

Finiteness of small factor analysis models

Mathias Drton · Han Xiao

Received: 22 April 2009 / Revised: 5 November 2009 / Published online: 7 April 2010
© The Institute of Statistical Mathematics, Tokyo 2010

Abstract We consider factor analysis models with one or two factors. Fixing the number of factors, we prove a finiteness result about the covariance matrix parameter space when the size of the covariance matrix increases. According to this result, there exists a distinguished matrix size starting at which one can determine whether a given covariance matrix belongs to the parameter space by determining whether all principal submatrices of the distinguished size belong to the corresponding parameter space. We show that the distinguished matrix size is four in the model with one factor and six with two factors.

Keywords Algebraic statistics · Graphical model · Multivariate normal distribution · Latent variables

1 Introduction

Suppose we observe a sample of multivariate normal random vectors and wish to test whether their covariance matrix is diagonal. The likelihood ratio test for this problem involves the determinant of the sample correlation matrix. Therefore, this test cannot be used if the sample size n is smaller than the number p of entries in the random vectors, because the sample correlation matrix will always be singular. A nice way around this problem was proposed in [Schott \(2005\)](#), where the sum of squared pairwise

M. Drton (✉) · H. Xiao
Department of Statistics, The University of Chicago, 5734 S. University Ave,
Chicago, IL 60637, USA
e-mail: drton@uchicago.edu

H. Xiao
e-mail: xiao@galton.uchicago.edu

sample correlations is used as a test statistics. The relevant distribution theory involves a central limit theorem in the paradigm where $n, p \rightarrow \infty$ such that $p/n \rightarrow c \in (0, \infty)$.

Why is it possible to prove such a limit theorem and which situations are candidates for development of similar results and associated statistical techniques? We believe that a fundamental aspect of these questions is a property of finiteness. In the above problem this property amounts to the ability to determine whether a covariance matrix is diagonal by verifying whether each principal 2×2 submatrix is diagonal. The squared sample correlations summed up in the test statistics of [Schott \(2005\)](#) do exactly that. In this paper we show that such finiteness structure arises more generally in factor analysis models.

The factor analysis model for p observed variables and with m factors is the set of multivariate normal distributions $\mathcal{N}_p(\mu, \Sigma)$ with arbitrary mean vector μ and a covariance matrix Σ in the set

$$F_{p,m} = \{ \Delta + \Gamma \Gamma^t : \Delta \text{ positive definite and diagonal, } \Gamma \in \mathbb{R}^{p \times m} \}.$$

For background on this classical statistical model see, for instance, [Anderson and Rubin \(1956\)](#) and [Harman \(1976\)](#). Note also that for $m = 0$ it is reasonable to define $F_{p,0}$ to be the set of positive definite and diagonal $p \times p$ matrices. In this paper we establish the following result that resolves part of a conjecture in [Drton et al. \(2007\)](#); see also open problem 7.8 in [Drton et al. \(2009\)](#).

Theorem 1 *Suppose $m \in \{0, 1, 2\}$, and let $\Sigma = (\sigma_{ij})$ be a positive definite matrix of size $p \times p$ with $p \geq 2(m+1)$. Then Σ is in $F_{p,m}$ if and only if the principal submatrix $\Sigma_{A,A} = (\sigma_{ij})_{i,j \in A}$ is in $F_{2(m+1),m}$ for each index set $A \subseteq \{1, \dots, p\}$ of size $2(m+1)$.*

As motivated above, this result is of statistical interest as it suggests that in a high-dimensional setting with large number of variables p a test of the factor analysis models can be carried out by testing smaller marginal hypotheses concerning only $2(m+1)$ variables. While our result provides the theoretical basis for such tests, an appropriate distribution theory still has to be worked out. This, however, is a topic beyond the scope of this note.

The remainder of the paper is devoted to the proof of [Theorem 1](#). In [Sect. 2](#), we outline our approach to the finiteness problem and resolve the one-factor case. In [Sect. 3](#), we tackle the more complicated case of two factors. Concluding remarks are given in [Sect. 4](#).

2 Approach to the problem and one-factor models

We first introduce some notational conventions. Suppose I and J are two index sets. In this paper I is typically of the form $[p] = \{1, 2, \dots, p\}$ or $\{2, 3, \dots, p\}$ with $p \in \mathbb{N}$. For any finite set A , let $|A|$ denote its cardinality. Suppose $\Lambda = (\lambda_{ij})_{i \in I, j \in J}$ is an $|I| \times |J|$ matrix. For $A \subset I$ and $B \subset J$, we let $\Lambda_{A,B} = (\lambda_{ij})_{i \in A, j \in B}$ denote the $|A| \times |B|$ submatrix. When using the complement of a set A as an index set we write $\setminus A$ as in

$$\Lambda_{\setminus A, \setminus B} := \Lambda_{I \setminus A, J \setminus B}.$$

As a further shorthand, we let $\setminus i := \setminus \{i\}$, $\Lambda_A := \Lambda_{A, J}$ and $\Lambda_{, B} := \Lambda_{I, B}$.

If $\Sigma \in F_{p, m}$, then the representation $\Sigma = \Delta + \Gamma \Gamma^t$ is not unique because we may multiply an orthogonal matrix to Γ from the right. However, the diagonal matrix Δ , and thus, the positive semi-definite and rank m matrix $\Gamma \Gamma^t$ may be unique. We will repeatedly use the following lemma to establish such uniqueness.

Lemma 1 *Let $p \geq 2m + 1$, and consider two $p \times p$ matrices $\Psi = (\psi_{ij})$ and $\Phi = (\phi_{ij})$ with the same off-diagonal entries and same rank m . For any $i \in [p]$, if we can find two disjoint subsets $A, B \subset [p] \setminus \{i\}$ of cardinality $|A| = |B| = m$ such that $\det(\Psi_{A, B}) \neq 0$, then $\psi_{ii} = \phi_{ii}$.*

Proof Since Ψ and Φ are both of rank m , the following two $(m + 1) \times (m + 1)$ minors are equal to zero:

$$\begin{vmatrix} \psi_{ii} & \Psi_{i, B} \\ \Psi_{A, i} & \Psi_{A, B} \end{vmatrix} = \begin{vmatrix} \phi_{ii} & \Phi_{i, B} \\ \Phi_{A, i} & \Phi_{A, B} \end{vmatrix} = 0.$$

The above two minors are entry-wise the same except for ψ_{ii} and ϕ_{ii} . Since $\det(\Psi_{A, B}) = \det(\Phi_{A, B}) \neq 0$, it follows that $\psi_{ii} = \phi_{ii}$. □

The next lemma will provide a way to give an induction-based proof of Theorem 1.

Lemma 2 *Let $p \geq 2m + 3$, and suppose that $\Sigma \in \mathbb{R}^{p \times p}$ is a positive definite matrix that has all $(p - 1) \times (p - 1)$ principal submatrices in $F_{p-1, m}$. Write*

$$\Sigma_{\setminus p, \setminus p} = \Delta + \Gamma \Gamma^t \quad \text{and} \quad \Sigma_{\setminus 1, \setminus 1} = D + G G^t$$

with Δ and D positive definite and diagonal, and $\Gamma, G \in \mathbb{R}^{(p-1) \times m}$. To see the correspondence with the original matrix Σ clearly, label the rows of D and G by $2, 3, \dots, p$. Then the following two conditions imply that Σ belongs to $F_{p, m}$:

- (i) *The two matrices Γ and G satisfy $\Gamma_{\setminus 1} = G_{\setminus p}$;*
- (ii) *There are disjoint subsets $B, C \subseteq ([p] \setminus \{1, p\})$ of cardinality $|B| = |C| = m$ such that $\det(\Sigma_{B, C}) \neq 0$.*

Proof From $\Delta = (\delta_{ij})$, $\Gamma = (\gamma_{ij})$, D and G form the matrices

$$\bar{\Lambda} = \begin{pmatrix} \delta_{11} & 0 \\ 0 & D \end{pmatrix} \quad \text{and} \quad \bar{\Gamma} = \begin{pmatrix} \Gamma_1 \\ G \end{pmatrix}.$$

Let $\bar{\Sigma} = \bar{\Lambda} + \bar{\Gamma} \bar{\Gamma}^t$, which is a matrix in $F_{p, m}$. We claim that $\Sigma = \bar{\Sigma} \in F_{p, m}$.

By assumption (i), Γ and G agree in $p - 2$ rows and it holds that $\bar{\sigma}_{ij} = \sigma_{ij}$ except possibly for the pairs $(i, j) = (1, p)$ and $(i, j) = (p, 1)$. In order to show that $\bar{\sigma}_{1p} = \sigma_{1p}$, we use the index sets B and C from condition (ii) to construct the following three $(m + 1) \times (m + 1)$ minors, which are all equal to zero:

$$\begin{vmatrix} \bar{\Sigma}_{1, C} & \bar{\sigma}_{1p} \\ \bar{\Sigma}_{B, C} & \bar{\Sigma}_{B, p} \end{vmatrix} = \begin{vmatrix} \Sigma_{1, C} & \bar{\sigma}_{1p} \\ \Sigma_{B, C} & \Sigma_{B, p} \end{vmatrix} = \begin{vmatrix} \Sigma_{1, C} & \sigma_{1p} \\ \Sigma_{B, C} & \Sigma_{B, p} \end{vmatrix} = 0. \tag{1}$$

The first minor is zero because $\{1\} \cup B$ and $C \cup \{p\}$ are two disjoint subsets of cardinality $m + 1$ and because $\bar{\Sigma}$ is in $F_{p,m}$. Recall that $\text{rank}(\bar{\Gamma}\bar{\Gamma}^t) = m$. The last minor is zero for the same reason. Recall that we assume that $p \geq 2m + 3$, whereas the third matrix in (1) comes from a principal submatrix of size $2m + 2$. The middle minor is zero because it is entry-wise equal to the first minor. Since by assumption $\det(\Sigma_{B,C}) \neq 0$, it follows from Lemma 1 that $\bar{\sigma}_{1p} = \sigma_{1p}$. \square

We are now ready to study the case of $m = 1$ factor.

Theorem 2 ($m = 1$) *A positive definite matrix $\Sigma \in \mathbb{R}^{p \times p}$ with $p \geq 4$ belongs to $F_{p,1} \setminus F_{p,0}$ if and only if every 4×4 principal submatrix belongs to $F_{4,1}$ and at least one 2×2 principal submatrix does not belong to $F_{2,0}$.*

Proof The ‘only if’ part is immediate. We prove the ‘if’ part by induction. The induction base is $p = 4$ in which case the statement is vacuous. Now suppose that $p \geq 5$ and that Σ has every principal 4×4 submatrix in $F_{4,1}$ and at least one non-zero off-diagonal entry. If no other off-diagonal entry is non-zero, then $\Sigma = \Delta + \Gamma\Gamma^t$ for a vector $\Gamma \in \mathbb{R}^p$ with exactly two non-zero entries. So, we may assume that Σ has at least two non-zeros entries. Without loss of generality, suppose that $\sigma_{23} \neq 0$ and $\sigma_{34} \neq 0$.

By the induction hypothesis, $\Sigma_{\setminus p, \setminus p}$ and $\Sigma_{\setminus 1, \setminus 1}$ belong to $F_{p-1,1}$. We are thus able to write

$$\Sigma_{\setminus p, \setminus p} = \Lambda + \Gamma\Gamma^t \quad \text{and} \quad \Sigma_{\setminus 1, \setminus 1} = D + GG^t$$

with $\Gamma, G \in \mathbb{R}^{p-1}$. Therefore, we have two representations of $\Sigma_{\setminus \{1,p\}, \setminus \{1,p\}}$, namely,

$$\Sigma_{\setminus \{1,p\}, \setminus \{1,p\}} = \Lambda_{\setminus 1, \setminus 1} + \Gamma_{\setminus 1}\Gamma_{\setminus 1}^t = D_{\setminus p, \setminus p} + G_{\setminus p}G_{\setminus p}^t.$$

Note that again we assign row indices to the matrices D and G based on the correspondence to the original matrix Σ , so we use $D_{\setminus p, \setminus p}$ and $G_{\setminus p, \setminus p}$ instead of $D_{\setminus (p-1), \setminus (p-1)}$ and $G_{\setminus (p-1), \setminus (p-1)}$ respectively. Since $\sigma_{23} = \gamma_2\gamma_3 \neq 0$ and $\sigma_{34} = \gamma_3\gamma_4 \neq 0$, we deduce that $\sigma_{24} = \gamma_2\gamma_4 \neq 0$. Using these three non-zero entries of Σ and applying Lemma 1, we know that $\Gamma_{\setminus 1}\Gamma_{\setminus 1}^t = G_{\setminus p}G_{\setminus p}^t$. Therefore, $\Gamma_{\setminus 1}$ and $G_{\setminus p}$ can only differ by a sign. Changing the sign of G if necessary, we obtain that $\Gamma_{\setminus 1} = G_{\setminus p}$ and thus Lemma 2 implies that the induction step goes through. \square

3 Two-factor models

If $\Gamma \in \mathbb{R}^{p \times m}$ has rank m and G is another matrix in $\mathbb{R}^{p \times m}$ that satisfies $GG^t = \Gamma\Gamma^t$, then $\Gamma = GQ$ for some orthogonal matrix Q (Anderson and Rubin 1956, Lemma 5.1). However, this holds more generally.

Lemma 3 *If $\Gamma, G \in \mathbb{R}^{p \times m}$ are matrices with dimensions $p \geq m$ that satisfy $GG^t = \Gamma\Gamma^t$, then $\Gamma = GQ$ for some orthogonal matrix Q .*

Proof Suppose Γ has rank $k < m$. Then $\text{rank}(G) = k$ also. There are now orthogonal transformations Q_1 and Q_2 such that $\Gamma Q_1 = (\Gamma', 0)$ and $G Q_2 = (G', 0)$, where Γ' and G' are full rank $p \times k$ matrices. According to the above fact, $\Gamma' = G' Q$ for some orthogonal $k \times k$ matrix Q . Let $Q_3 = \text{diag}(Q, I_{m-k})$. Then $Q_2 Q_3 Q_1^t$ is an orthogonal matrix and $\Gamma = G Q_2 Q_3 Q_1^t$. \square

By Lemma 3, if the matrix $\Sigma_{\setminus\{1,p\},\setminus\{1,p\}}$ has a unique representation as the sum of a positive definite and diagonal matrix plus a positive semi-definite matrix of rank m , then condition (i) in Lemma 2 can be satisfied by applying an orthogonal transformation. We thus need to study when this representation is unique and how uniqueness may fail for $m = 2$ factors. We begin our discussion by considering a 5×5 matrix. This prepares us for an induction step from $p = 6$ to $p = 7$ because in this step $\Sigma_{\setminus\{1,p\},\setminus\{1,p\}}$ is of size 5×5 .

Lemma 4 *Suppose Σ is a 5×5 positive definite matrix that can be written as*

$$\Sigma = \Delta + \Gamma \Gamma^t, \tag{2}$$

where Δ is positive definite and diagonal, and $\Gamma \in \mathbb{R}^{p \times 2}$ has rank 2. Let k be the largest integer n for which there is an index set $A \subset [5]$ of cardinality n with $\text{rank}(\Gamma_A) = 1$. Then under the assumptions that Σ has no representation like (2) with $\text{rank}(\Gamma) \leq 1$, and every row of Σ contains at least one non-zero off-diagonal entry, it holds that $k \leq 3$ and we have the following:

- (i) *If $k = 1$ or $k = 2$, then the representation in (2) is unique, that is, $\Gamma \Gamma^t = G G^t$ for any other representation $\Sigma = D + G G^t$.*
- (ii) *If $k = 3$, then after a permutation of rows we may assume that $\text{rank}(\Gamma_{[3]}) = 1$. Then there exists an orthogonal matrix Q such that*

$$\Gamma Q = \begin{pmatrix} \gamma'_{11} & \gamma'_{21} & \gamma'_{31} & \gamma_{41} & \gamma_{51} \\ 0 & 0 & 0 & \gamma'_{42} & \gamma'_{52} \end{pmatrix}^t$$

where primes indicate entries that are necessarily non-zero. Moreover, for any other representation $\Sigma = D + G G^t$, it holds that $G = (g_{ij})$, when brought into the same form as Γ by an orthogonal transformation, shares the first column with Γ and satisfies $g'_{42} g'_{52} = \gamma'_{42} \gamma'_{52}$.

Proof Since every row of Σ is assumed to contain a non-zero entry, no row of Γ is zero. Assuming that $\text{rank}(\Gamma) \leq 1$ is not possible in a representation of Σ , we must have that $k \leq 4$. If $k = 4$, by applying an orthogonal transform, we can write Γ as

$$\Gamma = \begin{pmatrix} \Gamma_{[4],1} & 0 \\ \gamma_{51} & \gamma_{52} \end{pmatrix}.$$

But then we have a contradiction to our assumptions because we can represent Σ as

$$\Sigma = \begin{pmatrix} \Delta_{[4]} & 0 \\ 0 & \delta_{55} + \gamma_{52}^2 \end{pmatrix} + \begin{pmatrix} \Gamma_{[4],1} \\ \gamma_{51} \end{pmatrix} \begin{pmatrix} \Gamma_{[4],1} \\ \gamma_{51} \end{pmatrix}^t.$$

Hence, we must have $k \leq 3$.

Now suppose there is another representation $\Sigma = D + GG^t$. Let $\Psi = \Gamma\Gamma^t$ and $\Phi = GG^t$. There are two cases.

- (a) $k = 1, 2$: If $k = 2$, we may assume $\text{rank}(\Gamma_{[2]}) = 1$. By applying an orthogonal transform, we can write Γ as

$$\Gamma^t = \begin{pmatrix} \gamma'_{11} & \gamma'_{21} & \gamma'_{31} & \gamma_{41} & \gamma_{51} \\ 0 & 0 & \gamma'_{32} & \gamma'_{42} & \gamma'_{52} \end{pmatrix}$$

where we again use primes to highlight non-zero entries. Since $k < 3$, the submatrix $\Gamma_{\{3,4,5\}}$ has rank 2, and thus we have enough non-zero 2×2 off-diagonal minors of Ψ and Φ to apply Lemma 1 and deduce uniqueness of the representation. If $k = 1$, then Lemma 1 applies immediately.

- (b) $k = 3$: Let us assume $\text{rank}(\Gamma_{[3]}) = 1$, in which case we can write Γ as

$$\Gamma^t = \begin{pmatrix} \gamma'_{11} & \gamma'_{21} & \gamma'_{31} & \gamma_{41} & \gamma_{51} \\ 0 & 0 & 0 & \gamma'_{42} & \gamma'_{52} \end{pmatrix}$$

Since the off-diagonal minor

$$\begin{vmatrix} \psi_{32} & \psi_{34} \\ \psi_{52} & \psi_{54} \end{vmatrix} \neq 0,$$

Lemma 1 yields that $\psi_{11} = \phi_{11}$. Similarly, we obtain that $\psi_{22} = \phi_{22}$ and $\psi_{33} = \phi_{33}$. It follows that $\text{rank}(G_{[3]}) = 1$, and thus we can write G as

$$G^t = \begin{pmatrix} \gamma'_{11} & \gamma'_{21} & \gamma'_{31} & g_{41} & g_{51} \\ 0 & 0 & 0 & g'_{42} & g'_{52} \end{pmatrix}.$$

Since $\psi_{14} = \gamma'_{11}\gamma_{41} = \phi_{14} = \gamma'_{11}g_{41}$ and $\psi_{15} = \gamma'_{11}\gamma_{51} = \phi_{15} = \gamma'_{11}g_{51}$, we know $\gamma_{41} = g_{41}$ and $\gamma_{51} = g_{51}$. Therefore,

$$G^t = \begin{pmatrix} \gamma'_{11} & \gamma'_{21} & \gamma'_{31} & \gamma_{41} & \gamma_{51} \\ 0 & 0 & 0 & g'_{42} & g'_{52} \end{pmatrix}.$$

For this case the representation is not unique, but since $\psi_{45} = \phi_{45}$ it must hold that $\gamma'_{42}\gamma'_{52} = g'_{42}g'_{52}$. □

Equipped with Lemma 4, we are able to prove finiteness for $m = 2$ factors.

Theorem 3 ($m = 2$) *A positive definite matrix $\Sigma \in \mathbb{R}^{p \times p}$ with $p \geq 6$ belongs to $F_{p,2} \setminus F_{p,1}$ if and only if every principal 6×6 submatrix belongs to $F_{6,2}$ and at least one 4×4 principal submatrix does not belong to $F_{4,1}$.*

Proof As in the proof of Theorem 2, only the induction step requires work. So suppose that every principal 6×6 submatrix of Σ is in $F_{6,2}$. By the induction hypothesis, all the $(p - 1) \times (p - 1)$ principal submatrices belong to $F_{p-1,2}$, and without loss of generality, we may assume that $\Sigma_{\{2, \dots, 6\}, \{2, \dots, 6\}}$ contains a 4×4 submatrix not in $F_{4,1}$.

Suppose that some row of $\Sigma_{\{2,\dots,6\},\{2,\dots,6\}}$, say the first one, has all off-diagonal entries equal to zero. Consider the representation $\Sigma_{\setminus p,\setminus p} = \Lambda + \Gamma\Gamma^t$. Then $\text{rank}(\Gamma_{\{3,4,5,6\} \times \{2\}}) = 2$, for otherwise $\Sigma_{\{2,\dots,6\},\{2,\dots,6\}}$ would be in $F_{5,1}$. It follows that $\Gamma_2 = (0, 0)$ and all the off-diagonal entries in the second row of Σ except for the last one are zero. By considering a representation of the matrix $\Sigma_{\setminus 1,\setminus 1}$ instead, we can deduce that in fact all of the off-diagonal entries in the second row of Σ are zero. Hence, the induction step goes through easily as we can insert a row of zeros into the matrix Γ in a representation $\Sigma_{\setminus 2,\setminus 2} = \Lambda + \Gamma\Gamma^t$.

In the remaining cases, we can assume that the matrix $\Sigma_{\{2,\dots,6\},\{2,\dots,6\}}$ satisfies the two conditions in Lemma 4. If $\Sigma_{\{2,\dots,6\},\{2,\dots,6\}}$ belongs to case (i) of Lemma 4, then the center matrix $\Sigma_{\setminus \{1,p\},\setminus \{1,p\}}$ has a unique representation. To see this, note that by Lemma 4, $\Sigma_{\{2,\dots,6\},\{2,\dots,6\}}$ has a unique representation, and that we can use a non-zero off-diagonal 2×2 minor from $\Sigma_{\{2,\dots,6\},\{2,\dots,6\}}$ to deduce the uniqueness of the representation of $\Sigma_{\setminus \{1,p\},\setminus \{1,p\}}$. Therefore, Lemma 2 implies that the induction step goes through in this case.

Now assume that $\Sigma_{\{2,\dots,6\},\{2,\dots,6\}}$ belongs to case (ii) of Lemma 4. We first write $\Sigma_{\setminus p,\setminus p} = \Lambda + \Gamma\Gamma^t$ and $\Sigma_{\setminus 1,\setminus 1} = D + GG^t$, where Γ and G are $(p - 1) \times 2$ matrices. By Lemma 4, Γ and G have the following typical forms

$$\Gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma'_{21} & 0 \\ \gamma'_{31} & 0 \\ \gamma'_{41} & 0 \\ \gamma_{51} & \gamma'_{52} \\ \gamma_{61} & \gamma'_{62} \\ \gamma_{71} & \gamma_{72} \\ \vdots & \vdots \\ \gamma_{p-1,1} & \gamma_{p-1,2} \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} \gamma'_{21} & 0 \\ \gamma'_{31} & 0 \\ \gamma'_{41} & 0 \\ \gamma_{51} & g'_{52} \\ \gamma_{61} & g'_{62} \\ g_{71} & g_{72} \\ \vdots & \vdots \\ g_{p-1,1} & g_{p-1,2} \\ g_{p,1} & g_{p,2} \end{pmatrix},$$

where we assigned row indices based on the correspondence to the rows in Σ .

If at least one of the entries $\gamma_{72}, \dots, \gamma_{p-1,2}$ or $g_{72}, \dots, g_{p-1,2}$ is non-zero, then Lemma 1 implies uniqueness of the representation of $\Sigma_{\setminus \{1,p\},\setminus \{1,p\}}$, which allows to apply Lemma 2. Otherwise, we only know that $\gamma'_{52}\gamma'_{62} = g'_{52}g'_{62}$. If $(\gamma'_{52}, \gamma'_{62}) = (g'_{52}, g'_{62})$ or $\gamma_{12} = 0$ or $g_{p,2} = 0$, then the induction step goes through. So we are left with the case, where $\gamma_{72} = \dots = \gamma_{p-1,2} = g_{72} = \dots = g_{p-1,2} = 0$, and $\gamma_{12} \neq 0$, $g_{p,2} \neq 0$. Multiplying the second column of G by -1 if necessary, we can assume that γ'_{52} and g'_{52} have the same sign. To complete the proof we will show that in this case $\gamma'_{52} = g'_{52}$.

Consider the submatrix $S = \Sigma_{\{1,3,4,5,6,p\},\{1,3,4,5,6,p\}}$. From the representation of $\Sigma_{\setminus p,\setminus p}$, we obtain that

$$S_{\setminus p, \setminus p} = \Delta_{\{1,3,4,5,6\}, \{1,3,4,5,6\}} + \begin{pmatrix} \gamma_{11} & \gamma'_{12} \\ \gamma'_{31} & 0 \\ \gamma'_{41} & 0 \\ \gamma_{51} & \gamma'_{52} \\ \gamma_{61} & \gamma'_{62} \end{pmatrix} \begin{pmatrix} \gamma_{11} & \gamma'_{12} \\ \gamma'_{31} & 0 \\ \gamma'_{41} & 0 \\ \gamma_{51} & \gamma'_{52} \\ \gamma_{61} & \gamma'_{62} \end{pmatrix}^t. \tag{3}$$

Similarly, the representation

$$S_{\setminus 1, \setminus 1} = D_{\{3,4,5,6,p\}, \{3,4,5,6,p\}} + \begin{pmatrix} \gamma'_{31} & 0 \\ \gamma'_{41} & 0 \\ \gamma_{51} & g'_{52} \\ \gamma_{61} & g'_{62} \\ g_{p1} & g'_{p2} \end{pmatrix} \begin{pmatrix} \gamma'_{31} & 0 \\ \gamma'_{41} & 0 \\ \gamma_{51} & g'_{52} \\ \gamma_{61} & g'_{62} \\ g_{p1} & g'_{p2} \end{pmatrix}^t. \tag{4}$$

is inherited from the representation of $\Sigma_{\setminus 1, \setminus 1}$. Since $S \in F_{6,2}$, it also has a representation $S = C + FF^t$ as in (2) with $C = (c_{ij})$ diagonal. We label the rows and columns of C by $\{1, 3, 4, 5, 6, p\}$ based on the correspondence to the rows in Σ . Using the structure of the two representations of $S_{\setminus 1, \setminus 1}$ and $S_{\setminus p, \setminus p}$ in (3) and (4), we can deduce via Lemma 1 that $\delta_{55} = c_{55} = d_{55}$; again we assign indices for δ_{55} and d_{55} according to the correspondence to rows in Σ . It follows that $\gamma_{51}^2 + \gamma_{52}^2 = \gamma_{51}^2 + g_{52}^2$. Having assumed that γ_{52} and g_{52} have the same sign, we conclude that $\gamma_{52} = g_{52}$. \square

4 Conclusion

Our main result, Theorem 1, shows that for $m \leq 2$ factors the covariance matrices of distributions in the factor analysis model possess a finiteness structure. This also of interest for recent work on the algebraic geometry of the two-factor model (Sullivant 2009). Our proof uses only linear algebra and shows that, in the covered cases, the distinguished matrix size for finiteness is $2(m + 1)$, that is, one can decide whether a covariance matrix belongs to $F_{p,m}$ by only looking at $2(m + 1) \times 2(m + 1)$ principal submatrices. In recent work concerning a closure of the model, Draisma (2010) shows that finiteness holds also for an arbitrary number of factors m . However, his method of proof does not provide the distinguished matrix size at which finiteness occurs. It is natural to conjecture that this matrix size is equal to $2(m + 1)$ in general. The following example clarifies that the distinguished matrix size cannot be smaller.

Example 1 Consider the matrix

$$\Sigma = \begin{pmatrix} 2I_{m+1} & I_{m+1} \\ I_{m+1} & 2I_{m+1} \end{pmatrix}$$

where I_{m+1} is an $(m + 1) \times (m + 1)$ identity matrix. Then every $(2m + 1) \times (2m + 1)$ principal submatrix belongs to $F_{2m+1,m}$. For example, the following matrix obtained by deleting one row and one column of Σ can be written as

$$\begin{pmatrix} 2I_m & 0 & I_m \\ 0 & 2 & 0 \\ I_m & 0 & 2I_m \end{pmatrix} = \begin{pmatrix} I_m & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & I_m \end{pmatrix} + \begin{pmatrix} I_m \\ 0 \\ I_m \end{pmatrix} \begin{pmatrix} I_m \\ 0 \\ I_m \end{pmatrix}^t.$$

Nevertheless, $\Sigma \notin F_{2m+2,m}$, because the off-diagonal block $\Sigma_{[m+1],[2m+2]\setminus[m+1]}$ has rank $(m + 1)$.

Acknowledgments This material is based upon work supported by the U.S. National Science Foundation under Grant No. DMS-0746265. Mathias Drton was also supported by an Alfred P. Sloan Fellowship.

References

- Anderson, T. W., Rubin, H. (1956). Statistical inference in factor analysis. In *Proceedings of the third Berkeley symposium on mathematical statistics and probability*, 1954–1955 (Vol. V, pp. 111–150). Berkeley: University of California Press.
- Draisma, J. (2010). Finiteness for the k-factor model and chirality varieties. *Advances in Mathematics*, 223(1), 243–256.
- Drton, M., Sturmfels, B., Sullivant, S. (2007). Algebraic factor analysis: Tetrads, pentads and beyond. *Probability Theory and Related Fields*, 138(3–4), 463–493.
- Drton, M., Sturmfels, B., Sullivant, S. (2009). *Lectures on algebraic statistics*. Basel: Birkhäuser.
- Harman, H. H. (1976). *Modern factor analysis* (revised ed.). Chicago: University of Chicago Press.
- Schott, J. R. (2005). Testing for complete independence in high dimensions. *Biometrika*, 92(4), 951–956.
- Sullivant, S. (2009). A Gröbner basis for the secant ideal of the second hypersimplex. *Journal of Commutative Algebra*, 1(2), 327–338.