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Proportional hazards regression under progressive Type-II censoring

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Abstract This paper proposes an inferential method for the semiparametric proportional hazards model for progressively Type-II censored data. We establish martingale properties of counting processes based on progressively Type-II censored data that allow to derive the asymptotic behavior of estimators of the regression parameter, the conditional cumulative hazard rate functions, and the conditional reliability functions. A Monte Carlo study and an example are provided to illustrate the behavior of our estimators and to compare progressive Type-II censoring sampling plans with classical Type-II right censoring sampling plan.

Keywords Counting processes · Martingales · Order statistics · Progressive censoring · Proportional hazards model · Reliability · Semiparametric

1 Introduction

Using counting processes approach, Bordes (2004) discussed the estimation of the cumulative hazard rate function and then the reliability/survival function based on

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a progressively Type-II censored sample. For the cumulative hazard rate function, a Nelson-Aalen type estimator was derived while for the survival function a Kaplan-Meier estimator was proposed. In this paper, we propose inference procedures for progressively censored non-homogeneous data arising from the proportional hazards model (see Cox 1972, 1975). That is, we develop inference based on a progressively censored sample of lifetime data observed along with covariate vectors. Progressively censored sampling plans are applied to a real data set in the last section of the paper.

A general account of theoretical developments and applications concerning progressive censoring is given in the book by Balakrishnan and Aggarwala (2000). Recently, many inferential results have been developed based on progressively censored data; see, for example, the works of Viveros and Balakrishnan (1994), Alvarez-Andrade and Bordes (2004), Balakrishnan and Bordes (2004), Balakrishnan et al. (2003, 2004a,b), Balasooriya and Balakrishnan (2000), Balasooriya et al. (2000), Basak and Balakrishnan (2003), Guilbaud (2001, 2004), and Ng et al. (2002, 2004). However, most of these methods in the context of progressive censoring are based on parametric models. Moreover, all these works are based on homogeneous data in the sense that a progressively censored sample is obtained from independent and identically distributed random variables.

The progressive censoring is of great importance in planning duration experiments in reliability studies. In many industrial experiments involving lifetimes of machines or units, experiments have to be terminated early and also the number of failures must be limited for various reasons (for example, when expensive items must be destroyed, when experiments are time-consuming and expensive, etc.). In addition, some lifetests require removal of functioning test specimens to collect degradation related information to failure time data. The samples that arise from such experiments are called censored samples. The planning of experiments with the aim of reducing the number of failures and the total duration of the experiment leads naturally to the well-known Type-II and Type-I right censoring schemes. For many references and historical notes on this subject, we refer the interested reader to Balakrishnan and Aggarwala (2000). The progressive censoring scheme described below has the same objectives, but it is constructed with the aim of moderating the loss of information by reducing the number of failures with respect to the full sample approach. Montanari and Cacciari (1988) reported results of progressively censored data aging tests on XLPE-insulated cable models under combined thermal-electrical stresses (covariates). In this experiment, live specimens were removed at selected times and/or at the time of breakdowns. The progressive censored reliability sampling plans were discussed by Balasooriya et al. (2000) and Balasooriya and Balakrishnan (2000), while Ng et al. (2004) discussed optimal sampling plans under Weibull lifetime distribution. Another potential application of our work deals with carcinogenicity studies. Indeed, in such studies, animals, usually rats or mice, are divided into two or more groups by randomization and a chemical is administrated at a constant daily dose rate for a major portion of the lifetime of the test animals. Because such an experiment is expensive and time consuming (till 2 years for one study) it is important to propose efficient design with reduced sample size and/or shortened study duration (see Ahn et al. 1998). Moreover, carcinogenicity studies involve sacrifice experiments leading to right censored lifetimes for some animals. Type-II progressively censored data could be obtained by scheduling sacrifice experiments at some observed times of death. There is also a more theoretical potential application of progressive censoring schemes. It could be used as an alternative sampling way in bootstrap methods with respect to the usual sampling *with* or *without* replacement. Especially when the bootstrapped subsamples have to be of small size (in order to reduce computational times).

Let T be a random lifetime and Z be a vector of covariates in $\mathbb{R}^p$. The proportional hazards model assumes that, conditional on Z, the hazard rate function of T is given by

$$\lambda(t; Z) = \exp(\beta_0^T Z)\lambda_0(t), \quad t \in \mathbb{R}^+, \tag{1.1}$$

where $\beta_0 \in \mathbb{R}^p$ is an unknown regression parameter vector and λ_0 is an unknown baseline hazard rate function. Let $(T_1, Z_1), \ldots, (T_n, Z_n)$ be n independent copies of (T, Z). A progressively Type-II censored sample is obtained in the following manner. First, suppose that we are given integers $r_1, \ldots, r_m$, chosen *a priori*, such that $r_1 + \cdots + r_m + m = n$. Consider the lifetimes $T_1, \ldots, T_n$ as the failure times of n units that are placed on test at time zero. We denote by $X_{(1)}$ the first failure time and by i_1 the number of the unit that failed. Immediately following the first failure, r_1 units numbered $i_2, \ldots, i_{r_1+1}$ are removed from the test at random (by sampling without replacement) and we denote $I_1 = \{i_1, \ldots, i_{r_1+1}\}$. Then, we denote by $X_{(2)}$ the second observed failure time, i_{r_1+2} the number of the corresponding unit, and r_2 surviving units numbered $i_{r_1+3}, \ldots, i_{r_1+r_2+2}$ are removed from the the test at random. We then denote $I_2 = \{i_{r_1+2}, \ldots, i_{r_1+r_2+2}\}$. This process continues until, at the time $X_{(m)}$ of the m-th observed failure of unit number $i_{r_1+\cdots+r_{m-1}+m}$, the surviving units $i_{r_1+\cdots+r_{m-1}+m+1}, \ldots, i_n$ are all removed from the experiment and we denote $I_m = \{i_{r_1+\cdots+r_{m-1}+m}, \ldots, i_n\}$. Note that this censoring scheme leads to a subsample $X_{(1)} < \cdots < X_{(m)}$ of the order statistics $T_{(1)} < \cdots < T_{(n)}$ obtained from $T_1, \ldots, T_n$. Note also that the sets of unit numbers $I_1, \ldots, I_m$ satisfy

$$\bigcup_{k=1}^{m} I_k = \{1, \ldots, n\} \quad \text{and} \quad I_k \cap I_l = \emptyset \ \text{ for } 1 \le k < l \le m.$$

For simplicity, we introduce the notations $\alpha_1 = 1$ and $\alpha_k = \sum_{j=1}^{k-1} r_j + k$ for $2 \le k \le m+1$. With this notation, we can then write for $1 \le k \le m$

$$I_k = \{i_{\alpha_k}, \ldots, i_{\alpha_{k+1}-1}\}.$$

Note that if all the r_i are equal to zero, then we observe the complete set of order statistics $T_{(1)}, \ldots, T_{(n)}$, whereas if $r_1 = \cdots = r_{m-1} = 0$, we observe only the first m order statistics $T_{(1)}, \ldots, T_{(m)}$. The latter case corresponds to the classical Type-II right censoring scheme which is thus generalized by the progressive Type-II censoring scheme. It is worth to note that order statistics resulting from a progressive Type-II censoring scheme are a special case of generalized order statistics introduced by Kamps (1995).

2 Basic properties of progressively censored samples

2.1 Joint law and spacings

First, note that conditional on $I_1, \dots, I_m$, the joint density of $(X_{(1)}, \dots, X_{(m)})$ is given by

$$
\begin{aligned}
&f_{X_{(1)},\dots,X_{(m)}}(x_1, \dots, x_m | I_1, \dots, I_m) \\
&\quad = c(I_1, \dots, I_m) \prod_{i=1}^{m} \lambda_0(x_i)\, [S_0(x_i)]^{\sum_{j\in I_i} W_j}\, 1(x_1 \le \dots \le x_m),
\end{aligned} \tag{2.1}
$$

where, for simplicity, we denote $W_j = \exp(\beta_0^T Z_j)$ and with a normalizing constant $c(I_1, \dots, I_m)$ defined by

$$
c(I_1, \dots, I_m) = \prod_{i=1}^{m} \left(\sum_{j \in I_i^m} W_j \right);
$$

here $I_i^m = I_i \cup I_{i+1} \cup \dots \cup I_m$ for $1 \le i \le m$.

Proposition 1 *Denoting Λ_0 for the cumulative baseline hazard rate function, conditional on $I_1, \dots, I_m$, the spacings $E_1, \dots, E_m$ defined by*

$$
E_i = \left(\sum_{j \in I_i^m} W_j \right) \left\{ \Lambda_0(X_{(i)}) - \Lambda_0(X_{(i-1)}) \right\}, \quad 1 \le i \le m,
$$

where $X_{(0)} = 0$, are independent and identically distributed as standard exponential.

Proof Because Λ_0 is an increasing function, applying the same progressive Type-II censoring scheme to $(T_i, Z_i)_{1\le i\le n}$ and to $(\Lambda_0(T_i), Z_i)_{1\le i\le n}$, we get respectively the progressively censored samples $(X_{(i)}, I_i)_{1\le i\le m}$ and $(\Lambda_0(X_{(i)}), I_i)_{1\le i\le m}$. Since, conditional on Z_i, $\Lambda_0(T_i)$ is model (1.1) with a constant baseline hazard rate function equal to 1, it follows by (2.1) that, conditional on $I_1, \dots, I_m$, the joint density of $\big(\Lambda_0(X_{(1)}), \dots, \Lambda_0(X_{(m)})\big)$ is equal to

$$
\begin{aligned}
&f_{\Lambda_0(X_{(1)}),\dots,\Lambda_0(X_{(m)})}(x_1, \dots, x_m | I_1, \dots, I_m) \\
&\quad = c(I_1, \dots, I_m) \exp\left(-\sum_{i=1}^{m} \sum_{j \in I_i^m} W_j x_i \right) 1(x_1 \le \dots \le x_m).
\end{aligned}
$$

Now, inverting the C^1-diffeomorphism that links $(E_1, \dots, E_m)$ to $(\Lambda_0(X_{(1)}), \dots, \Lambda_0(X_{(m)}))$, we obtain

$$\Lambda_0(X_{(i)}) = \sum_{j=1}^{i} \frac{E_j}{\sum_{k\in I_j^m} W_k} \quad \text{for } 1 \le i \le m.$$

Then it follows that conditional on $I_1, \ldots, I_m$, the joint density of $(E_1, \ldots, E_m)$ is equal to

$$f_{E_1,\ldots,E_m}(x_1, \ldots, x_m | I_1, \ldots, I_m) = \exp\left(-\sum_{i=1}^{m} x_i\right) 1(x_1 \ge 0, \ldots, x_m \ge 0),$$

which completes the proof. □

Proposition 2 *Assume that t is a real number such that $\Lambda_0(t) < +\infty$, $\omega = \sup_{n\ge1} W_n < +\infty$, and $r = \sup_{n\ge1} r_n < +\infty$. Then, we have $P(X_{(m)} \le t | I_1, \ldots, I_m) \to 0$ as $n \to +\infty$.*

Proof Note that $n \to +\infty$ implies that $m \to +\infty$. Moreover, we have

$$\begin{aligned} P(X_{(m)} \le t | I_1, \ldots, I_m) &= P(\Lambda_0(X_{(m)}) \le \Lambda_0(t) | I_1, \ldots, I_m) \\ &= P\left(\sum_{j=1}^{m} \frac{E_j}{\sum_{k\in I_j^m} W_k} \le \Lambda_0(t) \,\middle|\, I_1, \ldots, I_m\right) \\ &\le P\left(\sum_{j=1}^{m} \frac{E_j}{\sum_{k=j}^{m}(r_k+1)} \le \omega\Lambda_0(t) \,\middle|\, I_1, \ldots, I_m\right) \\ &\le P\left(\sum_{j=1}^{m} \frac{E_j}{m-j+1} \le (r+1)\omega\Lambda_0(t) \,\middle|\, I_1, \ldots, I_m\right). \end{aligned}$$

Now, since

$$E\left(\sum_{j=1}^{m} \frac{E_j}{m-j+1} \,\middle|\, I_1, \ldots, I_m\right) = \sum_{j=1}^{m} \frac{1}{m-j+1} \to +\infty \quad \text{as } n \to +\infty,$$

we obtain the desired result. □

2.2 Counting processes considerations

Let us consider the filtration $\mathbb{F} = (\mathcal{F}_t; t \ge 0)$ defined by

$$\mathcal{F}_t = \sigma\{(X_{(i)}, I_i); 1 \le i \le m \text{ and } X_{(i)} \le t\}, \quad t \ge 0,$$

and the counting processes N_i, $\mathbb{F}$-adapted, defined by

$$N_i(t) = 1(X_{(i)} \le t), \quad t \ge 0,$$

where $1(\cdot)$ is the set characteristic function. Let us denote $N = \sum_{i=1}^{m} N_i$.

Now, let us define for $1 \leq i \leq n$ the random variables (Y_i, δ_i) as

$$\begin{cases} Y_i = X_{(k)} & \text{if } i \in I_k, \\ \delta_i = 1 & (i \in \{i_{\alpha_1}, \ldots, i_{\alpha_m}\}) \end{cases}.$$

Then for $i = 1, \ldots, n$, we define counting processes $\bar{N}_i$ by

$$\bar{N}_i(t) = 1(Y_i \leq t; \delta_i = 1), \quad t \geq 0.$$

Clearly, counting processes $\bar{N}_1, \ldots, \bar{N}_n$ are $\mathbb{F}$-adapted and the following obvious relations hold:

$$N_i = \sum_{j \in I_i} \bar{N}_j, \quad \forall i \in \{1, \ldots, m\}.$$

Proposition 3 *Processes $\bar{M}_i$ and M_j defined for $1 \leq i \leq n$, $1 \leq j \leq m$ and $t \geq 0$ by*

$$\bar{M}_i(t) = \bar{N}_i(t) - \int_0^t 1(Y_i \geq s) \exp(\beta_0^T Z_i) \lambda_0(s) \mathrm{d}s$$

and

$$M_i(t) = N_i(t) - \int_0^t \sum_{j \in I_i} \exp(\beta_0^T Z_j) 1(X_{(i)} \geq s) \lambda_0(s) \mathrm{d}s,$$

are $\mathbb{F}$-martingales.

Proof By standard results on counting processes (see, for example, Aven and Jensen 1999; Andersen et al. 1993), it is sufficient to find the $\mathbb{F}$-intensity $\bar{\lambda}_i$ of the $\bar{N}_i$'s to derive the martingale property of $\bar{M}_i$. Then the martingale property of M_i follows by summation. Recall that for $t > 0$, we have

$$\bar{\lambda}_i(t) = \lim_{h \searrow 0} E\left[\bar{N}_i(t+h) - \bar{N}_i(t) | \mathcal{F}_{t-}\right] / h. \tag{2.2}$$

Let $h > 0$ be a real number. Then, because $1(Y_i \geq t)$ is $\mathbb{F}$-predictable, we have

$$\begin{aligned} &E\left[\bar{N}_i(t+h) - \bar{N}_i(t) | \mathcal{F}_{t-}\right] \\ &\quad = E\left[(\bar{N}_i(t+h) - \bar{N}_i(t)) 1(Y_i < t) | \mathcal{F}_{t-}\right] + E\left[(\bar{N}_i(t+h) - \bar{N}_i(t)) 1(Y_i \geq t) | \mathcal{F}_{t-}\right] \\ &\quad = 1(Y_i \geq t) E\left[\bar{N}_i(t+h) - \bar{N}_i(t) | \mathcal{F}_{t-}\right] \\ &\quad = 1(Y_i \geq t) P(Y_i \in (t, t+h], \delta_i = 1 | \mathcal{F}_{t-}) \\ &\quad = 1(Y_i \geq t) \sum_{k=0}^{m-1} P(Y_i \in (t, t+h], \delta_i = 1, X_{(k)} < t \leq X_{(k+1)} | \mathcal{F}_{t-}) \end{aligned}$$

$$= 1(Y_i \geq t)\sum_{k=0}^{m-1} P(X_{(k+1)} \in (t, t+h], i_{\alpha_{k+1}} = i|\mathcal{F}_{X_{(k)}}, X_{(k)} < t \leq X_{(k+1)})$$
$$\times P(X_{(k)} < t \leq X_{(k+1)}|\mathcal{F}_{t-})$$
$$= 1(Y_i \geq t)\sum_{k=0}^{m-1} \frac{P(X_{(k+1)} \in [t, t+h], i_{\alpha_{k+1}} = i|\mathcal{F}_{X_{(k)}})}{P(X_{(k+1)} \geq t|\mathcal{F}_{X_{(k)}})} P(X_{(k)} < t \leq X_{(k+1)}|\mathcal{F}_{t-}).$$

The right, hand side of the last equality, along with (2.2), yields

$$\bar{\lambda}_i(t) = 1(Y_i \geq t)\sum_{k=0}^{m-1} \frac{f_{X_{(k+1)}, i_{\alpha_{k+1}}}(t, i|\mathcal{F}_{X_{(k)}})}{\int_t^{+\infty} f_{X_{(k+1)}}(u|\mathcal{F}_{X_{(k)}})du} P(X_{(k)} < t \leq X_{(k+1)}|\mathcal{F}_{t-}). \quad (2.3)$$

Denoting by $f_{X_{(1)},\ldots,X_{(k+1)},i_{\alpha_{k+1}}}(x_1, \ldots, x_{k+1}, i|I_1^k)$ the joint distribution function of the random vector $(X_{(1)}, \ldots, X_{(k+1)}, i_{\alpha_{k+1}})$ conditionally on I_1^k, we have

$$f_{X_{(1)},\ldots,X_{(k+1)},i_{\alpha_{k+1}}}(x_1, \ldots, x_{k+1}, i|I_1^k)$$
$$= c\prod_{i=1}^{k} W_{\alpha_i}\lambda_0(x_i)[S_0(x_i)]^{\sum_{j\in I_i} W_j}$$
$$\times W_i\lambda_0(x_{k+1})[S_0(x_{k+1})]^{\sum_{j\in I_{k+1}^m} W_j} 1(x_1 < \cdots < x_{k+1}, i \in I_{k+1}^m),$$

where c is a normalizing constant. Then, we have

$$f_{X_{(1)},\ldots,X_{(k)}}(x_1, \ldots, x_k|I_1^k)$$
$$= \sum_{i=1}^{n}\int_{x_k}^{+\infty} f_{X_{(1)},\ldots,X_{(k+1)},i_{\alpha_{k+1}}}(x_1, \ldots, x_{k+1}, i|I_1^k)\mathrm{d}x_{k+1}$$
$$= c\prod_{i=1}^{k} W_{\alpha_i}\lambda_0(x_i)[S_0(x_i)]^{\sum_{j\in I_i} W_j}\left(\sum_{j\in I_{k+1}^m} W_j\right)[S_0(x_k)]^{\sum_{j\in I_{k+1}^m} W_j} 1(x_1 \leq \cdots \leq x_k).$$

Since $\mathcal{F}_{X_{(k)}} = \sigma\{X_{(1)}, \ldots, X_{(k)}, I_1^k\}$, we have

$$
\begin{aligned}
f_{X_{(k+1)},i_{\alpha_{k+1}}}(t,i|\mathcal{F}_{X_{(k)}}) &= \frac{f_{X_{(1)},\ldots,X_{(k+1)},i_{\alpha_{k+1}}}(x_1,\ldots,x_{k+1},i|I_1^k)}{f_{X_{(1)},\ldots,X_{(k)}}(x_1,\ldots,x_k|I_1^k)} \\
&= \frac{W_i}{\sum_{j\in I_{k+1}^m} W_j}\lambda_0(t)\left[\frac{S_0(t)}{S_0(x_k)}\right]^{\sum_{j\in I_{k+1}^m} W_j} 1(t\geq x_k, i\in I_{k+1}^m), \\
f_{X_{(k+1)}}(t|\mathcal{F}_{X_{(k)}}) &= \sum_{i=1}^n f_{X_{(k+1)},i_{\alpha_{k+1}}}(t,i|\mathcal{F}_{X_{(k)}}) \qquad (2.4) \\
&= \lambda_0(t)\left[\frac{S_0(t)}{S_0(x_k)}\right]^{\sum_{j\in I_{k+1}^m} W_j} 1(t\geq x_k),
\end{aligned}
$$

and

$$
\int_t^{+\infty} f_{X_{(k+1)}}(u|\mathcal{F}_{X_{(k)}})\mathrm{d}u = \frac{1}{\sum_{j\in I_{k+1}^m} W_j}\left[\frac{S_0(t)}{S_0(x_k)}\right]^{\sum_{j\in I_{k+1}^m} W_j}. \qquad (2.5)
$$

Finally, Eqs. (2.3), (2.4) and (2.5) yield

$$
\bar{\lambda}_i(t) = W_i\lambda_0(t)1(Y_i\geq t), \quad t\geq 0.
$$

With the above intensity process for $\bar{N}_i$, we obtain the desired result. □

3 Estimators and their properties

3.1 Estimators of unknown parameters

Earlier in the Introduction section, we remarked that the progressive Type-II censoring scheme leads to a random partition $\mathfrak{I}_1,\ldots,\mathfrak{I}_m$ of $\{1,\ldots,n\}$, such that $\mathrm{Card}(\mathfrak{I}_i) = r_i + 1$ for $1\leq i\leq m$. We denoted by $I_1,\ldots,I_m$ the observed partition once the experiment is finished. Then we are able to calculate the probability of observing the partition $I_1,\ldots,I_m$ as

$$
\begin{aligned}
&P(\mathfrak{I}_1 = I_1,\ldots,\mathfrak{I}_m = I_m) \\
&\quad= \int_0^{+\infty} W_{\alpha_1}\lambda_0(x_1)[S_0(x_1)]^{\sum_{j\in I_1} W_j}\int_{x_1}^{+\infty} W_{\alpha_2}\lambda_0(x_2)[S_0(x_2)]^{\sum_{j\in I_2} W_j}\ldots \\
&\qquad\cdots\int_{x_{m-1}}^{+\infty} W_{\alpha_m}\lambda_0(x_m)[S_0(x_m)]^{\sum_{j\in I_m} W_j}\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_m \\
&\quad= \prod_{i=1}^m\frac{W_{\alpha_i}}{\sum_{j\in I_i^m} W_j} = \prod_{i=1}^m\frac{\exp(\beta_0^T Z_{\alpha_i})}{\sum_{j\in I_i^m}\exp(\beta_0^T Z_j)}.
\end{aligned}
$$

Hence it is natural to estimate the unknown regression parameter β_0 by the value of β that makes the above probability to observe the partition $I_1,\ldots,I_m$ maximal since this probability depends no longer on the unknown functional parameter λ_0. Taking the logarithm of this probability, it follows that the estimator $\hat{\beta}_n$ of β_0 is then given by

$$\hat{\beta}_n = \arg\max_{\beta \in \mathbb{R}^p} C_n(\beta), \tag{3.1}$$

where

$$C_n(\beta) = \sum_{i=1}^{m} \left\{ \beta^T Z_{\alpha_i} - \log\left(\sum_{j \in I_i^m} \exp(\beta^T Z_j) \right) \right\}.$$

Remark 1 It is easy to see that the above estimator coincides with the Cox (1972) estimator when there is no censoring, i.e., when all the r_i are fixed as 0 in the progressive censoring scheme.

Using the martingale property of Proposition 3, we have

$$\sum_{i=1}^{m} \mathrm{d}M_i(s) = \sum_{i=1}^{m} \mathrm{d}N_i(s) - \sum_{i=1}^{m} \left(\sum_{j \in I_i} \exp(\beta_0^T Z_j) \right) 1(X_{(i)} \geq s)\mathrm{d}s$$

which, upon neglecting the martingale part, leads to a pseudo-estimator $\hat{\Lambda}_0(t; \beta_0)$ of Λ_0 defined by

$$\hat{\Lambda}_0(t; \beta_0) = \int_0^t \frac{\mathrm{d}N(s)}{S^{(0)}(s; \beta_0)} \quad \forall t \geq 0,$$

where $S^{(0)}(s; \beta_0) = \sum_{i=1}^{m} \sum_{j \in I_i} \exp(\beta_0^T Z_j) 1(X_{(i)} \geq s)$. Finally, we estimate Λ_0 by replacing β_0 by $\hat{\beta}_n$ in $\hat{\Lambda}_0(t; \beta_0)$, and then, Λ_0 is estimated by a Breslow-type estimator $\hat{\Lambda}_0$ defined by

$$\hat{\Lambda}_0(t) = \sum_{1 \leq i \leq m; X_{(i)} \leq t} \frac{1}{\sum_{j \in I_i^m} \exp(\hat{\beta}_n^T Z_j)}, \quad t \geq 0. \tag{3.2}$$

Remark 2 An estimating function for β_0 can also be derived using the likelihood function of the counting process N (see Andersen et al. 1993). We show that such an estimating function depends on $\lambda_0(X_{(i)})$ which are, therefore, replaced by the jumps of the $\hat{\Lambda}_0$ at points $X_{(i)}$. The estimating function so obtained is, up to an additive term that does not depend on unknown parameters, the same as C_n.

Then, the cumulative hazard rate function $\Lambda(t; Z)$ of a duration T conditionally on Z that satisfies model (1.1) is naturally estimated by $\hat{\Lambda}(\cdot; Z)$ defined by

$$\hat{\Lambda}(t; Z) = \exp(\hat{\beta}_n^T Z)\hat{\Lambda}_0(t), \quad t \geq 0. \tag{3.3}$$

Finally, we need to estimate the survival function $S(\cdot; Z)$ of T conditionally on Z. Because the natural link between $\Lambda(\cdot; Z)$ and $S(\cdot; Z)$ is the integral-product, the estimator, denoted by $\hat{S}(\cdot; Z)$, is defined by

$$\hat{S}(t; Z) = \prod_{1 \le i \le m; X_{(i)} \le t} \left(1 - \frac{\exp(\hat{\beta}_n^T Z)}{\sum_{j \in I_i^m} \exp(\hat{\beta}_n^T Z_j)} \right), \quad t \ge 0. \tag{3.4}$$

3.2 Asymptotic properties of the estimators

First, we introduce notation and conditions under which we are able to obtain asymptotic results. For any $\mathbb{R}^p$-valued column vector Z, we define $Z^{\otimes k}$ equal to 1, Z and ZZ^T for k equal to 0, 1 and 2, respectively. For $0 \le k \le 2$, we define for $(t, \beta) \in \mathbb{R}^+ \times \mathbb{R}^p$, random functions

$$S^{(k)}(t, \beta) = \sum_{i=1}^{m} \left(\sum_{j \in I_i} Z_j^{\otimes k} \exp(\beta^T Z_j) \right) 1(X_{(i)} \ge t).$$

We also define, for $(t, \beta) \in \mathbb{R}^+ \times \mathbb{R}^p$, quantities

$$E(s, \beta) = \frac{S^{(1)}(t, \beta)}{S^{(0)}(t, \beta)} \quad \text{and} \quad V(s, \beta) = \frac{S^{(2)}(t, \beta)}{S^{(0)}(t, \beta)} - E^{\otimes 2}(s, \beta).$$

Condition 1 (a) τ is a real number such that $\int_0^\tau \lambda_0(s) \mathrm{d}s < +\infty$.
(b) The sequence $(r_i)_{i \ge 1}$ is uniformly bounded.
(c) There exists a neighborhood $\mathcal{B}$ of β_0 and scalar, p-vector and $p \times p$-matrix functions $s^{(0)}$, $s^{(1)}$ and $s^{(2)}$, respectively, defined on $\mathcal{B} \times [0, \tau]$ such that for $k \in \{0, 1, 2\}$:

$$\sup_{(t,\beta) \in [0,\tau] \times \mathcal{B}} \left\| \frac{1}{n} S^{(k)}(t, \beta) - s^{(k)}(t, \beta) \right\| \xrightarrow{P} 0 \quad \text{as } n \to +\infty,$$

where $\| \cdot \|$ denotes the Euclidean-norm.
(d) For $k \in \{0, 1, 2\}$, $\beta \mapsto s^{(k)}(t, \beta)$ is a continuous function on $\mathcal{B}$ uniformly in $t \in [0, \tau]$.
(e) $s^{(1)}(t, \beta) = \dfrac{\partial s^{(0)}}{\partial \beta}(t, \beta)$ and $s^{(2)}(t, \beta) = \dfrac{\partial^2 s^{(0)}}{\partial \beta \partial \beta^T}(t, \beta)$ for $(t, \beta) \in [0, \tau] \times \mathcal{B}$.
(f) $t \mapsto s^{(0)}(s, \beta_0)$ is bounded away from 0 on $[0, \tau]$.
(g) The matrix $\Sigma(\tau)$ defined by

$$\Sigma(\tau) = \int_0^\tau v(s, \beta_0) s^{(0)}(s, \beta_0) \lambda_0(s) \mathrm{d}s,$$

where

$$v(s,\beta_0) = \frac{s^{(2)}(s,\beta_0)}{s^{(0)}(s,\beta_0)} - e^{\otimes 2}(s,\beta_0) \quad \text{and} \quad e(s,\beta_0) = \frac{s^{(1)}(s,\beta_0)}{s^{(0)}(s,\beta_0)},$$

is positive definite.

(h) $\sup_{n\geq 1} \|Z_n\| < +\infty$.

Remark 3 The boundedness of functions $s^{(k)}(t,\beta)$ on $[0,\tau]\times\mathcal{B}$ follows from Condition 1 (h).

Theorem 1 *Under Condition 1, as $m \to +\infty$, we have*

(i) *$\hat{\beta}_n$ converges in probability to β_0.*
(ii) *$\sqrt{n}(\hat{\beta}_n - \beta_0)$ converges in distribution to a centered Gaussian random vector with variance-covariance matrix $\Sigma^{-1}(\tau)$.*
(iii) *$\Sigma(\tau)$ is consistently (in probability) estimated by*

$$\hat{\Sigma}(\tau) = \frac{1}{n}\sum_{i=1}^{m} V(X_{(i)}, \hat{\beta}_n).$$

The proof of this theorem is adapted from the proof of Andersen and Gill (1982) (see also Andersen et al. 1993). As a consequence, we only sketch the proof pointing out what makes the difference from the proof of Andersen and Gill.

Proof Because of the following obvious identity

$$\sum_{i=1}^{m}\int_0^t Z_{\alpha_i}\mathrm{d}N_i(s) - \sum_{i=1}^{n}\int_0^t Z_i\mathrm{d}\bar{N}_i(s) = 0, \quad t\in[0,\tau],$$

we can remark that

$$\begin{aligned} C_n(\beta) = C_n(\tau,\beta) &= \sum_{i=1}^{m}\int_0^{\tau}\left(\beta^T Z_{\alpha_i} - \log(S^{(0)}(s,\beta))\right)\mathrm{d}N_i(s), \\ &= \sum_{i=1}^{n}\int_0^{\tau}\left(\beta^T Z_i - \log(S^{(0)}(s,\beta))\right)\mathrm{d}\bar{N}_i(s). \end{aligned}$$

We then write

$$\mathcal{U}_n(\tau,\beta) = \frac{\partial C_n}{\partial\beta}(\tau,\beta) = \sum_{i=1}^{n}\int_0^{\tau}(Z_i - E(s,\beta))\,\mathrm{d}\bar{N}_i(s),$$

and

$$\mathcal{I}_n(\tau,\beta) = \frac{\partial^2 C_n}{\partial\beta\partial\beta^T}(t,\beta) = -\int_0^{\tau} V(s,\beta)\mathrm{d}N(s).$$

So, the proof of (i) follows the proof of Theorem VII.2.1 of Andersen et al. (1993, p. 497) and is therefore omitted. To obtain result (ii), we remark that

$$\frac{1}{\sqrt{n}}\mathcal{U}_n(\tau,\hat{\beta}_n)-\mathcal{U}_n(\tau,\beta_0)=-\frac{1}{\sqrt{n}}\mathcal{U}_n(\tau,\beta_0)=\sqrt{n}(\hat{\beta}_n-\beta_0)\mathcal{I}_n(\tau,\bar{\beta}_n)/n, \quad (3.5)$$

where $\bar{\beta}_n$ lies in the line segment with extremities β_0 and $\hat{\beta}_n$. Because $\hat{\beta}_n$ converges in probability to β_0, $\bar{\beta}_n$ converges in probability to β_0 too. Following the proof of Theorem VII.2.2 in Andersen et al. (1993, pp. 498–500), we show that the process $(n^{-1/2}\mathcal{U}_n(t,\beta_0); t\in[0,\tau])$ converges in distribution to a Gaussian martingale $\mathcal{G}$ with variance–covariance matrix $\Sigma(t)$ in $(D[0,\tau])^p$ and that $n^{-1}\mathcal{I}_n(\tau,\bar{\beta}_n)$ converges in probability to $\Sigma(\tau)$ as n tends to infinity. This proves results (ii) and (iii). □

Theorem 2 *Under Condition 1,* $\sqrt{n}\left(\hat{\Lambda}(\cdot,Z)-\Lambda(\cdot,Z)\right)$ *converges weakly to a centered Gaussian process in* $D[0,\tau]$ *with variance function* σ^2 *defined, for* $t\in[0,\tau]$, *as*

$$\sigma^2(t,Z)=\exp(2\beta_0^TZ)\left(\int_0^t\frac{\lambda_0(s)}{s^{(0)}(s,\beta_0)}\mathrm{d}s\right.$$
$$\left.+\int_0^t(Z-e(s,\beta_0))^T\lambda_0(s)\mathrm{d}s\times\Sigma^{-1}(\tau)\times\int_0^t(Z-e(s,\beta_0))\lambda_0(s)\mathrm{d}s\right),$$

and uniformly consistently estimated on $[0,\tau]$ *by*

$$\hat{\sigma}^2(t,Z)=\exp(2\hat{\beta}_n^TZ)\left(\int_0^t\frac{\mathrm{d}N(s)}{(S^{(0)}(s,\hat{\beta}_n))^2}+\int_0^t\left(\frac{Z-E(s,\hat{\beta}_n)}{S^{(0)}(s,\hat{\beta}_n)}\right)^T\mathrm{d}N(s)\right.$$
$$\left.\times\hat{\Sigma}^{-1}(\tau)\times\int_0^t\left(\frac{Z-E(s,\hat{\beta}_n)}{S^{(0)}(s,\hat{\beta}_n)}\right)\mathrm{d}N(s)\right).$$

Proof Following the proof of Theorem VII.2.3 of Andersen et al. (1993, p. 503), we first show that $\mathcal{Y}_n$, defined for $t\in[0,\tau]$, by

$$\mathcal{Y}_n(t)=\sqrt{n}\left(\hat{\Lambda}_0(t)-\Lambda_0(t)\right)+\sqrt{n}\left(\hat{\beta}_n-\beta_0\right)\int_0^t e(s,\beta_0)\lambda_0(s)\mathrm{d}s$$

and $\sqrt{n}\left(\hat{\beta}_n-\beta_0\right)$, are asymptotically independent. Indeed, we have

$$\sqrt{n}\left(\hat{\Lambda}_0(t)-\Lambda_0(t)\right)=\sqrt{n}\int_0^t\left(\frac{1}{S^{(0)}(s,\hat{\beta}_n)}-\frac{1}{S^{(0)}(s,\beta_0)}\right)\mathrm{d}N(s) \quad (3.6)$$
$$+\sqrt{n}\int_0^t\frac{1(S^{(0)}(s,\beta_0)>0)}{S^{(0)}(s,\beta_0)}\mathrm{d}M(s) \quad (3.7)$$
$$-\sqrt{n}\int_0^t 1(S^{(0)}(s,\beta_0)=0)\lambda_0(s)\mathrm{d}s. \quad (3.8)$$

By a Taylor expansion around β_0 of the right-hand side of (3.6) and using the consistency of $\hat{\beta}_n$, Conditions 1 (c)–(f) and the Lenglart inequality, we show that this term is asymptotically equivalent, uniformly in $t \in [0, \tau]$, to $-\sqrt{n}\left(\hat{\beta}_n - \beta_0\right) \int_0^t e(s, \beta_0)\lambda_0(s)\mathrm{d}s$. On the other hand, it is easy to show that for all $\varepsilon > 0$ we have

$$P\left(\sqrt{n}\int_0^t 1(S^{(0)}(s, \beta_0) = 0)\lambda_0(s)\mathrm{d}s > \varepsilon\right) \leq P(X_{(m)} < t).$$

Since Conditions 1 (a)–(b) and (h) imply that assumptions of Proposition 2 are fulfilled, we get that term (3.8) converges in probability to 0, uniformly in $t \in [0, \tau]$. Finally, writing $\tilde{\mathcal{Y}}_n(t)$ the term in (3.7), we obtain that $\mathcal{Y}_n$ and $\tilde{\mathcal{Y}}_n$ are asymptotically equivalent uniformly on $[0, \tau]$. Applying the Rebolledo theorem (see Andersen et al. 1993), we show that $\tilde{\mathcal{Y}}_n$ converges weakly to a Gaussian martingale with variance function η^2 defined for $t \in [0, \tau]$ by

$$\eta^2(t) = \int_0^t \frac{\lambda_0(s)}{s^{(0)}(s, \beta_0)}\mathrm{d}s.$$

Moreover, it is straightforward to show that for all $n \geq 1$ we have

$$\left\langle n^{-1/2}\mathcal{U}_n(\cdot, \beta_0), \tilde{\mathcal{Y}}_n(\cdot)\right\rangle = 0.$$

It then follows that $\tilde{\mathcal{Y}}_n$ and $\mathcal{U}_n(\tau, \beta_0)$ are asymptotically independent. Using (3.5) and Conditions 1 (a) and (c)–(g), we show that $\sqrt{n}\left(\hat{\beta}_n - \beta_0\right)$ and $\Sigma^{-1}(\tau)\mathcal{U}_n(\tau, \beta_0)$ are asymptotically equivalent. Therefore, $\mathcal{Y}_n$ and $\sqrt{n}\left(\hat{\beta}_n - \beta_0\right)$ are asymptotically independent. Now, by a Taylor expansion, we have

$$\begin{aligned}
&\sqrt{n}\left(\hat{\Lambda}(t, Z) - \Lambda(t, Z)\right)\\
&\quad = \exp(\hat{\beta}_n^T Z)\sqrt{n}\left(\hat{\Lambda}_0(t) - \Lambda_0(t)\right) + \Lambda_0(t)Z^T\exp(\bar{\beta}_n^T Z)\sqrt{n}\left(\hat{\beta}_n - \beta_0\right),
\end{aligned}$$

where $\bar{\beta}_n$ belongs to the line segment with extremities β_0 and $\hat{\beta}_n$. Because both $\hat{\beta}_n$ and $\bar{\beta}_n$ are $\sqrt{n}$-consistent estimators of β_0, we obtain that uniformly in $t \in [0, \tau]$

$$\begin{aligned}
&\sqrt{n}\left(\hat{\Lambda}(t, Z) - \Lambda(t, Z)\right)\\
&\quad = \exp(\beta_0^T Z)\left[\mathcal{Y}_n(t) + \int_0^t (Z - e(s, \beta_0))^T \lambda_0(s)\mathrm{d}s \times \sqrt{n}\left(\hat{\beta}_n - \beta_0\right)\right] + o_P(1)\\
&\quad = \exp(\beta_0^T Z)\left[\tilde{\mathcal{Y}}_n(t) + \int_0^t (Z - e(s, \beta_0))^T \lambda_0(s)\mathrm{d}s \times \sqrt{n}\left(\hat{\beta}_n - \beta_0\right)\right] + o_P(1).
\end{aligned}$$

From the last equality, it follows that $\sqrt{n}\left(\hat{\Lambda}(\cdot, Z) - \Lambda(\cdot, Z)\right)$ converges weakly to a Gaussian process with variance function $\sigma^2(\cdot, Z)$. The estimator $\hat{\sigma}^2(\cdot, Z)$ is obtained by replacing all the unknown quantities β_0, $s^{(0)}(\cdot, \beta_0)$, $e(\cdot, \beta_0)$, $\Sigma(\tau)$ and $d\Lambda_0$ in $\sigma^2(\cdot, Z)$ by $\hat{\beta}_n$, $S^{(0)}(\cdot, \hat{\beta}_n)$, $E(\cdot, \hat{\beta}_n)$, $\hat{\Sigma}(\tau)$ and $d\hat{\Lambda}_0$ respectively. Its uniform consistency is obtained using the Lenglart inequality and Conditions 1 (a) and (c)–(g). □

Corollary 1 *Under Condition 1,* $\sqrt{n}(\hat{S}(\cdot, Z) - S(\cdot, Z))$ *converges weakly in* $D[0, \tau]$ *to a Gaussian process with variance function* θ^2 *defined for* $t \in [0, \tau]$ *as* $\theta^2(t, Z) = S^2(t, Z)\sigma^2(t, Z)$. *Moreover,* $\theta^2(\cdot, Z)$ *is consistently estimated by* $\hat{\theta}^2(t, Z) = \hat{S}^2(t, Z)$ $\hat{\sigma}^2(t, Z)$ *uniformly on* $[0, \tau]$.

Proof This is an immediate consequence of compact differentiability of the *product-integral*, the functional delta-method (see Gill 1989 or Andersen et al. 1993) and Theorem 2. □

4 Numerical study and an example

4.1 A Monte Carlo study

In this section, we present the results of a Monte Carlo study based on $K = 500$ simulated samples of size n. These samples are progressively Type-II censored, Type-II censored, and uncensored. The covariates are random variables having Bernoulli distribution with parameter 1/2. The distribution of a random lifetime T, conditional on a covariate Z, is defined by the hazard rate function $\lambda(t; Z) = \exp(\beta_0 Z)(t/3)^2$ for $t \geq 0$, where the regression parameter $\beta_0 \in \{-2, 0, 2\}$ has to be estimated. For each set of simulated samples (see Table 1), we calculate the empirical mean and the empirical standard deviation (within parentheses) of the K estimates of β_0, and in addition we calculate the empirical mean of the standard deviation estimates (within brackets). We consider, for two sample sizes ($n = 99$ and $n = 300$), the three following sampling schemes:

(A) Progressive Type-II censoring sampling plan: $m = n/3$ and all the r_i's equal to 2.
(B) Type-II censoring sampling plan: $m = n/3$.
(C) Complete data.

Note that the two censored sampling plans lead to 66% of censoring. In general, we see from Table 1 that in the case of censored sampling plans (A) and (B), there

Table 1 Statistical analysis of 500 estimates of β_0 for two sample size and three sampling schemes

Sample size	Scheme	$\beta_0 = -2$	$\beta_0 = 0$	$\beta_0 = 2$
	(A)	−2.064 (0.514) [0.479]	−0.005 (0.354) [0.354]	2.039 (0.476) [0.471]
$n = 99$	(B)	−2.062 (0.503) [0.506]	−0.015 (0.380) [0.355]	2.069 (0.521) [0.510]
	(C)	−2.031 (0.301) [0.290]	−0.003 (0.219) [0.206]	2.032 (0.279) [0.289]
	(A)	−2.010 (0.261) [0.262]	0.004 (0.206) [0.201]	1.984 (0.261) [0.261]
$n = 300$	(B)	−2.017 (0.283) [0.279]	0.002 (0.195) [0.201]	2.019 (0.286) [0.280]
	(C)	−2.008 (0.171) [0.162]	0.008 (0.122) [0.117]	2.007 (0.168) [0.162]

is a loss of efficiency in estimators due to the loss of information that occurs due to censoring.

However, comparing the censoring schemes (A) and (B), which have the same number of failures, we can see that the standard deviations corresponding to the progressively Type-II censored data are generally less than (or at least comparable) the standard deviations corresponding to the usual Type-II censored sample. We can also observe that results for progressively Type-II right censored sampling plans yield better results for the mean and the standard deviation of the estimators than those based on the usual Type-II censored sampling plans which facilitates in identifying significant effect of covariates (i.e., for $\beta_0 \neq 0$). Furthermore, as the sample size increases, the gain in efficiency of the estimates become quite evident.

These simulations also show that the mean of K estimated standard deviations (values within parentheses) are close to the empirical standard deviation of the K estimates of β_0 (values within brackets), for all the sample size, sampling plan, and values of β_0.

Finally, we recall that the progressive Type-II censored sampling plan adopted in Table 1 have not been defined in order to optimize the efficiency of the regression parameter estimator. The determination of such optimal progressive Type-II censored sampling plans in the semiparametric framework remains as an open problem.

4.2 A practical example

We consider a data set of times to breakdown (in Minutes) of insulated fluids tested under high test voltages given by Nelson (1990, Table 3.1, p. 129). Insulated fluids were tested at seven high voltages: $26, 28, \ldots, 38\,\text{kV}$, which are the values of the unique covariate in this case. For each voltage level, several times to breakdown were measured resulting in $n = 76$ failure times from $0.09'$ to $2323.7'$. These data were analyzed with the R software. The test for proportional hazards assumption in these data resulted in a p-value of 0.93, whereas testing that the covariate is statistically not significant via the likelihood ratio test produced a p-value of 10^{-14}. Other standard graphical methods also reveal that these data are correctly fitted by the proportional hazards model.

Now, we compare several progressive Type-II censoring schemes on these data. We set $m = 30$, and consider the following four sampling plans:

(I) $r_1 = r_2 = r_3 = 12$, $r_4 = 10$ and $r_5 = \cdots = r_{30} = 0$;
(II) $r_1 = \cdots = r_{23} = 2$ and $r_{24} = \cdots = r_{30} = 0$;
(III) $r_1 = \cdots = r_{26} = 0$, $r_{27} = 10$ and $r_{28} = r_{29} = r_{30} = 12$;
(IV) $r_1 = \cdots = r_{29} = 0$ and $r_{30} = 46$.

Note that Scheme (IV) corresponds to the usual Type-II censoring scheme, and so the data set resulting from this scheme simply corresponds to the smallest 30 order statistics. Censoring schemes (I)–(III) lead to random samples. In these cases, for each censoring scheme, we computed the average of $N = 1000$ estimates of the regression parameter β and its standard deviation obtained by applying N times the censoring scheme on the complete data. The results so obtained are summarized in Table 2.

Table 2 Insulated fluid data: comparison of several censoring schemes

Censoring scheme	$\hat{\beta}$	SD of $\hat{\beta}$
Complete data	0.401	0.058
(I)	0.398	0.092
(II)	0.411	0.091
(III)	0.467	0.094
(IV)	0.481	0.095

Results in Table 2 reveal first of all that there is a gain in using a Type-II progressive censoring scheme as compared to the usual Type-II censoring scheme. We see that Schemes (I)–(III) yield better results than Scheme (IV) especially for the estimate of the regression parameter (providing in general estimates close to the complete sample results). Also, there is a reduction in the standard error for Schemes (I)–(II) as compared to (IV), although the reduction is only marginal. It is also clear from Table 2 that the censoring scheme introduces some variations in the results of the estimates. Yet again, we need to emphasise here that we have not investigated the problem of choosing an optimal progressively censored sampling plan in this semiparametric setup.

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