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Nonparametric inference in multiplicative intensity model by discrete time observation

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Abstract This paper deals with nonparametric inference problems in the multiplicative intensity model for counting processes. We propose a Nelson–Aalen type estimator based on discrete observation. The functional asymptotic normality of the estimator is proved. The limit process is the same as that in the continuous observation case, thus the proposed estimator based on discrete observation has the same properties as the Nelson–Aalen estimator based on continuous observation. For example, the asymptotic efficiency of proposed estimator is valid based on less information than the continuous observation case. A Kaplan–Meier type estimator is also discussed. Nonparametric goodness of fit test is considered, and an asymptotically distribution free test is proposed.

Keywords Counting process · Discrete observation · Multiplicative intensity model · Weak convergence

1 Introduction

The multiplicative intensity model for counting processes is a simple, but very useful model in applied statistics, especially in survival analysis. Andersen et al. (1993) gives a good introduction of the model in their Chap. IV. It was Aalen (1978) who opened the window to the asymptotic theory for the model. Using the theory of martingales with continuous time parameter, he derived the functional asymptotic normality of Nelson–Aalen's estimator. Following his approach, many authors presented a lot of fruitful

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results in the 1980s. However, the real data is of course discrete random variables; the failure time data is given only per minutes, hours, days or weeks. So statisticians have to approximate the stochastic integral by using discrete data. Since the time parameter of martingale is continuous (i.e. $[0, T]$), the validity of this approximation has been an open problem for a long time. Some researchers have been trying to solve the problem, and new methods such as interval censoring have been proposed. See Sun (2006) and references therein. In this paper, we study the asymptotic theory of some discrete versions of Nelson–Aalen's and Kaplan–Meier's estimators. The limit processes of the estimators are the same as those in the continuous observation case. Our result guarantees the validity for the approximation of the stochastic integral.

Let us recall the multiplicative intensity model. For every $n \in \mathbb{N}$, let $t \rightsquigarrow N_t^n$ be a counting process defined on a stochastic basis $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t\in[0,T]}, P^n)$, where $T > 0$ is a fixed time. We suppose that the intensity of the counting process N^n is given by

$$\alpha(t)Y_t^n,$$

where α is a deterministic function and Y^n is a predictable process. In survival analysis, the value Y_t^n typically denotes the number of individuals which are at risk at time t. Assuming $n^{-1}Y^n$ converges to a deterministic function y as $n \to \infty$, one can develop the asymptotic theory for Nelson–Aalen's and Kaplan–Meier's estimators.

Our setting is as follows.

Sampling scheme For some $r \leq 1$, the processes $t \rightsquigarrow N_t^n$ and $t \rightsquigarrow Y_t^n$ are observed only at times $0 = t_0^n < t_1^n < \cdots < t_{m(n)}^n = T$ such that $h_n = o(n^{-1/2r})$ as $n \to \infty$, where $h_n = \max_{1\leq i\leq m(n)} |t_i^n - t_{i-1}^n|$. ◊

Some similar problems on this kind of *high frequency* discretization schemes have already been considered for diffusion processes. However, *functional asymptotics* for counting processes based on *high frequency data* have not yet been studied sufficiently. Indeed, even the important problems for basic Nelson–Aalen's and Kaplan–Meier's estimators are still open.

In Sect. 2, we review the theory of Nelson–Aalen's estimator in the continuous observation case. Our result for the discrete observation case is given in Sect. 3. We mention a result for Kaplan–Meier's estimator in Sect. 4. In Sect. 5, we consider a nonparametric goodness of fit test. It is shown that our test is asymptotically distribution free. In the last section, we give some concluding remarks.

Throughout this article, all limit notations mean that we take the limit as $n \to \infty$. We denote by $\xrightarrow{p}$ and $\overset{P^n}{\Longrightarrow}$ the convergence in probability and the weak convergence, respectively. We denote by $\ell^\infty[0, T]$ the space of bounded functions on $[0, T]$, and equip the space with the uniform metric.

2 Nelson–Aalen's estimator by continuous observation

First of all, let us make some conditions for the function α and the predictable process Y^n, which are a little stronger than the original version of Aalen (1978).

Condition C *The function α is bounded, non-negative, measurable function satisfying $\int_0^T \alpha(s)\mathrm{d}s > 0$. There exists a measurable function y, satisfying $0 < \inf_t y(t) \le \sup_t y(t) < \infty$, such that $\sup_{t\in[0,T]} |n^{-1}Y_t^n - y(t)| \xrightarrow{p} 0$.* $\Diamond$

We are interested in estimating the function $x \mapsto A(x)$ defined by

$$A(x) = \int_0^x \alpha(s)\mathrm{d}s = \int_0^T 1_{[0,x]}(s)\alpha(s)\mathrm{d}s.$$

We define the predictable process $t \rightsquigarrow Y_t^{n-}$ by

$$Y_t^{n-} = \begin{cases} 1/Y_t^n & \text{if } Y_t^n \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Nelson–Aalen estimator $x \rightsquigarrow \widehat{A}^n(x)$ is defined by

$$\widehat{A}^n(x) = \int_0^x Y_s^{n-}\mathrm{d}N_s^n = \int_0^T 1_{[0,x]}(s)Y_s^{n-}\mathrm{d}N_s^n.$$

To show the functional asymptotic normality of the estimator, we introduce the following processes $x \rightsquigarrow A^n(x)$ and $x \rightsquigarrow M_x^n$:

$$\begin{aligned} A^n(x) &= \int_0^x \alpha(s)I_s^n\mathrm{d}s, \quad \text{where } I_s^n = Y_s^{n-}Y_s^n; \\ M_x^n &= \sqrt{n}(\widehat{A}^n(x) - A^n(x)) \\ &= \sqrt{n}\int_0^x Y_s^{n-}[\mathrm{d}N_s^n - \alpha(s)Y_s^n\mathrm{d}s]. \end{aligned}$$

Lemma 1 *Under Condition C, it holds that $\sup_{x\in[0,T]} \sqrt{n}|A^n(x) - A(x)| \xrightarrow{p} 0$.*

Lemma 2 *Under Condition C, it holds that $M^n \overset{P^n}{\Longrightarrow} G$ in $\ell^\infty[0,T]$, where $x \rightsquigarrow G(x)$ is a Wiener process with the covariance given by*

$$EG(x)G(x') = \int_0^{x\wedge x'} \frac{\alpha(s)}{y(s)}\mathrm{d}s.$$

Lemma 1 is trivial. Lemma 2 is a simple application of the martingale central limit theorem. Combining these lemmas, we obtain the following theorem.

Theorem 3 **(Aalen (1978))** *Under Condition C, it holds that $\sqrt{n}(\widehat{A}^n - A) \overset{P^n}{\Longrightarrow} G$ in $\ell^\infty[0,T]$, where G is a process appearing in the limit of Lemma 2.*

3 Nelson–Aalen's estimator by discrete observation

We make the following condition.

Condition D *Let $r \leq 1$ be that in Sampling Scheme. There exists a constant $D > 0$ such that*

$$nE^n|Y_t^{n-} - Y_s^{n-}| \leq D|t-s|^r$$

for all sufficiently large n and all $t, s \in [0, T]$ such that $|t-s| \leq 1$. ◊

We propose the Nelson–Aalen type estimator $x \rightsquigarrow \widetilde{A}^n(x)$ based on discrete observation given by

$$\widetilde{A}^n(x) = \sum_{i=1}^{m(n)} 1_{[0,x]}(t_i^n) Y_{t_{i-1}^n}^{n-} [N_{t_i^n}^n - N_{t_{i-1}^n}^n].$$

We introduce the following processes $x \rightsquigarrow L^n(x)$ and $x \rightsquigarrow Z^n(x)$:

$$L^n(x) = \sqrt{n}(\widetilde{A}^n(x) - A^n(x));$$
$$Z^n(x) = \sqrt{n} \sum_{i=1}^{m(n)} 1_{[0,x]}(t_i^n) \int_{t_{i-1}^n}^{t_i^n} Y_s^{n-}[\mathrm{d}N_s^n - \alpha(s) Y_s^n \mathrm{d}s].$$

Lemma 4 *Under Conditions C and D, it holds that* $\sup_{x\in[0,T]} |L^n(x) - Z^n(x)| \xrightarrow{p} 0.$

Lemma 5 *Under Condition C, it holds that* $\sup_{x\in[0,T]} |Z^n(x) - M_x^n| \xrightarrow{p} 0.$

The proofs of the above lemmas will be given later in this section. As a consequence of Lemmas 1, 2, 4 and 5, we obtain the main result of the paper.

Theorem 6 *Under Conditions C and D, it holds that* $\sqrt{n}(\widetilde{A}^n - A) \overset{P^n}{\Longrightarrow} G$ *in* $\ell^\infty[0,T]$, *where G is a process appearing in the limit of Lemma 2.*

proof of Lemma 4 First notice that

$$\sup_{x\in[0,T]} |L^n(x) - Z^n(x)| \leq \sqrt{n} \sum_{i=1}^{m(n)} \int_{t_{i-1}^n}^{t_i^n} |Y_{t_{i-1}^n}^{n-} - Y_s^{n-}| \mathrm{d}N_s^n$$
$$+ \sqrt{n} \sup_{x\in[0,T]} \left| \int_0^x \alpha(s) I_s^n \mathrm{d}s - \sum_{i=1}^{m(n)} 1_{[0,x]}(t_i^n) \right.$$
$$\left. \times \int_{t_{i-1}^n}^{t_i^n} \alpha(s) I_s^n \mathrm{d}s \right| =: (I) + (II).$$

Notice that the term (I) does not depend on x any more. Now, introduce the predictable time

$$S^n = \inf\{s \in [0,T]; n^{-1}\alpha(s)Y_s^n \geq H\}, \quad \text{where } H = \sup_{s\in[0,T]} \alpha(s)\{y(s)+1\}, \quad (1)$$

and its announcing sequence $\{S_p^n\}$ (see I.2.16 of Jacod and Shriyaev 2003). Then we have $\sup_{s\in[0,S_p^n]} \alpha(s)Y_s^n \leq nH$. Since $P^n(\lim_p S_p^n = S^n) = 1$ and $P^n(S^n = T) \to 1$, it is sufficient to evaluate the following quantity:

$$\begin{aligned}
&\sqrt{n}\sum_{i=1}^{m(n)} E^n\left(\int_{t_{i-1}^n\wedge S_p^n}^{t_i^n\wedge S_p^n} |Y_{t_{i-1}^n}^{n-} - Y_s^{n-}| \mathrm{d}N_s^n\right) \\
&= \sqrt{n}\sum_{i=1}^{m(n)} E^n\left(\int_{t_{i-1}^n\wedge S_p^n}^{t_i^n\wedge S_p^n} |Y_{t_{i-1}^n}^{n-} - Y_s^{n-}|\alpha(s)Y_s^n \mathrm{d}s\right) \\
&\leq H\sqrt{n}\sum_{i=1}^{m(n)} E^n\left(\int_{t_{i-1}^n\wedge S_p^n}^{t_i^n\wedge S_p^n} n|Y_{t_{i-1}^n}^{n-} - Y_s^{n-}| \mathrm{d}s\right) \\
&\leq H\sqrt{n}\sum_{i=1}^{m(n)} \int_{t_{i-1}^n}^{t_i^n} D|t_{i-1}^n - s|^r \mathrm{d}s \\
&\leq HDT\sqrt{n}h_n^r.
\end{aligned}$$

Indeed, it follows from Lebesgue's convergence theorem that

$$\begin{aligned}
&\sqrt{n}\sum_{i=1}^{m(n)} E^n\left(\int_{t_{i-1}^n\wedge S^n}^{t_i^n\wedge S^n} |Y_{t_{i-1}^n}^{n-} - Y_s^{n-}| \mathrm{d}N_s^n\right) \\
&= \lim_p \sqrt{n}\sum_{i=1}^{m(n)} E^n\left(\int_{t_{i-1}^n\wedge S_p^n}^{t_i^n\wedge S_p^n} |Y_{t_{i-1}^n}^{n-} - Y_s^{n-}| \mathrm{d}N_s^n\right) \\
&\leq HDT\sqrt{n}h_n^r,
\end{aligned}$$

and for any $\varepsilon > 0$

$$\begin{aligned}
P^n(|(I)| > \varepsilon) &\leq P^n(S^n < T) + \frac{1}{\varepsilon}\sqrt{n}\sum_{i=1}^{m(n)} E^n\left(\int_{t_{i-1}^n\wedge S^n}^{t_i^n\wedge S^n} |Y_{t_{i-1}^n}^{n-} - Y_s^{n-}| \mathrm{d}N_s^n \cdot 1_{\{S^n=T\}}\right) \\
&\leq P^n(S^n < T) + \frac{1}{\varepsilon}HDT\sqrt{n}h_n^r \to 0.
\end{aligned}$$

As for (II), we don't have to consider the point $x = T$. Notice that, for $x \in [t_{i-1}^n, t_i^n)$,

$$\int_0^x \alpha(s)I_s^n \mathrm{d}s - \sum_{i=1}^{m(n)} 1_{[0,x]}(t_i^n)\int_{t_{i-1}^n}^{t_i^n} \alpha(s)I_s^n \mathrm{d}s = \int_{t_{i-1}^n}^x \alpha(s)I_s^n \mathrm{d}s.$$

So we have

$$(II) \leq \max_{1 \leq i \leq m(n)} \sqrt{n} \int_{t_{i-1}^n}^{t_i^n} \alpha(s) I_s^n \mathrm{d}s \leq \sqrt{n} h_n \sup_{s \in [0,T]} \alpha(s) \to 0.$$

The proof is finished. □

proof of Lemma 5 Notice that $Z^n(T) = M_T^n$. For $x \in [t_{i-1}^n, t_i^n)$, it holds that

$$Z^n(x) - M_x^n = \sqrt{n} \int_{t_{i-1}^n}^{x} Y_s^{n-} [\mathrm{d}N_s^n - \alpha(s) Y_s^n \mathrm{d}s].$$

So we will show that

$$\max_{1 \leq i \leq m(n)} \sup_{u \in [t_{i-1}^n, t_i^n]} \left| \sqrt{n} \int_{t_{i-1}^n}^{u} Y_s^{n-} [\mathrm{d}N_s^n - \alpha(s) Y_s^n \mathrm{d}s] \right| \xrightarrow{p} 0.$$

Now, introduce the predictable time

$$S^n = \inf\{s \in [0, T]; nY_s^{n-}(\alpha(s) \vee 1) \geq H\}, \text{ where } H = \sup_{s \in [0,T]} \{(\alpha(s) \vee 1)/y(s)\} + 1,$$

and its announcing sequence $\{S_p^n\}$. By the same reason as that around (1), it is sufficient to evaluate the following value: using Burkholder-Davis-Gundy's inequality, there exists a constant $c_p > 0$ depending only on $p = 4$ such that

$$\begin{aligned}
& E^n \left(\max_{1 \leq i \leq m(n)} \sup_{u \in [t_{i-1}^n, t_i^n]} \left| \sqrt{n} \int_{t_{i-1}^n}^{u \wedge S_p^n} Y_s^{n-} [\mathrm{d}N_s^n - \alpha(s) Y_s^n \mathrm{d}s] \right|^4 \right) \\
& \leq \sum_{i=1}^{m(n)} E^n \left(\sup_{u \in [t_{i-1}^n, t_i^n]} \left| \sqrt{n} \int_{t_{i-1}^n}^{u \wedge S_p^n} Y_s^{n-} [\mathrm{d}N_s^n - \alpha(s) Y_s^n \mathrm{d}s] \right|^4 \right) \\
& \leq c_4 \sum_{i=1}^{m(n)} E^n \left| n \int_{t_{i-1}^n}^{t_i^n \wedge S_p^n} |Y_s^{n-}|^2 \mathrm{d}N_s^n \right|^2 \\
& \leq 4c_4 \sum_{i=1}^{m(n)} E^n \left| n \int_{t_{i-1}^n}^{t_i^n \wedge S_p^n} |Y_s^{n-}|^2 [\mathrm{d}N_s^n - \alpha(s) Y_s^n \mathrm{d}s] \right|^2 \\
& \quad + 4c_4 \sum_{i=1}^{m(n)} E^n \left| n \int_{t_{i-1}^n}^{t_i^n \wedge S_p^n} |Y_s^{n-}|^2 \alpha(s) Y_s^n \mathrm{d}s \right|^2 \\
& =: 4c_4 \{(I) + (II)\}.
\end{aligned}$$

(In the above computation, we used the inequality $x^2 = |(x-y)+y|^2 \le |2\max\{|x-y|, |y|\}|^2 \le 4\{|x-y|^2 + y^2\}$.) Now, it follows from Doob's inequality that

$$\begin{aligned}
(I) &\le 4n^2 \sum_{i=1}^{m(n)} E^n \int_{t_{i-1}^n}^{t_i^n \wedge S_p^n} |Y_s^{n-}|^4 \alpha(s) Y_s^n \mathrm{d}s \\
&= 4n^2 \sum_{i=1}^{m(n)} E^n \int_{t_{i-1}^n}^{t_i^n \wedge S_p^n} |Y_s^{n-}|^3 \alpha(s) \mathrm{d}s \\
&\le 4n^{-1} H^3 \sum_{i=1}^{m(n)} E^n \int_{t_{i-1}^n}^{t_i^n \wedge S_p^n} \alpha(s) \mathrm{d}s \\
&\le 4n^{-1} H^3 \int_0^T \alpha(s) \mathrm{d}s \\
&\to 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(II) &= \sum_{i=1}^{m(n)} E^n \left| n \int_{t_{i-1}^n}^{t_i^n \wedge S_p^n} Y_s^{n-} \alpha(s) \mathrm{d}s \right|^2 \\
&\le \sum_{i=1}^{m(n)} \left| \int_{t_{i-1}^n}^{t_i^n \wedge S_p^n} H \mathrm{d}s \right|^2 \\
&\le \sum_{i=1}^{m(n)} \left| \int_{t_{i-1}^n}^{t_i^n} H \mathrm{d}s \right|^2 \\
&= \sum_{i=1}^{m(n)} |t_i^n - t_{i-1}^n|^2 H^2 \\
&\le T h_n H^2 \to 0.
\end{aligned}$$

So the proof of the lemma has been established. □

4 Kaplan–Meier's estimator by discrete observation

In this section we are interested in estimating the survival function

$$S(x) = \exp\left(-\int_0^x \alpha(s) \mathrm{d}s \right).$$

The Kaplan–Meier estimator based on the continuous observation is

$$\widehat{S}^n(x) = \prod_{s \le x} (1 - \Delta \widehat{A}^n(s)).$$

A natural definition of Kaplan–Meier's estimator based on the discrete observation is

$$\widetilde{S}^n(x) = \prod_{s \le x} (1 - \Delta \widetilde{A}^n(s)).$$

Using the functional delta method, together with Theorem 6, we have that $\sqrt{n}(\widetilde{S}^n - S) \overset{P^n}{\Longrightarrow} -S \cdot G$ in $\ell^\infty[0, T]$, where G is a process appearing in Lemma 2. See the proof for Theorem IV.3.2 of Andersen et al. (1993) and Chap. 3.9 of van der Vaart and Wellner (1996).

5 Goodness of fit test

In this section, we consider the nonparametric goodness of fit test problem for α. Let us fix some notations. We denote by P^n_α the probability measure under which the intensity of N^n is $\alpha(s)Y^n_s$. For given α_0, we denote $A_0(x) = \int_0^x \alpha_0(s)\mathrm{d}s$ and the y_0 instead of y appearing in Condition C. We wish to test

$H_0 : \alpha = \alpha_0$ versus $H_1 : \alpha \in \mathcal{A}$,

where $\mathcal{A} = \{\alpha;$ Conditions C and D are satisfied and $A(x) \neq A_0(x)$ for some $x\}$.

Due to Theorem 6 and the continuous mapping theorem, it holds that

$$\sup_{x \in [0,T]} \sqrt{n}|\widetilde{A}^n(x) - A_0(x)| \overset{P^n_{\alpha_0}}{\Longrightarrow} \sup_{t \in [0, \Sigma_0^2]} |B_t| =^d \Sigma_0 \sup_{t \in [0,1]} |B_t|,$$

where $t \rightsquigarrow B_t$ is a standard Brownian motion,

$$\Sigma_0 = \sqrt{\int_0^T \frac{\alpha_0(s)}{y_0(s)} \mathrm{d}s},$$

and the notation "$=^d$" means that the distributions are the same.

In order to obtain an asymptotically distribution free test, we need a consistent estimator for Σ_0. For this purpose, we propose the following estimator:

$$\widetilde{\Sigma}^n = \sqrt{\sum_{i=1}^{m(n)} n|Y^{n-}_{t^n_{i-1}}|^2[N^n_{t^n_i} - N^n_{t^n_{i-1}}]}.$$

Lemma 7 *Suppose that α_0 satisfies Conditions C and D. Under H_0, it holds that $\widetilde{\Sigma}^n \xrightarrow{p} \Sigma_0$.*

The proof will be given later. Due to this lemma, we have the asymptotically distribution free test.

Theorem 8 *Suppose that α_0 satisfies Conditions C and D. Under H_0, it holds that*

$$\frac{\sup_{x\in[0,T]}\sqrt{n}|\widetilde{A}^n(x)-A_0(x)|}{\widetilde{\Sigma}^n} \overset{P^n_{\alpha_0}}{\Longrightarrow} \sup_{t\in[0,1]}|B_t|.$$

See Khmaladze and Shinjikashvili (2001) and references therein for explicite/numerical expression for the distribution of the limit $\sup_{t\in[0,1]}|B_t|$ which is necessary for computing p-values.

We also have the consistency of the test.

Theorem 9 *For $\alpha\in\mathcal{A}$, it holds that*

$$\frac{\sup_{x\in[0,T]}\sqrt{n}|\widetilde{A}^n(x)-A_0(x)|}{\widetilde{\Sigma}^n} \neq O_{P^n_\alpha}(1).$$

proof of Lemma 7 We write

$$\begin{aligned}|\widetilde{\Sigma}^n|^2 &= n\sum_{i=1}^{m(n)}\int_{t^n_{i-1}}^{t^n_i}(|Y^{n-}_{t^n_{i-1}}|^2-|Y^{n-}_s|^2)\mathrm{d}N^n_s + n\int_0^T|Y^{n-}_s|^2\mathrm{d}N^n_t\\ &=: (I)+(II).\end{aligned}$$

Let us consider the term (II). By Lenglart's inequality, it is easy to see that

$$n\int_0^T|Y^{n-}_s|^2[\mathrm{d}N^n_s-\alpha_0(s)Y^n_s\mathrm{d}s]\overset{p}{\longrightarrow}0,$$

while

$$n\int_0^T|Y^{n-}_s|^2\alpha_0(s)Y^n_s\mathrm{d}s = n\int_0^T Y^{n-}_s\alpha_0(s)\mathrm{d}s\overset{p}{\longrightarrow}\Sigma_0^2.$$

So $(II)\overset{p}{\longrightarrow}\Sigma_0^2$.

As for the term (I), introducing the predictable time S^n given by

$$S^n=\inf\{s\in[0,T];n^{-1}\alpha_0(s)Y^n_s\geq H\}\wedge\inf\{s\in[0,T];nY^{n-}_s\geq H\},$$

and its announcing sequence $\{S^n_p\}$. Since $P^n_{\alpha_0}(\lim_p S^n_p=S^n)=1$ and $P^n_{\alpha_0}(S^n=T)\to 1$ for sufficiently large H, it is sufficient to evaluate the following quantity.

$$\begin{aligned}&nE^n_{\alpha_0}\left(\sum_{i=1}^{m(n)}\int_{t^n_{i-1}\wedge S^n_p}^{t^n_i\wedge S^n_p}||Y^{n-}_{t^n_{i-1}}|^2-|Y^{n-}_s|^2|\mathrm{d}N^n_s\right)\\ &=nE^n_{\alpha_0}\left(\int_{t^n_{i-1}\wedge S^n_p}^{t^n_i\wedge S^n_p}||Y^{n-}_{t^n_{i-1}}|^2-|Y^{n-}_s|^2|\alpha_0(s)Y^n_s ds\right)\end{aligned}$$

$$\le \sup_s \alpha_0(s)\cdot Hn^2 \sum_{i=1}^{m(n)} E^n_{\alpha_0}\left(\int_{t^n_{i-1}\wedge S^n_p}^{t^n_i\wedge S^n_p} ||Y^{n-}_{t^n_{i-1}}|^2 - |Y^{n-}_s|^2|\mathrm{d}s\right)$$
$$= \sup_s \alpha_0(s)\cdot Hn^2 \sum_{i=1}^{m(n)} E^n_{\alpha_0}\left(\int_{t^n_{i-1}\wedge S^n_p}^{t^n_i\wedge S^n_p} |Y^{n-}_{t^n_{i-1}} - Y^{n-}_s||Y^{n-}_{t^n_{i-1}} + Y^{n-}_s|\mathrm{d}s\right)$$
$$\le \sup_s \alpha_0(s)\cdot 2H^2 n \sum_{i=1}^{m(n)} E^n_{\alpha_0}\left(\int_{t^n_{i-1}\wedge S^n_p}^{t^n_i\wedge S^n_p} |Y^{n-}_{t^n_{i-1}} - Y^{n-}_s|\mathrm{d}s\right)$$
$$\le \sup_s \alpha_0(s)\cdot 2H^2 \sum_{i=1}^{m(n)} \int_{t^n_{i-1}}^{t^n_i} D|t^n_{i-1} - s|^r \mathrm{d}s$$
$$\le \sup_s \alpha_0(s)\cdot 2H^2 DTh^r_n \to 0.$$

Thus $(I) \xrightarrow{p} 0$, and the proof is completed. □

proof of Theorem 9 We can write

$$\sup_{x\in[0,T]} \sqrt{n}|\widetilde{A}^n(x) - A_0(x)| \ge \sup_{x\in[0,T]} \sqrt{n}|A(x) - A_0(x)| - \sup_{x\in[0,T]} \sqrt{n}|\widetilde{A}^n(x) - A(x)|.$$

By Theorem 6, we have $\sup_{x\in[0,T]}\sqrt{n}|\widetilde{A}^n(x) - A(x)| = O_{P_\alpha}(1)$. Since $A(x) \neq A_0(x)$ for some x, it holds that $\sup_{x\in[0,T]}\sqrt{n}|A(x) - A_0(x)| \to \infty$. Since $\widetilde{\Sigma}^n \xrightarrow{p} (\int_0^T \alpha(s)/y(s)\mathrm{d}s)^{1/2}$ under the probability measure P^n_α, the assertion follows. □

6 Concluding remarks

We have established the functional asymptotic normality of the estimator $\widetilde{A}^n$. The limit is the same as that for Nelson–Aalen's estimator $\widehat{A}^n$ based on the continuous observation. Hence all asymptotic procedures which have been successful for $\widehat{A}^n$ work also for our estimator. See Sect. IV.1 of Andersen et al. (1993). It should be noted that our estimator is asymptotically efficient, in the sense of convolution and asymptotic minimax theorems in the space $\ell^\infty[0,T]$ based on less information than the continuous observation case. See Chap. VIII of Andersen et al. (1993), Chap. 3.11 of van der Vaart and Wellner (1996), and Sect. 4.1.2 of Nishiyama (2000).

What we have done is to give the validity of the approximation of the stochastic integral which people have been *believing* for a long time. The pair of the assumptions

$$h_n = o(n^{-1/2r}) \quad \text{and} \quad nE^n|Y^{n-}_t - Y^{n-}_s| \le D|t-s|^r \quad \text{for some } r \le 1$$

gives us an answer. One may think of "low frequency" cases where $h_n = O(n^{-1/2r})$ with $r \ge 1$. It is supposed that asymptotically distribution free results, or even consistency results, could not be proved in such cases. We leave these problems for future works.

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