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Limiting behavior of relative Rényi entropy in a non-regular location shift family

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Abstract We calculate the limiting behavior of relative Rényi entropy between adjacent two probability distribution in a non-regular location-shift family which is generated by a probability distribution whose support is an interval or a half-line. This limit can be regarded as a generalization of Fisher information, and seems closely related to information geometry and large deviation theory.

Keywords Relative Rényi entropy · α-divergence · Information geometry · Non-regular location shift family

1 Introduction

In a regular distribution family, Cramér-Rao inequality gives the bounds of the first order coefficient of the mean square error and the exponential decreasing rate of error probability with an infinitesimal radius in the large deviation evaluation, and the maximum likelihood estimator (MLE) attains under the regularity conditions.

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Therefore, Fisher information can be regarded as the amount of accessible information. However, in a non-regular location shift family that is generated by a distribution of $\mathbb{R}$ whose support is not $\mathbb{R}$ (e.g., a Weibull distribution, gamma distribution, or beta distribution), the Fisher information diverges and cannot be defined. Therefore, in order to characterize the bound of asymptotic performance in estimation, we need a information quantity generalizing Fisher information.

For this purpose, Akahira et al. (1995) proposed the limit of the Hellinger affinity $-\log \int p_\theta^{\frac{1}{2}}(\omega) p_{\theta+\epsilon}^{\frac{1}{2}}(\omega)\, d\omega$ as a substitute information quantity. This value is obtained by a transformation from the Hellinger distance. Moreover, Akahira (1996) proposed the limit of the relative Rényi entropy (Chernoff's distance) $I^s(p\|q) := -\log \int p^s(\omega) q^{1-s}(\omega)\, d\omega$ $(0 < s < 1)$ as a substitute information quantity for a non-regular location shift family while the relative Rényi entropy is discussed by several authors (Chernoff 1952; Hoeffding 1965). When the probability distribution function satisfies boundary conditions, Akahira (1996) calculated the limit of $I^\alpha(p_\theta \| p_{\theta+\epsilon})/\epsilon^2$. However, when the probability distribution function does not satisfy the boundary conditions, Akahira et al. (1995) derived only the order of $I^{1/2}(p_\theta \| p_{\theta+\epsilon})$. That is, they obtain the relation between the order and the behavior of distribution function on the neighborhood of the boundary. However, it has been an open problem to calculate its coefficient with arbitrary $\alpha \in (0, 1)$. In this paper, we calculate the asymptotic behavior of $I^\alpha(p_\theta \| p_{\theta+\epsilon})$ based on the behavior of distribution function on the neighborhood of the boundary in the case of location shift family.

This calculation has the following two meanings, i.e., a statistical meaning and a geometrical meaning. As is shown by Chernoff's formula (Chernoff 1952) and Hoeffding's formula (Hoeffding 1965), the asymptotic error exponents in simple hypothesis testing are characterized by the relative Rényi entropy $I^s(p\|q)$. Using these bounds, Hayashi (2006) gives the following statistical meaning of the obtained calculation. In the large deviation evaluation, the exponential decreasing rates of error probability with an infinitesimal radius can be upperly bounded by the function of the limit of $I^\alpha(p_\theta \| p_{\theta+\epsilon})/g(\epsilon)$, where $g(\epsilon)$ is the order function. That is, the upper bounds contain minimizations of the limit concerning α. This relation is summarized in Sect. 4.

As a geometrical meaning, the relative Rényi entropy $I^s(p\|q)$ is linked with α-divergence $D^\alpha(p\|q) := \frac{4}{1-\alpha^2}\left(1 - \int_\Omega p^{\frac{1-\alpha}{2}}(\omega) q^{\frac{1-\alpha}{2}}(\omega)\, d\omega\right)$, which was introduced by Amari et al. (2000) from an information geometrical viewpoint, by the monotone transformation $x \mapsto -\log\left(1 - \frac{1-\alpha^2}{4} x\right)$. Since α-divergence is a special case of f-divergence introduced by Csiszár (1967), which satisfies the information processing inequality, the relative Rényi entropy satisfies the information processing inequality

$$I^s(p\|q) \geq I^s(p \circ f^{-1} \| q \circ f^{-1})$$

for any map f. When the Kullback-Leibler divergence is finite, the relative Rényi entropies are connected with the Kullback-Leibler divergence by the relation

$$D(p\|q) = \lim_{s\to 1} \frac{1}{s(1-s)} I^s(p\|q) = \lim_{s\to 0} \frac{1}{s(1-s)} I^s(q\|p). \tag{1}$$

Thus, the relative Rényi entropies are suitable as substitutes for the Kullback-Leibler divergence.

As is known, if a one-parameter distribution family $\mathcal{S} := \{p_\theta | \theta \in \Theta \subset \mathbb{R}\}$ satisfies suitable regularity conditions, Kullback-Leibler divergence is closely related to the Fisher information J_θ defined by (3) as

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^2} D(p_{\theta+\epsilon}\|p_\theta) = \frac{1}{2} J_\theta \tag{2}$$

$$J_\theta := \int_\Omega \left(\frac{\partial \log \frac{\partial p_\theta}{\partial p_{\theta_0}}(\omega)}{\partial \theta} \right)^2 p_\theta(\mathrm{d}\omega). \tag{3}$$

However, when the support depends on the parameter θ, the equation does not hold because the divergence is infinite. As was shown by Akahira (1996), under suitable regularity conditions, the equation

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^2 s(1-s)} I^s(p_\theta\|p_{\theta+\epsilon}) = \frac{1}{2} J_\theta \tag{4}$$

holds. As the examples in Sects. 2 and 3 show, there are cases where relation (4) holds, but Eq. (2) does not. The above facts indicate that the limit of the relative Rényi entropy is a suitable substitute for the Fisher information in a non-regular location shift family from a geometrical viewpoint.

Moreover, in a regular family, since Fisher information is well-defined, the Riemann metric can be naturally defined on every tangent space. However, in a non-regular location shift family, as was pointed out by Amari (1984), the natural metric on the tangent space is not a Riemann metric, but a general Minkowski metric. Such a manifold with a general Minkowski metric on every tangent space is called a Finsler space. In order to treat the asymptotic behavior of the MLE, Amari (1984) proposed to regard a non-regular location shift family as a Finsler space with the Minkowski metric $F(\theta) := \lim_{\epsilon\to 0} \frac{1}{\epsilon} H(p_\theta\|p_{\theta+\epsilon})^{\frac{1}{\kappa}}$, where H is the Hellinger distance and κ is a real positive number. Unfortunately, the relation between the MLE and this Minkowski metric has not been adequately clarified, and the value of this Minkowski metric has not been calculated. Our result for the case $s = \frac{1}{2}$ gives the value of this Minkowski metric.

This paper is organized as follows. As a main result, Sect. 2 gives the limit of relative Rényi entropy in the interval support case, which is proven in Sect. 5. As another main result, Sect. 3 gives that in the half-line support case, which is proven in Sect. 6. Section 4 gives a summary of large deviation bound in the non-regular case given in Hayashi (2006).

2 Interval support case

In this section, we discuss the location shift family generated by a C^3 continuous probability density function f whose support is an open interval $(a, b) \subset \mathbb{R}$. We assume conditions (5) and (6) for f:

$$f_1(x) := f(a+x) \cong A_1 x^{\kappa_1 - 1} \quad \text{as } x \to +0 \tag{5}$$
$$f_2(x) := f(b-x) \cong A_2 x^{\kappa_2 - 1} \quad \text{as } x \to +0, \tag{6}$$

where $\kappa_1, \kappa_2 > 0$, and $f(x) \cong g(x)$ ($g(x)$ is a polynomial with real powers of x) as $x \to +0$ means that $\frac{f(x)-g(x)}{x^\epsilon} \to 0$ as $x \to +0$ and ϵ is the maximum power of $g(x)$. In addition, if $\kappa_i \neq 1$, we assume the following conditions:

$$f_i'(x) \cong A_i(\kappa_i - 1)x^{\kappa_i - 2} \quad \text{as} \quad x \to +0 \tag{7}$$
$$f_i''(x) \cong A_i(\kappa_i - 1)(\kappa_i - 2)x^{\kappa_i - 3} \quad \text{as} \quad x \to +0 \quad \text{if } \kappa_i \neq 2 \tag{8}$$
$$x f_i''(x) \to 0 \quad \text{as} \quad x \to +0 \quad \text{if } \kappa_i = 2, \tag{9}$$

where $g'(x)$ and $g''(x)$ are the first and second derivatives of $f(x)$ with respect to x. If $\kappa_i = 1$, we assume the existence of the limits $\lim_{x\to+0} f_i'(x)$ and $\lim_{x\to+0} f_i''(x)$. If $\kappa_i > 2$, we assume that

$$J_f := \int_a^b f^{-1}(x)(f')^2(x)\,\mathrm{d}x < \infty. \tag{10}$$

For example, when f is the beta distribution $f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ whose support is (0, 1), the above conditions are satisfied and we have

$$\kappa_1 = \alpha, \quad \kappa_2 = \beta, \quad A_1 = A_2 = \frac{1}{B(\alpha,\beta)}. \tag{11}$$

In this paper, we denote the beta function by $B(x, y)$. Then, we have the following theorem.

Theorem 1 *Assume that $\kappa := \kappa_1 = \kappa_2$, we obtain the following relations*:

$$\lim_{\epsilon\to+0} \frac{I^s(f_\theta \| f_{\theta+\epsilon})}{\epsilon^\kappa} = \begin{cases} \frac{1-\kappa}{\kappa}\,(A_1 s B(s+\kappa(1-s), 1-\kappa) \\ \quad + A_2(1-s)B(1-s+\kappa s, 1-\kappa)) & 0<\kappa<1 \\ A_1 s + A_2(1-s) & \kappa = 1 \\ \frac{A_1 s(1-s(\kappa-1))B(s+\kappa(1-s),2-\kappa)}{\kappa} & 1<\kappa<2 \\ \quad + \frac{A_2(1-s)(1-(1-s)(\kappa-1))B(1-s+\kappa s,2-\kappa)}{\kappa} & \end{cases} \tag{12}$$
$$\lim_{\epsilon\to+0} \frac{I^s(f_\theta \| f_{\theta+\epsilon})}{-\epsilon^2 \log \epsilon} = \frac{(A_1 + A_2)s(1-s)}{2} \qquad \kappa = 2$$
$$\lim_{\epsilon\to+0} \frac{I^s(f_\theta \| f_{\theta+\epsilon})}{\epsilon^2} = \frac{s(1-s)}{2} J_f \qquad 2 < \kappa,$$

where $f_\theta(x) := f(x-\theta)$. These convergences are uniform for $0 < s < 1$. If $\kappa_1 < \kappa_2$, substituting $\kappa := \kappa_1$, $A_2 := 0$, we obtain the above equations.

The uniformity of $0 < s < 1$ is essential for the discussion in Hayashi (2006). The above theorem in cases (ii) and The case $\kappa > 2$ is an example where relation (4) holds, but relation (2) does not. Note that when $0 < \kappa < 2$, in general, the equation $\lim_{\epsilon\to+0} \frac{I^s(f_\theta\|f_{\theta+\epsilon})}{\epsilon^\kappa} = \lim_{\epsilon\to-0} \frac{I^s(f_\theta\|f_{\theta+\epsilon})}{|\epsilon|^\kappa}$ does not hold.

Next, we introduce two quantities for any $c \in (a, b)$ as

$$I_s^-(c, f, \epsilon) := \int_a^c f^{1-s}(x) f^s(x+\epsilon)\,\mathrm{d}x - \int_a^c f(x)\,\mathrm{d}x - f(c)s\epsilon - \frac{s}{2} f'(c)\epsilon^2,$$

$$I_s^+(c, f, \epsilon) := \int_c^{b-\epsilon} f^{1-s}(x) f^s(x+\epsilon)\,\mathrm{d}x - \int_c^b f(x)\,\mathrm{d}x + f(c)s\epsilon + \frac{s}{2} f'(c)\epsilon^2.$$

Lemma 1 *We obtain the following relations*:

$$\lim_{\epsilon\to+0} \frac{I_s^-(c,f,\epsilon)}{\epsilon^{\kappa_1}} = \begin{cases} -\frac{1-\kappa_1}{\kappa_1} A_1 s B(s+\kappa_1(1-s), 1-\kappa_1) & 0 < \kappa_1 < 1 \\ -A_1 s & \kappa_1 = 1 \\ -\frac{A_1 s(1-s(\kappa_1-1))B(s+\kappa_1(1-s), 2-\kappa_1)}{\kappa_1} & 1 < \kappa_1 < 2 \end{cases}$$
$$\lim_{\epsilon\to+0} \frac{I_s^-(c,f,\epsilon)}{-\epsilon^2\log\epsilon} = -\frac{A_1 s(1-s)}{2} \qquad \kappa_1 = 2 \tag{13}$$
$$\lim_{\epsilon\to+0} \frac{I_s^-(c,f,\epsilon)}{\epsilon^2} = -\frac{s(1-s)}{2} J_{f,c}^- \qquad 2 < \kappa_1$$

and

$$\lim_{\epsilon\to+0} \frac{I_s^+(c,f,\epsilon)}{\epsilon^{\kappa_2}} = \begin{cases} \frac{1-\kappa_2}{\kappa_2} (-A_2(1-s)B(1-s+\kappa_2 s, 1-\kappa_2))) & 0 < \kappa_2 < 1 \\ -A_2(1-s) & \kappa_2 = 1 \\ -\frac{A_2(1-s)(1-(1-s)(\kappa_2-1))B(1-s+\kappa_2 s, 2-\kappa_2)}{\kappa_2} & 1 < \kappa_2 < 2 \end{cases}$$
$$\lim_{\epsilon\to+0} \frac{I_s^+(c,f,\epsilon)}{-\epsilon^2\log\epsilon} = -\frac{A_2 s(1-s)}{2} \qquad \kappa_2 = 2 \tag{14}$$
$$\lim_{\epsilon\to+0} \frac{I_s^+(c,f,\epsilon)}{\epsilon^2} = -\frac{s(1-s)}{2} J_{f,c}^+ \qquad 2 < \kappa_2,$$

where J_f^- and J_f^+ are defined as

$$J_{f,c}^- := \int_a^c f^{-1}(x)(f'(x))^2\,\mathrm{d}x, \quad J_{f,c}^+ := \int_c^b f^{-1}(x)(f'(x))^2\,\mathrm{d}x.$$

These convergences are uniform for $0 < s < 1$.

Lemma 1 is proven in Appendix 5.

Proof of Theorem 1 Since $I^s(f_\theta\|f_{\theta+\epsilon}) = I^s(f_{-\epsilon}\|f_0)$, $I^s(f_\theta\|f_{\theta+\epsilon}) = -\log(1 + I_s^+(c, f, \epsilon) + I_s^-(c, f, \epsilon))$. Thus, $\lim_{\epsilon\to+0} \frac{I^s(f_\theta\|f_{\theta+\epsilon})}{\epsilon^\kappa} = \lim_{\epsilon\to+0} \frac{-I_s^+(c,f,\epsilon) - I_s^-(c,f,\epsilon))}{\epsilon^\kappa}$. Therefore, Lemma 1 yields equation (12). □

Akahira (1996) proved the following proposition.

Proposition 1 *When*

$$\lim_{x\to a+0} f(x) = \lim_{x\to b-0} f(x) = c, \qquad (15)$$
$$\lim_{x\to a+0} f'(x) = -\lim_{x\to b-0} f'(x) = h,$$

and the condition (10) *holds, then,*

$$I^s(f_\theta|f_{\theta+\epsilon}) \cong c\epsilon + \frac{c^2 - h + s(1-s)J_f}{2}\epsilon^2 \text{ as } \epsilon \to +0.$$

At first glance, the above proposition contains our case (iv). However, in case (iv), the relation (10) does not hold, i.e., J_f diverges. That is, our result with case (iv) does not contradict with the above proposition. Further, our case (ii) with (15) coincides with the above proposition in the first order. Also, our case (v) coincides with the above proposition with $c = h = 0$. Note that the above proposition does not treat the uniformity of the convergence concerning s while our theorem treats it.

3 Half-line support case

In this section, we discuss the case where the support is the half-line $(0, \infty)$ and the probability density function f is C^3 continuous. Similarly to (5) and (6), we assume that

$$f(x) \cong Ax^{\kappa-1} \text{ as } x \to 0. \qquad (16)$$

When $\kappa \neq 1$, we assume the following conditions:

$$f'(x) \cong A_i(\kappa-1)x^{\kappa-2} \quad \text{as} \quad x \to +0 \qquad (17)$$
$$f''(x) \cong A_i(\kappa-1)(\kappa-2)x^{\kappa-3} \quad \text{as} \quad x \to +0 \text{ if } \kappa \neq 2 \qquad (18)$$
$$xf''(x) \to 0 \quad \text{as} \quad x \to +0 \text{ if } \kappa = 2. \qquad (19)$$

When $\kappa = 1$, we assume the existence of the limits $\lim_{x\to+0} f'(x)$ and $\lim_{x\to+0} f''(x)$. In addition, we assume that there exist real numbers $c > 0$ and $\epsilon > 0$ such that $f(x)$ is monotone decreasing for $x > c$ and

$$\int_c^\infty f^{-1}(x)(f'(x))^2\, dx < \infty \qquad (20)$$

$$\lim_{\epsilon\to+0} \lim_{R\to\infty} \sup_{x,y>R:|x-y|<\epsilon} \frac{f(y)}{f(x)} < +\infty \qquad (21)$$

$$\lim_{\epsilon\to+0} \lim_{R\to\infty} \sup_{x,y>R:|x-y|<\epsilon} \frac{f'(y)}{f'(x)} < +\infty \qquad (22)$$

$$\lim_{\epsilon\to+0} \lim_{R\to\infty} \sup_{x,y>R:|x-y|<\epsilon} \frac{f''(y)}{f''(x)} < +\infty, \qquad (23)$$

and there exists $R > 0$ such that

$$f''(x) \geq 0 \text{ for } x \geq R. \tag{24}$$

For example, when f is Weibull distribution $f(x) = \alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$, the above conditions are satisfied and we have

$$\kappa = \alpha, \quad A = \alpha\beta. \tag{25}$$

When f is gamma distribution $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$, the above conditions are satisfied and

$$\kappa = \alpha, \quad A = \frac{\beta^\alpha}{\Gamma(\alpha)}. \tag{26}$$

Now, we obtain the following theorem.

Theorem 2 *We obtain*

$$\begin{aligned}
\lim_{\epsilon\to+0}\frac{I^s(f_\theta\|f_{\theta+\epsilon})}{\epsilon^\kappa} &= \begin{cases} \frac{1-\kappa}{\kappa}(AsB(s+\kappa(1-s),1-\kappa) & 0 < \kappa < 1 \\ As & \kappa = 1 \\ \frac{As(1-s(\kappa-1))B(s+\kappa(1-s),2-\kappa)}{\kappa} & 1 < \kappa < 2 \end{cases} \\
\lim_{\epsilon\to+0}\frac{I^s(f_\theta\|f_{\theta+\epsilon})}{-\epsilon^2\log\epsilon} &= \frac{As(1-s)}{2} \qquad \kappa = 2 \\
\lim_{\epsilon\to+0}\frac{I^s(f_\theta\|f_{\theta+\epsilon})}{\epsilon^2} &= \frac{s(1-s)}{2}J_f \qquad 2 < \kappa,
\end{aligned} \tag{27}$$

where

$$J_f := \int_0^\infty f^{-1}(x)(f')^2(x)\,dx. \tag{28}$$

These convergences are uniform for $0 < s < 1$.

For a real number $c > 0$ satisfying (20)–(23), we define

$$\tilde{I}_s^+(c,f,\epsilon) := \int_c^\infty f^{1-s}(x)f^s(x+\epsilon)\,dx - \int_c^i nfty f(x)\,dx + f(c)s\epsilon + \frac{s}{2}f'(c)\epsilon^2.$$

Lemma 2 *We obtain*

$$\lim_{\epsilon\to+0}\frac{\tilde{I}_s^+(c,f,\epsilon)}{\epsilon^2} = -\frac{s(1-s)}{2}J_{f,c}^+ \tag{29}$$

and the convergence of (29) *is uniform for* $0 < s < 1$, *where*

$$J_{f,c}^+ := \int_c^\infty f^{-1}(x)(f'(x))^2\,dx.$$

Lemma 2 is proven in Appendix 6.

Proof of Theorem 2 Similarly to Theorem 1, it follows from Lemma 1 and Lemma 2 that $I^s(f_\theta \| f_{\theta+\epsilon}) = -\log(1 + \tilde{I}_s^+(c, f, \epsilon) + I_s^-(c, f, \epsilon))$, which yields Eq. (27). □

4 Relation between main results and large deviation theory

We will outline a relation between Theorems 1 and 2 and large deviation theory only for a location shift family $\{f_\theta(x) := f(x-\theta) | \theta \in \mathbb{R}\}$, where f satisfies the conditions given in Sects. 2 or 3. This relation was discussed by Hayashi (2006) more precisely. As generalizations of Bahadur's large deviation type bound, we define the following quantities:

$$\alpha_1(\theta) := \limsup_{\epsilon \to +0} \frac{1}{g(\epsilon)} \sup_{\mathbf{T}} \inf_{\theta-\epsilon \le \theta' \le \theta+\epsilon} \beta(\mathbf{T}, \theta', \epsilon)$$

$$\alpha_2(\theta) := \sup_{\mathbf{T}} \liminf_{\epsilon \to +0} \frac{1}{g(\epsilon)} \inf_{\theta-\epsilon \le \theta' \le \theta+\epsilon} \beta(\mathbf{T}, \theta', \epsilon)$$

$$\beta(\mathbf{T}, \theta, \epsilon) := \liminf \frac{-1}{n} \log p_\theta^n\{|T_n - \theta| > \epsilon\},$$

where $\mathbf{T} = \{T_n\}$ is a sequence of estimators, i.e. every T_n is a function from the data set $\mathbb{R}^n$ to the parameter set $\mathbb{R}$. Also, $g(\epsilon)$ is chosen by

$$g(\epsilon) = \begin{cases} \epsilon^\kappa & 0 < \kappa < 2 \\ -\epsilon^2 \log \epsilon & \kappa = 2 \\ \epsilon^2 & \kappa > 2. \end{cases}$$

As Ibragimov (1981) pointed out, when KL-divergence is infinite, there exists a super efficient estimator $\mathbf{T}$ such that $\beta(\mathbf{T}, \theta, \epsilon)$ and $\lim_{\epsilon \to +0} \frac{1}{g(\epsilon)} \beta(\mathbf{T}, \theta, \epsilon)$ are infinite at one point θ. Therefore, we need to take the infimum $\inf_{\theta-\epsilon \le \theta' \le \theta+\epsilon}$ into account. Of course, in a regular case, as was proven by Hayashi (2006), the two bounds $\alpha_1(\theta)$ and $\alpha_2(\theta)$ coincide.

If the convergence $\lim_{\epsilon \to 0} \frac{I^s(p_{\theta-\epsilon/2} \| p_{\theta+\epsilon/2})}{g(\epsilon)}$ is uniform for $s \in (0, 1)$ and $\theta \in K$ for any compact set $K \subset \mathbb{R}$, these quantities are evaluated as

$$\alpha_1(\theta) \le \overline{\alpha}_1(\theta) := \begin{cases} 2^\kappa \sup_{0<s<1} I_{g,\theta}^s & \text{if } 0 < \kappa < 2 \\ 4 \sup_{0<s<1} I_{g,\theta}^s & \text{if } \kappa \ge 2 \end{cases}$$

$$\alpha_2(\theta) \le \overline{\alpha}_2(\theta) := \begin{cases} \sup_{0<s<1} \frac{I_{g,\theta}^s}{s(1-s)} \left(s^{\frac{1}{\kappa-1}} + (1-s)^{\frac{1}{\kappa-1}} \right)^{\kappa-1} & \text{if } 0 < \kappa < 1 \\ 2 I_{g,\theta}^{\frac{1}{2}} & \text{if } \kappa = 1 \\ \inf_{0<s<1} \frac{I_{g,\theta}^s}{s(1-s)} \left(s^{\frac{1}{\kappa-1}} + (1-s)^{\frac{1}{\kappa-1}} \right)^{\kappa-1} & \text{if } 2 > \kappa > 1 \\ \inf_{0<s<1} \frac{I_{g,\theta}^s}{s(1-s)} & \text{if } 2 \le \kappa, \end{cases}$$

where $I^s_{g,\theta}$ are defined by

$$I^s_{g,\theta} := \lim_{\epsilon\to+0} \frac{I^s(p_{\theta-\epsilon/2}\|p_{\theta+\epsilon/2})}{g(\epsilon)} \quad 1 \geq s \geq 0.$$

Note that the uniformity of the convergence concerning $0 < s < 1$ is necessary for deriving the above inequalities. In Hayashi (2006), these inequalities were proven and the attainability of bounds $\overline{\alpha}_1(\theta)$ and $\overline{\alpha}_2(\theta)$ were discussed.

5 Proof of Lemma 1

5.1 Asymptotic behavior of $I^-_s(c, f, \epsilon)$

In the following, when the limit $\lim_{\epsilon\to+0} g(x+\epsilon)$ ($\lim_{\epsilon\to+0} g(x-\epsilon)$) exists for a function g, we denote it by $g(x+0)$ ($g(x-0)$), respectively. Our situation is divided into five cases: (i) $0 < \kappa_1 < 1$, (ii) $\kappa_1 = 1$, (iii) $1 < \kappa_1 < 2$, (iv) $\kappa_1 = 2$, and (v) $\kappa_1 > 2$. First, we discuss cases (ii) and (v).

$$\begin{aligned}
&\int_a^c f^{1-s}(x) f^s(x+\epsilon)\,\mathrm{d}x \\
&\quad = \int_a^c f^{1-s}(x)\left[f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right)\right]\mathrm{d}x \\
&\qquad + \int_a^c \left(f(x) + f^{1-s}(x)(f^s)'(x)\epsilon + f^{1-s}(x)(f^s)''(x)\frac{\epsilon^2}{2}\right)\mathrm{d}x
\end{aligned} \tag{30}$$

The second term is calculated by

$$\begin{aligned}
&\int_a^c \left(f(x) + f^{1-s}(x)(f^s)'(x)\epsilon + f^{1-s}(x)(f^s)''(x)\frac{\epsilon^2}{2}\right)\mathrm{d}x \\
&\quad = \int_a^c \left(f(x) + f'(x)s\epsilon + \left(\frac{s(s-1)}{2} f^{-1}(x)(f'(x))^2 + \frac{s}{2} f''(x)\right)\epsilon^2\right)\mathrm{d}x \\
&\quad = \int_a^c f(x)\,\mathrm{d}x + f(c)\epsilon s + f'(c)s\frac{\epsilon^2}{2} - f(a+0)s\epsilon \\
&\qquad + \left(\frac{s(s-1)}{2}\int_a^c f^{-1}(x)(f'(x))^2\,\mathrm{d}x - \frac{s}{2} f'(a+0)\right)\epsilon^2.
\end{aligned} \tag{31}$$

The term

$$\frac{1}{\epsilon^2}\int_a^c f^{1-s}(x)\left[f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right)\right]\mathrm{d}x \tag{32}$$

goes to 0 uniformly for $0 < s < 1$ as $\epsilon \to +0$ as follows. In case (ii), $\kappa_1 = 1$, the term $\left[f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right)\right]/\epsilon^2$ goes to 0 uniformly on $(a, c]$. This speed does not depends on s. This is because $\lim_{x\to+0} f_1(x) < +\infty$.

That is, the above argument has been shown. In case (v), $\kappa_1 > 2$, this convergence is uniform on any compact subset of $(a, c]$. We choose a sufficiently small number $\delta > 0$. From condition (8), we have $0 \le (f^s)''(x) \le f''(y)$ for $a < y < x < a - \delta$. Then, there exists $t(\epsilon, x) \in [0, 1]$ such that

$$\begin{aligned}
&\frac{1}{\epsilon^2}\left|\int_a^{a+\delta} f^{1-s}(x)\left[f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right)\right] \mathrm{d}x\right| \\
&= \frac{1}{2}\left|\int_a^{a+\delta} f^{1-s}(x)\left[(f^s)''(x + t(\epsilon, x)\epsilon) - (f^s)''(x)\right] \mathrm{d}x\right|. \\
&\le \frac{1}{2}(\sup_{s\in[0,1],x\in[a,c]} f^{1-s}(x)) \int L_a^{a+\delta} f''(x)\, \mathrm{d}x \\
&= \frac{1}{2}(\sup_{s\in[0,1],x\in[a,c]} f^{1-s}(x))(f'(a+\delta) - f'(a+0)) < \infty.
\end{aligned}$$

Thus, for any $\epsilon' > 0$, there exists a postive real number δ' such that the above value is less than ϵ'. Now, by choosing a sufficiently small number $\epsilon > 0$ independently of s, we have

$$\frac{1}{\epsilon^2}\int_{a+\delta}^{c} f^{1-s}(x)\left[f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right)\right] \mathrm{d}x < \epsilon'. \tag{33}$$

This implies that (32) goes to 0 uniformly concerning s. Therefore, in case (ii), since $f(a+0) = A_1$, we obtain (13) and the uniformity for $0 < s < 1$. In case (v), since $f(a+0) = f'(a+0) = 0$, we obtain (13) and the uniformity for $\kappa_1 > 2$.

Next, we discuss cases (i), (iii), and (iv). We can calculate $I_s^-(c, f, \epsilon)$ as

$$\begin{aligned}
&\int_a^c f^{1-s}(x) f^s(x+\epsilon)\, \mathrm{d}x \\
&= \int_{a+\delta}^{c} f^{1-s}(x)\left[f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right)\right] \mathrm{d}x \\
&\quad + \int_{a+\delta}^{c}\left(f(x) + f^{1-s}(x)(f^s)'(x)\epsilon + f^{1-s}(x)(f^s)''(x)\frac{\epsilon^2}{2}\right) \mathrm{d}x \\
&\quad + \int_a^{a+\delta} f^{1-s}(x) f^s(x+\epsilon)\, \mathrm{d}x
\end{aligned} \tag{34}$$

In the following, we discuss only case (i). Concerning the second term of (34), we have

$$\int_{a+\delta}^{c}\left(f(x) + f^{1-s}(x)(f^s)'(x)\epsilon + f^{1-s}(x)(f^s)''(x)\frac{\epsilon^2}{2}\right) \mathrm{d}x + \int_a^{a+\delta} f(x)\, \mathrm{d}x$$

$$
\begin{aligned}
&= \int_a^c f(x)\,\mathrm{d}x + \left(\int_{a+\delta}^c f'(x)\,\mathrm{d}x\right) s\epsilon + \frac{s(s-1)}{2}\left(\int_{a+\delta}^c f^{-1}(x)(f'(x))^2\,\mathrm{d}x\right)\epsilon^2 \\
&\quad + \frac{s}{2}\left(\int_{a+\delta}^c f''(x)\,\mathrm{d}x\right) \\
&= \int_a^c f(x)\,\mathrm{d}x + (f(c) - f(a+\delta))s\epsilon + (f'(c) - f'(a+\delta))\frac{s}{2}\epsilon^2 \\
&\quad + \frac{s(s-1)}{2}\int_{a+\delta}^c f^{-1}(x)(f'(x))^2\,\mathrm{d}x\epsilon^2 \\
&= \int_a^c f(x)\,\mathrm{d}x + f(c)s\epsilon + \frac{s}{2}sf'(c)\epsilon^2 \\
&\quad - f(a+\delta)s\epsilon - f'(a+\delta)\frac{s}{2}\epsilon^2 + \frac{s(s-1)}{2}\left(\int_{a+\delta}^c f^{-1}(x)(f'(x))^2\,\mathrm{d}x\right)\epsilon^2
\end{aligned}
\tag{35}
$$

Concerning the third term of (34), we can calculate

$$
\begin{aligned}
&\int_a^{a+\delta} f^{1-s}(x) f^s(x+\epsilon)\,\mathrm{d}x - \int_a^{a+\delta} f(x)\,\mathrm{d}x \\
&= \int_a^{a+\delta} \left(f^{1-s}(x) f^s(x+\epsilon) - f(x)\right)\,\mathrm{d}x \\
&= \int_a^{a+\delta}\int_0^{\epsilon} f^{1-s}(x)(f^s)'(x+y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^{\epsilon}\int_0^{\frac{\delta}{y}} s\frac{f_1^{1-s}(yz)}{f_1^{1-s}(y(z+1))}\frac{f_1'(y(z+1))}{f_1'(y)}\,\mathrm{d}z y f_1'(y)\,\mathrm{d}y,
\end{aligned}
\tag{36}
$$

where we set $z = \frac{x-a}{y}$. Note that when y is small enough, $f_1'(y)$ and $f_1(y(z+1))$ is positive because of (5) and (7). Since

$$
\int_0^{\infty} \frac{z^{(\kappa_1-1)(1-s)}}{(1+z)^{(\kappa_1-1)(1-s)+2-\kappa_1}}\,\mathrm{d}z = B(\kappa_1 + s - \kappa_1 s, 1-\kappa_1),
$$

using (5) and (7), we can prove that for any $\epsilon' > 0$ there exist real numbers $\delta > 0$ and $\epsilon > 0$ independently for s such that

$$
\left|\int_0^{\frac{\delta}{y}} \frac{f_1^{1-s}(yz)}{f_1^{1-s}(y(z+1))}\frac{f_1'(y(z+1))}{f_1'(y)}\,\mathrm{d}z - B(\kappa_1 + s - \kappa_1 s, 1-\kappa_1)\right| < \epsilon' \tag{37}
$$

for any y satisfying $\epsilon > y > 0$. For any $\epsilon' > 0$, there exists a real $\epsilon > 0$ such that

$$
\left|\frac{\int_0^{\epsilon} y f_1'(y)\,\mathrm{d}y}{\epsilon^{\kappa_1}} - A_1\frac{\kappa_1 - 1}{\kappa_1}\right| < \epsilon'. \tag{38}
$$

Therefore,

$$
\begin{aligned}
&\left| \frac{\int_0^{\epsilon} \int_0^{\frac{\delta}{y}} s \frac{f_1^{1-s}(yz)}{f_1^{1-s}(y(z+1))} \frac{f_1'(y(z+1))}{f_1'(y)} \mathrm{d}z y f_1'(y)\, \mathrm{d}y}{\epsilon^{\kappa_1}} + A_1 B(\kappa_1 + s - \kappa_1 s, 1 - \kappa_1) \frac{s(1-\kappa_1)}{\kappa_1} \right| \\
&\le \left| \int_0^{\frac{\delta}{y}} s \frac{f_1^{1-s}(yz)}{f_1^{1-s}(y(z+1))} \frac{f_1'(y(z+1))}{f_1'(y)} \mathrm{d}z - sB(\kappa_1 + s - \kappa_1 s, 1-\kappa_1) \right| \frac{\int_0^{\epsilon} y f_1'(y) \mathrm{d}y}{\epsilon^{\kappa_1}} \\
&\quad + sB(\kappa_1 + s - \kappa_1 s, 1 - \kappa_1) \left| \frac{\int_0^{\epsilon} y f_1'(y)\, \mathrm{d}y}{\kappa_1} + A_1 \frac{(1-\kappa_1)}{\kappa_1} \right| \epsilon^{\kappa_1} \\
&< \epsilon' \left(A_1 \frac{(1-\kappa_1)}{\kappa_1} + sB(\kappa_1 + s - \kappa_1 s, 1-\kappa_1) + \epsilon' \right) \\
&\le \epsilon' \left(A_1 \frac{(1-\kappa_1)}{\kappa_1} + \epsilon' + \sup_{0<s<1} sB(\kappa_1 + s - \kappa_1 s, 1 - \kappa_1) \right) \le C_0 \epsilon', \qquad (39)
\end{aligned}
$$

where we choose $C_0 = \left(A_1 \frac{(1-\kappa_1)}{\kappa_1} + 1 + \sup_{0<s<1} sB(\kappa_1 + s - \kappa_1 s, 1 - \kappa_1) \right)$. Therefore, relations (34), (35), (36), and (39) yield

$$
\begin{aligned}
&\frac{\left| I_s^-(c, f, \epsilon) - A_1 B(\kappa_1 + s - \kappa_1 s, 1 - \kappa_1) \frac{s(1-\kappa_1)}{\kappa_1} \epsilon^{\kappa_1} \right|}{\epsilon^{\kappa_1}} \\
&\quad \le \frac{1}{\epsilon^{\kappa_1}} \left[\int_{a+\delta}^{c} f^{1-s}(x) \left(f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x) \frac{\epsilon^2}{2} \right) \right) \mathrm{d}x \right. \\
&\qquad \left. + \left| -f(a+\delta)s\epsilon - f'(a+\delta) \frac{s}{2} \epsilon^2 + \frac{s(s-1)}{2} \left(\int_{a+\delta}^{c} f^{-1}(x)(f'(x))^2\, \mathrm{d}x \right) \epsilon^2 \right| \right] \\
&\qquad + C_0 \epsilon'. \qquad (40)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{\left| I_s^-(c, f, \epsilon) - A_1 B(\kappa_1 + s - \kappa_1 s, 1 - \kappa_1) \frac{s(1-\kappa_1)}{\kappa_1} \epsilon^{\kappa_1} \right|}{\epsilon^{\kappa_1}} \\
&\quad \le \frac{1}{\epsilon^{\kappa}} \left(C_1 \epsilon + C_2 \epsilon^2 + \epsilon' \epsilon^2 \right) + C_0 \epsilon',
\end{aligned}
$$

where $C_1 := |f(a+\delta)|$, $C_2 := \frac{|f'(a+\delta)|}{2}$. Note that the constants C_1 and C_2 are independent of $1 > s > 0$. We obtain (13) and the uniformity in case (i).

Next, we discuss cases (iii), ($1 < \kappa_1 < 2$) and (iv), ($\kappa_1 = 2$). Concerning the second term of (34), we can calculate

$$\begin{aligned}
&\int_{a+\delta}^{c} \left(f(x) + f^{1-s}(x)(f^s)'(x)\epsilon + f^{1-s}(x)(f^s)''(x)\frac{\epsilon^2}{2} \right) \mathrm{d}x \\
&\quad + \int_{a}^{a+\delta} f(x) + f^{1-s}(x)(f^s)'(x)\,\mathrm{d}x \\
&= \int_a^c \left(f(x) + f'(x)s\epsilon \right) \mathrm{d}x + \frac{s(s-1)}{2}\left(\int_{a+\delta}^{c} f^{-1}(x)(f'(x))^2\,\mathrm{d}x \right)\epsilon^2 \\
&\quad + \frac{s}{2}\left(\int_{a+\delta}^{c} f''(x)\,\mathrm{d}x \right)\epsilon^2 \\
&= \int_a^c f(x)\,\mathrm{d}x + (f(c) - f(a+0))s\epsilon + (f'(c) - f'(a+\delta))s\frac{\epsilon^2}{2} \\
&\quad + \frac{s(1-s)}{2}\left(\int_{a+\delta}^{c} f^{-1}(x)(f'(x))^2\,\mathrm{d}x \right)\epsilon^2.
\end{aligned} \tag{41}$$

Concerning the last term of (34), we have

$$\begin{aligned}
&\int_a^{a+\delta} f^{1-s}(x) f^s(x+\delta)\,\mathrm{d}x - \int_a^{a+\delta} f(x) + f^{1-s}(x)(f^s)'(x)\,\mathrm{d}x \\
&= \int_a^{a+\delta}\int_0^{\epsilon}\int_0^{y_1} f^{1-s}(x)(f^s)''(x+y_2)\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x \\
&= \int_0^{\epsilon}\int_0^{y_1}\int_0^{\frac{\delta}{y_2}} \left[s\frac{f^{1-s}(y_2 z)}{f^{1-s}(y_2(z+1))}\frac{f''(y_2(z+1))}{f''(y_2)}\frac{f''(y_2)f(y_2)}{(f')^2(y_2)} \right. \\
&\quad \left. + s(s-1)\frac{f^{1-s}(y_2 z)}{f^{1-s}(y_2(z+1))}\frac{f(y_2)}{f(y_2(z+1))}\frac{(f')^2(y_2(z+1))}{(f')^2(y_2)} \right] \\
&\quad \times \mathrm{d}z\frac{(f')^2(y_2)}{f(y_2)} y_2\,\mathrm{d}y_2\,\mathrm{d}y_1.
\end{aligned} \tag{42}$$

In the following, we consider only case (iii). Since

$$\int_0^{\infty} z^{(1-s)(\kappa_1-1)}(1+z)^{s(\kappa_1-1)-2}\,\mathrm{d}z = B(1+(1-s)(\kappa_1-1), 2-\kappa_1),$$

using (5), (7), and (8), we can show that for any $\epsilon' > 0$, there exist real numbers $\delta > 0$ and $\epsilon > 0$ such that

$$\begin{aligned}
&\left| \left[\int_0^{\frac{\delta}{y_2}} \left(s\frac{f^{1-s}(y_2 z)}{f^{1-s}(y_2(z+1))}\frac{f''(y_2(z+1)}{f''(y_2)}\frac{f''(y_2)f(y_2)}{(f')^2(y_2)} \right. \right. \right. \\
&\quad \left. + s(s-1)\frac{f^{1-s}(y_2 z)}{f^{1-s}(y_2(z+1))}\frac{f(y_2)}{f(y_2(z+1))}\frac{(f')^2(y_2(z+1))}{(f')^2(y_2)} \right) \mathrm{d}z
\end{aligned}$$

$$-B(1+(1-s)(\kappa_1-1), 2-\kappa_1)\frac{s(\kappa_1-2+(s-1)(\kappa_1-1))}{\kappa_1-1}\Bigg]\Bigg| < \epsilon' \tag{43}$$

for $\epsilon > \forall y_2 > 0$. For any $\epsilon' > 0$, there exists a real number $\epsilon > 0$ such that

$$\frac{\left|\int_0^\epsilon \int_0^{y_1} \frac{(f')^2(y_2)}{f(y_2)} y_2 \, \mathrm{d}y_2 \, \mathrm{d}y_1 - \frac{\kappa_1 - 1}{\kappa_1}\epsilon^{\kappa_1}\right|}{\epsilon_1^{\kappa}} < \epsilon'. \tag{44}$$

Similarly to (39), it follows from (42)–(44) that we can choose a constant C_0 such that

$$\begin{aligned}&\left|\frac{\int_a^{a+\delta}\int_0^\epsilon \int_0^{y_1} f^{1-s}(x)(f^s)''(x+y_2)\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x}{\epsilon^{\kappa_1}}\right.\\ &\qquad\left.+\frac{A_1 B(1+(1-s)(\kappa_1-1), 2-\kappa_1)\frac{s(2-\kappa_1+(1-s)(\kappa_1-1))}{\kappa_1}}{\epsilon^{\kappa_1}}\right|\\ &< C_0\epsilon.\end{aligned} \tag{45}$$

From (34), (41), (42), and (44), we can evaluate

$$\begin{aligned}&\frac{\left|I_s^-(c,f,\epsilon)+A_1 B(1+(1-s)(\kappa_1-1),2-\kappa_1)\frac{s(2-\kappa_1+(1-s)(\kappa_1-1))}{\kappa_1}\epsilon^{\kappa_1}\right|}{\epsilon^{\kappa_1}}\\ &\le \frac{1}{\epsilon^{\kappa_1}}\left[\int_{a+\delta}^c f^{1-s}(x)\left|f^s(x+\epsilon)-\left(f^s(x)+(f^s)'(x)\epsilon+(f^s)''(x)\frac{\epsilon^2}{2}\right)\right|\mathrm{d}x,\right.\\ &\quad\left.+\left|-f'(a+\delta)s\frac{\epsilon^2}{2}+\frac{s(1-s)}{2}\left(\int_{a+\delta}^c f^{-1}(x)(f'(x))^2\,\mathrm{d}x\right)\epsilon^2\right|\right]+C_0\epsilon'.\end{aligned} \tag{46}$$

Note that $f(a+0)=0$. Thus, from (33), we can choose a constant C_2 independently for $0 < s < 1$ such that

$$\begin{aligned}&\frac{\left|I_s^-(c,f,\epsilon)+A_1 B(1+(1-s)(\kappa_1-1),2-\kappa_1)\frac{s(2-\kappa_1+(1-s)(\kappa_1-1))}{\kappa_1}\epsilon^{\kappa_1}\right|}{\epsilon^{\kappa_1}}\\ &\le \frac{1}{\epsilon^{\kappa_1}}\left(C_2\epsilon^2+\epsilon'\epsilon^2\right)+C_0\epsilon'.\end{aligned}$$

Thus, we obtain (13) and the uniformity in case (iii).

In the following, we discuss case (iv). Using the conditions (5), (7), and (9), we can prove that for any $\epsilon' > 0$, there exist real numbers $\delta > 0$ and $\epsilon > 0$ such that

$$\left| \frac{\int_0^{\frac{\delta}{y_2}} s \frac{f^{1-s}(y_2 z)}{f^{1-s}(y_2(z+1))} \frac{f''(y_2(z+1))}{f''(y_2)} \frac{f''(y_2) f(y_2)}{(f')^2(y_2)} \mathrm{d}z}{-\log y_2} + \frac{\int_0^{\frac{\delta}{y_2}} s(s-1) \frac{f^{1-s}(y_2 z)}{f^{1-s}(y_2(z+1))} \frac{f(y_2)}{f(y_2(z+1))} \frac{(f')^2(y_2(z+1))}{(f')^2(y_2)} \mathrm{d}z}{-\log y_2} + \frac{s(1-s)(-\log y_2)}{-\log y_2} \right| < \epsilon' \tag{47}$$

for $\epsilon > y_2 > 0$. For any $\epsilon' > 0$, there exists a real number $\epsilon > 0$ such that

$$\frac{\left| \int_0^{\epsilon} \int_0^{y_1} -\log y_2 \frac{(f')^2(y_2)}{f(y_2)} y_2 \, \mathrm{d}y_2 \, \mathrm{d}y_1 - A_1(-\frac{1}{2}\epsilon^2 \log \epsilon) \right|}{-\epsilon^2 \log \epsilon} < \epsilon'. \tag{48}$$

Similarly to (39), we can choose a constant C_0 such that

$$\frac{\left| \int_a^{a+\delta} \int_0^{\epsilon} \int_0^{y_1} f^{1-s}(x)(f^s)''(x+y_2) \, \mathrm{d}y_2 \, \mathrm{d}y_1 \, \mathrm{d}x + A_1 \frac{s(1-s)}{2} \epsilon^2 (-\log \epsilon) \right|}{-\epsilon^2 \log \epsilon} < C_0 \epsilon' \tag{49}$$

From (34), (41), (42), and (49), we can evaluate this as

$$\begin{aligned} &\frac{\left| I_s^-(c, f, \epsilon) + A_1 \frac{s(1-s)}{2} \epsilon^2 (-\log \epsilon) \right|}{\epsilon^2 (-\log \epsilon)} \\ &\leq \frac{1}{\epsilon^2(-\log \epsilon)} \left[\int_{a+\delta}^{c} f^{1-s}(x) \left| f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x) \frac{\epsilon^2}{2} \right) \right| \mathrm{d}x \right. \\ &\quad \left. + \left| -f'(a+\delta) s \frac{\epsilon^2}{2} + \frac{s(1-s)}{2} \left(\int_{a+\delta}^{c} f^{-1}(x)(f'(x))^2 \, \mathrm{d}x \right) \epsilon^2 \right| \right] + C_0 \epsilon'. \end{aligned} \tag{50}$$

Note that $f(a+0) = 0$. Thus, from (33), we can choose a constant C_2 independently for $1 > s > 0$ such that

$$\frac{\left| I_s^-(c, f, \epsilon) + A_1 \frac{s(1-s)}{2} \epsilon^2 (-\log \epsilon) \right|}{\epsilon^2 (-\log \epsilon)} \leq \frac{1}{\epsilon^2 (-\log \epsilon)} \left(C_2 \epsilon^2 + \epsilon' \epsilon^2 \right) + C_0 \epsilon'.$$

Thus, we obtain (13) and the uniformity in case (iv).

5.2 Asymptotic behavior of $I_s^+(c, f, \epsilon)$

As in subsection 5.1, our situation is divided into five cases: (i) $0 < \kappa_2 < 1$, (ii) $\kappa_2 = 1$, (iii) $1 < \kappa_2 < 2$, (iv) $\kappa_2 = 2$, and (v) $\kappa_2 > 2$. First, we consider cases (ii) and (v).

$$
\begin{aligned}
&\int_c^{b-\epsilon} f^{1-s}(x) f^s(x+\epsilon)\,\mathrm{d}x \\
&\quad = \int_c^{b-\epsilon} f^{1-s}(x)\left[f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right)\right]\mathrm{d}x \\
&\qquad + \int_c^{b-\epsilon} \left(f(x) + f'(x)s\epsilon + f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2}\epsilon^2 + f''(x)\frac{s}{2}\epsilon^2\right)\mathrm{d}x \\
&\quad = \int_c^{b} \left(f(x) + f'(x)s\epsilon + f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2}\epsilon^2 + f''(x)\frac{s}{2}\epsilon^2\right)\mathrm{d}x \\
&\qquad - \int_{b-\epsilon}^{b} \left(f(b) + f'(b)(x-b) + f'(b)s\epsilon\right)\mathrm{d}x \\
&\qquad + \int_c^{b-\epsilon} f^{1-s}(x)\left[f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right)\right]\mathrm{d}x \\
&\qquad - \int_{b-\epsilon}^{b} \left(f(x) + f'(x)s\epsilon + f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2}\epsilon^2 + f''(x)\frac{s}{2}\epsilon^2\right) \\
&\qquad - \left(f(b-0) + f'(b-0)(x-b) + f'(b-0)s\epsilon\right)\mathrm{d}x.
\end{aligned}
$$

The first and second terms are calculated as

$$
\begin{aligned}
&\int_c^{b} \left(f(x) + f'(x)s\epsilon + f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2}\epsilon^2 + f''(x)\frac{s}{2}\epsilon^2\right)\mathrm{d}x \\
&\quad - \int_{b-\epsilon}^{b} \left(f(b-0) + f'(b)(x-b) + f'(b-0)s\epsilon\right)\mathrm{d}x \\
&= \int_c^b f(x)\,\mathrm{d}x + (f(b-0) - f(c))s\epsilon + (f'(b-0) - f'(c))\frac{s}{2}\epsilon^2 \\
&\quad + \left(\int_c^b f^{-1}(x)(f')^2(x)\,\mathrm{d}x\right)\frac{s(s-1)}{2}\epsilon^2 \\
&\quad - \int_{b-\epsilon}^{b} \left(f(b-0) + f'(b-0)(x-b) + f'(b)s\epsilon\right)\mathrm{d}x \\
&= \int_c^b f(x)\,\mathrm{d}x - f(c)s\epsilon - f'(c)\frac{s}{2}\epsilon^2 + f(b)s\epsilon + f'(b)\frac{s}{2}\epsilon^2 \\
&\quad + \left(\int_c^b f^{-1}(x)(f')^2(x)\,\mathrm{d}x\right)\frac{s(s-1)}{2}\epsilon^2 - f(b-0)\epsilon + f'(b-0)\frac{\epsilon^2}{2} - f'(b-0)s\epsilon^2 \\
&= \int_c^b f(x)\,\mathrm{d}x - f(c)s\epsilon - f'(c)\frac{s}{2}\epsilon^2
\end{aligned}
$$

$$+f(b-0)(s-1)\epsilon+f'(b-0)(1-s)\frac{\epsilon^2}{2}+\left(\int_c^b f^{-1}(x)(f')^2(x)\,\mathrm{d}x\right)\frac{s(s-1)}{2}\epsilon^2.$$

The term

$$\begin{aligned}&\frac{1}{\epsilon^2}\left[\left|\int_c^{b-\epsilon} f^{1-s}(x)\left[f^s(x+\epsilon)-\left(f^s(x)+(f^s)'(x)\epsilon+(f^s)''(x)\frac{\epsilon^2}{2}\right)\right]\mathrm{d}x\right|\right.\\&+\left|\int_{b-\epsilon}^b\left(f(x)+f'(x)s\epsilon+f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2}\epsilon^2+f''(x)\frac{s}{2}\epsilon^2\right)\right.\\&\left.\left.-\left(f(b)+f'(b)(x-b)+f'(b)s\epsilon\right)\mathrm{d}x\right|\right]\end{aligned}$$

goes to 0 uniformly for $0 < s < 1$ as $\epsilon \to +0$ as follows. The second term is bounded by

$$\begin{aligned}&\frac{1}{\epsilon^2}\left|\sup_{x\in(b-\epsilon,b)}|f'(x)-f'(b)|\int_{b-\epsilon}^b(|(x-b)|+s\epsilon)\,\mathrm{d}x\right.\\&\left.+\frac{s}{2}\epsilon^2(f'(b)-f(b-\epsilon))+\frac{s(s-1)}{2}\epsilon^2\int_{b-\epsilon}^b f^{-1}(x)(f')^2(x)\,\mathrm{d}x\right|\\&=\sup_{x\in(b-\epsilon,b)}|f'(x)-f'(b)|\left(s+\frac{1}{2}\right)+\frac{s}{2}(f'(b)-f(b-\epsilon))\\&+\frac{s(s-1)}{2}\int_{b-\epsilon}^b f^{-1}(x)(f')^2(x)\,\mathrm{d}x\\&\le\sup_{x\in(b-\epsilon,b)}|f'(x)-f'(b)|\left(1+\frac{1}{2}\right)+\frac{1}{2}(f'(b)-f(b-\epsilon))+\frac{1}{2}\int_{b-\epsilon}^b f^{-1}(x)(f')^2(x)\,\mathrm{d}x,\end{aligned}$$

which goes to 0 as $\epsilon \to +0$. Similarly to (33), we can show that the first term goes to 0 uniformly concerning s. In particular, in case (v), we can show this by dividing the first term into two integrals $\int_c^{b-\delta}$ and $\int_{b-\delta}^{b-\epsilon}$. In the following, we will use the convergence concerning the part $\int_c^{b-\delta}$. That is, for any $\delta > 0$ and any $\epsilon' > 0$, there exists a real number $\epsilon > 0$ such that

$$\frac{1}{\epsilon^2}\left|\int_c^{b-\delta} f^{1-s}(x)\left[f^s(x+\epsilon)-\left(f^s(x)+(f^s)'(x)\epsilon+(f^s)''(x)\frac{\epsilon^2}{2}\right)\right]\mathrm{d}x\right|<\epsilon'. \tag{51}$$

Thus, from the existence of $f_2'(0)$ and the relation $f(b-0) = A_2$, we obtain (14) and the uniformity in case (ii). From (10) and the relations $f(b-0) = f'(b-0) = 0$, we obtain (14) and the uniformity in case (v).

Next, we consider cases (i), (iii), and (iv).

$$\begin{aligned}
&\int_c^{b-\epsilon} f^{1-s}(x) f^s(x+\epsilon)\,\mathrm{d}x \\
&= \int_c^{b-\delta} f^{1-s}(x)\left[f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right)\right]\mathrm{d}x \\
&\quad + \int_c^{b-\delta} f(x) + \epsilon f^{1-s}(x)(f^s)'(x) + \frac{\epsilon^2}{2} f^{1-s}(x)(f^s)''(x)\,\mathrm{d}x \\
&\quad + \int_{b-\delta}^{b-\epsilon} f^{1-s}(x) f^s(x+\epsilon)\,\mathrm{d}x.
\end{aligned} \tag{52}$$

In the following, we discuss only case (i). we have

$$\begin{aligned}
&\int_c^{b-\delta} f(x) + \epsilon f^{1-s}(x)(f^s)'(x) + \frac{\epsilon^2}{2} f^{1-s}(x)(f^s)''(x)\,\mathrm{d}x + \int_{b-\delta+\epsilon}^{b} f(x)\,\mathrm{d}x \\
&= \int_c^b f(x)\,\mathrm{d}x - \int_{b-\delta}^{b-\delta+\epsilon} f(x)\,\mathrm{d}x \\
&\quad + \left(\int_c^{b-\delta} f^{1-s}(x)(f^s)'(x)\,\mathrm{d}x\right)\epsilon + \left(\int_c^{b-\delta} f^{1-s}(x)(f^s)''(x)\,\mathrm{d}x\right)\frac{\epsilon^2}{2} \\
&= \int_c^b f(x)\,\mathrm{d}x - \int_{b-\delta}^{b-\delta+\epsilon} f(x)\,\mathrm{d}x + \left(\int_c^{b-\delta} f'(x)\,\mathrm{d}x\right)s\epsilon \\
&\quad + \left(\frac{s(s-1)}{2}\int_c^{b-\delta}(s-1) f^{-1}(x)(f'(x))^2\,\mathrm{d}x + \frac{s}{2}\int_c^{b-\delta} f''(x)\,\mathrm{d}x\right)\epsilon^2 \\
&= \int_c^b f(x)\,\mathrm{d}x - \int_{b-\delta}^{b-\delta+\epsilon} f(x)\,\mathrm{d}x + (f(b-\delta) - f(c))s\epsilon + (f'(b-\delta) - f'(c))\frac{s}{2}\epsilon^2 \\
&\quad + \frac{s(s-1)}{2}\left(\int_c^{b-\delta} f^{-1}(x)(f'(x))^2\,\mathrm{d}x\right)\epsilon^2.
\end{aligned} \tag{53}$$

Letting $z := \frac{b-x}{y}$, we have

$$\begin{aligned}
&\int_{b-\delta}^{b-\epsilon} f^{1-s}(x) f^s(x+\epsilon)\,\mathrm{d}x - \int_{b-\delta+\epsilon}^{b} f(x)\,\mathrm{d}x \\
&= \int_{b-\delta+\epsilon}^{b}\left(f^{1-s}(x-\epsilon) f^s(x) - f(x)\right)\mathrm{d}x \\
&= \int_{b-\delta+\epsilon}^{b} f^s(x)\int_{-\epsilon}^{0}(f^{1-s})'(x+y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^{\epsilon}\int_0^{\frac{\delta-\epsilon}{y}} \frac{f_2^s(yz)}{f_2^s(y(z+1))}\frac{f_2'(y(z+1))}{f_2'(y)}\,\mathrm{d}z y f_2'(y)\,\mathrm{d}y.
\end{aligned} \tag{54}$$

Similarly to (39), we can prove that for any $\epsilon' > 0$, there exist real numbers $\delta > 0$ and $\epsilon > 0$ such that

$$\left| \frac{\int_0^{\frac{\delta-\epsilon}{y}} \frac{f_2^s(yz)}{f_2^s(y(z+1))} \frac{f_2'(y(z+1))}{f_2'(y)} \mathrm{d}z}{\epsilon^{\kappa_2}} + B(\kappa_2 + s - \kappa_2 s, 1-\kappa_2)\frac{s(1-\kappa_2)}{\kappa_2} \right| < \epsilon' \quad (55)$$

for any y satisfying $\epsilon > y > 0$. Therefore, from (51)–(55), similarly to (40) we can choose constants C_1 and C_2 independently for s, such that

$$\begin{aligned} &\frac{\left| I_s^+(c, f, \epsilon) + A_2 B(\kappa_2 + (1-s) - \kappa_2(1-s), 1-\kappa_2)\frac{(1-s)(1-\kappa_2)}{\kappa_2}\epsilon^{\kappa_2} \right|}{\epsilon^{\kappa_2}} \\ &< \frac{1}{\kappa_2}\left(C_1\epsilon + C_2\epsilon^2 + \epsilon'\epsilon^2 \right) + \epsilon'. \end{aligned} \quad (56)$$

Thus, we obtain (14) and the uniformity in case (i).

Next, we consider cases (iii) and (iv). Concerning the second term of (52), we have

$$\begin{aligned} &\int_c^{b-\delta} f(x) + \epsilon f^{1-s}(x)(f^s)'(x) + \frac{\epsilon^2}{2} f^{1-s}(x)(f^s)''(x)\,\mathrm{d}x \\ &\quad + \int_{b-\delta+\epsilon}^{b} f(x) - f^s(x)(f^{1-s})'(x)\epsilon\,\mathrm{d}x \\ &= \int_c^b \left(f(x) + f'(x)s\epsilon\right)\mathrm{d}x + \left(\int_c^{b-\delta} f^{1-s}(x)(f^s)''(x)\,\mathrm{d}x \right)\frac{\epsilon^2}{2} \\ &\quad - \int_{b-\delta}^{b} \left(f(x) + f'(x)s\epsilon\right)\mathrm{d}x + \int_{b-\delta+\epsilon}^{b} \left(f(x) - f^s(x)(f^{1-s})'(x)\epsilon\right)\mathrm{d}x \\ &= \int_c^b f(x)\,\mathrm{d}x + f(b-0)s\epsilon - f(c)s\epsilon + \left(\int_c^{b-\delta} f^{1-s}(x)(f^s)''(x)\,\mathrm{d}x \right)\frac{\epsilon^2}{2} \\ &\quad - \int_{b-\delta}^{b} \left(f(x) + f'(x)s\epsilon\right)\mathrm{d}x + \int_{b-\delta+\epsilon}^{b} \left(f(x) - f^s(x)(f^{1-s})'(x)\epsilon\right)\mathrm{d}x. \end{aligned} \quad (57)$$

We can evaluate this as

$$\begin{aligned} &\left| - \int_{b-\delta}^{b} \left(f(x) + f'(x)s\epsilon\right)\mathrm{d}x + \int_{b-\delta+\epsilon}^{b} \left(f(x) - f^s(x)(f^{1-s})'(x)\epsilon\right)\mathrm{d}x \right| \\ &= \left| - \int_{b-\delta}^{b-\delta+\epsilon} f(x)\,\mathrm{d}x - \int_{b-\delta}^{b-\delta+\epsilon} f'(x)s\epsilon\,\mathrm{d}x - \int_{b-\delta+\epsilon}^{b} f'(x)\epsilon\,\mathrm{d}x \right| \\ &= \left| \int_{b-\delta}^{b-\delta+\epsilon} f(b-\delta+\epsilon) - f(x) - f'(x)s\epsilon\,\mathrm{d}x \right| \\ &\le \int_{b-\delta}^{b-\delta+\epsilon} |f(b-\delta+\epsilon) - f(x)| + |f'(x)|s\epsilon\,\mathrm{d}x \end{aligned}$$

$$\leq \max_{0\leq t\leq 1} |f'(b-\delta+\epsilon t)|\frac{3}{2}\epsilon^2. \tag{58}$$

Concerning the third term of (52), we have

$$\begin{aligned}
&\int_{b-\delta}^{b-\epsilon} f^{1-s}(x) f^s(x+\epsilon)\,\mathrm{d}x + \int_{b-\delta+\epsilon}^{b} -f(x) + f^s(x)(f^{1-s})'(x)\epsilon\,\mathrm{d}x \\
&= \int_{b-\delta+\epsilon}^{b} f^s(x)\left(f^{1-s}(x-\epsilon) - f^{1-s}(x) + (f^{1-s})'(x)\epsilon\right)\mathrm{d}x \\
&= \int_{b-\delta+\epsilon}^{b} f^s(x)\left(-\int_{-\epsilon}^{0} (f^{1-s})'(x+y_1) - (f^{1-s})'(x)\,\mathrm{d}y_1\right)\mathrm{d}x \\
&= \int_{b-\delta+\epsilon}^{b} f^s(x)\left(-\int_{-\epsilon}^{0}\int_{0}^{y_1} (f^{1-s})''(x+y_2)\,\mathrm{d}y_2\,\mathrm{d}y_1\right)\mathrm{d}x \\
&= \int_0^{\epsilon}\int_0^{y_1}\int_0^{\frac{\delta-\epsilon}{y_2}} \left[(1-s)\frac{f_2^s(y_2 z)}{f_2^s(y_2(z+1))}\frac{f_2''(y_2(z+1))}{f''(y_2)}\frac{f_2''(y_2)f(y_2)}{(f_2')^2(y_2)}\right. \\
&\quad \left. + s(s-1)\frac{f_2^s(y_2 z)}{f_2^s(y_2(z+1))}\frac{f_2(y_2)}{f_2(y_2(z+1))}\frac{(f_2')^2(y_2(z+1))}{(f_2')^2(y_2)}\right]\mathrm{d}z\,\frac{(f_2')^2(y_2)}{f_2(y_2)}y_2\,\mathrm{d}y_2\,\mathrm{d}y_1
\end{aligned} \tag{59}$$

Similarly to (45), in case (iii), we can prove that for any $\epsilon' > 0$ there exist real numbers $\delta > 0$ and $\epsilon > 0$ such that

$$\begin{aligned}
&\left|\frac{\int_{b-\delta+\epsilon}^{b} f^s(x)\left(-\int_{-\epsilon}^{0}\int_0^{y_1}(f^{1-s})''(x+y_2)\,\mathrm{d}y_2\,\mathrm{d}y_1\right)\mathrm{d}x}{\epsilon^{\kappa_2}}\right. \\
&\qquad \left. + \frac{A_2 B(1+s(\kappa_2-1), 2-\kappa_2)\frac{(1-s)(2-\kappa_2+s(\kappa_2-1))\epsilon^{\kappa_2}}{\kappa_2}}{\epsilon^{\kappa_2}}\right| \\
&< \epsilon'
\end{aligned} \tag{60}$$

Similarly to (46), from (51), (52), (57)–(60), we can choose a constant C_2 independently for s such that

$$\begin{aligned}
&\frac{\left|I_s^+(c,f,\epsilon) + A_2 B(1+s(\kappa_2-1), 2-\kappa_2)\frac{(1-s)(2-\kappa_2+s(\kappa_2-1))}{\kappa_2}\epsilon^{\kappa_2}\right|}{\epsilon^{\kappa_2}} \\
&< \frac{1}{\epsilon^{\kappa_2}}\left(C_2\epsilon^2 + \epsilon'\epsilon^2\right) + \epsilon'.
\end{aligned} \tag{61}$$

Thus, we obtain (14) and the uniformity in case (iii). Similarly to (49), in case (iv), we can prove that

$$\frac{\left|\int_{b-\delta+\epsilon}^{b} f^s(x)\left(-\int_{-\epsilon}^{0}\int_0^{y_1}(f^{1-s})''(x+y_2)\,\mathrm{d}y_2\,\mathrm{d}y_1\right)\,\mathrm{d}x + A_2\frac{s(1-s)}{2}\epsilon^2(-\log\epsilon)\right|}{\epsilon^2(-\log\epsilon)} < \epsilon'. \tag{62}$$

Similarly to (50), from (51), (52), (57)–(59), and (62), we can choose a constant C_2 independently for s such that

$$\frac{\left|I_s^+(c, f, \epsilon) + A_2\frac{s(1-s)}{2}\epsilon^2(-\log\epsilon)\right|}{\epsilon^2(-\log\epsilon)} < \frac{1}{\epsilon^2(-\log\epsilon)}\left(C_2\epsilon^2 + \epsilon'\epsilon^2\right) + \epsilon'. \tag{63}$$

Thus, we obtain (14) and the uniformity in case (iv).

6 Proof of Lemma 2

We can calculate

$$\begin{aligned}
&\int_c^\infty f^{1-s}(x)f^s(x+\epsilon)\,\mathrm{d}x \\
&= \int_c^\infty f^{1-s}(x)\left(f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right)\right)\mathrm{d}x \\
&\quad + \int_c^\infty f^{1-s}(x)\left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right)\mathrm{d}x.
\end{aligned} \tag{64}$$

The second term of (64) is calculated as

$$\begin{aligned}
&\int_c^\infty f^{1-s}(x)\left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right)\mathrm{d}x \\
&= \int_c^\infty f(x) + f'(x)s\epsilon + f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2}\epsilon^2 + f''(x)\frac{s}{2}\epsilon^2\,\mathrm{d}x \\
&= \int_c^\infty f(x)\,\mathrm{d}x + \int_c^\infty f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2}\,\mathrm{d}x - f(c)s\epsilon - f'(c)\frac{s}{2}\epsilon^2\,\mathrm{d}x.
\end{aligned} \tag{65}$$

We choose a sufficiently large number R such that $f''(x) \geq 0$ for $x \geq R$. Now, we put three real numbers $E_0 := \sup_{x,y>R:|x-y|<\epsilon}\frac{f(y)}{f(x)}$, $E_1 := \sup_{x,y>R:|x-y|<\epsilon}\frac{f'(y)}{f'(x)}$, and $E_2 := \sup_{x,y>R:|x-y|<\epsilon}\frac{f''(y)}{f''(x)}$. Choosing a number $t(x,\epsilon) \in [0, 1]$ suitably, we obtain

$$\frac{1}{\epsilon^2}\left|\int_R^{\infty} f^{1-s}(x) f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right) \mathrm{d}x\right|$$
$$= \frac{1}{2}\left|\int_R^{\infty} f^{1-s}(x)((f^s)''(x + t(x,\epsilon)\epsilon) - (f^s)''(x))\,\mathrm{d}x\right|$$
$$\leq \frac{1}{2}\int_R^{\infty} f^{1-s}(x)[(E_1^2 E_0^{2-s} - 1)s(s-1)f^{s-2}(x)(f')^2(x)$$
$$+(E_2 E_0^{1-s} - 1)s f^{s-1}(x) f''(x)]\,\mathrm{d}x$$
$$\leq \frac{1}{2}\int_R^{\infty}\left[\frac{E_1^2 E_0^2 - 1)}{2} f^{-1}(x)(f')^2(x)dx + \frac{1}{2}(E_2 E_0 - 1)(f'(+\infty) - f'(R))\right], \tag{66}$$

For any $\epsilon' > 0$, we choose a sufficiently large number R such that the right hand side of (66) is less than ϵ'. Similarly to (33), the term $\frac{1}{\epsilon^2}\left|\int_c^R f^{1-s}(x) f^s(x+\epsilon) - \left(f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2}\right) \mathrm{d}x\right|$ goes to 0 uniformly concerning s. Therefore, we obtain (29) and the uniformity for $0 < s < 1$.

7 Conclusion

We have calculated the limit of the relative Rényi entropy. As mentioned in Sect. 4, this calculation plays an important role in large deviation type asymptotic theory. On the other hand, we conjecture that these limits characterize the asymptotic behavior of the MLE. This relation, though, is expected to be clarified. From the information geometrical viewpoint, this result indicates that a non-regular family gives a new geometrical structure.

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