

$$\max_{\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{Q}) = \min\left\{ r\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{X}'\mathbf{V}\mathbf{Q} & \mathbf{X}'\mathbf{V}\mathbf{X} \end{bmatrix} - r(\mathbf{V}\mathbf{X}), \ \ r(\mathbf{Q}) \right\}, \tag{19}$$

$$\min_{\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{Q}) = r\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{X}'\mathbf{V}\mathbf{Q} & \mathbf{X}'\mathbf{V}\mathbf{X} \end{bmatrix} - r\begin{bmatrix} \mathbf{K} \\ \mathbf{X}'\mathbf{V}\mathbf{X} \end{bmatrix}. \tag{20}$$

Hence,

(a) *The rank of the product* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{Q}$ *is invariant if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or*

$$r\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{X}'\mathbf{V}\mathbf{Q} & \mathbf{X}'\mathbf{V}\mathbf{X} \end{bmatrix} = r\begin{bmatrix} \mathbf{K} \\ \mathbf{X}'\mathbf{V}\mathbf{X} \end{bmatrix} + r(\mathbf{Q}).$$

(b) *The product* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{Q}$ *is unique if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or* $\mathbf{Q} = \mathbf{0}$.

Applying Theorem 1 to $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$, $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}$ and $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}$ leads to the following corollary.

Corollary 1 *Let* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ *be as given in* (7). *Then,*

(a) *The maximal and minimal ranks of* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ *are given by*

$$\max_{\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}) = r(\mathbf{K}), \tag{21}$$

$$\min_{\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}) = r(\mathbf{K}) + r(\mathbf{V}\mathbf{X}) - r\begin{bmatrix} \mathbf{K} \\ \mathbf{V}\mathbf{X} \end{bmatrix}. \tag{22}$$

(b) *The rank of* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ *is invariant if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$.

(c) $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ *is unique if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$, *in which case,* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}} = \mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}$.

(d) *The maximal and minimal ranks of the product* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}$ *are given by*

$$\max_{\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}) = \min\{ r(\mathbf{K}), \ \ r(\mathbf{X}) \}, \tag{23}$$

$$\min_{\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}) = r(\mathbf{K}) + r(\mathbf{V}\mathbf{X}) - r\begin{bmatrix} \mathbf{K} \\ \mathbf{V}\mathbf{X} \end{bmatrix}. \tag{24}$$

(e) *The rank of* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}$ *is invariant if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or* $r(\mathbf{K}) + r(\mathbf{V}\mathbf{X}) = r[\,\mathbf{K}', \mathbf{X}'\,\mathbf{V}\,] + r(\mathbf{X})$.

(f) *The product* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}$ *is unique if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or* $\mathbf{X} = \mathbf{0}$.

(g) *The maximal and minimal ranks of* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{\Sigma}$ *are given by*

$$\max_{\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{\Sigma}) = \min\left\{ r\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{X}'\mathbf{V}\mathbf{\Sigma} & \mathbf{X}'\mathbf{V}\mathbf{X} \end{bmatrix} - r(\mathbf{V}\mathbf{X}), \quad r(\mathbf{\Sigma}) \right\}, \tag{25}$$

$$\min_{\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{\Sigma}) = r\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{X}'\mathbf{V}\mathbf{\Sigma} & \mathbf{X}'\mathbf{V}\mathbf{X} \end{bmatrix} - r\begin{bmatrix} \mathbf{K} \\ \mathbf{X}'\mathbf{V}\mathbf{X} \end{bmatrix}. \tag{26}$$

(h) *The product* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{\Sigma}$ *is unique if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or* $\mathbf{\Sigma} = \mathbf{0}$.

(i) *The rank of* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{\Sigma}$ *is invariant if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or*

$$r\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{X}'\mathbf{V}\mathbf{\Sigma} & \mathbf{X}'\mathbf{V}\mathbf{X} \end{bmatrix} = r\begin{bmatrix} \mathbf{K} \\ \mathbf{X}'\mathbf{V}\mathbf{X} \end{bmatrix} + r(\mathbf{\Sigma}).$$

3 Existence of unbiased WLSE($\mathbf{K}\boldsymbol{\beta}$) and its properties

One of the most important and desirable properties of WLSE($\mathbf{K}\boldsymbol{\beta}$) is its unbiasedness for $\mathbf{K}\boldsymbol{\beta}$ under (2). It can be seen from (8) that the estimator WLSE($\mathbf{K}\boldsymbol{\beta}$) is unbiased for $\mathbf{K}\boldsymbol{\beta}$ under (2) if and only if

$$\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}\boldsymbol{\beta} = \mathbf{K}\boldsymbol{\beta}. \tag{27}$$

Lemma 6 *Let* WLSE($\mathbf{K}\boldsymbol{\beta}$) *be as given in* (6). *Then the estimator is unbiased for* $\mathbf{K}\boldsymbol{\beta}$ *under* (2) *if and only if* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X}\boldsymbol{\beta} = \mathbf{K}\boldsymbol{\beta}$. *In particular, if*

$$\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X} = \mathbf{K}, \tag{28}$$

then the estimator WLSE($\mathbf{K}\boldsymbol{\beta}$) *in* (6) *is unbiased for* $\mathbf{K}\boldsymbol{\beta}$ *under* (2).

Solving the equation in (28), we obtain the following result.

Theorem 2 *Let* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ *be as given in* (7). *Then,*

(a) *There exists a* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ *such that* (28) *holds if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}')$, *namely,* $\mathbf{K}\boldsymbol{\beta}$ *is estimable under* (2).

(b) *Under* (a), *the general expression of* $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ *satisfying* (28) *can be written as*

$$\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} = \tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} + \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{X}^{+} + \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{W}\mathbf{E}_{\mathbf{X}}, \tag{29}$$

or equivalently,

$$\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} = \mathbf{K}\mathbf{X}^{-} + \tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}(\mathbf{I}_n - \mathbf{X}\mathbf{X}^{-}), \tag{30}$$

where $\tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} = \mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}$, $\mathbf{W} \in \mathbb{R}^{p\times n}$ *is arbitrary, and* $\mathbf{X}^{-}$ *is any generalized inverse of* $\mathbf{X}$.

More properties of the projection matrix $\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ in (29) are given below.

Theorem 3 *Let $\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ be as given in (29). Then,*

(a) $\mathbf{Z} \in \{\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}\}$ *if and only if* $\mathbf{ZX} = \mathbf{K}$ *and* $r[\mathbf{Z}',\ \mathbf{VX}\,] = r(\mathbf{X})$.

(b) $r(\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}) = r(\mathbf{K})$.

(c) $\mathscr{R}(\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}) = \mathscr{R}(\mathbf{K})$.

(d) $\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ *is unique for any* $\mathbf{X}^-$ *if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or* $r(\mathbf{X}) = n$. *In this case,*
  (i) $\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} = \mathbf{K}(\mathbf{X}'\mathbf{VX})^+\mathbf{X}'\mathbf{V}$ *if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$,
  (ii) $\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} = \mathbf{KX}^+$ *if* $r(\mathbf{X}) = n$.

(e) $\max\limits_{\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}) = \min\left\{ r\begin{bmatrix} \mathbf{X}'\mathbf{VX} & \mathbf{X}'\mathbf{V}\boldsymbol{\Sigma} \\ \mathbf{K} & \mathbf{0} \end{bmatrix} - r(\mathbf{VX}),\ \ r\begin{bmatrix} \mathbf{X} & \boldsymbol{\Sigma} \\ \mathbf{K} & \mathbf{0} \end{bmatrix} - r(\mathbf{X}) \right\}.$

(f) $\min\limits_{\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}) = r\begin{bmatrix} \mathbf{X}'\mathbf{VX} & \mathbf{X}'\mathbf{V}\boldsymbol{\Sigma} \\ \mathbf{K} & \mathbf{0} \end{bmatrix} + r\begin{bmatrix} \mathbf{X} & \boldsymbol{\Sigma} \\ \mathbf{K} & \mathbf{0} \end{bmatrix} - r\begin{bmatrix} \mathbf{X} & \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K} \\ \mathbf{VX} & \mathbf{0} & \mathbf{VX} \end{bmatrix}.$

(g) $\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}$ *is unique if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or* $\mathscr{R}(\boldsymbol{\Sigma}) \subseteq \mathscr{R}(\mathbf{X})$.

Applying Theorems 2 and 3 to (11) gives the following results.

Corollary 2 *Let $\mathbf{P}_{\mathbf{X};\mathbf{V}}$ be as given in (11). Then,*

(a) *There always exists a* $\mathbf{P}_{\mathbf{X};\mathbf{V}}$ *such that*

$$\mathbf{P}_{\mathbf{X};\mathbf{V}}\mathbf{X} = \mathbf{X} \tag{31}$$

holds. Moreover, the general expression of $\mathbf{P}_{\mathbf{X};\mathbf{V}}$ *satisfying* (31) *can be written as*

$$\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}} = \tilde{\mathbf{P}}_{\mathbf{X};\mathbf{V}} + \mathbf{XF}_{\mathbf{VX}}\mathbf{X}^+ + \mathbf{XF}_{\mathbf{VX}}\mathbf{WE}_{\mathbf{X}}, \tag{32}$$

or equivalently,

$$\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}} = \mathbf{XX}^- + \tilde{\mathbf{P}}_{\mathbf{X};\mathbf{V}}(\mathbf{I}_n - \mathbf{XX}^-), \tag{33}$$

where $\tilde{\mathbf{P}}_{\mathbf{X};\mathbf{V}} = \mathbf{X}(\mathbf{X}'\mathbf{VX})^+\mathbf{X}'\mathbf{V}$, $\mathbf{W} \in \mathbb{R}^{p\times n}$ *is arbitrary, and* $\mathbf{X}^-$ *is any generalized inverse of* $\mathbf{X}$.

(b) $\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}$ *is idempotent for any* $\mathbf{X}^-$.

(c) $r(\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}) = r(\mathbf{X})$.

(d) $\mathscr{R}(\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}) = \mathscr{R}(\mathbf{X})$.

(e) $\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}$ *is unique for any* $\mathbf{X}^-$ *if and only if* $\mathscr{R}(\mathbf{X}') = \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or* $r(\mathbf{X}) = n$. *In this case,* $\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}} = \mathbf{X}(\mathbf{X}'\mathbf{VX})^+\mathbf{X}'\mathbf{V}$ *for* $\mathscr{R}(\mathbf{X}') = \mathscr{R}(\mathbf{X}'\mathbf{V})$, *or* $\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}} = \mathbf{I}_n$ *for* $r(\mathbf{X}) = n$.

(f) *The product* $\mathbf{V}\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}$ *is unique for any* $\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}$, *and*

$$\mathbf{V}\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}} = \mathbf{VX}(\mathbf{X}'\mathbf{VX})^+\mathbf{X}'\mathbf{V} = \mathbf{V}^{1/2}\mathbf{P}_{\mathbf{V}^{1/2}\mathbf{X}}\mathbf{V}^{1/2},$$
$$\mathscr{R}(\mathbf{V}\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}) = \mathscr{R}(\mathbf{VX}),\ \ r(\mathbf{V}\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}) = r(\mathbf{X}'\mathbf{V}\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}) = r(\mathbf{VX}),$$

where $\mathbf{V}^{1/2}$ *is the nnd square root of* $\mathbf{V}$.

(g) $\max_{\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}} r(\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}) = \min\{r(\mathbf{X}) + r(\boldsymbol{\Sigma}\mathbf{V}\mathbf{X}) - r(\mathbf{V}\mathbf{X}),\ \ r(\boldsymbol{\Sigma})\}$.

(h) $\min_{\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}} r(\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}) = r(\mathbf{X}) + r(\boldsymbol{\Sigma}\mathbf{V}\mathbf{X}) + r(\boldsymbol{\Sigma}) - r\begin{bmatrix} \mathbf{X} & \boldsymbol{\Sigma} \\ \mathbf{V}\mathbf{X} & \mathbf{0} \end{bmatrix}$.

(i) $\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}$ *is unique if and only if* $\mathscr{R}(\mathbf{X}') = \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or* $\mathscr{R}(\boldsymbol{\Sigma}) \subseteq \mathscr{R}(\mathbf{X})$.

Let $\mathbf{W} = \mathbf{0}$. Then (32) reduces to

$$\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}} = \tilde{\mathbf{P}}_{\mathbf{X};\mathbf{V}} + \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{X}^{+} = \mathbf{P}_{\mathbf{X}} + \tilde{\mathbf{P}}_{\mathbf{X};\mathbf{V}}(\mathbf{I}_n - \mathbf{P}_{\mathbf{X}}). \tag{34}$$

Some properties of the projector $\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}$ in (34) are given below.

Theorem 4 *Let* $\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}$ *be as given in* (34)*. Then the following statements are equivalent:*

(a) $\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}} = \hat{\mathbf{P}}'_{\mathbf{X};\mathbf{V}}$.
(b) $\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}} = \mathbf{P}_{\mathbf{X}}$.
(c) $r[\,\mathbf{V}\mathbf{X},\ \mathbf{X}\,] = r(\mathbf{X})$, *i.e.*, $\mathscr{R}(\mathbf{V}\mathbf{X}) \subseteq \mathscr{R}(\mathbf{X})$.

Some properties of WLSE($\mathbf{K}\boldsymbol{\beta}$) and WLSE($\mathbf{X}\boldsymbol{\beta}$) corresponding to $\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ and $\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}$ in (29) and (32) are given in the following two theorems.

Theorem 5 *Assume* $\mathbf{K}\boldsymbol{\beta}$ *is estimable under* (2)*, and let* $\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}$ *be as given in* (29)*. Then,*

(a) *The corresponding estimator* WLSE($\mathbf{K}\boldsymbol{\beta}$) *in* (6) *can be written as*

$$\text{WLSE}(\mathbf{K}\boldsymbol{\beta}) = \hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{y} = (\tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} + \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{X}^{+} + \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{W}\mathbf{E}_{\mathbf{X}})\mathbf{y}, \tag{35}$$

where $\tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} = \mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}$*, and* $\mathbf{W} \in \mathbb{R}^{p\times n}$ *is arbitrary.*

(b) *The expectation and the covariance matrix of* WLSE($\mathbf{K}\boldsymbol{\beta}$) *in* (35) *are given by*

$$E[\text{WLSE}(\mathbf{K}\boldsymbol{\beta})] = \mathbf{K}\boldsymbol{\beta} \ \ \textit{and} \ \ \text{Cov}[\text{WLSE}(\mathbf{K}\boldsymbol{\beta})] = \sigma^2\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}\hat{\mathbf{P}}'_{\mathbf{K};\mathbf{X};\mathbf{V}}. \tag{36}$$

(c) WLSE($\mathbf{K}\boldsymbol{\beta}$) *in* (35) *is unique if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or* $\mathbf{y} \in \mathscr{R}(\mathbf{X})$*. In this case,*
   (i) WLSE($\mathbf{K}\boldsymbol{\beta}$) $= \mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{y}$ *if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$,
   (ii) WLSE($\mathbf{K}\boldsymbol{\beta}$) $= \mathbf{K}\mathbf{X}^{+}\mathbf{y}$ *if* $\mathbf{y} \in \mathscr{R}(\mathbf{X})$.

(d) $$\max r(\text{Cov}[\text{WLSE}(\mathbf{K}\boldsymbol{\beta})]) = \min\left\{ r\begin{bmatrix} \mathbf{X}'\mathbf{V}\mathbf{X} & \mathbf{X}'\mathbf{V}\boldsymbol{\Sigma} \\ \mathbf{K} & \mathbf{0} \end{bmatrix} - r(\mathbf{V}\mathbf{X}),\ r\begin{bmatrix} \mathbf{X} & \boldsymbol{\Sigma} \\ \mathbf{K} & \mathbf{0} \end{bmatrix} - r(\mathbf{X}) \right\}.$$

(e) $$\min r(\text{Cov}[\text{WLSE}(\mathbf{K}\boldsymbol{\beta})]) = r\begin{bmatrix} \mathbf{X}'\mathbf{V}\mathbf{X} & \mathbf{X}'\mathbf{V}\boldsymbol{\Sigma} \\ \mathbf{K} & \mathbf{0} \end{bmatrix} + r\begin{bmatrix} \mathbf{X} & \boldsymbol{\Sigma} \\ \mathbf{K} & \mathbf{0} \end{bmatrix} - r\begin{bmatrix} \mathbf{X} & \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K} \\ \mathbf{V}\mathbf{X} & \mathbf{0} & \mathbf{V}\mathbf{X} \end{bmatrix}.$$

(f) Cov[WLSE($\mathbf{K}\boldsymbol{\beta}$)] *is unique if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or* $\mathscr{R}(\boldsymbol{\Sigma}) \subseteq \mathscr{R}(\mathbf{X})$.

Equations (35) and (36) reveal such an unexpected fact that for any given weight matrix $\mathbf{V}$ and any estimable parametric functions $\mathbf{K}\boldsymbol{\beta}$, there always exists a WLSE($\mathbf{K}\boldsymbol{\beta}$) that is unbiased for $\mathbf{K}\boldsymbol{\beta}$ under (2). In such case, we can construct the weight matrix $\mathbf{V}$ and the arbitrary matrix $\mathbf{W}$ in (35) such that the unbiased WLSE($\mathbf{K}\boldsymbol{\beta}$) has some prescribed properties.

Theorem 6 *Let* $\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}$ *be as given in* (32). *Then,*

(a) *The corresponding estimator* WLSE($\mathbf{X}\boldsymbol{\beta}$) $= \hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}\mathbf{y}$ *can be written as*

$$\mathrm{WLSE}(\mathbf{X}\boldsymbol{\beta}) = \hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}\mathbf{y} = (\,\tilde{\mathbf{P}}_{\mathbf{X};\mathbf{V}} + \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{X}^{+} + \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{W}\mathbf{E}_{\mathbf{X}}\,)\mathbf{y}, \tag{37}$$

where $\tilde{\mathbf{P}}_{\mathbf{X};\mathbf{V}} = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}$, *and* $\mathbf{W} \in \mathbb{R}^{p\times n}$ *is arbitrary.*

(b) *The expectation and the covariance matrix of* WLSE($\mathbf{X}\boldsymbol{\beta}$) *in* (37) *are given by*

$$E[\mathrm{WLSE}(\mathbf{X}\boldsymbol{\beta})] = \mathbf{X}\boldsymbol{\beta} \quad \textit{and} \quad \mathrm{Cov}[\mathrm{WLSE}(\mathbf{X}\boldsymbol{\beta})] = \sigma^2\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}\hat{\mathbf{P}}'_{\mathbf{X};\mathbf{V}}. \tag{38}$$

(c) WLSE($\mathbf{X}\boldsymbol{\beta}$) *in* (37) *is unique if and only if* $\mathscr{R}(\mathbf{X}') = \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or* $\mathbf{y} \in \mathscr{R}(\mathbf{X})$. *In this case,*
  (i) $\mathrm{WLSE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{y}$ *if* $\mathscr{R}(\mathbf{X}') = \mathscr{R}(\mathbf{X}'\mathbf{V})$,
  (ii) $\mathrm{WLSE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{P}_{\mathbf{X}}\mathbf{y}$ *if* $\mathbf{y} \in \mathscr{R}(\mathbf{X})$.

(d) $\max r(\mathrm{Cov}[\mathrm{WLSE}(\mathbf{X}\boldsymbol{\beta})]) = \min\{\, r(\boldsymbol{\Sigma}\mathbf{V}\mathbf{X}) + r(\mathbf{X}) - r(\mathbf{V}\mathbf{X}),\ \ r(\boldsymbol{\Sigma})\,\}$.

(e) $\min r(\mathrm{Cov}[\mathrm{WLSE}(\mathbf{X}\boldsymbol{\beta})]) = r(\boldsymbol{\Sigma}\mathbf{V}\mathbf{X}) + r(\mathbf{X}) + r(\boldsymbol{\Sigma}) - r\begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{X} \\ \mathbf{0} & \mathbf{V}\mathbf{X} \end{bmatrix}$.

(f) $\mathrm{Cov}[\mathrm{WLSE}(\mathbf{X}\boldsymbol{\beta})]$ *is unique if and only if* $\mathscr{R}(\mathbf{X}') = \mathscr{R}(\mathbf{X}'\mathbf{V})$ *or* $\mathscr{R}(\boldsymbol{\Sigma}) \subseteq \mathscr{R}(\mathbf{X})$.

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Appendix

Proof of Theorem 1 It is easily seen from (7) that the product $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{Q}$ can be written as

$$\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{Q} = \mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{Q} + \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U}\mathbf{Q}, \tag{39}$$

where the matrix $\mathbf{U}$ is arbitrary. Recall that elementary block matrix operations (EBMOs) do not change the rank of a matrix. Applying (15) and (16) and EMBOs to this expression gives

$$\begin{aligned}
\max_{\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\,\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{Q}\,) &= \max_{\mathbf{U}} r[\,\mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{Q} + \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U}\mathbf{Q}\,] \\
&= \min\{\, r[\,\mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{Q},\ \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\,],\ \ r(\mathbf{Q})\,\}, \\
\min_{\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\,\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{Q}\,) &= \min_{\mathbf{U}} r[\,\mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{Q} + \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U}\mathbf{Q}\,] \\
&= r[\,\mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{Q},\ \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\,] - r(\mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}),
\end{aligned}$$

where by (13)

$$r[\,\mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{Q},\ \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\,] = r\begin{bmatrix} \mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{Q} & \mathbf{K} \\ \mathbf{0} & \mathbf{V}\mathbf{X} \end{bmatrix} - r(\mathbf{V}\mathbf{X})$$
$$= r\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ -\mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{Q} & \mathbf{V}\mathbf{X} \end{bmatrix} - r(\mathbf{V}\mathbf{X})$$
$$= r\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{X}'\mathbf{V}\mathbf{Q} & \mathbf{X}'\mathbf{V}\mathbf{X} \end{bmatrix} - r(\mathbf{V}\mathbf{X}).$$

Hence we have (19) and (20). Result (a) follows from (19) and (20). Result (b) is obvious from (39). □

Proof of Theorem 2 Substituting (7) into (28) and simplifying by (18) gives

$$\mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U}\mathbf{X} = \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}. \tag{40}$$

From Lemma 5, the equation in (40) is solvable for $\mathbf{U}$ if and only if

$$r\begin{bmatrix} \mathbf{X} \\ \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}} \end{bmatrix} = r(\mathbf{X}). \tag{41}$$

Applying EBMOs gives

$$r\begin{bmatrix} \mathbf{X} \\ \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}} \end{bmatrix} = r\begin{bmatrix} \mathbf{X} \\ \mathbf{K} - \mathbf{K}(\mathbf{V}\mathbf{X})^{+}(\mathbf{V}\mathbf{X}) \end{bmatrix} = r\begin{bmatrix} \mathbf{X} \\ \mathbf{K} \end{bmatrix},$$

so that (41) is equivalent to $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}')$. In this case, it can derived from Lemma 5 that the general solution of (40) is

$$\mathbf{U} = \mathbf{X}^{+} + [\,\mathbf{I}_n - (\mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}})^{+}(\mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}})\,]\mathbf{U}_1 + \mathbf{U}_2\mathbf{E}_{\mathbf{X}}, \tag{42}$$

where $\mathbf{U}_1,\ \mathbf{U}_2 \in \mathbb{R}^{p\times n}$ are arbitrary. Substituting (42) into (7) yields

$$\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} = \tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} + \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{X}^{+} + \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U}_2\mathbf{E}_{\mathbf{X}}, \tag{43}$$

establishing (29). Eq. (30) is obtained by setting $\mathbf{U} = \mathbf{X}^{-}$ in (7):

$$\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} = \tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} + [\,\mathbf{K} - \mathbf{K}(\mathbf{V}\mathbf{X})^{+}(\mathbf{V}\mathbf{X})\,]\mathbf{X}^{-} = \mathbf{K}\mathbf{X}^{-} + \tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}(\,\mathbf{I}_n - \mathbf{X}\mathbf{X}^{-}\,).$$

In fact, recall that the general expression of $\mathbf{X}^{-}$ can be written as $\mathbf{X}^{-} = \mathbf{X}^{+} + \mathbf{F}_{\mathbf{X}}\mathbf{U}_1 + \mathbf{U}_2\mathbf{E}_{\mathbf{X}}$, where $\mathbf{U}_1$ and $\mathbf{U}_2$ are arbitrary matrices. Substituting this $\mathbf{X}^{-}$ into (30) yields

$$\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} = \tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} + \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{X}^{-} = \tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} + \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{X}^{+} + \mathbf{K}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U}_2\mathbf{E}_{\mathbf{X}},$$

which is the same as (29). □

Proof of Theorem 3 It can be seen from (29) that $\mathbf{Z} \in \{\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}\}$ if and only if the following matrix equation

$$\tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} + \mathbf{K}\mathbf{F}_{\mathbf{VX}}\mathbf{X}^{+} + \mathbf{K}\mathbf{F}_{\mathbf{VX}}\mathbf{W}\mathbf{E}_{\mathbf{X}} = \mathbf{Z} \tag{44}$$

is solvable for $\mathbf{W}$. By Lemma 5, the equation in (44) is solvable for $\mathbf{W}$ if and only if

$$r[\,\mathbf{G},\ \mathbf{K}\mathbf{F}_{\mathbf{VX}}\,] = r(\mathbf{K}\mathbf{F}_{\mathbf{VX}}) \quad \text{and} \quad r\begin{bmatrix} \mathbf{G} \\ \mathbf{E}_{\mathbf{X}} \end{bmatrix} = r(\mathbf{E}_{\mathbf{X}}), \tag{45}$$

where $\mathbf{G} = \mathbf{Z} - \tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} - \mathbf{K}\mathbf{F}_{\mathbf{VX}}\mathbf{X}^{+}$. It can be derived from (12), (13), (18) and EMBOs that $r(\mathbf{X}\mathbf{F}_{\mathbf{VX}}) = r(\mathbf{X}) - r(\mathbf{VX})$, $r(\mathbf{E}_{\mathbf{X}}) = n - r(\mathbf{X})$, and

$$\begin{aligned}
r[\,\mathbf{G},\ \mathbf{K}\mathbf{F}_{\mathbf{VX}}\,] &= r\begin{bmatrix} \mathbf{Z} - \tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} - \mathbf{K}\mathbf{F}_{\mathbf{VX}}\mathbf{X}^{+} & \mathbf{K} \\ \mathbf{0} & \mathbf{VX} \end{bmatrix} - r(\mathbf{VX}) \\
&= r\begin{bmatrix} \mathbf{Z} & \mathbf{K} \\ \mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} & \mathbf{VX} \end{bmatrix} - r(\mathbf{VX}) \\
&= r\begin{bmatrix} \mathbf{Z} & \mathbf{K} - \mathbf{ZX} \\ \mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} & \mathbf{0} \end{bmatrix} - r(\mathbf{VX}) \\
&= r\begin{bmatrix} \mathbf{Z} & \mathbf{K} - \mathbf{ZX} \\ \mathbf{X}'\mathbf{V} & \mathbf{0} \end{bmatrix} - r(\mathbf{VX}), \\
r\begin{bmatrix} \mathbf{G} \\ \mathbf{E}_{\mathbf{X}} \end{bmatrix} &= r\begin{bmatrix} \mathbf{Z} - \tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} - \mathbf{K}\mathbf{F}_{\mathbf{VX}}\mathbf{X}^{+} & \mathbf{0} \\ \mathbf{I}_n & \mathbf{X} \end{bmatrix} - r(\mathbf{X}) \\
&= r\begin{bmatrix} \mathbf{0} & -\mathbf{ZX} + \mathbf{K}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{VX} + \mathbf{K}\mathbf{F}_{\mathbf{VX}}\mathbf{X}^{+}\mathbf{X} \\ \mathbf{I}_n & \mathbf{0} \end{bmatrix} - r(\mathbf{X}) \\
&= r\begin{bmatrix} \mathbf{0} & \mathbf{K} - \mathbf{ZX} \\ \mathbf{I}_n & \mathbf{0} \end{bmatrix} - r(\mathbf{X}) \\
&= n + r(\mathbf{K} - \mathbf{ZX}) - r(\mathbf{X}).
\end{aligned}$$

Hence (45) is equivalent to $r\begin{bmatrix} \mathbf{Z} & \mathbf{K} - \mathbf{ZX} \\ \mathbf{X}'\mathbf{V} & \mathbf{0} \end{bmatrix} = r(\mathbf{X})$ and $\mathbf{ZX} = \mathbf{K}$, establishing (a). It follows from $\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}\mathbf{X} = \mathbf{K}$ that $r(\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}) \geq r(\mathbf{K})$. Also note from (29) that $r(\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}) \leq r(\mathbf{K})$. Thus $r(\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}) = r(\mathbf{K})$ for any $\mathbf{W}$ in (29), as required for (b). The range equality in (c) is obvious from (29) and (b). Result (d) follows directly from (29). Let $\mathbf{G} = \tilde{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}} + \mathbf{K}\mathbf{F}_{\mathbf{VX}}\mathbf{X}^{+}$. Then it can be derived from (15), (16) and EBMOs that

$$\max_{\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}) = \max_{\mathbf{W}} r(\,\mathbf{G}\boldsymbol{\Sigma} + \mathbf{K}\mathbf{F}_{\mathbf{VX}}\mathbf{W}\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}\,)$$
$$= \min\left\{ r[\,\mathbf{G}\boldsymbol{\Sigma},\ \mathbf{K}\mathbf{F}_{\mathbf{VX}}\,],\ r\begin{bmatrix} \mathbf{G}\boldsymbol{\Sigma} \\ \mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma} \end{bmatrix} \right\}, \tag{46}$$

$$\min_{\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}} r(\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}) = \min_{\mathbf{W}} r(\,\mathbf{G}\boldsymbol{\Sigma} + \mathbf{K}\mathbf{F}_{\mathbf{VX}}\mathbf{W}\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}\,)$$
$$= r[\,\mathbf{G}\boldsymbol{\Sigma},\ \mathbf{K}\mathbf{F}_{\mathbf{VX}}\,] + r\begin{bmatrix} \mathbf{G}\boldsymbol{\Sigma} \\ \mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma} \end{bmatrix} - r\begin{bmatrix} \mathbf{G}\boldsymbol{\Sigma} & \mathbf{K}\mathbf{F}_{\mathbf{VX}} \\ \mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma} & \mathbf{0} \end{bmatrix}. \tag{47}$$

Applying (12), (13), (18) and simplifying by EMBOs gives rise to

$$r[\,\mathbf{G}\boldsymbol{\Sigma},\ \mathbf{K}\mathbf{F}_{\mathbf{VX}}\,] = r\begin{bmatrix} \mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\boldsymbol{\Sigma} + \mathbf{K}\mathbf{F}_{\mathbf{VX}}\mathbf{X}^{+}\boldsymbol{\Sigma} & \mathbf{K} \\ \mathbf{0} & \mathbf{VX} \end{bmatrix} - r(\mathbf{VX})$$
$$= r\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ -\mathbf{VX}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\boldsymbol{\Sigma} & \mathbf{VX} \end{bmatrix} - r(\mathbf{VX})$$
$$= r\begin{bmatrix} \mathbf{X}'\mathbf{V}\mathbf{X} & \mathbf{X}'\mathbf{V}\boldsymbol{\Sigma} \\ \mathbf{K} & \mathbf{0} \end{bmatrix} - r(\mathbf{VX}),$$
$$r\begin{bmatrix} \mathbf{G}\boldsymbol{\Sigma} \\ \mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma} \end{bmatrix} = r\begin{bmatrix} \mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\boldsymbol{\Sigma} + \mathbf{K}\mathbf{F}_{\mathbf{VX}}\mathbf{X}^{+}\boldsymbol{\Sigma} & \mathbf{0} \\ \boldsymbol{\Sigma} & \mathbf{X} \end{bmatrix} - r(\mathbf{X})$$
$$= r\begin{bmatrix} \mathbf{X} & \boldsymbol{\Sigma} \\ \mathbf{K} & \mathbf{0} \end{bmatrix} - r(\mathbf{X}),$$
$$r\begin{bmatrix} \mathbf{G}\boldsymbol{\Sigma} & \mathbf{K}\mathbf{F}_{\mathbf{VX}} \\ \mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma} & \mathbf{0} \end{bmatrix} = r\begin{bmatrix} \mathbf{K}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\boldsymbol{\Sigma} + \mathbf{K}\mathbf{F}_{\mathbf{VX}}\mathbf{X}^{+}\boldsymbol{\Sigma} & \mathbf{K} & \mathbf{0} \\ \boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X} \\ \mathbf{0} & \mathbf{VX} & \mathbf{0} \end{bmatrix} - r(\mathbf{VX}) - r(\mathbf{X})$$
$$= r\begin{bmatrix} \mathbf{0} & \mathbf{K} & \mathbf{0} \\ \boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X} \\ -\mathbf{VX}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\boldsymbol{\Sigma} & \mathbf{VX} & \mathbf{0} \end{bmatrix} - r(\mathbf{VX}) - r(\mathbf{X})$$
$$= r\begin{bmatrix} \mathbf{X} & \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K} \\ \mathbf{VX} & \mathbf{0} & \mathbf{VX} \end{bmatrix} - r(\mathbf{VX}) - r(\mathbf{X}).$$

Substituting these rank equalities into (46) and (47) gives (e) and (f). It can be derived from (29) that $\hat{\mathbf{P}}_{\mathbf{K};\mathbf{X};\mathbf{V}}\boldsymbol{\Sigma}$ is unique if and only if $\mathbf{K}\mathbf{F}_{\mathbf{VX}} = \mathbf{0}$ or $\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma} = \mathbf{0}$. Hence (g) follows. □

Proof of Corollary 2 Results (a), (c), (d), (e), (g), (h) and (i) follow from Theorem 3. Note that

$$(\mathbf{X}\mathbf{X}^{-})^2 = \mathbf{X}\mathbf{X}^{-} \quad \text{and} \quad \mathbf{X}\mathbf{X}^{-}\tilde{\mathbf{P}}_{\mathbf{X};\mathbf{V}}(\,\mathbf{I}_n - \mathbf{X}\mathbf{X}^{-}\,) = \tilde{\mathbf{P}}_{\mathbf{X};\mathbf{V}}(\,\mathbf{I}_n - \mathbf{X}\mathbf{X}^{-}\,).$$

Thus it is easy to derive from (30) that

$$\hat{\mathbf{P}}^2_{\mathbf{X};\mathbf{V}} = [\,\mathbf{X}\mathbf{X}^- + \tilde{\mathbf{P}}_{\mathbf{X};\mathbf{V}}(\,\mathbf{I}_n - \mathbf{X}\mathbf{X}^-\,)\,]^2 = \mathbf{X}\mathbf{X}^- + \tilde{\mathbf{P}}_{\mathbf{X};\mathbf{V}}(\,\mathbf{I}_n - \mathbf{X}\mathbf{X}^-\,) = \hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}},$$

establishing (b). Result (f) follows from (29). □

Proof of Theorem 4 Since both $\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}$ and $\hat{\mathbf{P}}'_{\mathbf{X};\mathbf{V}}$ are idempotent, it can be derived from (14) that

$$\begin{aligned} r(\,\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}} - \hat{\mathbf{P}}'_{\mathbf{X};\mathbf{V}}\,) &= 2r[\,\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}},\ \hat{\mathbf{P}}'_{\mathbf{X};\mathbf{V}}\,] - 2r(\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}) \\ &= 2r[\,\mathbf{X},\ \mathbf{P}_{\mathbf{X}} + (\,\mathbf{I}_n - \mathbf{P}_{\mathbf{X}}\,)\mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^+\mathbf{X}'\,] - 2r(\mathbf{X}) \\ &= 2r[\,\mathbf{X},\ \mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^+\mathbf{X}'\,] - 2r(\mathbf{X}) \\ &= 2r[\,\mathbf{X},\ \mathbf{V}\mathbf{X}\,] - 2r(\mathbf{X}). \end{aligned}$$

It can also be derived from (14) that

$$\begin{aligned} r(\,\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}} - \mathbf{P}_{\mathbf{X}}\,) &= r[\,\hat{\mathbf{P}}'_{\mathbf{X};\mathbf{V}},\ \mathbf{P}_{\mathbf{X}}\,] + r[\,\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}},\ \mathbf{P}_{\mathbf{X}}\,] - r(\hat{\mathbf{P}}_{\mathbf{X};\mathbf{V}}) - r(\mathbf{X}) \\ &= r[\,\mathbf{P}_{\mathbf{X}} + (\,\mathbf{I}_n - \mathbf{P}_{\mathbf{X}}\,)\mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^+\mathbf{X}',\ \mathbf{P}_{\mathbf{X}}\,] - r(\mathbf{X}) \\ &= r[\,\mathbf{V}\mathbf{X},\ \mathbf{X}\,] - r(\mathbf{X}). \end{aligned}$$

Thus (a), (b) and (c) are equivalent. □

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