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Uniformly robust tests in errors-in-variables models

Longcheen Huwang · Y. H. Steve Huang · Yi-Hua Tina Wang

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Abstract Consider an ordinary errors-in-variables model. The true level $\alpha_n(\boldsymbol{\theta}^*)$ of a test at nominal level α and sample size n is said to be pointwise robust if $\alpha_n(\boldsymbol{\theta}^*) \rightarrow \alpha$ as $n \rightarrow \infty$ for each parameter $\boldsymbol{\theta}^*$. Let Ω^* be a set of values of $\boldsymbol{\theta}^*$. Define $\alpha_n = \sup_{\boldsymbol{\theta}^* \in \Omega^*} \alpha_n(\boldsymbol{\theta}^*)$. The test is said to be uniformly robust over Ω^* if $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$. Corresponding definitions apply to the coverage probabilities of confidence sets. It is known that all existing large-sample tests for the parameters of the errors-in-variables model are pointwise robust. However, they might not be uniformly robust over certain null parameter spaces. In this paper, we construct uniformly robust tests for testing the vector coefficient parameter and vector slope parameter in the functional errors-in-variables model. These tests are established through constructing the confidence sets for the same parameters in the model with similar desirable property. Power comparisons based on simulation studies between the proposed tests and some existing tests in finite samples are also presented.

Keywords Errors-in-variables · Pointwise robust · Uniformly robust · Confidence coefficient · Projected test

L. Huwang (✉) · Y.-H. Tina Wang
Institute of Statistics, National Tsing Hua University, Hsinchu, Taiwan
e-mail: huwang@stat.nthu.edu.tw

Y.-H. Tina Wang
e-mail: d937004@oz.nthu.edu.tw

Y. H. Steve Huang
Department of Mathematics, Tamkang University, Tamsui, Taiwan
e-mail: huang@math.tku.edu.tw

1 Introduction

Consider an ordinary errors-in-variables model

$$Y_i = \beta_0 + \mathbf{u}_i'\boldsymbol{\beta}_1 + \epsilon_i, \quad \mathbf{Z}_i = \mathbf{u}_i + \boldsymbol{\tau}_i, \ \ i = 1, \ldots, n(> p+1), \tag{1}$$

where $\mathbf{u}_i = (u_{i1}, \ldots, u_{ip})'$ are the true but unobserved explanatory variables, $\boldsymbol{\beta} = (\beta_0, \ \boldsymbol{\beta}_1')' = (\beta_0, \beta_1, \ldots, \beta_p)'$ is the coefficient parameter, and ϵ_i, which may consist of both measurement and equation errors, are i.i.d. $N(0, \sigma_\epsilon^2)$, $\sigma_\epsilon^2 > 0$. It is assumed that the measurement errors $\boldsymbol{\tau}_i = (\tau_{i1}, \ldots, \tau_{ip})'$ in $\mathbf{Z}_i$ are i.i.d. $N(\mathbf{0}, \Sigma_{\tau\tau})$, where $\Sigma_{\tau\tau}$ is known and $\boldsymbol{\tau}_i$ are independent of $\mathbf{u}_i$ and ϵ_i. The knowledge of $\Sigma_{\tau\tau}$ can come from either a set of independent data or the repeated observations made on $\mathbf{u}_i$. Let $\boldsymbol{\theta} = (\boldsymbol{\beta}', \boldsymbol{\phi}')' \in \Omega$ be the vector consisting of all parameters, where $\boldsymbol{\beta}$ is the parameter of interest, $\boldsymbol{\phi}$ is the nuisance parameter which consists of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ in the functional model (see definition below) and σ_ϵ^2, and Ω is the corresponding parameter space. In model (1) if $\mathbf{u}_i$ are fixed unknown parameters, it is said to have a functional model. On the other hand, if $\mathbf{u}_i$ are random variables, (1) is called a structural model. Errors-in-variables model arises in many applications. Surveys of results can be found in Moran (1971), Kendall and Stuart (1979), Anderson (1984), Fuller (1987), Cheng and Van Ness (1994), Carroll et al. (1995), etc.

In this paper, we consider the problem of testing the hypotheses $H_{10} : \boldsymbol{\beta} = \boldsymbol{\beta}^*$ versus $H_{11} : \boldsymbol{\beta} \neq \boldsymbol{\beta}^*$ and $H_{20} : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_1^*$ versus $H_{21} : \boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_1^*$, where $\boldsymbol{\beta}^*$ and $\boldsymbol{\beta}_1^*$ are two arbitrarily specified vectors. Let $A(\boldsymbol{\beta}^*)$ be the acceptance region of a level α test for testing $H_{10} : \boldsymbol{\beta} = \boldsymbol{\beta}^*$ versus $H_{11} : \boldsymbol{\beta} \neq \boldsymbol{\beta}^*$. Then the true level of this test is defined by

$$\alpha_n(\boldsymbol{\theta}^*) = P_{\boldsymbol{\theta} = \boldsymbol{\theta}^*}[(\mathbf{Y}', \mathbf{Z}')' \notin A(\boldsymbol{\beta}^*)], \tag{2}$$

where $\boldsymbol{\theta}^* = (\boldsymbol{\beta}^{*\prime}, \boldsymbol{\phi}')'$ and $(\mathbf{Y}', \mathbf{Z}') = (Y_1, \ldots, Y_n, \mathbf{Z}_1', \ldots, \mathbf{Z}_n')$. A test for H_{10} is said to be uniformly robust over the null parameter space Ω^* if

$$\alpha_n = \sup_{\boldsymbol{\theta}^* \in \Omega^*} \alpha_n(\boldsymbol{\theta}^*) \to \alpha \quad \text{as } n \to \infty, \tag{3}$$

where Ω^* is a subspace of Ω with $\boldsymbol{\beta}$ being fixed at $\boldsymbol{\beta}^*$. The above $\sup_{\boldsymbol{\theta}^* \in \Omega^*} \alpha_n(\boldsymbol{\theta}^*)$ is usually called the size of the test H_{10}. It is interesting that for any fixed sample size n (no matter how large), there exist infinitely many $\Omega^{*\prime}s$ (each different $\boldsymbol{\beta}^*$ of $\boldsymbol{\beta}$ corresponds to a different Ω^*) such that for testing H_{10} versus H_{11} (or H_{20} versus H_{21}), all existing large-sample tests have $\sup_{\boldsymbol{\theta}^* \in \Omega^*} \alpha_n(\boldsymbol{\theta}^*)$ close to 1. The reason for this phenomenon is due to a general theorem in the interval estimation of (1) by Gleser and Hwang (1987). For each sample point $(\mathbf{y}', \mathbf{z}')'$, let $S[(\mathbf{y}', \mathbf{z}')']$ denote the set of parameter values

$$S[(\mathbf{y}', \mathbf{z}')'] = \{\boldsymbol{\beta} : (\mathbf{y}', \mathbf{z}')' \in A(\boldsymbol{\beta}), \boldsymbol{\theta} \in \Omega\}. \tag{4}$$

Then it is true (Lehmann 1986, p. 91, (18)) that

$$P_{\boldsymbol{\theta}}\{\boldsymbol{\beta} \in S[(\mathbf{Y}', \mathbf{Z}')']\} = P_{\boldsymbol{\theta}}\{(\mathbf{Y}', \mathbf{Z}')' \in A(\boldsymbol{\beta})\}. \tag{5}$$

Gleser and Hwang showed that for errors-in-variables models, Fieller problem, and many other problems, any almost surely finite diameter confidence set has a zero confidence coefficient for a fixed sample size n, no matter how large it is. Here the diameter of a confidence set is defined to be the supremum distance between any two points in this set and the confidence coefficient is defined to be the infimum of the coverage probabilities over the parameter space. Since all existing large-sample confidence sets (corresponding to large-sample tests) for $\boldsymbol{\beta}$ have finite diameters almost surely, they have zero confidence coefficients, i.e., $\inf_{\boldsymbol{\theta}\in\Omega} P_{\boldsymbol{\theta}}\{\boldsymbol{\beta} \in S[(\mathbf{Y}', \mathbf{Z}')']\} = 0$ for all finite diameter confidence sets $S[(\mathbf{Y}', \mathbf{Z}')']$. Consequently, for all existing large-sample tests, by (5) we have

$$1 - \sup_{\boldsymbol{\theta}\in\Omega} \alpha_n(\boldsymbol{\theta}) = 1 - \sup_{\boldsymbol{\theta}\in\Omega} P_{\boldsymbol{\theta}}\{(\mathbf{Y}', \mathbf{Z}')' \notin A(\boldsymbol{\beta})\} = \inf_{\boldsymbol{\theta}\in\Omega} P_{\boldsymbol{\theta}}\{\boldsymbol{\beta} \in S[(\mathbf{Y}', \mathbf{Z}')']\} = 0.$$

Note that the above $\sup_{\boldsymbol{\theta}\in\Omega} \alpha_n(\boldsymbol{\theta})$ is not the size of a large-sample test since the parameter $\boldsymbol{\beta}$ can vary in the parameter space Ω. Due to the above result that $\sup_{\boldsymbol{\theta}\in\Omega} \alpha_n(\boldsymbol{\theta}) = 1$, there exist infinitely many $\Omega^{*'}s$ defined in (3) such that the size $\sup_{\boldsymbol{\theta}^*\in\Omega^*} \alpha_n(\boldsymbol{\theta}^*)$ of the test $H_{10} : \boldsymbol{\beta} = \boldsymbol{\beta}^*$ is close to 1 no matter how large the sample size n is. However, it is worth noting that we are unable to prove $\sup_{\boldsymbol{\theta}^*\in\Omega^*} \alpha_n(\boldsymbol{\theta}^*) = 1$ for a large-sample test by using Gleser and Hwang's theorem since $\boldsymbol{\beta}^*$ is a fixed vector in Ω^*. Also note that Huwang (1996) considered the simplest ($p = 1$) structural errors-in-variables model of (1) and constructed a confidence set for $\boldsymbol{\beta}$ which does not have finite diameter almost surely and hence Gleser and Hwang's theorem does not apply. Huwang's confidence set is shown to have the infimum coverage probability converging to the nominal level uniformly over the parameter space Ω. Therefore, the corresponding test does not have $\sup_{\boldsymbol{\theta}\in\Omega} \alpha_n(\boldsymbol{\theta}) = 1$. The present work proposes new tests for testing the main parameters $\boldsymbol{\beta}$ and $\boldsymbol{\beta}_1$ in the functional errors-in-variables model for any dimension p (structural model can be dealt similarly). These tests are shown to satisfy (3) and hence are uniformly robust over the null parameter space Ω^*. By (5), the confidence set can be obtained from the totality of parameter values for which the hypothesis is accepted. As a result, the proposed tests can be used to construct confidence sets which have confidence coefficients converging to the nominal levels as $n \to \infty$. Therefore, Huwang's (1996) result for $p = 1$ in the structural model is a special case of the current work. Based on the theorem of Gleser and Hwang, the issue of lack of uniform robustness does not only exist in the case where $\Sigma_{\tau\tau}$ is known but also in other assumptions of identifying the model (1). In this paper, we only consider the case where $\Sigma_{\tau\tau}$ is known as the form of identifying the model and use the approach that starts with a consistent estimator of the key parameter concerned and then calculate the variance of a function of the consistent estimator and the parameter. We then derive some test statistic by dividing the function of the consistent estimator and the parameter by its estimated variance. However, the crucial point is not to estimate the key parameter in the denominator of the ratio, but to use the hypothesized value of the test, which can be attributed to Huwang (1996) or more originally to Hwang (1995) although different models are considered there. As for the other forms of identifying the model (1), for example, if the model assumes that the ratios of the variances of ϵ_i and τ_{ij}, $1 \le j \le p$, are known, it might be very difficult to use the above approach to prove uniform robustness.

The rest of the paper is organized as follows. Section 2 proposes two uniformly robust tests for testing $H_{10} : \boldsymbol{\beta} = \boldsymbol{\beta}^*$ versus $H_{11} : \boldsymbol{\beta} \neq \boldsymbol{\beta}^*$ and $H_{20} : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_1^*$ versus $H_{21} : \boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_1^*$ in the functional errors-in-variables model. In Sect. 3, simulation studies are employed to compare the true test levels and power of some pointwise robust tests and the proposed uniformly robust tests. A brief conclusion is given in Sect. 4. Some technical proofs are in the Appendix.

2 Main results

2.1 Functional model

In this subsection, we assume that the $\mathbf{u}_i$ in (1) are fixed unknown parameters. Namely, we have a functional model. From now on, let $\boldsymbol{\delta}_i = (0, \boldsymbol{\tau}_i')'$, $\mathbf{x}_i = (1, \mathbf{u}_i')'$, and $\mathbf{W}_i = (1, \mathbf{Z}_i')' = \mathbf{x}_i + \boldsymbol{\delta}_i$. We also let

$$\mathbf{M}_{YY} = \frac{1}{n}\sum_1^n Y_i^2, \quad \mathbf{M}_{WW} = \frac{1}{n}\sum_1^n \mathbf{W}_i\mathbf{W}_i', \quad \mathbf{M}_{xx} = \frac{1}{n}\sum_1^n \mathbf{x}_i\mathbf{x}_i',$$
$$\mathbf{M}_{WY} = \frac{1}{n}\sum_1^n \mathbf{W}_i' Y_i, \quad \mathbf{M}_{YW} = \frac{1}{n}\sum_1^n Y_i\mathbf{W}_i, \tag{6}$$

where $\mathbf{M}_{xx}$ is assumed to be nonsingular and $\lim_{n\to\infty}\mathbf{M}_{xx} = \mathbf{M}^0$, a nonsingular matrix. Then, it is easy to show that

$$E\begin{pmatrix} \mathbf{M}_{YY} & \mathbf{M}_{WY} \\ \mathbf{M}_{YW} & \mathbf{M}_{WW} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\beta}'\mathbf{M}_{xx}\boldsymbol{\beta} + \sigma_\epsilon^2 & \boldsymbol{\beta}'\mathbf{M}_{xx} \\ \mathbf{M}_{xx}\boldsymbol{\beta} & \mathbf{M}_{xx} + \Sigma_{\delta\delta} \end{pmatrix}, \tag{7}$$

where $\Sigma_{\delta\delta}$ is the known covariance matrix of $\boldsymbol{\delta}_i$ (since $\Sigma_{\tau\tau}$ is known). Subsequently, a reasonable consistent estimator of $\boldsymbol{\beta}$ can be defined as

$$\hat{\boldsymbol{\beta}} = (\mathbf{M}_{WW} - \Sigma_{\delta\delta})^{-1}\mathbf{M}_{YW}. \tag{8}$$

Now, let $v_i = \epsilon_i - \boldsymbol{\delta}_i'\boldsymbol{\beta} = Y_i - \mathbf{W}_i'\boldsymbol{\beta}$ and $\mathbf{M}_{vW} = n^{-1}\sum_1^n v_i\mathbf{W}_i = \mathbf{M}_{YW} - \mathbf{M}_{WW}\boldsymbol{\beta}$. By (8) and $E\mathbf{M}_{vW} = -\Sigma_{\delta\delta}\boldsymbol{\beta}$, we have

$$\begin{aligned}\hat{\boldsymbol{\beta}} &= (\mathbf{M}_{WW} - \Sigma_{\delta\delta})^{-1}[(\mathbf{M}_{WW} - \Sigma_{\delta\delta})\boldsymbol{\beta} + \mathbf{M}_{YW} - (\mathbf{M}_{WW} - \Sigma_{\delta\delta})\boldsymbol{\beta}] \\ &= \boldsymbol{\beta} + (\mathbf{M}_{WW} - \Sigma_{\delta\delta})^{-1}[\mathbf{M}_{YW} - (\mathbf{M}_{WW} - \Sigma_{\delta\delta})\boldsymbol{\beta}],\end{aligned} \tag{9}$$

and

$$(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \mathbf{M}_{vW} - E\mathbf{M}_{vW} = \frac{1}{n}\sum_{i=1}^{n}[v_i\mathbf{W}_i - E(v_i\mathbf{W}_i)]. \quad (10)$$

Now for testing $H_{10} : \boldsymbol{\beta} = \boldsymbol{\beta}^*$ versus $H_{11} : \boldsymbol{\beta} \neq \boldsymbol{\beta}^*$, we shall construct a uniformly robust test as follows. Firstly, we calculate the covariance matrix of $(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$, which will be denoted by $\mathbf{Q}/n$. Secondly, we show that

$$n(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'(\mathbf{M}_{WW} - \Sigma_{\delta\delta})\mathbf{Q}^{-1}(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{L} \chi^2_{p+1} \quad (11)$$

uniformly over the parameter space Ω. And finally, a matrix $\hat{\mathbf{Q}}$, which is a function of $Y_1, \ldots, Y_n, \mathbf{W}_1, \ldots, \mathbf{W}_n$, and $\boldsymbol{\beta}$, will be defined and

$$n(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\mathbf{Q}}^{-1} - \mathbf{Q}^{-1})(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{P} 0$$

uniformly over Ω will be shown. Combining all these three parts, we have

$$n(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'(\mathbf{M}_{WW} - \Sigma_{\delta\delta})\hat{\mathbf{Q}}^{-1}(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{L} \chi^2_{p+1}$$

uniformly over the parameter space Ω. From this result, it is easy to construct a test for $H_{10} : \boldsymbol{\beta} = \boldsymbol{\beta}^*$ versus $H_{11} : \boldsymbol{\beta} \neq \boldsymbol{\beta}^*$ which satisfies (3), namely a uniformly robust test over the null parameter space Ω^* (see Theorem 2 for details).

By definition, the covariance matrix of $v_i\mathbf{W}_i - E(v_i\mathbf{W}_i)$ equals

$$\begin{aligned}
\mathrm{Cov}[v_i\mathbf{W}_i - E(v_i\mathbf{W}_i)] &= E[(v_i\mathbf{W}_i - E(v_i\mathbf{W}_i))(v_i\mathbf{W}_i - E(v_i\mathbf{W}_i))'] \\
&= E[(v_i\mathbf{x}_i + v_i\boldsymbol{\delta}_i - \Sigma_{v\delta})(v_i\mathbf{x}_i + v_i\boldsymbol{\delta}_i - \Sigma_{v\delta})'] \\
&= \sigma_v^2\mathbf{x}_i\mathbf{x}_i' + E(v_i^2\mathbf{x}_i\boldsymbol{\delta}_i') + E(v_i^2\boldsymbol{\delta}_i\mathbf{x}_i') + E(v_i^2\boldsymbol{\delta}_i\boldsymbol{\delta}_i') - \Sigma_{v\delta}\Sigma_{v\delta}' \\
&= \sigma_v^2\mathbf{x}_i\mathbf{x}_i' + E(v_i^2\boldsymbol{\delta}_i\boldsymbol{\delta}_i') - \Sigma_{v\delta}\Sigma_{v\delta}', \qquad (12)
\end{aligned}$$

where $\Sigma_{v\delta} = E(v_i\boldsymbol{\delta}_i) = -\Sigma_{\delta\delta}\boldsymbol{\beta}$ and σ_v^2 is the variance of v_i. In the right side of the last second line in (12), $E(v_i^2\mathbf{x}_i\boldsymbol{\delta}_i')$ and $E(v_i^2\boldsymbol{\delta}_i\mathbf{x}_i')$ are zero matrices because each entry of them is of the form

$$E(x_{ij}\delta_{ij'}v_i^2) = x_{ij}E(\delta_{ij'}v_i^2) = 0, \quad 1 \leq j, j' \leq p+1,$$

due to that $(\boldsymbol{\delta}_i', v_i)'$ has a multivariate normal distribution

$$\begin{pmatrix} \boldsymbol{\delta}_i \\ v_i \end{pmatrix} \sim N\left[\mathbf{0}, \begin{pmatrix} \Sigma_{\delta\delta} & \Sigma_{v\delta} \\ \Sigma_{v\delta}' & \sigma_v^2 \end{pmatrix}\right] \quad (13)$$

and hence $(\delta_{ij'}, v_i)'$ has a bivariate normal distribution with zero third moment. Subsequently, the conditional distribution of $\boldsymbol{\delta}_i$ given v_i is

$$\boldsymbol{\delta}_i \mid v_i \sim N(\mathbf{a}v_i, \mathbf{B}), \tag{14}$$

where $\mathbf{a} = \Sigma_{v\delta}/\sigma_v^2 = -\Sigma_{\delta\delta}\boldsymbol{\beta}/\sigma_v^2$ and $\mathbf{B} = \Sigma_{\delta\delta} - \Sigma_{v\delta}\Sigma'_{v\delta}/\sigma_v^2 = \Sigma_{\delta\delta} - \Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}/\sigma_v^2$. Based on (14), we have

$$\begin{aligned}
E(v_i^2\boldsymbol{\delta}_i\boldsymbol{\delta}_i') &= E\begin{bmatrix} v_i^2\delta_{i1}^2 & v_i^2\delta_{i1}\delta_{i2} & \cdots & v_i^2\delta_{i1}\delta_{i(p+1)} \\ \cdot & v_i^2\delta_{i2}^2 & \cdots & v_i^2\delta_{i2}\delta_{i(p+1)} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & v_i^2\delta_{i(p+1)}^2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_v^2\sigma_{11} + 2\phi_{v1}^2, & \sigma_v^2\sigma_{12} + 2\phi_{v1}\phi_{v2}, & \cdots & \sigma_v^2\sigma_{1\,(p+1)} + 2\phi_{v1}\phi_{v(p+1)} \\ \cdot & \sigma_v^2\sigma_{22} + 2\phi_{v2}^2, & \cdots & \sigma_v^2\sigma_{2\,(p+1)} + 2\phi_{v2}\phi_{v(p+1)} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \sigma_v^2\sigma_{(p+1)(p+1)} + 2\phi_{v(p+1)}^2 \end{bmatrix} \\
&= \sigma_v^2\Sigma_{\delta\delta} + 2\Sigma_{v\delta}\Sigma'_{v\delta},
\end{aligned} \tag{15}$$

where σ_{ij} is the (i, j)th element of $\Sigma_{\delta\delta}$, $i, j = 1, \ldots, p+1$ and ϕ_{vk} is the kth element of $\Sigma_{v\delta}$, $k = 1, \ldots, p+1$. Substituting (15) into (12), we have

$$\text{Cov}[v_i\mathbf{W}_i - E(v_i\mathbf{W}_i)] = \sigma_v^2(\mathbf{x}_i\mathbf{x}_i' + \Sigma_{\delta\delta}) + \Sigma_{v\delta}\Sigma'_{v\delta}.$$

Subsequently, the covariance matrix of $(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ in (10) is given by

$$\begin{aligned}
\text{Cov}[(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})] &= \frac{1}{n}[\sigma_v^2(\mathbf{M}_{xx} + \Sigma_{\delta\delta}) + \Sigma_{v\delta}\Sigma'_{v\delta}] \\
&= \frac{1}{n}[\sigma_v^2(\mathbf{M}_{xx} + \Sigma_{\delta\delta}) + \Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}] = \frac{\mathbf{Q}}{n},
\end{aligned} \tag{16}$$

where $\mathbf{Q} = \sigma_v^2(\mathbf{M}_{xx} + \Sigma_{\delta\delta}) + \Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}$ is positive definite since $\mathbf{M}_{xx}$ is so.

To prove (11), it suffices to show that $\mathbf{Q}^{-\frac{1}{2}}\mathbf{S} \xrightarrow{L} N(\mathbf{0}, \mathbf{I}_{p+1})$ uniformly over the parameter space Ω, where $\mathbf{S} = \sqrt{n}(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ and $\mathbf{I}_{p+1}$ is the $p+1$ by $p+1$ identity matrix. Because a moment generating function uniquely determines a cumulative distribution function, and vice versa if the moment generating function exists, it is straightforward to show that $\mathbf{Q}^{-\frac{1}{2}}\mathbf{S} \xrightarrow{L} N(\mathbf{0}, \mathbf{I}_{p+1})$ uniformly over the parameter space Ω if the moment generating function of $\mathbf{Q}^{-\frac{1}{2}}\mathbf{S}$

$$M(\mathbf{t}) = E[\exp(\mathbf{S}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t})] \rightarrow e^{\mathbf{t}'\mathbf{t}/2} \tag{17}$$

uniformly over the parameter space Ω for any $\mathbf{t} \in R^{p+1}$. So we will prove (17) in the following.

Theorem 1 *Under the model* (1), *assume that a functional errors-in-variables model holds with* $\mathbf{M}_{xx}$ *being nonsingular. Then the moment generating function* $M(\mathbf{t}) = E[\exp(\mathbf{S}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t})] \to e^{\mathbf{t}'\mathbf{t}/2}$ *uniformly over the parameter space* Ω *for any* $\mathbf{t} \in R^{p+1}$. *Consequently,* $\mathbf{Q}^{-\frac{1}{2}}\mathbf{S} \xrightarrow{L} N(\mathbf{0}, \mathbf{I}_{p+1})$ (*or* (11) *holds*) *uniformly over the parameter space* Ω.

Proof By the definition of $\mathbf{S}$ and (10),

$$
\begin{aligned}
M(\mathbf{t}) &= E\left\{\exp\left\{\frac{1}{\sqrt{n}}\sum_1^n[v_i(\mathbf{x}_i' + \boldsymbol{\delta}_i') - E(v_i(\mathbf{x}_i' + \boldsymbol{\delta}_i'))]\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\right\}\right\} \\
&= \prod_1^n E\{E\{\exp[(v_i\mathbf{x}_i' + v_i\boldsymbol{\delta}_i' - \mathbf{c})\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}/\sqrt{n}] \mid v_i\}\} \\
&= \prod_1^n E\{\exp[(\mathbf{x}_i'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v/\sqrt{n})v_i^* + (\mathbf{a}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v^2/\sqrt{n} \\
&\quad + \mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v^2/2n)v_i^{*2} - \mathbf{c}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}/\sqrt{n}]\},
\end{aligned} \tag{18}
$$

where $v_i^* = v_i/\sigma_v$, $\mathbf{a} = -\Sigma_{\delta\delta}\boldsymbol{\beta}/\sigma_v^2$, $\mathbf{B} = \Sigma_{\delta\delta} - \Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}/\sigma_v^2$ and $\mathbf{c} = E(v_i\boldsymbol{\delta}_i') = E[v_iE(\boldsymbol{\delta}_i'|v_i)] = E(\mathbf{a}'v_i^2) = \mathbf{a}'\sigma_v^2$. Here the third equality of (18) and $\mathbf{c}$ are obtained based on (13), (14), and the moment generating function of $\boldsymbol{\delta}_i|v_i \sim N(\mathbf{a}v_i, \mathbf{B})$. Define

$$
\eta = \frac{\mathbf{a}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v^2}{\sqrt{n}} + \frac{\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v^2}{2n}. \tag{19}
$$

From (18), we have

$$
\begin{aligned}
M(\mathbf{t}) &= \prod_1^n E\{\exp\{\eta[v_i^* + \mathbf{x}_i'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v/(2\sqrt{n}\eta)]^2 \\
&\quad -\sigma_v^2\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{x}_i\mathbf{x}_i'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}/(4n\eta) - \mathbf{c}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}/\sqrt{n}\}\} \\
&= \prod_1^n\{(1-2\eta)^{-\frac{1}{2}}\exp\{\sigma_v^2\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{x}_i\mathbf{x}_i'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}/[2n(1-2\eta)] - \mathbf{c}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}/\sqrt{n}\}\} \\
&= \exp\left\{-\frac{n}{2}\log(1-2\eta) + \sigma_v^2\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{M}_{xx}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}/[2(1-2\eta)] - \sqrt{n}\mathbf{c}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\right\},
\end{aligned} \tag{20}
$$

where the second equality is based on the fact that $[v_i^* + \mathbf{x}_i'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v/(2\sqrt{n}\eta)]^2$ has a noncentral chi-square distribution with one degree of freedom, noncentrality $\xi = \sigma_v^2\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{x}_i\mathbf{x}_i'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}/(4n\eta^2)$, and moment generating function $(1-2\eta)^{-1/2}\exp\{-\xi/2+$

$\xi/[2(1-2\eta)]\}$. By a Taylor expansion,

$$\begin{aligned}\frac{n}{2}\log(1-2\eta) &= \frac{n}{2}\left(-2\eta-2\eta^2-\frac{8}{3}\eta^{*3}\right)\\ &= -\sqrt{n}\mathbf{a}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v^2-\frac{1}{2}\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v^2-\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{a}\mathbf{a}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v^4\\ &\quad -\frac{1}{\sqrt{n}}\mathbf{a}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v^4-\frac{1}{4n}(\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v^2)^2-\frac{4}{3}n\eta^{*3},\end{aligned} \tag{21}$$

where η^* is a number and $0 \le |\eta^*| \le |\eta|$. Plugging (21) in the right side of the last equality in (20), we have

$$\begin{aligned}M(\mathbf{t}) = \exp\Bigg\{&\frac{1}{2}\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{Q}^*\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}+\frac{1}{\sqrt{n}}\mathbf{a}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v^4\\ &+\frac{1}{4n}(\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v^2)^2+\frac{4}{3}n\eta^{*3}\Bigg\},\end{aligned} \tag{22}$$

where

$$\mathbf{Q}^* = \sigma_v^2\left(\mathbf{B}+2\sigma_v^2\mathbf{a}\mathbf{a}'+\frac{\mathbf{M}_{xx}}{1-2\eta}\right) = \sigma_v^2\left(\frac{\mathbf{M}_{xx}}{1-2\eta}+\Sigma_{\delta\delta}\right)+\Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}. \tag{23}$$

Now by Lemmas 1 and 2 in the Appendix and that $n\mid\eta^{*3}\mid\le n\mid\eta^3\mid = O_p(n^{-1/2})\to 0$, for any fixed $\mathbf{t}\in R^{p+1}$,

$$\exp\left\{\frac{1}{\sqrt{n}}\mathbf{a}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v^4+\frac{1}{4n}(\mathbf{t}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\mathbf{t}\sigma_v^2)^2+\frac{4}{3}n\eta^{*3}\right\}\to 1$$

uniformly over the parameter space Ω. As a consequence, to show $M(\mathbf{t})\to e^{\mathbf{t}'\mathbf{t}/2}$ uniformly over the parameter space Ω, it suffices to prove that $\mathbf{Q}^{-\frac{1}{2}}\mathbf{Q}^*\mathbf{Q}^{-\frac{1}{2}}\to\mathbf{I}_{p+1}$ uniformly over the parameter space Ω. Comparing the matrices $\mathbf{Q}$ and $\mathbf{Q}^*$ in (16) and (23), respectively, this is equivalent to show that $\eta\to 0$ uniformly over the parameter space Ω. By the definition of η and Lemmas 1 and 2 in the Appendix, the result follows immediately. □

By the result of Theorem 1, which is equivalent to (11), we construct a uniformly robust test for testing $H_{10}:\boldsymbol{\beta}=\boldsymbol{\beta}^*$ versus $H_{11}:\boldsymbol{\beta}\ne\boldsymbol{\beta}^*$.

Theorem 2 *Under the model* (1)*, assume that a functional errors-in-variables model holds with* $\mathbf{M}_{xx}$ *being nonsingular. Then*

$$\Gamma_1 = n(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'(\mathbf{M}_{WW}-\Sigma_{\delta\delta})\hat{\mathbf{Q}}^{-1}(\mathbf{M}_{WW}-\Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})\xrightarrow{L}\chi^2_{p+1} \tag{24}$$

uniformly over the parameter space Ω*, where*

$$\hat{\mathbf{Q}} = \hat{\sigma}_v^2 \mathbf{M}_{WW} + \Sigma_{\delta\delta} \boldsymbol{\beta}\boldsymbol{\beta}' \Sigma_{\delta\delta}, \ \hat{\sigma}_v^2 = \frac{1}{n} \sum_1^n (Y_i - \mathbf{W}_i' \boldsymbol{\beta})^2. \tag{25}$$

Consequently, for testing $H_{10} : \boldsymbol{\beta} = \boldsymbol{\beta}^*$ *versus* $H_{11} : \boldsymbol{\beta} \neq \boldsymbol{\beta}^*$*, the test which rejects* H_{10} *if* $\Gamma_1|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*} > \chi^2_{p+1,\alpha}$*, where* $P(\chi^2_{p+1} > \chi^2_{p+1,\alpha}) = \alpha$ *is a level* α *uniformly robust test over the null parameter space* Ω^**.*

Proof To show (24), by Theorem 1 it suffices to prove that

$$\mathbf{S}'(\hat{\mathbf{Q}}^{-1} - \mathbf{Q}^{-1})\mathbf{S} = \mathbf{S}'\mathbf{Q}^{-\frac{1}{2}}\mathbf{Q}^{\frac{1}{2}}(\hat{\mathbf{Q}}^{-1} - \mathbf{Q}^{-1})\mathbf{Q}^{\frac{1}{2}}\mathbf{Q}^{-\frac{1}{2}}\mathbf{S} \xrightarrow{P} 0 \tag{26}$$

uniformly over the parameter space Ω, where $\mathbf{S} = \sqrt{n}(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$. Also by Theorem 1, (26) is established if we can show that

$$\mathbf{Q}^{\frac{1}{2}}(\hat{\mathbf{Q}}^{-1} - \mathbf{Q}^{-1})\mathbf{Q}^{\frac{1}{2}} = \mathbf{Q}^{\frac{1}{2}}\hat{\mathbf{Q}}^{-1}\mathbf{Q}^{\frac{1}{2}} - \mathbf{I}_{p+1} \xrightarrow{P} \mathbf{0},$$

or equivalently

$$\mathbf{Q}^{-\frac{1}{2}}\hat{\mathbf{Q}}\mathbf{Q}^{-\frac{1}{2}} - \mathbf{I}_{p+1} = \mathbf{Q}^{-\frac{1}{2}}(\hat{\mathbf{Q}} - \mathbf{Q})\mathbf{Q}^{-\frac{1}{2}} \xrightarrow{P} \mathbf{0}$$

uniformly over the parameter space Ω. Since $n\hat{\sigma}_v^2/\sigma_v^2 \sim \chi_n^2$, we have $\hat{\sigma}_v^2 = \sigma_v^2 + \sigma_v^2 O_p(n^{-\frac{1}{2}})$, where $O_p(n^{-\frac{1}{2}})$ does not depend on any parameters. Subsequently, by (16) and (25) we have

$$\begin{aligned}\mathbf{Q}^{-\frac{1}{2}}(\hat{\mathbf{Q}} - \mathbf{Q})\mathbf{Q}^{-\frac{1}{2}} &= O_p(n^{-\frac{1}{2}})\sigma_v^2\mathbf{Q}^{-\frac{1}{2}}\mathbf{M}_{WW}\mathbf{Q}^{-\frac{1}{2}} \\ &\quad + \sigma_v^2\mathbf{Q}^{-\frac{1}{2}}[\mathbf{M}_{WW} - (\mathbf{M}_{xx} + \Sigma_{\delta\delta})]\mathbf{Q}^{-\frac{1}{2}}.\end{aligned}$$

Using the results of Lemmas 3 and 4 in the Appendix, the theorem follows. □

Remark 1 In fact, the result of Theorem 2 which shows that both $inf_{\boldsymbol{\theta}\in\Omega}\,\alpha_n(\boldsymbol{\theta})$ and $\sup_{\boldsymbol{\theta}\in\Omega}\alpha_n(\boldsymbol{\theta})$ converge to α as $n \to \infty$ is stronger than (3). Namely, the convergence of $\alpha_n(\boldsymbol{\theta})$ is not only uniform in all parameters in Ω^* (where $\boldsymbol{\beta}$ is fixed at $\boldsymbol{\beta}^*$) but also uniform in the parameter $\boldsymbol{\beta}$.

Note that $\hat{\mathbf{Q}}$ in (25) is a "pseudo" estimator of $\mathbf{Q}$ because the parameter $\boldsymbol{\beta}$ therein is not estimated. Since it is difficult to accurately estimate $\boldsymbol{\beta}$ over the parameter space Ω, leaving $\boldsymbol{\beta}$ unestimated in $\hat{\mathbf{Q}}$ would reduce the error in "estimating" $\mathbf{Q}$. Also note that $\hat{\mathbf{Q}}$ is nonsingular almost surely since $\mathbf{M}_{WW}$ is so.

In Theorem 3 below we show that the power of the uniformly robust test proposed in Theorem 2 goes to 1 asymptotically.

Theorem 3 *For testing H_{10} : $\boldsymbol{\beta} = \boldsymbol{\beta}^*$ versus H_{11} : $\boldsymbol{\beta} \neq \boldsymbol{\beta}^*$, at any $\boldsymbol{\beta} = \boldsymbol{\beta}^1 \neq \boldsymbol{\beta}^*$, the test that rejects H_{10} if $\Gamma_1|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*} > \chi^2_{p+1,\alpha}$ where Γ_1 is defined in (24) has power $P_{\boldsymbol{\beta}^1}(\Gamma_1|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*} > \chi^2_{p+1,\alpha}) \to 1$ as $n \to \infty$.*

Proof See Appendix. □

Now we proceed to construct a uniformly robust test for testing the hypothesis H_{20} : $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_1^*$ versus H_{21} : $\boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_1^*$. Let $\hat{\sigma}_v'^2 = (n-1)^{-1}\sum_{i=1}^n [Y_i - \bar{Y} - \sum_{j=1}^p \beta_j (W_{i(j+1)} - \bar{W}_{\cdot j+1})]^2$, where $\bar{W}_{\cdot j+1} = n^{-1}\sum_{i=1}^n W_{i(j+1)}$, $j = 0, \ldots, p$. Then

$$\hat{\sigma}_v'^2 = \frac{1}{n-1}\sum_{i=1}^n [Y_i - \mathbf{W}_i'\boldsymbol{\beta} - (\bar{Y} - \bar{\mathbf{W}}'\boldsymbol{\beta})]^2 = \frac{1}{n-1}\sum_{i=1}^n (v_i - \bar{v})^2, \tag{27}$$

where $\bar{\mathbf{W}} = (\bar{W}_{\cdot 1}, \ldots, \bar{W}_{\cdot p+1})'$, $v_i = Y_i - \mathbf{W}_i'\boldsymbol{\beta}$, and $\bar{v} = n^{-1}\sum_1^n v_i$ (cf. $\hat{\sigma}_v^2 = n^{-1}\sum_1^n v_i^2$ defined in (25)). Note that $(n-1)\hat{\sigma}_v'^2/\sigma_v^2 \sim \chi^2_{n-1}$ and $\hat{\sigma}_v'^2$ is a function of $\boldsymbol{\beta}_1 = (\beta_1, \ldots, \beta_p)'$ only, not of β_0. Also, since the first entry of $\boldsymbol{\delta}_i$ is 0, $\Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}$ is a function of $\boldsymbol{\beta}_1$ only, not of β_0. As a result,

$$\hat{\mathbf{Q}}' = \hat{\sigma}_v'^2 \mathbf{M}_{WW} + \Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta} \tag{28}$$

is a function of $\boldsymbol{\beta}_1$ only, not of β_0. Now, by an argument similar to that of Theorem 2, the following corollary can be easily established.

Corollary 4 *Suppose that in (1) we have a functional errors-in-variables model with $\mathbf{M}_{xx}$ being nonsingular. Then as $n \to \infty$,*

$$\Gamma_2 = n(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'(\mathbf{M}_{WW} - \Sigma_{\delta\delta})\hat{\mathbf{Q}}'^{-1}(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{L} \chi^2_{p+1} \tag{29}$$

uniformly over the parameter space Ω. □

Since $\hat{\mathbf{Q}}'$ and $\mathbf{M}_{WW} - \Sigma_{\delta\delta}$ are both nonsingular almost surely, let

$$(\mathbf{M}_{WW} - \Sigma_{\delta\delta})\hat{\mathbf{Q}}'^{-1}(\mathbf{M}_{WW} - \Sigma_{\delta\delta}) = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{pmatrix}^{-1},$$

where $\mathbf{R}_{22}$ is a p by p matrix. By this and Corollary 4 the following theorem holds immediately.

Theorem 5 *Under the same assumptions of Corollary 4,*

$$\Gamma_3 = n(\hat{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_1)'\mathbf{R}_{22}^{-1}(\hat{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_1) \xrightarrow{L} \chi^2_p \tag{30}$$

uniformly over the parameter space Ω, where $\boldsymbol{\beta}_1 = (\beta_1, \ldots, \beta_p)'$. Therefore, for testing H_{20} : $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_1^*$ versus H_{21} : $\boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_1^*$, the test which rejects H_{20} if $\Gamma_3|_{\boldsymbol{\beta}_1=\boldsymbol{\beta}_1^*} > \chi^2_{p,\alpha}$, where $P(\chi^2_p > \chi^2_{p,\alpha}) = \alpha$ is a level α uniformly robust test over the null parameter space Ω^*.

Remark 2 Under the model (1), suppose that $\mathbf{u}_i$, $i = 1, \ldots, n$, are i.i.d. vector random variables (i.e. consider a structural model) with nonsingular covariance matrix and continuous distribution function. Then the results similar to those in the functional model can be easily obtained.

2.2 Remarks about the uniformly robust tests for scalar parameters and subsets of β_1

Note that the approach used previously to construct the uniformly robust tests for testing the values of $\boldsymbol{\beta}$ and $\boldsymbol{\beta}_1$ can not be applied to test the hypothesis $H_{30} : \beta_0 = \beta_0^*$ versus $H_{31} : \beta_0 \neq \beta_0^*$. Due to the fact that the variance of $\hat{\beta}_0$ is a function of $\boldsymbol{\beta}_1$ and other parameters, not of β_0, if all parameters in this variance are estimated by their consistent estimators and an approximate pivot is derived by the usual way, the pivot will not converge to χ_1^2 uniformly over the parameter space Ω. This is due to the reason that the confidence interval for β_0 has an almost surely finite diameter and hence a zero confidence level by Gleser and Hwang's theorem.

For $1 \leq j \leq p$, to test the hypothesis $H_{40} : \beta_j = \beta_j^*$ versus $H_{41} : \beta_j \neq \beta_j^*$, note that the variance of $\hat{\beta}_j$ is a function of $\beta_1, \ldots, \beta_p$. If all parameters except β_j in this variance are estimated by their consistent estimators and an approximate pivot is derived, we conjecture that the resultant test is not uniformly robust although the corresponding confidence interval may not have an almost surely finite diameter. The reason for this conjecture is that in the functional model if for some k, $1 \leq k \neq j \leq p$, u_{ik}, $i = 1, \ldots, n$, do not vary much, it is difficult to accurately estimate β_k and hence the variance of $\hat{\beta}_j$. As a result, this could lead to an inferior test for $H_{40} : \beta_j = \beta_j^*$. A possible way to resolve this problem is to employ the projection method. Note that a conservative confidence interval for β_j (or β_0) can be constructed by projecting a uniformly robust confidence set for $\boldsymbol{\beta}$ onto the β_j (or β_0) axis. As a result, the corresponding test of this projected confidence interval is a conservative one. We will compare this projected confidence interval with some other confidence interval in the next section. Furthermore, it is also possible to use the projection method to construct a uniformly robust test for testing any subset of $\boldsymbol{\beta}_1$, which corresponds to the accurate unobserved explanatory variables while the complement corresponds to the inaccurate ones. However, it is still expected that the test is conservative. Consequently, to construct less conservative tests in such situations are needed in the future research.

3 Performance comparisons

In this section, we use statistical simulation to compare the performance of the pointwise robust test, the modified test (the former is given in Theorem 2.2.1 of Fuller (1987) and the later is a modification of the former suggested in Theorem 2.5.2 of the same book), and the uniformly robust test. We assume a structural model of (1)

$$Y_i = 1 + \mathbf{u}_i'(2, 3)' + \epsilon_i, \mathbf{Z}_i = \mathbf{u}_i + \boldsymbol{\tau}_i, \quad i = 1, \ldots, n \tag{31}$$

with $\mathbf{u}_i \sim N_2(\mathbf{0}, \Sigma_{uu})$, $\epsilon_i \sim N(0, \sigma_\epsilon^2)$, $\tau_i \sim N_2(\mathbf{0}, \Sigma_{\tau\tau})$, and $\Sigma_{\tau\tau} = \Sigma_{uu}(r^{-1} - 1)$

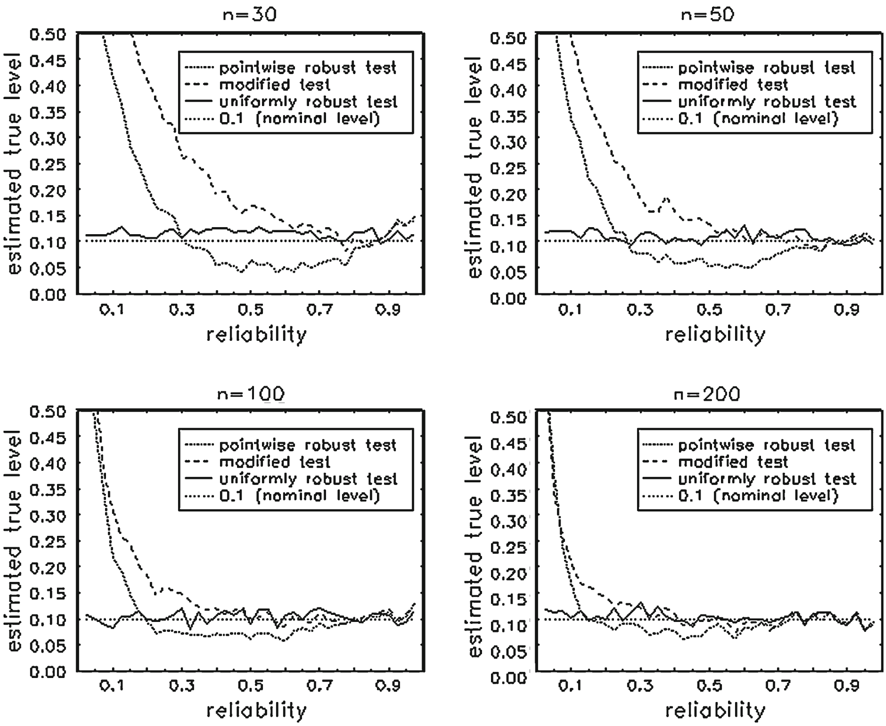

Fig. 1 Estimated true levels of the pointwise robust, modified and uniformly robust tests with nominal level $\alpha = 0.1$ and replication=1,000. Model (31) is assumed with $\Sigma_{uu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_\epsilon^2 = 0.5$

(r is called the reliability of $\mathbf{Z}_i$ if $\mathbf{u}_i$ is a univariate variable). We let the nominal level $\alpha = 0.1$, $\Sigma_{uu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $\sigma_\epsilon^2 = 0.5$ in Fig. 1. We also let $r = 0.025l$, $l = 1, \ldots, 39$, and $n = 30, 50, 100, 200$ in the figure. For testing $H_{10} : (\beta_0, \beta_1, \beta_2)' = (1, 2, 3)'$ versus $H_{11} : (\beta_0, \beta_1, \beta_2)' \neq (1, 2, 3)'$, 1000 replicates were generated for each value of the parameters and n independently, and then the estimated true levels of the three tests are computed. All level curves are plotted by connecting the estimated true levels sequentially. From the figure, we observe that for fixed values of n and of other parameters, the estimated true levels of the pointwise robust test and the modified test deviate considerably from the nominal level 0.1 when r is small and approach 1 as r goes to 0. This provides the evidence of lack of uniform robustness for the pointwise robust test and the modified test. When the values of other parameters are fixed, the estimated true levels of the pointwise robust test and the modified test wiggle around the nominal level as r increases for a fixed n. And for the values of all parameters being fixed, the wiggle of the estimated true level lessens in size as the sample n becomes large. In contrast, for any value of n, the estimated true level of the uniformly robust test is very close to the nominal level 0.1 for all values of the parameters considered. This is consistent with the theoretical result proved in Sect. 2.

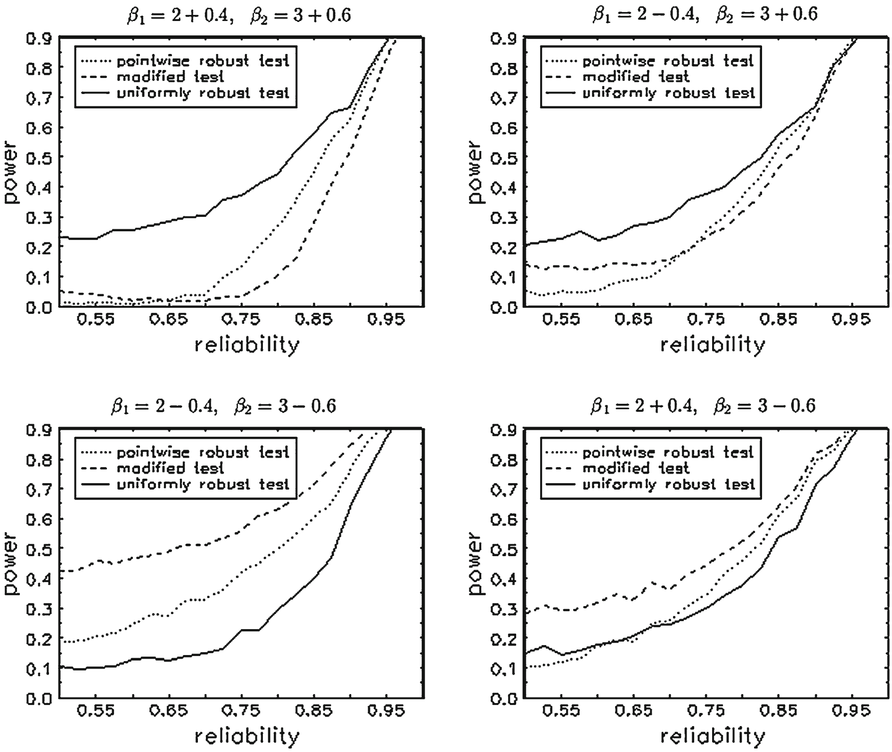

Fig. 2 Power of the pointwise robust, modified and uniformly robust tests at different values of $(\beta_1, \beta_2)'$ for testing $H_{20} : (\beta_1, \beta_2)' = (2, 3)'$ based on 1,000 replications for nominal level $\alpha = 0.1$ and $n = 30$. Model (31) is assumed with $\Sigma_{uu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_\epsilon^2 = 0.5$

In Fig. 2, we compare the power for testing $H_{20} : (\beta_1, \beta_2)' = (2, 3)'$ versus $H_{21} : (\beta_1, \beta_2)' \neq (2, 3)'$ of the three tests based on 1000 simulation replicates at four different values of $(\beta_1, \beta_2)'$ for $r \geq 0.5$ (under these cases the three estimated true levels are roughly close to the nominal level), $n = 30$ and nominal level $\alpha = 0.1$. When $\beta_1 = 2.4$ and $\beta_2 = 3.6$ or when $\beta_1 = 1.6$ and $\beta_2 = 3.6$, the uniformly robust test outperforms the other two tests. In particular, the power of both the pointwise robust test and the modified test is less than the nominal level 0.1 for $0.5 \leq r \leq 0.72$ when $\beta_1 = 2.4$ and $\beta_2 = 3.6$. On the other hand, the modified test gives the best results among the three tests when $\beta_1 = 1.6$ and $\beta_2 = 2.4$ or when $\beta_1 = 2.4$ and $\beta_2 = 2.4$. This indicates that there has a trade-off on power between the uniformly robust test and the modified test. For the uniformly robust test, it seems to have a tendency that when other parameters are fixed, the power decreases as (β_1, β_2) shrinks towards $(0, 0)$. Similar results are observed for $n = 50$ and presented in Fig. 3. Besides Figs. 2 and 3, we also simulated the power of the three tests at several different values of $(\beta_1, \beta_2)'$. Results similar to those in Figures 2 and 3 are obtained and hence are not reported.

As suggested by one of the referees, in Fig. 4 (also based on 1,000 simulation replicates) we compare the coverage probability and length of the confidence interval

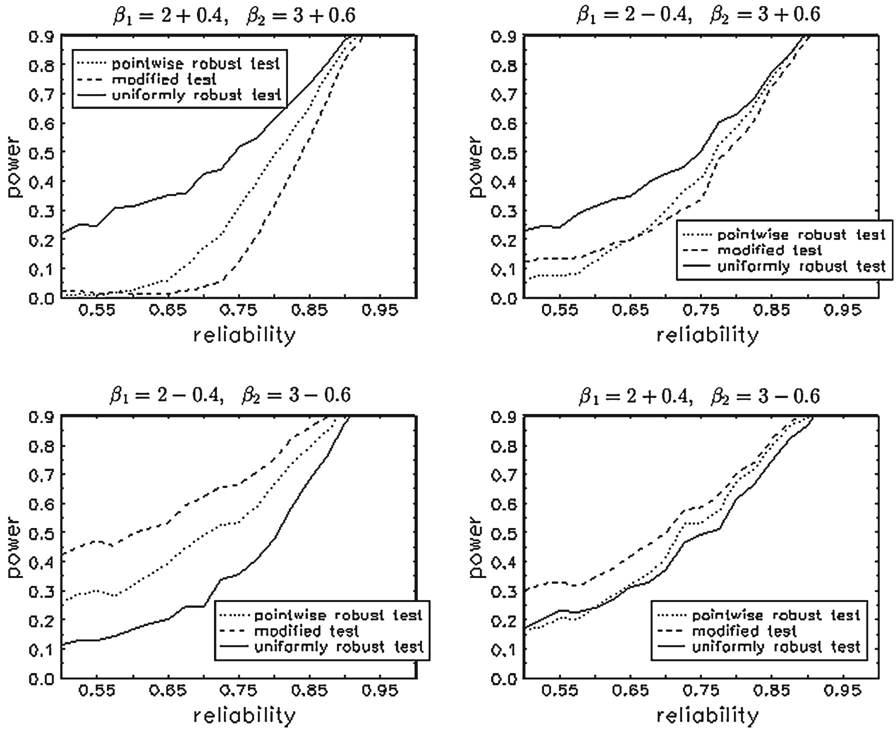

Fig. 3 Power of the pointwise robust, modified and uniformly robust tests at different values of $(\beta_1, \beta_2)'$ for testing $H_{20} : (\beta_1, \beta_2)' = (2, 3)'$ based on 1,000 replications for nominal level $\alpha = 0.1$ and $n = 50$. Model (31) is assumed with $\Sigma_{uu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_\epsilon^2 = 0.5$

for the scalar parameter β_2 inverted by the modified test and of that projected by the confidence set of $(\beta_1, \beta_2)'$ inverted by the uniformly robust test for $n = 50$ and 100 and confidence level $1 - \alpha = 0.9$. From the figure, we observe that the confidence interval inverted by the modified test has coverage probability wiggling around the confidence level, while the projected confidence interval always has coverage probability greater than the confidence level (as mentioned in Sect. 2.2, this projected confidence interval is conservative). For length comparison, since the confidence set for $(\beta_1, \beta_2)'$ inverted by the uniformly robust test has a positive probability with infinite diameter, subsequently the projected confidence interval for β_2 has infinite length with a positive probability as well. Hence, it is not appropriate to directly compare the average lengths of both confidence intervals for β_2. Instead, we present respectively the first and the third quartiles of the 1,000 lengths of both intervals. As expected both the first and the third quartiles of the lengths of the confidence interval inverted by the modified test are smaller than those of the projected confidence interval and the differences lessen as the reliability r increases. In view of the above results, in practice if we know that the reliability r is large enough (for instance $r \geq 0.5$), it might be preferable to use the modified test for testing the scalar parameter β_2 and construct the confidence interval for β_2 from the test.

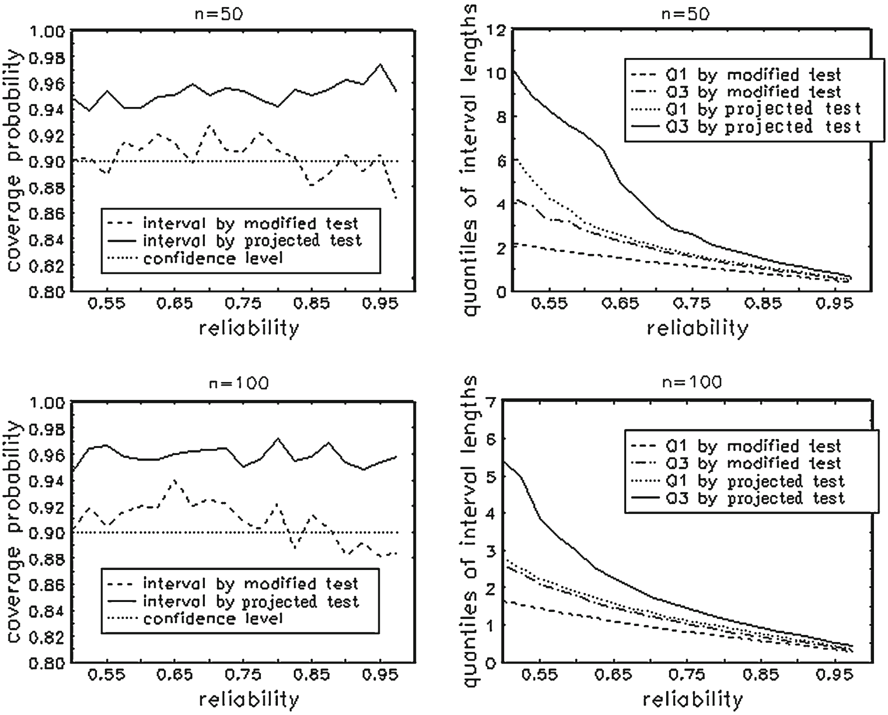

Fig. 4 Coverage probabilities and quartiles of lengths of the confidence intervals derived from the modified and the projected tests based on 1,000 replications for confidence level = 0.9 and $n = 50$ and 100. Model (31) is assumed with $\Sigma_{uu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_\epsilon^2 = 0.5$

4 Conclusion

In this paper, we propose uniformly robust tests for testing the vector coefficient parameter and vector slope parameter in the functional errors-in-variables model (structural model can be dealt similarly). These tests are derived through constructing the confidence sets for the same parameters in the model with similar desirable property. Unlike the existing asymptotic tests, these uniformly robust tests have estimated true levels reasonably close to the nominal levels for any parameter configurations and sample sizes. Their powers are also comparable to those of the existing asymptotic tests.

Appendix

In order to prove Theorem 1, we need the following two lemmas.

Lemma 1 *The absolute value of every element in the vector* $\mathbf{a}'\mathbf{Q}^{-\frac{1}{2}}\sigma_v^2$ *is bounded above by* $\sqrt{p+1}$*, the square root of the dimension of* $\boldsymbol{\beta}$.

Proof From (14) and (16), we have

$$\begin{aligned} \| \mathbf{a}'\mathbf{Q}^{-\frac{1}{2}}\sigma_v^2 \|^2 &= \boldsymbol{\beta}'\Sigma_{\delta\delta}[\sigma_v^2(\mathbf{M}_{xx}+\Sigma_{\delta\delta})+\Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}]^{-1}\Sigma_{\delta\delta}\boldsymbol{\beta} \\ &= \text{trace of } \mathbf{T} = \text{ sum of the eigenvalues of } \mathbf{T}, \end{aligned} \tag{32}$$

where $\mathbf{T} = \Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}[\sigma_v^2(\mathbf{M}_{xx}+\Sigma_{\delta\delta})+\Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}]^{-1}$. Let λ be any eigenvalue of $\mathbf{T}$ and $\mathbf{e}$ is the corresponding eigenvector. Then

$$\Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}\mathbf{e}^* = \lambda[\sigma_v^2(\mathbf{M}_{xx}+\Sigma_{\delta\delta})+\Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}]\mathbf{e}^*, \tag{33}$$

where $\mathbf{e}^* = [\sigma_v^2(\mathbf{M}_{xx}+\Sigma_{\delta\delta})+\Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}]^{-1}\mathbf{e}$. Multiplying $\mathbf{e}^{*'}$ on the left of both sides of (33), we conclude that $0 \le \lambda \le 1$ because $\mathbf{M}_{xx}$ is positive definite. Now from (32), the result follows immediately. □

Lemma 2 *The absolute value of every element in the matrix* $\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\sigma_v^2$ *is bounded above by* $\sqrt{p+1}$.

Proof For any two matrices $\mathbf{M}_1$ and $\mathbf{M}_2$, let the notation $\mathbf{M}_1 \le \mathbf{M}_2$ denote that $\mathbf{M}_2 - \mathbf{M}_1$ is positive semi-definite. Since $\Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}/\sigma_v^2$ is positive semi-definite, by the definition of $\mathbf{B}$ in (18) we have

$$\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\sigma_v^2 \le \mathbf{Q}^{-\frac{1}{2}}\Sigma_{\delta\delta}\mathbf{Q}^{-\frac{1}{2}}\sigma_v^2. \tag{34}$$

Let λ be any eigenvalue of $\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\sigma_v^2$ and $\mathbf{e}$ is the corresponding eigenvector. Then

$$\mathbf{B}\mathbf{e}^* = \lambda\frac{\mathbf{Q}}{\sigma_v^2}\mathbf{e}^*, \tag{35}$$

where $\mathbf{e}^* = \mathbf{Q}^{-\frac{1}{2}}\mathbf{e}$. Multiplying $\mathbf{e}^{*'}$ on the left of both sides of (35), we have $0 \le \lambda \le 1$ because $\mathbf{Q}/\sigma_v^2 - \mathbf{B} = \mathbf{M}_{xx} + 2\Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}/\sigma_v^2$ is positive definite. Moreover, because $\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\sigma_v^2$ is symmetric, there exists an orthogonal matrix $\mathbf{L}$ such that

$$\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\sigma_v^2 = \mathbf{L}\mathbf{D}\mathbf{L}', \tag{36}$$

where $\mathbf{D}$ is a diagonal matrix with diagonal elements being the eigenvalues of $\mathbf{Q}^{-\frac{1}{2}}$ $\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\sigma_v^2$. Let γ_{ij} be the (i, j)th element of $\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\sigma_v^2$. Then $\gamma_{ij} = \boldsymbol{\xi}_i'\mathbf{L}\mathbf{D}\mathbf{L}'\boldsymbol{\xi}_j$, where $\boldsymbol{\xi}_i = (0, \ldots, 0, 1, 0, \ldots, 0)'$ is the ith unit vector in R^{p+1}, $i = 1, \ldots, p+1$. Consequently,

$$\begin{aligned} |\gamma_{ij}| \le \| \mathbf{D}\mathbf{L}'\boldsymbol{\xi}_i \| \| \mathbf{L}'\boldsymbol{\xi}_j \| &= (\boldsymbol{\xi}_i'\mathbf{L}\mathbf{D}^2\mathbf{L}'\boldsymbol{\xi}_i)^{\frac{1}{2}}(\boldsymbol{\xi}_j'\mathbf{L}\mathbf{L}'\boldsymbol{\xi}_j)^{\frac{1}{2}} \\ &= \text{ square root of the } (i, i)\text{th element of } \mathbf{L}\mathbf{D}^2\mathbf{L}'. \end{aligned}$$

Since $\mathbf{LD}^2\mathbf{L}'$ is nonnegative definite, trace$(\mathbf{LD}^2\mathbf{L}') \geq$ the (i, i)th element of $\mathbf{LD}^2\mathbf{L}'$. Moreover, trace$(\mathbf{LD}^2\mathbf{L}') =$ trace$(\mathbf{D}^2)$, which equals the sum of squares of the eigenvalues of $\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\sigma_v^2$. The result follows because any eigenvalue λ of $\mathbf{Q}^{-\frac{1}{2}}\mathbf{B}\mathbf{Q}^{-\frac{1}{2}}\sigma_v^2$ is between 0 and 1. □

Lemma 3 *The absolute value of every element in the matrix $\sigma_v^2\mathbf{Q}^{-\frac{1}{2}}\mathbf{M}_{WW}\mathbf{Q}^{-\frac{1}{2}}$ is bounded above by $\sqrt{p+1}(p+3-l) + O_p(n^{-\frac{1}{2}})$, where $l-1$ is the number of elements in $\boldsymbol{\delta}_i$ equal to 0 and $O_p(n^{-\frac{1}{2}}) \xrightarrow{P} 0$ uniformly over the parameter space Ω.*

Proof By the definition of $\mathbf{M}_{WW}$ in (7), it follows that $\sigma_v^2\mathbf{Q}^{-\frac{1}{2}}\mathbf{M}_{WW}\mathbf{Q}^{-\frac{1}{2}} = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$, where

$$\mathbf{T}_1 = (\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}\mathbf{M}_{xx}(\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}, \quad \mathbf{T}_2 = (\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}\mathbf{M}_{\delta\delta}(\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}},$$

$$\mathbf{T}_3 = (\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}\mathbf{M}_{(x\delta)}(\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}, \quad \mathbf{M}_{\delta\delta} = \frac{1}{n}\sum_1^n \boldsymbol{\delta}_i\boldsymbol{\delta}_i',$$

$$\mathbf{M}_{(x\delta)} = \frac{1}{n}\sum_1^n(\mathbf{x}_i\boldsymbol{\delta}_i' + \boldsymbol{\delta}_i\mathbf{x}_i'). \tag{37}$$

By an argument similar to that of Lemma 2, the absolute value of every element in $\mathbf{T}_1$ is bounded above by $\sqrt{p+1}$. Let λ be any eigenvalue of $\mathbf{T}_2$, and thus $|\ \mathbf{M}_{\delta\delta} - \lambda\sigma_v^{-2}\mathbf{Q}\ | = 0$. Subsequently, by the definition of $\mathbf{Q}$ in (16) there exists a nonzero vector $\mathbf{e} \in R^{p+1}$ such that

$$\mathbf{e}'\mathbf{M}_{\delta\delta}\mathbf{e} - \lambda\mathbf{e}'\left(\mathbf{M}_{xx} + \Sigma_{\delta\delta} + \frac{1}{\sigma_v^2}\Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}\right)\mathbf{e} = 0,$$

and hence

$$\lambda = \frac{\mathbf{e}'\mathbf{M}_{\delta\delta}\mathbf{e}}{\mathbf{e}'(\mathbf{M}_{xx} + \Sigma_{\delta\delta} + \sigma_v^{-2}\Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta})\mathbf{e}}. \tag{38}$$

Note that if $\mathbf{e}'\Sigma_{\delta\delta}\mathbf{e} = 0$, it is easy to show that $\mathbf{e}'\mathbf{M}_{\delta\delta}\mathbf{e} = 0$ as well and hence $\lambda = 0$. Otherwise,

$$\lambda \leq \frac{\mathbf{e}'\mathbf{M}_{\delta\delta}\mathbf{e}}{\mathbf{e}'\Sigma_{\delta\delta}\mathbf{e}}. \tag{39}$$

Suppose, without loss of generality, that $\boldsymbol{\delta}_i' = (0, \boldsymbol{\tau}_i') = (0, \ldots, 0, \delta_{il}, \ldots, \delta_{i(p+1)})$ where $\delta_{ij} \neq 0$ for some l, $1 < l \leq p+1$ and $j = l, \ldots, p+1$ (i.e. $\boldsymbol{\tau}_i$ in (1) could be a degenerate multivariate normal distribution). Let $\boldsymbol{\delta}_{i*}' = (\delta_{il}, \ldots, \delta_{i(p+1)})$, $\mathbf{e}_*' = (e_l, \ldots, e_{p+1})$, $\mathbf{M}_{\delta\delta*} = n^{-1}\sum_1^n \boldsymbol{\delta}_{i*}\boldsymbol{\delta}_{i*}'$, and $\Sigma_{\delta\delta*}$ be the covariance matrix of $\boldsymbol{\delta}_{i*}$. Then by (39) and the fact that $\mathbf{M}_{\delta\delta*}$ and $\Sigma_{\delta\delta*}$ are both symmetric and positive definite, we have

$$\lambda \leq \frac{\mathbf{e}'_*\mathbf{M}_{\delta\delta*}\mathbf{e}_*}{\mathbf{e}'_*\Sigma_{\delta\delta*}\mathbf{e}_*} \leq \text{ maximum eigenvalue of } \mathbf{M}_{\delta\delta*}\Sigma_{\delta\delta*}^{-1}$$
$$= \text{ maximum eigenvalue of } \Sigma_{\delta\delta*}^{-\frac{1}{2}}\mathbf{M}_{\delta\delta*}\Sigma_{\delta\delta*}^{-\frac{1}{2}}. \tag{40}$$

Furthermore, since the matrix $\Sigma_{\delta\delta*}^{-\frac{1}{2}}\mathbf{M}_{\delta\delta*}\Sigma_{\delta\delta*}^{-\frac{1}{2}}$ is symmetric and positive definite, the maximum eigenvalue of the matrix is less than or equal to the trace of the matrix. Now

$$\text{trace}(\Sigma_{\delta\delta*}^{-\frac{1}{2}}\mathbf{M}_{\delta\delta*}\Sigma_{\delta\delta*}^{-\frac{1}{2}}) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=l}^{p+1}\rho_{ij}^2 = (p+2-l) + O_p(n^{-\frac{1}{2}}), \tag{41}$$

where $\boldsymbol{\rho}_i = (\rho_{il}, \ldots, \rho_{i(p+1)})' = \Sigma_{\delta\delta*}^{-\frac{1}{2}}\boldsymbol{\delta}_{i*}$, $\sum_{j=l}^{p+1}\rho_{ij}^2 \sim \chi^2_{p+2-l}$, and $O_p(n^{-\frac{1}{2}})$ does not depend on any parameter. By an argument similar to the proof of Lemma 2, the absolute value of every element in $\mathbf{T}_2$ is less than or equal to $\sqrt{p+1}$ times the absolute value of the maximum eigenvalue of $\mathbf{T}_2$. Combining this result, (40) and (41), the absolute value of every element in $\mathbf{T}_2$ is bounded above by $(p+2-l)\sqrt{p+1} + O_p(n^{-\frac{1}{2}})$, where $O_p(n^{-\frac{1}{2}})$ does not depend on any parameter.

Next, we shall show that $\mathbf{T}_3$ is a matrix with all elements of order $O_p(n^{-\frac{1}{2}})$, where $O_p(n^{-\frac{1}{2}}) \xrightarrow{P} 0$ uniformly over the parameter space Ω. Since

$$\mathbf{T}_3 = (\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}\frac{1}{n}\sum_1^n(\mathbf{x}_i\boldsymbol{\delta}'_i + \boldsymbol{\delta}_i\mathbf{x}'_i)(\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}},$$

it suffices to show that every element in the matrix

$$(\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}\frac{1}{n}\sum_1^n\mathbf{x}_i\boldsymbol{\delta}'_i(\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}} = \frac{1}{n}\sum_1^n[(\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}\mathbf{x}_i][(\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}\boldsymbol{\delta}_i]' \tag{42}$$

has order $O_p(n^{-\frac{1}{2}})$ which converges to 0 in probability over the parameter space Ω. let $\Sigma^* = (\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}\Sigma_{\delta\delta}(\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}$ denote the covariance matrix of $(\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}\boldsymbol{\delta}_i$. By an argument similar to that of $\mathbf{T}_2$, it follows that the absolute value of every element in Σ^* is bounded above by $(p+2-l)\sqrt{p+1}$. Furthermore, the jth column of (42), $1 \leq j \leq p+1$, has a normal distribution

$$N\left[\mathbf{0}, \frac{1}{n^2}(\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}\sum_1^n\mathbf{x}_i\mathbf{x}'_i(\sigma_v^{-2}\mathbf{Q})^{-\frac{1}{2}}\sigma^*_{jj}\right],$$

where σ^*_{jj} denotes the jth diagonal element of Σ^*. The covariance matrix of the above normal distribution equals $n^{-1}\mathbf{T}_1\sigma^*_{jj}$, where the absolute value of every element in the matrix is bounded by $n^{-1}(p+2-l)(p+1)$ (recall that the absolute value of every element in $\mathbf{T}_1$ is bounded above by $\sqrt{p+1}$). Consequently, every element of the jth column of (42) is of $O_p(n^{-\frac{1}{2}})$ which converges to 0 in probability uniformly over the parameter space Ω, so every element in $\mathbf{T}_3$ has the same property. Combining the results for $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$ together, the lemma is established. □

Lemma 4 *The matrix $\sigma_v^2\mathbf{Q}^{-\frac{1}{2}}[\mathbf{M}_{WW}-(\mathbf{M}_{xx}+\Sigma_{\delta\delta})]\mathbf{Q}^{-\frac{1}{2}}$ converges to zero matrix in probability uniformly over the parameter space Ω.*

Proof By the definition of $\mathbf{M}_{WW}$ in (7), the matrix $\sigma_v^2\mathbf{Q}^{-\frac{1}{2}}[\mathbf{M}_{WW}-(\mathbf{M}_{xx}+\Sigma_{\delta\delta})]\mathbf{Q}^{-\frac{1}{2}}$ can be written as

$$\sigma_v^2\mathbf{Q}^{-\frac{1}{2}}\mathbf{M}_{(x\delta)}\mathbf{Q}^{-\frac{1}{2}}+\sigma_v^2\mathbf{Q}^{-\frac{1}{2}}(\mathbf{M}_{\delta\delta}-\Sigma_{\delta\delta})\mathbf{Q}^{-\frac{1}{2}}, \tag{43}$$

where $\mathbf{M}_{\delta\delta}$ and $\mathbf{M}_{(x\delta)}$ are defined in (37). By the result in the proof of Lemma 3, every element in $\sigma_v^2\mathbf{Q}^{-\frac{1}{2}}\mathbf{M}_{(x\delta)}\mathbf{Q}^{-\frac{1}{2}}$ (defined as $\mathbf{T}_3$ in the proof of Lemma 3) is of order $O_P(n^{-\frac{1}{2}})$, where $O_p(n^{-\frac{1}{2}})\xrightarrow{P}0$ uniformly over the parameter space Ω. Rewrite the second term in (43) as

$$\sigma_v\mathbf{Q}^{-\frac{1}{2}}\Sigma_{\delta\delta}^{\frac{1}{2}}(\Sigma_{\delta\delta}^{-\frac{1}{2}}\mathbf{M}_{\delta\delta}\Sigma_{\delta\delta}^{-\frac{1}{2}}-\mathbf{I}_{p+1}^*)\Sigma_{\delta\delta}^{\frac{1}{2}}\mathbf{Q}^{-\frac{1}{2}}\sigma_v, \tag{44}$$

where

$$\Sigma_{\delta\delta}^{\frac{1}{2}}=\begin{pmatrix}0 & 0\\ 0 & \Sigma_{\delta\delta*}^{\frac{1}{2}}\end{pmatrix},\quad \Sigma_{\delta\delta}^{-\frac{1}{2}}=\begin{pmatrix}0 & 0\\ 0 & \Sigma_{\delta\delta*}^{-\frac{1}{2}}\end{pmatrix},\quad \mathbf{I}_{p+1}^*=\begin{pmatrix}0 & 0\\ 0 & \mathbf{I}_{p+2-l}\end{pmatrix}.$$

Here $\Sigma_{\delta\delta*}$ is the covariance matrix of the nonzero components vector $\boldsymbol{\delta}_{i*}=(\delta_{il},\ldots,\delta_{i\,p+1})'$ in $\boldsymbol{\delta}_i$ (suppose, without loss of generality, that $\boldsymbol{\delta}_i'=(0,\ldots,0,\delta_{il},\ldots,\delta_{i(p+1)})$ where $\delta_{ij}\neq 0$ for some $1<l\le p+1$ and $l\le j\le p+1$). To show that $\Sigma_{\delta\delta}^{-\frac{1}{2}}\mathbf{M}_{\delta\delta}\Sigma_{\delta\delta}^{-\frac{1}{2}}\xrightarrow{P}\mathbf{I}_{p+1}^*$ uniformly over the parameter space Ω, it suffices to prove that $\Sigma_{\delta\delta*}^{-\frac{1}{2}}\mathbf{M}_{\delta\delta*}\Sigma_{\delta\delta*}^{-\frac{1}{2}}\xrightarrow{P}\mathbf{I}_{p+2-l}$ uniformly over the parameter space Ω, where $\mathbf{M}_{\delta\delta*}=n^{-1}\sum_1^n\boldsymbol{\delta}_{i*}\boldsymbol{\delta}_{i*}'$. Recall that

$$\Sigma_{\delta\delta*}^{-\frac{1}{2}}\mathbf{M}_{\delta\delta*}\Sigma_{\delta\delta*}^{-\frac{1}{2}}=\frac{1}{n}\sum_{i=1}^{n}(\Sigma_{\delta\delta*}^{-\frac{1}{2}}\boldsymbol{\delta}_{i*})(\Sigma_{\delta\delta*}^{-\frac{1}{2}}\boldsymbol{\delta}_{i*})'\sim \text{Wishart}\left(\frac{1}{n}\mathbf{I}_{p+2-l},n\right). \tag{45}$$

Therefore, $\Sigma_{\delta\delta}^{-\frac{1}{2}}\mathbf{M}_{\delta\delta}\Sigma_{\delta\delta}^{-\frac{1}{2}}-\mathbf{I}_{p+1}^*=O_p(n^{-\frac{1}{2}})$ and

$$\Sigma_{\delta\delta}^{-\frac{1}{2}}\mathbf{M}_{\delta\delta}\Sigma_{\delta\delta}^{-\frac{1}{2}}\xrightarrow{P}\mathbf{I}_{p+1}^* \tag{46}$$

uniformly over the parameter space Ω.

Because $\mathbf{Q} = \sigma_v^2(\mathbf{M}_{xx} + \Sigma_{\delta\delta}) + \Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}$, $\mathbf{M}_{xx} + \Sigma_{\delta\delta}$ is positive definite, and $\sigma_v^{-2}\Sigma_{\delta\delta}\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_{\delta\delta}$ is positive semidefinite, we have

$$\| \sigma_v \mathbf{Q}^{-\frac{1}{2}} \Sigma_{\delta\delta}^{\frac{1}{2}} \boldsymbol{\xi}_i \|^2 = \boldsymbol{\xi}_i' \Sigma_{\delta\delta}^{\frac{1}{2}} \mathbf{Q}^{-1} \Sigma_{\delta\delta}^{\frac{1}{2}} \sigma_v^2 \boldsymbol{\xi}_i \le \boldsymbol{\xi}_i' \Sigma_{\delta\delta}^{\frac{1}{2}} (\mathbf{M}_{xx} + \Sigma_{\delta\delta})^{-1} \Sigma_{\delta\delta}^{\frac{1}{2}} \boldsymbol{\xi}_i, \quad (47)$$

where $\boldsymbol{\xi}_i = (0, \ldots, 0, 1, 0, \ldots, 0)'$ is the ith unit vector in R^{p+1}. Since $\Sigma_{\delta\delta}^{\frac{1}{2}}\mathbf{Q}^{-1}\Sigma_{\delta\delta}^{\frac{1}{2}}\sigma_v^2$ is symmetric and positive semidefinite,

$$\begin{aligned} \text{every eigenvalue of } \Sigma_{\delta\delta}^{\frac{1}{2}}\mathbf{Q}^{-1}\Sigma_{\delta\delta}^{\frac{1}{2}}\sigma_v^2 \le \ & \text{trace}\left(\Sigma_{\delta\delta}^{\frac{1}{2}}\mathbf{Q}^{-1}\Sigma_{\delta\delta}^{\frac{1}{2}}\sigma_v^2\right) \\ \le \ & \text{trace}\left[\Sigma_{\delta\delta}^{\frac{1}{2}}(\mathbf{M}_{xx} + \Sigma_{\delta\delta})^{-1}\Sigma_{\delta\delta}^{\frac{1}{2}}\right] \quad (\text{by } (47)) \\ = \ & \text{sum of the eigenvalues of } \Sigma_{\delta\delta}(\mathbf{M}_{xx}+\Sigma_{\delta\delta})^{-1}. \end{aligned} \quad (48)$$

Let λ be any eigenvalue of $\Sigma_{\delta\delta}(\mathbf{M}_{xx} + \Sigma_{\delta\delta})^{-1}$. Then there exists a nonzero vector $\mathbf{e} \in R^{p+1}$ such that

$$\mathbf{e}'\Sigma_{\delta\delta}\mathbf{e} - \lambda\mathbf{e}'(\mathbf{M}_{xx} + \Sigma_{\delta\delta})\mathbf{e} = 0. \quad (49)$$

From this, we have $0 \le \lambda \le 1$. Combining this and (48), we conclude that every eigenvalue of $\Sigma_{\delta\delta}^{\frac{1}{2}}\mathbf{Q}^{-1}\Sigma_{\delta\delta}^{\frac{1}{2}}\sigma_v^2 \le p+1$. By an argument similar to the proof of Lemma 2, the absolute value of every element in $\Sigma_{\delta\delta}^{\frac{1}{2}}\mathbf{Q}^{-1}\Sigma_{\delta\delta}^{\frac{1}{2}}\sigma_v^2$ is less than or equal to the square root of the sum of squares of the eigenvalues of the same matrix, which is less than or equal to $(p + 1)^{\frac{3}{2}}$. As a consequence, the absolute value of every element in $\mathbf{Q}^{-\frac{1}{2}}\sigma_v\Sigma_{\delta\delta}^{\frac{1}{2}}$ is less than or equal to $(p + 1)^{\frac{3}{4}}$. Combining this and (46), we have $\sigma_v^2\mathbf{Q}^{-\frac{1}{2}}(\mathbf{M}_{\delta\delta} - \Sigma_{\delta\delta})\mathbf{Q}^{-\frac{1}{2}} = O_p(n^{-\frac{1}{2}})$, where $O_p(n^{-\frac{1}{2}})$ does not depend on any parameter. Putting all results together, we establish the lemma. □

Proof of Theorem 3 Here we only give a sketchy proof without going into details. Note that

$$\begin{aligned} \Gamma_1|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*} &= n[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^1) + (\boldsymbol{\beta}^1 - \boldsymbol{\beta}^*)]'(\mathbf{M}_{WW} - \Sigma_{\delta\delta})\hat{\mathbf{Q}}^{-1}|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*} \\ &\quad \times(\mathbf{M}_{WW} - \Sigma_{\delta\delta})[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^1) + (\boldsymbol{\beta}^1 - \boldsymbol{\beta}^*)] \\ &= n(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^1)'(\mathbf{M}_{WW} - \Sigma_{\delta\delta})\hat{\mathbf{Q}}^{-1}|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^1) \\ &\quad +2n(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^1)'(\mathbf{M}_{WW} - \Sigma_{\delta\delta})\hat{\mathbf{Q}}^{-1}|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*} \\ &\quad \times(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\boldsymbol{\beta}^1 - \boldsymbol{\beta}^*) + n(\boldsymbol{\beta}^1 - \boldsymbol{\beta}^*)'(\mathbf{M}_{WW} - \Sigma_{\delta\delta})\hat{\mathbf{Q}}^{-1}|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*} \\ &\quad \times(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\boldsymbol{\beta}^1 - \boldsymbol{\beta}^*), \end{aligned}$$

where

$$\hat{\mathbf{Q}}|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*} = \hat{\sigma}_v^2|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}\mathbf{M}_{WW} + \Sigma_{\delta\delta}\boldsymbol{\beta}^*\boldsymbol{\beta}^{*\prime}\Sigma_{\delta\delta} \to \sigma_v^{*2}(\mathbf{M}^0 + \Sigma_{\delta\delta}) + \Sigma_{\delta\delta}\boldsymbol{\beta}^*\boldsymbol{\beta}^{*\prime}\Sigma_{\delta\delta} \equiv \mathbf{Q}^*,$$

which is positive definite since

$$\hat{\sigma}_v^2|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*} = \frac{1}{n}\sum_1^n (Y_i - \mathbf{W}_i'\boldsymbol{\beta}^*)^2 \to \sigma_v^{*2} > 0 \quad \text{as } n \to \infty.$$

Firstly, it is easy to show that

$$\sqrt{n}(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^1) \to N(\mathbf{0}, \mathbf{Q}^1),$$

where $\mathbf{Q}^1 = \sigma_v^2(\mathbf{M}^0 + \Sigma_{\delta\delta})\Sigma_{\delta\delta}\boldsymbol{\beta}^1\boldsymbol{\beta}^{1\prime}\Sigma_{\delta\delta}$. Consequently,

$$\hat{\mathbf{Q}}^{-\frac{1}{2}}|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}\sqrt{n}(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^1) \to N(\mathbf{0}, \mathbf{Q}^{*-\frac{1}{2}}\mathbf{Q}^1\mathbf{Q}^{*-\frac{1}{2}})$$

and

$$n(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^1)'(\mathbf{M}_{WW} - \Sigma_{\delta\delta})\hat{\mathbf{Q}}^{-1}|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^1) = O_p(1).$$

Secondly, observe that

$$2n(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^1)'(\mathbf{M}_{WW} - \Sigma_{\delta\delta})\hat{\mathbf{Q}}^{-1}|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\boldsymbol{\beta}^1 - \boldsymbol{\beta}^*) = 2\sqrt{n}(\boldsymbol{\beta}^1 - \boldsymbol{\beta}^*)'$$
$$(\mathbf{M}_{WW} - \Sigma_{\delta\delta})\hat{\mathbf{Q}}^{-\frac{1}{2}}|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}\hat{\mathbf{Q}}^{-\frac{1}{2}}|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}\sqrt{n}(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^1) = \sqrt{n}O_p(1).$$

Finally,

$$n(\boldsymbol{\beta}^1 - \boldsymbol{\beta}^*)'(\mathbf{M}_{WW} - \Sigma_{\delta\delta})\hat{\mathbf{Q}}^{-1}|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}(\mathbf{M}_{WW} - \Sigma_{\delta\delta})(\boldsymbol{\beta}^1 - \boldsymbol{\beta}^*) = nc_n,$$

where $c_n \to c > 0$. Combining all above together, we have

$$P_{\boldsymbol{\beta}^1}(\Gamma_1|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*} > \chi^2_{p+1,\alpha}) = P_{\boldsymbol{\beta}^1}(O_p(1) + \sqrt{n}O_p(1) + nc_n > \chi^2_{p+1,\alpha}) \to 1 \quad \text{as } n \to \infty.$$

□

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