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Goodness of fit test for ergodic diffusion processes

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Abstract A goodness of fit test for the drift coefficient of an ergodic diffusion process is presented. The test is based on the score marked empirical process. The weak convergence of the proposed test statistic is studied under the null hypothesis and it is proved that the limit process is a continuous Gaussian process. The structure of its covariance function allows to calculate the limit distribution and it turns out that it is a function of a standard Brownian motion and so exact rejection regions can be constructed. The proposed test is asymptotically distribution free and it is consistent under any simple fixed alternative.

Keywords Consistent test · Empirical process · Asymptotically distribution free tests

1 Introduction

Goodness of fit tests play an important role in theoretical and applied statistics. They allow to verify the correspondence between the proposed theoretical models and real data. Such kind of tests are really useful if they are distribution free, in the sense that

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their distribution do not depend on the underlying model. This fact is considered their principal advantage because it permits to construct exact rejection regions. For example if $X^n = (X_1, \ldots, X_n)$ are n independent random variables with distribution function F, to test the simple hypothesis $F = F_0$ against any other alternative we can introduce the well known Kolmogorov-Smirnov statistic, $\Delta_n(X^n) = \sup_x \sqrt{T}|\hat{F}_n(x) - F_0(x)|$ where $\hat{F}_n(x) = \frac{1}{n}\sum_{j=1}^n \mathbf{1}_{\{X_j \le x\}}$ and as usual, $\mathbf{1}_A$ denote the indicator function on a set A. This test is asymptotically distribution-free in the sense that the limit distribution do not depend on F_0 and is consistent against any fixed alternative (see, for example, Durbin 1973).

In this paper we study the similar problem of goodness of fit test when the basic model is a diffusion process and we present a test statistic that is asymptotic distribution free. Let X be an ergodic diffusion process on $\mathbb{R}$, solution of a stochastic differential equation, that is a strong Markov process with continuous sample paths which satisfies

$$\mathrm{d}X_t = S_0(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad \text{for } t > 0, \tag{1}$$

with some random initial value X_0, where S_0 and σ are some functions and $W_t, t > 0$ is a Wiener process. Diffusion processes of this type are widely used as models in many different fields such as biology, physics, economics and finance. Despite the fact of their importance in applications, few works are devoted to the goodness of fit test for diffusions. So the construction of goodness of fit tests for such kind of model is very important and needs very detailed studies.

Kutoyants (2004) discusses some possibilities of the construction of such tests. In particular, he considers the Kolmogorov–Smirnov statistics $\Delta_T(X^T) = \sup_x \sqrt{T}|\hat{F}_T(x) - F_{S_0}(x)|$, based on the continuous observation $X^T = \{X_t : 0 \le t \le T\}$ solution of (1). Here F_{S_0} denote the invariant distribution function of the diffusion process solution of (1) and $\hat{F}_T(x) = \frac{1}{T}\int_0^T \mathbf{1}_{\{X_t \le x\}}\mathrm{d}t$ is the empirical distribution function. The goodness of fit test based on the Kolmogorov–Smirnov statistics is asymptotically consistent and the asymptotic distribution under the null hypothesis follows from the weak convergence of the empirical process to a suitable Gaussian process. Note that the Kolmogorov–Smirnov statistics for ergodic diffusion process was studied in Fournie (1992), see also Fournie and Kutoyants (1993) for more details, while the weak convergence of the empirical process was proved in Negri (1998) (see Van der Vaart and Van Zanten 2005 for further developments). However, due to the structure of the covariance of the limit process, the Kolmogorov–Smirnov statistics is not asymptotically distribution free in diffusion process models. More recently Dachian and Kutoyants (2007) have proposed a modification of the Kolmogorov–Smirnov statistics for diffusion models that became asymptotically distribution free, but their approach is different from the one presented here. Moreover our test statistics can discriminate alternative hypotheses which cannot be treated by their approach.

In this work we present a goodness of fit test for a diffusion process model based on the statistic $\sup_x |V_T(x)|$ where

$$V_T(x) = \frac{1}{\sqrt{T}}\int_0^T \mathbf{1}_{(-\infty,x]}(X_t)\frac{1}{\sigma(X_t)}(\mathrm{d}X_t - S_0(X_t)\mathrm{d}t).$$

Following Koul and Stute (1999), we call V_T the *score marked empirical process*. We prove that the test based on the statistic $\sup_x |V_T(x)|$ is asymptotically distribution free and it is consistent against any alternative $S = S_1 \neq S_0$.

We study the weak convergence of the score marked empirical process under the null hypothesis and we prove that the limit process is a continuous Gaussian process. The structure of its covariance function allows us to calculate the limit distribution of the proposed statistic. It turns out that it is a functional of a standard Brownian motion with known distribution and so we can construct exact rejection regions.

Koul and Stute (1999) proposed such kind of statistics based on a class of empirical process constructed on certain residuals to check some parametric models for time series. They studied their large sample behavior under the null hypotheses and present a martingale transformation of the underlying process that makes tests based on it asymptotically distribution free. Some considerations on consistency have also been done. For the same model studied here the problem of testing different parametric forms of the drift coefficient is well developed (see for example Lin'kov 1981 or Kutoyants 2004 and references therein).

The work is organized as follows. In the next section, we present the model of ergodic diffusion process solution of a stochastic differential equation, its properties and some general conditions and assumptions. In Sect. 3 we prove the weak convergence for a general class of regular diffusion processes with finite speed measure. This result is interesting by itself and include the result on the weak convergence of the proposed statistic as a particular case. Section 4 is devoted to the presentation of the goodness of fit test for the model presented in Sect. 2 and to the study of the score marked empirical process statistics under the null hypothesis. Finally, in Sect. 5 we study the behavior of the proposed statistics under the alternative hypotheses and we prove that the test is consistent against any other alternative.

2 Preliminaries

Given a general stochastic basis, that is, a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and a filtration $\{\mathcal{A}_t\}_{t\geq 0}$ of $\mathcal{A}$, let us consider a one dimensional diffusion process solution of the following stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = S(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t \\ X_0 = \xi, \end{cases} \tag{2}$$

where $\{W_t : t \geq 0\}$ is a standard Wiener process, and the initial value $X_0 = \xi$ is independent of W_t, $t \geq 0$. The drift coefficient S will be supposed unknown to the observer and the diffusion coefficient σ^2 will be a known positive function. Let us introduce the following condition.

$\mathcal{ES}$. *The function S is locally bounded, the function σ^2 is continuous and bounded and for some constant $A > 0$, the condition $xS(x) + \sigma(x)^2 \leq A(1 + x^2)$, $x \in \mathbb{R}$, holds.*

If condition $\mathcal{ES}$ holds true, then the equation (2) has an unique weak solution (see Durrett 1996, p. 210). The *scale function* of a diffusion process solution of the stochastic differential equation (2) is defined by

$$p(x)=\int_0^x \exp\left\{-2\int_0^y \frac{S(v)}{\sigma^2(v)}\,\mathrm{d}v\right\}\mathrm{d}y.$$

The *speed measure* of the diffusion process (2) is defined by $m_S(\mathrm{d}x)=\frac{1}{\sigma(x)^2p'(x)}\mathrm{d}x$. Let us introduce the following condition:

$\mathcal{RP}$. *The scale function is such that*

$$\lim_{x\to\pm\infty} p(x)=\pm\infty$$

and the speed measure m_S is finite.

If the condition $\mathcal{RP}$ is satisfied then the process $\{X_t : t\geq 0\}$, weak solution of (2), has the ergodic property (see for example Gikhman and Skorohod 1972 or Durrett 1996), that is, there exists an unique invariant probability measure μ_S such that for every measurable function $g\in\mathcal{L}_1(\mu_S)$ we have with probability one,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g(X_t)\mathrm{d}t=\int_{\mathbb{R}} g(z)\mu_S(\mathrm{d}z).$$

Moreover the invariant measure μ_S has a density given by

$$f_S(y)=\frac{1}{m_S(\mathbb{R})\sigma(y)^2}\exp\left\{2\int_0^y\frac{S(v)}{\sigma(v)^2}\mathrm{d}v\right\},$$

where

$$m_S(\mathbb{R})=\int_{-\infty}^{\infty}\frac{1}{\sigma(x)^2}\exp\left\{2\int_0^x\frac{S(v)}{\sigma^2(v)}\,\mathrm{d}v\right\}\,\mathrm{d}x$$

is finite.

3 A limit theorem

In this section, we present a theorem on the weak convergence of a stochastic process that is interesting by itself. Let us consider a diffusion process $X=\{X_t : t\geq 0\}$ on an open interval $I\subseteq\mathbb{R}$, that is a strong Markov process with continuous sample paths taking values on I, not necessarily solution of a differential stochastic equation of type (2). Assume that X is regular, which implies that the scale function p and the speed measure m of the diffusion are well defined (see Rogers and Williams 2000). Under the assumption that the speed measure m is finite, and denoting by μ the normalized speed measure, $\mu=\frac{m}{m(I)}$, it follows that the diffusion process X is positive recurrent and it has the ergodic property, with μ as invariant measure. For every $x\in I$ and $t\geq 0$, the diffusion local time for the diffusion X with respect to the speed measure (see Van der Vaart and Van Zanten 2005; Itô and McKean 1965) in the point x at time

t is denoted as $l_t^X(x)$. The random function $x \to l_t^X(x)$ can be chosen continuos and has compact support. The main theorem for diffusion local time is the *occupation time* formula. For a diffusion process it can be written as

$$\int_0^t h(X_s)\mathrm{d}s = \int_I l_t^X(x)h(x)m(\mathrm{d}x), \tag{3}$$

for every measurable function $h : I \to \mathbb{R}$ (see Rogers and Williams 2000). If the measure m is finite then it holds that

$$\frac{1}{t}\sup_{x\in I} l_t^X(x) = O_{\mathbf{P}}(1). \tag{4}$$

See Theorem 4.2 of Van der Vaart and Van Zanten (2005) and also Van Zanten (2003).

Let a standard Wiener process W be given on the same stochastic basis where X is defined. Let us consider the process $M = \{M_t(\psi) : t \geq 0, \psi \in \mathcal{F}\}$ defined by

$$M_t(\psi) = \int_0^t \psi(X_s)\mathrm{d}W_s,$$

where ψ belongs to a countable class $\mathcal{F}$ of elements of $\mathcal{L}^2(I, m(\mathrm{d}x))$. To measure the distance between functions in $\mathcal{F}$ we use the semimetric ρ

$$\rho(\psi, \varphi) = \sqrt{\int_I |\psi(y) - \varphi(y)|^2 m(\mathrm{d}y)}.$$

For every $\psi \in \mathcal{F}$ the process $M(\psi) = \{M_t(\psi) : t \geq 0\}$ is a continuous local martingale. Following Nishiyama (1999), Definition 2.1, the quadratic ρ-modulus for the process $V = \{V_t(\psi) : t \geq 0, \psi \in \mathcal{F}\}$, where $V_t(\psi) = \frac{1}{\sqrt{t}}M_t(\psi)$, is defined as

$$||V||_{\rho,t} = \sup_{\rho(\psi,\varphi)>0} \frac{\sqrt{\frac{1}{t}\langle M(\psi) - M(\varphi)\rangle_t}}{\rho(\psi, \varphi)}.$$

Here $\langle M\rangle = \{\langle M\rangle_t : t \geq 0\}$ denotes the quadratic variation process of a continuous local martingales M.

Let us denote with $N(\epsilon, \mathcal{F}, \rho)$ the smallest number of closed balls, with ρ-radius $\epsilon > 0$, which cover the set $\mathcal{F}$.

Theorem 1 *Let X be a regular diffusion process on I with finite speed measure. Let $\mathcal{F} \subset \mathcal{L}^2(I, m(\mathrm{d}x))$ be countable. Then for all δ, K and $t > 0$ it holds that*

$$\mathbf{E} \sup_{\rho(\psi,\varphi)<\delta} |V_t(\psi) - V_t(\varphi)|\mathbf{1}_{\{\xi_t \leq K\}} \leq cK \int_0^\delta \sqrt{\log N(\epsilon, \mathcal{F}, \rho)}\mathrm{d}\epsilon,$$

where $c > 0$ is an universal constant and $\{\xi_t : t > 0\}$ is a stochastic process which satisfies $\xi_t = O_{\mathbf{P}}(1)$, as t goes to infinity.

Proof We have

$$\frac{1}{t}\langle M\rangle_t = \frac{1}{t}\int_0^t \psi(X_s)^2 \mathrm{d}s$$

and from the occupation formula (3) for the diffusion local time of a diffusion process we can write

$$\frac{1}{t}\int_0^t \psi(X_s)^2 \mathrm{d}s = \frac{1}{t}\int_I \psi(x)^2 l_T^X(x) m(\mathrm{d}x).$$

So we have

$$\|V\|_{\rho,t} = \sup_{\rho(\psi,\varphi)>0} \frac{\sqrt{\frac{1}{t}\int_I (\psi(x)-\varphi(x))^2 l_T^X(x) m(\mathrm{d}x)}}{\rho(\psi,\varphi)} \le \sqrt{\frac{1}{t}\sup_{x\in I} l_T^X(x)}.$$

Now recalling (4), the result follows from Theorem 2.3 in Nishiyama (1999) if we pose $\xi_t = \sqrt{\frac{1}{t}\sup_{x\in I} l_t^X(x)}$. □

Let us denote with $\ell^\infty(\mathcal{F})$ the space of bounded functions $\mathcal{Z}:\mathcal{F}\to\mathbb{R}$ equipped with the uniform norm $||\mathcal{Z}||_\infty = \sup_{\varphi\in\mathcal{F}}|\mathcal{Z}(\varphi)|$.

On the space $\ell^\infty(\mathcal{F})$ we introduce the Gaussian process $\{\Gamma(\psi):\psi\in\mathcal{F}\}$ with mean zero and covariance function given by

$$g(\psi,\varphi) = \int_I \psi(z)\varphi(z)\mu(\mathrm{d}z).$$

Theorem 2 *Let X be a regular diffusion process on I with finite speed measure. Let $\mathcal{F}\subset\mathcal{L}^2(I, m(\mathrm{d}x))$ be countable and $\int_0^1\sqrt{\log N(\epsilon,\mathcal{F},\rho)}\mathrm{d}\epsilon$ be finite. Then the family of stochastic maps $\{V_t(\psi):\psi\in\mathcal{F}\}$ weakly converges on the space $\ell^\infty(\mathcal{F})$, as t goes to infinity, to the Gaussian process $\{\Gamma(\psi):\psi\in\mathcal{F}\}$.*

Proof The convergence of the finite dimensional laws $(V_t(\psi_1),\ldots,V_t(\psi_k))$ to the law of $(\Gamma(\psi_1),\ldots,\Gamma(\psi_k))$, for every finite k follows from the central limit theorem for stochastic integrals (see Kutoyants 2004). The tightness follows from Theorem 1, assuming that $\int_0^1\sqrt{\log N(\epsilon,\mathcal{F},\rho)}\mathrm{d}\epsilon$ is finite (see also Nishiyama 2000). □

4 Goodness of fit test

Let us introduce our testing problem. Suppose that we observe the process $\{X_t : 0\le t\le T\}$, solution of the stochastic differential equation (2) and we wish to test the null hypothesis

$$H_0 : S = S_0$$

against any alternative $H_1 : S = S_1$ where S_1 satisfies the following condition $\mathcal{C}$: For some $x \in \mathbb{R}$ it holds

$$\int_{-\infty}^{+\infty} \mathbf{1}_{(-\infty,x]}(y)(S_0(y) - S_1(y)) f_{S_1}(y)\mathrm{d}y \neq 0.$$

We suppose that S_0 and S_1 belong to the class $\mathcal{S}_\sigma$ defined for a fixed function σ as

$$\mathcal{S}_\sigma = \{S : \text{conditions } \mathcal{ES} \text{ and } \mathcal{RP} \text{ are fulfilled}\}.$$

For every $x \in \mathbb{R}$, let us introduce the *score marked empirical process*

$$\begin{aligned} V_T(x) &= \frac{1}{\sqrt{T}} \int_0^T \mathbf{1}_{(-\infty,x]}(X_t) \frac{1}{\sigma(X_t)} (\mathrm{d}X_t - S_0(X_t)\mathrm{d}t) \\ &= \frac{1}{\sqrt{T}} \int_0^T \mathbf{1}_{(-\infty,x]}(X_t) \mathrm{d}W_t. \end{aligned}$$

The process $\{V_T(x) : x \in \mathbb{R}\}$ takes values in $C_B(\mathbb{R})$, the space of the continuous bounded function on $\mathbb{R}$. Let us introduce in this space the σ-algebra of borel set $\mathcal{B}$ generated by the open sets of $C_B(\mathbb{R})$ induced by the norm $||f|| = \sup_{x\in\mathbb{R}} |f(x)|$ for every $f \in C_B(\mathbb{R})$.

Let us introduce our test procedure. Fix a number $\varepsilon \in (0, 1)$ and let us consider the class of *asymptotic test of level* $1 - \varepsilon$ or *size* ε. Given any statistical decision function $\phi_T = \phi_T(X^T)$, the expected value of $\phi_T(X^T)$ is the probability to reject H_0 having the observation $X^T = \{X_t : 0 \leq t \leq T\}$. Let us denote by $\mathbf{E}_S^T$ the mathematical expectation with respect to the measures $\mathbf{P}_S^T$ induced by the process $\{X_t : 0 \leq t \leq T\}$ in the space $C[0, T]$ (the space of all the continuos functions on $[0, T]$). We define the class of all the tests of asymptotic level $1 - \varepsilon$ as

$$\mathcal{K}_\varepsilon = \left\{ \phi_T : \limsup_{T\to+\infty} \mathbf{E}_{S_0}^T \phi_T(X^T) \leq \varepsilon \right\}.$$

The power function of the test based on ϕ_T is the probability of the true decision under H_1, and is given by

$$\beta_t(\phi_T) = \mathbf{E}_{S_1}^T \phi_T(X^T).$$

A test procedure is consistent if

$$\lim_{T\to+\infty} \mathbf{E}_{S_1}^T \phi_T(X^T) = 1.$$

Let us introduce in the space $(C_B(\mathbb{R}), \mathcal{B})$ the Gaussian process $\{\Gamma(x) : x \in \mathbb{R}\}$ with mean zero and covariance function given by

$$\mathbf{E}(\Gamma(x)\Gamma(y)) = \int_{-\infty}^{x\wedge y} f_{S_0}(y)\mathrm{d}y = F_{S_0}(x \wedge y).$$

Since the function F_{S_0} is nondecreasing and nonnegative the limit process $\{\Gamma(x) : x \in \mathbb{R}\}$ admits the following representation in distribution

$$\Gamma(x) = B(F_{S_0}(x)) \tag{5}$$

for every $x \in \mathbb{R}$, where B denote a standard Brownian motion on the positive real line. The weak convergence of the process $\{V_T(x) : x \in \mathbb{R}\}$ to the process $\{\Gamma(x) : x \in \mathbb{R}\}$ in the space $(C_B(\mathbb{R}), \mathcal{B})$ is immediate if we apply Theorem 2 to the functions $\psi(y) = \mathbf{1}_{(-\infty,x]}(y)$. Here we remark that the class $\mathcal{F}$ in Theorem 2 has to be countable. However the process $\{V_T(x) : x \in \mathbb{R}\}$ is continuous in x, so in the current situation we can consider such class of functions. This result on the weak convergence of process V_T and the continuous mapping theorem gives the following convergence in distribution

$$\sup_{x\in\mathbb{R}} |V_T(x)| \Rightarrow \sup_{x\in\mathbb{R}} |\Gamma(x)|.$$

Moreover the representation (5) yield the following equality in distribution

$$\sup_{x\in\mathbb{R}} |\Gamma(x)| = \sup_{0\le t\le 1} |B(t)|.$$

We will consider the following statistical decision function

$$\phi_T^* = \mathbf{1}_{\{\sup_{x\in\mathbb{R}} |V_T(x)|>c_\varepsilon\}},$$

where the *critical value* c_ε is defined by

$$\mathbf{P}\left(\sup_{0\le t\le 1} |B(t)| > c_\varepsilon\right) = \varepsilon.$$

So we have proved that $\phi_T^* \in \mathcal{K}$ and that the test is asymptotically distribution free. In order to make the introduced statistical procedure useful, we have to study the asymptotic properties of the statistics $\sup_x |V_T(x)|$ under the alternative hypotheses. This is done in the next section.

5 Consistency of the test

This section is devoted to the study of the asymptotic beaviour of the test statitistic $\sup_{x\in\mathbb{R}} |V_T(x)|$ under the alternative hypotheses. We have shown in the previous section that the proposed statistics is asymptotically distribution free. Now we prove that the proposed test procedure is also consistent against any alternative $S = S_1$ belonging to the class

$$\mathcal{H}_1 = \left\{S_1 : \int_{-\infty}^{x} (S_0(y) - S_1(y)) f_{S_1}(y) \mathrm{d}y \neq 0, \text{ for some } x \in \mathbb{R}\right\}$$

Theorem 3 *Let S_0 and S_1 belong to $\mathcal{S}_\sigma$ and the condition $\mathcal{C}$ be satisfied. Then the test based on the statistical decision function*

$$\phi_T^* = \mathbf{1}_{\{\sup_{x\in\mathbb{R}} |V_T(x)|>c_\varepsilon\}}$$

is consistent against any alternative belonging to the class $\mathcal{H}_1$.

Proof To prove the consistency it is enough to show that, under H_1

$$\mathbf{P}\left(\lim_{T\to+\infty} \sup_{x\in\mathbb{R}} |V_T(x)| = +\infty\right) = 1.$$

We can write

$$\sup_{x\in\mathbb{R}} |V_T(x)| \geq \sqrt{T} \sup_{x\in\mathbb{R}} |A_T(x)| - \sup_{x\in\mathbb{R}} |V_T^1(x)|,$$

where $V_T^1(x)$ and $A_T(x)$ are given as follows. Under H_1, by Theorem 2 the process

$$V_T^1(x) = \frac{1}{\sqrt{T}} \int_0^T \mathbf{1}_{(-\infty,x]}(X_t) \frac{1}{\sigma(X_t)} (\mathrm{d}X_t - S_1(X_t)\mathrm{d}t)$$

weakly converges to the corresponding Gaussian process so the limit process is tight. On the other hand,

$$A_T(x) = \frac{1}{T} \int_0^T \mathbf{1}_{(-\infty,x]}(X_t) \frac{1}{\sigma(X_t)} (S_0(X_t) - S_1(X_t))\mathrm{d}t$$

converges a.s. to

$$A(x) = \int_{-\infty}^{+\infty} \mathbf{1}_{(-\infty,x]}(y) \frac{1}{\sigma(y)} (S_0(y) - S_1(y)) f_{S_1}(y)\mathrm{d}y.$$

If the condition $\mathcal{C}$ is satisfied we have

$$\lim_{T\to+\infty} \sqrt{T} \sup_{x\in\mathbb{R}} |A_T(x)| = +\infty \quad \text{a.s.}$$

and the test is consistent. □

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