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Estimating the intensity of a cyclic Poisson process in the presence of linear trend

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Abstract We construct and investigate a consistent kernel-type nonparametric estimator of the intensity function of a cyclic Poisson process in the presence of linear trend. It is assumed that only a single realization of the Poisson process is observed in a bounded window. We prove that the proposed estimator is consistent when the size of the window indefinitely expands. The asymptotic bias, variance, and the mean-squared error of the proposed estimator are also computed. A simulation study shows that the first order asymptotic approximations to the bias and variance of the estimator are not accurate enough. Second order terms for bias and variance were derived in order to be able to predict the numerical results in the simulation. Bias reduction of our estimator is also proposed.

Keywords Cyclic Poisson process · Intensity function · Linear trend · Nonparametric estimation · Consistency · Bias · Variance · Mean-squared error

1 Introduction and main results

Let X be a Poisson point process on $[0, \infty)$ with (unknown) locally integrable intensity function λ which is assumed to consist of two components, namely a periodic or cyclic component with period $\tau > 0$ and a (unknown) linear trend component. In other words,

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for any point $s \in [0, \infty)$, we can write the intensity function λ as

$$\lambda(s) = \lambda_c(s) + as \tag{1}$$

where $\lambda_c(s)$ is a periodic function with period τ and a denotes the slope of the linear trend. In the present paper, we do not assume any (parametric) form of λ_c except that it is periodic. That is we assume that the equality

$$\lambda_c(s + k\tau) = \lambda_c(s) \tag{2}$$

holds for all $s \in [0, \infty)$ and $k \in \mathbf{Z}$. Here we consider a Poisson point process on $[0, \infty)$ instead of, for instance, on $\mathbf{R}$ because λ has to satisfy (1) and must be nonnegative. For the same reason we also restrict our attention to the case $a \geq 0$. The present paper aims at extending previous work for the purely cyclic case, i.e. $a = 0$, (cf. Helmers et al. 2003, 2005; Kutoyants 1984; Sect. 2.3 of Kutoyants 1998) to the more general model (1).

Furthermore, let $W_1, \ W_2, \ldots$ be a sequence of intervals $[0, |W_n|]$, $n = 1, 2, \ldots$, such that the size or the Lebesgue measure $|W_n|$ of W_n is finite for each fixed $n \in \mathbf{N}$, but

$$|W_n| \to \infty \tag{3}$$

as $n \to \infty$.

Suppose now that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the Poisson process X defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ (cf. (1)) is observed, though only within a bounded interval, called 'window' $W \subset [0, \infty)$. Our goal in this paper is to construct a consistent nonparametric estimator of λ_c at a given point $s \in [0, \infty)$ from a single realization $X(\omega)$ of the Poisson process X observed in $W := W_n$. We also compute the asymptotic bias, variance, and the mean-squared error of the proposed estimator.

We will assume throughout that s is a Lebesgue point of λ, that is we have $\lim_{h\downarrow 0} \frac{1}{2h} \int_{-h}^{h} |\lambda(s + x) - \lambda(s)| \mathrm{d}x = 0$ (e.g. see Wheeden and Zygmund 1977, pp. 107–108), which automatically means that s is a Lebesgue point of λ_c as well. This assumption is a mild one since the set of all Lebesgue points of λ is dense in $\mathbf{R}$, whenever λ is assumed to be locally integrable.

To begin with we will suppose that the period τ is known, but the slope a and the function λ_c on $[0, \tau)$ are both unknown. Since λ_c is periodic with period τ, the problem of estimating λ_c at a given $s \in [0, \infty)$ can be reduced to the problem of estimating λ_c at a given $s \in [0, \tau)$. In this situation we may define estimators of respectively a and λ_c, at a given point $s \in [0, \tau)$, as follows:

$$\hat{a}_n := \frac{2X(W_n)}{|W_n|^2} \tag{4}$$

and

$$\tilde{\lambda}_{c,n}(s) := \frac{1}{\ln\left(\frac{|W_n|}{\tau}\right)} \sum_{k=1}^{\infty} \frac{1}{k} \frac{X([s + k\tau - h_n, s + k\tau + h_n] \cap W_n)}{2h_n} - \hat{a}_n \left(s + \frac{|W_n|}{\ln(\frac{|W_n|}{\tau})} \right), \tag{5}$$

where h_n is a sequence of positive real numbers converging to 0, that is,

$$h_n \downarrow 0 \tag{6}$$

as $n \to \infty$.

We note in passing that if, instead of estimating $\lambda_c(s)$, one is interested in estimating $\lambda(s)$ at a given point s, then $\lambda(s)$ can be estimated by

$$\hat{\lambda}_n(s) = \tilde{\lambda}_{c,n}(s) + \hat{a}_n s. \tag{7}$$

To obtain the estimator $\hat{a}_n$ of a it suffices to note that

$$\mathbf{E}X(W_n) = \frac{a}{2}|W_n|^2 + \mathcal{O}(|W_n|)$$

as $n \to \infty$, which directly yields the estimator given in (4). Note also that if X were a Poisson process with intensity $\lambda(s) = as$, then $\hat{a}_n$ would be the maximum likelihood estimator of a.

Next, we describe the idea behind the construction of the kernel-type estimator $\tilde{\lambda}_{c,n}(s)$ of $\lambda_c(s)$. By (1) and (2) we have, for any point $s \in [0, \tau)$ and $k \in \mathbf{N}$, that

$$\lambda_c(s) = \lambda_c(s + k\tau) = \lambda(s + k\tau) - a(s + k\tau). \tag{8}$$

Let $B_h(x) := [x - h, x + h]$ and $L_n := \sum_{k=1}^{\infty} k^{-1}\mathbf{I}(s + k\tau \in W_n)$. By (8), we can write

$$\begin{aligned}
\lambda_c(s) &= \frac{1}{L_n}\sum_{k=1}^{\infty}\frac{1}{k}\left(\lambda_c(s + k\tau)\right)\mathbf{I}(s + k\tau \in W_n) \\
&= \frac{1}{L_n}\sum_{k=1}^{\infty}\frac{1}{k}\left(\lambda(s + k\tau) - a(s + k\tau)\right)\mathbf{I}(s + k\tau \in W_n) \\
&= \frac{1}{L_n}\sum_{k=1}^{\infty}\frac{1}{k}\left(\lambda(s + k\tau)\right)\mathbf{I}(s + k\tau \in W_n) - as - \frac{a\tau}{L_n}\sum_{k=1}^{\infty}\mathbf{I}(s + k\tau \in W_n).
\end{aligned} \tag{9}$$

By the assumption that s is a Lebesgue point of λ and (6), we have

$$\begin{aligned}
\lambda_c(s) &\approx \frac{1}{L_n}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{|B_{h_n}(s + k\tau)|}\int_{B_{h_n}(s+k\tau)\cap W_n}\lambda(x)\mathrm{d}x - as - \frac{a|W_n|}{L_n} \\
&= \frac{1}{L_n}\sum_{k=1}^{\infty}\frac{1}{k}\frac{\mathbf{E}X(B_{h_n}(s + k\tau) \cap W_n)}{2h_n} - a\left(s + \frac{|W_n|}{L_n}\right) \\
&\approx \frac{1}{L_n}\sum_{k=1}^{\infty}\frac{1}{k}\frac{X(B_{h_n}(s + k\tau) \cap W_n)}{2h_n} - a\left(s + \frac{|W_n|}{L_n}\right).
\end{aligned} \tag{10}$$

Here we also have used the fact that

$$\frac{a\tau}{L_n}\sum_{k=1}^{\infty}\mathbf{I}(s+k\tau\in W_n)=\frac{a\tau}{L_n}\left(\frac{|W_n|}{\tau}+\zeta_n\right)=\frac{a|W_n|}{L_n}+\frac{a\tau\zeta_n}{L_n}\approx\frac{a|W_n|}{L_n}$$

(cf. (39)), where $|\zeta_n|\le 1$ for all $n\ge 1$. From the second $\approx$ in (10) and noting that $L_n \sim \ln(|W_n|/\tau)$ as $n\to\infty$, we see that

$$\begin{aligned}\bar{\lambda}_{c,n}(s)=&\frac{1}{\ln\left(\frac{|W_n|}{\tau}\right)}\sum_{k=1}^{\infty}\frac{1}{k}\frac{X([s+k\tau-h_n,s+k\tau+h_n]\cap W_n)}{2h_n}\\&-a\left(s+\frac{|W_n|}{\ln\left(\frac{|W_n|}{\tau}\right)}\right)\end{aligned}\tag{11}$$

can be viewed as an estimator of $\lambda_c(s)$, provided both the period τ and the slope a of the linear trend are assumed to be known. If a is unknown, we replace a by $\hat{a}_n$ (cf. (4)) and one obtains the estimator of $\lambda_c(s)$ given in (5).

Lemma 1 *Suppose that the intensity function λ satisfies (1) and is locally integrable. Then we have*

$$\mathbf{E}\left(\hat{a}_n\right)=a+\frac{2\theta}{|W_n|}+\mathcal{O}\left(\frac{1}{|W_n|^2}\right)\tag{12}$$

and

$$\operatorname{Var}\left(\hat{a}_n\right)=\frac{2a}{|W_n|^2}+\mathcal{O}\left(\frac{1}{|W_n|^3}\right)\tag{13}$$

as $n\to\infty$, where $\theta=\tau^{-1}\int_0^{\tau}\lambda_c(s)\mathrm{d}s$, the global intensity of the cyclic component λ_c. Hence, by (3), $\hat{a}_n$ is a consistent estimator of a; its mean-squared error (MSE) is given by

$$\operatorname{MSE}\left(\hat{a}_n\right)=\frac{4\theta^2+2a}{|W_n|^2}+\mathcal{O}\left(\frac{1}{|W_n|^3}\right)\tag{14}$$

as $n\to\infty$.

Theorem 1 *Suppose that the intensity function λ satisfies (1) and is locally integrable. If, in addition, $h_n\downarrow 0$ and*

$$h_n\ln|W_n|\to\infty,\tag{15}$$

then

$$\tilde{\lambda}_{c,n}(s)\xrightarrow{p}\lambda_c(s)\tag{16}$$

as $n \to \infty$ provided s is a Lebesgue point of λ_c. In other words, $\tilde{\lambda}_{c,n}(s)$ is a consistent estimator of $\lambda_c(s)$. In addition the MSE of $\tilde{\lambda}_{c,n}(s)$ converges to 0, as $n \to \infty$.

We note that Lemma 1 and Theorem 1 together imply that the estimator $\hat{\lambda}_n(s)$ in (7) is a consistent estimator of $\lambda(s)$.

Theorem 2 *Suppose that the intensity function λ satisfies (1) and is locally integrable. If $h_n \downarrow 0$ and (15) holds true, then*

$$\operatorname{Var}\left(\tilde{\lambda}_{c,n}(s)\right) = \frac{a\tau}{2h_n \ln(|W_n|/\tau)} + o\left(\frac{1}{h_n \ln|W_n|}\right) \tag{17}$$

as $n \to \infty$, provided s is a Lebesgue point of λ_c. If, in addition, λ_c has finite second derivative λ_c'' at s and

$$h_n^2 \ln|W_n| \to \infty, \tag{18}$$

then we have

$$\mathbf{E}\tilde{\lambda}_{c,n}(s) = \lambda_c(s) + \frac{\lambda_c''(s)}{6} h_n^2 + o(h_n^2) \tag{19}$$

as $n \to \infty$.

Note that (19) yields the classical bias term $(\lambda_c''(s)h_n^2)/6$ for kernel estimates of $\lambda_c(s)$ (cf. Helmers et al. 2005) in the purely cyclic case, where $a = 0$. On the other hand, the asymptotic approximation to the variance given in (17) differs from the classical formula $(\lambda_c(s)\tau)/(2h_n|W_n|)$ (cf. Helmers et al. 2005) in two ways: in the denominator of (17) we have $\ln(|W_n|/\tau)$ instead of $|W_n|$, implying that in the presence of linear trend one needs a much larger window (larger by a factor $|W_n|/\ln(|W_n|/\tau)$) than in the purely cyclic case $a = 0$. This is a simple consequence of the 'weight' $1/k$ we have used in definition (5) of $\tilde{\lambda}_{c,n}(s)$. The second difference is the factor a in the numerator of (17), replacing $\lambda_c(s)$ (cf. Theorem 3.2 of Helmers et al. 2005) in the purely cyclic case, when no linear trend is assumed to be present. In a way this tells us that the linear trend will dominate the variability of $\tilde{\lambda}_{c,n}(s)$.

It is clear from the proof of Theorem 2 that, in case a would be known, we obtain the same leading term $(a\tau)/(2h_n \ln(|W_n|/\tau))$ for the asymptotic approximation to the variance of the estimator of $\lambda_c(s)$ (e.g. see (51)). This is due to the fact that the variance of $\hat{a}_n(s + |W_n|/(\ln(|W_n|/\tau)))$ in (5) is of smaller order than the leading term on the r.h.s. of (17).

In the second part of Theorem 2 we assume that $h_n^2 \ln|W_n| \to \infty$, as $n \to \infty$. In fact, without assuming $h_n^2 \ln|W_n| \to \infty$, we can only prove that the remainder term on the r.h.s. of (19) is of order $o(h_n^2) + \mathcal{O}((\ln|W_n|)^{-1})$, as $n \to \infty$. Since the second term on the r.h.s. of (19) is exactly of order $\mathcal{O}(h_n^2)$, it is therefore natural to require $(\ln|W_n|)^{-1} = o(h_n^2)$, which is equivalent to the assumption $h_n^2 \ln|W_n| \to \infty$, as $n \to \infty$.

Remark 1 If either a or $\theta = \tau^{-1}\int_0^\tau \lambda_c(s)\mathrm{d}s$ (the global intensity of the cyclic component λ_c) is known, then the linear trend in (1) can be replaced by a trend as^b, for some known $b \in (0, 1)$. In the case a is known, $\lambda_c(s)$ can be estimated by

$$\tilde{\lambda}_{c,n,b}(s) := \frac{1}{L_{n,b}}\sum_{k=1}^{\infty}\frac{1}{k^b}\frac{X([s+k\tau-h_n, s+k\tau+h_n]\cap W_n)}{2h_n} - a\left(s+\frac{|W_n|}{L_{n,b}}\right), \tag{20}$$

where $L_{n,b} := \sum_{k=1}^{\infty} k^{-b}\mathbf{I}(s + k\tau \in W_n)$. In the case that a is unknown but θ is known, $\lambda_c(s)$ can be estimated by (20) with a is replaced by

$$\hat{a}_{n,b} := \frac{(1+b)X(W_n)}{|W_n|^{1+b}} - \frac{(1+b)\theta}{|W_n|^b}. \tag{21}$$

A calculation similar to the one given in the proof of Theorem 1 can now be used to show that $\tilde{\lambda}_{c,n,b}(s)$ is a consistent estimator of $\lambda_c(s)$ in both cases. In the case that a is unknown, knowledge of θ is needed to correct for the bias of the estimator of a (cf. (21)). Without bias correction, the bias of $\tilde{\lambda}_{c,n,b}(s)$ will not vanish, as $n \to \infty$, which of course implies that $\tilde{\lambda}_{c,n,b}(s)$ will not be consistent in estimating $\lambda_c(s)$. Furthermore, we have

$$\mathrm{Var}\left(\tilde{\lambda}_{c,n,b}(s)\right) = \mathcal{O}\left(\frac{1}{h_n|W_n|^{1-b}}\right) \tag{22}$$

as $n \to \infty$. Note that, in the case $b = 1$ we have $\mathrm{Var}(\tilde{\lambda}_{c,n}(s)) = \mathcal{O}((h_n \ln|W_n|)^{-1})$ (cf. (17)), whereas for the case $b = 0$ (purely cyclic case) the variance of the estimator of $\lambda_c(s)$ is of order $\mathcal{O}((h_n|W_n|)^{-1})$ (cf. Helmers et al. 2005), while the quantity in (22) is in between. This reflects the fact that the smaller the value of b, the larger is the part of the data set $X(\omega)$ which can be used to estimate $\lambda_c(s)$.

Finally, let us consider the intensity function λ in (1) with linear trend replaced by as^b, for some known $b > 1$, and consider our estimator of $\lambda_c(s)$ in (20). Because in the case $b > 1$, $L_{n,b} = \mathcal{O}(1)$ as $n \to \infty$, it follows that $\mathrm{Var}(\tilde{\lambda}_{c,n,b}(s))$ does not converge to zero, as $n \to \infty$, and the estimator $\tilde{\lambda}_{c,n,b}(s)$ is not consistent in estimating $\lambda_c(s)$. Hence, in this case, our kernel type estimation of $\lambda_c(s)$ fails.

In a way, the preceding argument tells us that adding a linear trend to the cyclic component (cf. (1)) is a 'border case': trends increasing slower than linear can be handled by our kernel estimation method, though at the price of knowing either a or θ. However, for trends increasing faster than linear, for instance a quadratic trend, our estimation procedure does not work, i.e. consistent estimation of the intensity of the cyclic component appears to be impossible by the proposed method.

Next we turn to the important case that the period τ is unknown. This will be the situation one typically encounters in statistical practice. In order to be able to construct an estimator of λ_c for the case τ is unknown, we require a consistent estimator of τ.

Let $\hat{\tau}_n$ be a consistent estimator of τ. Then we modify our estimator of λ_c in (5) by replacing τ by $\hat{\tau}_n$ to obtain

$$\hat{\lambda}_{c,n}(s) := \frac{1}{\ln\left(\frac{|W_n|}{\hat{\tau}_n}\right)} \sum_{k=1}^{\infty} \frac{1}{k} \frac{X([s+k\hat{\tau}_n - h_n, s+k\hat{\tau}_n + h_n] \cap W_n)}{2h_n} - \hat{a}_n \left(s + \frac{|W_n|}{\ln\left(\frac{|W_n|}{\hat{\tau}_n}\right)} \right). \quad (23)$$

The construction of estimators $\hat{\tau}_n$ of the period τ of a cyclic Poisson process with desired accuracy (cf. (25), (28) or (30)), using only a single realization from X, is outside the scope of the present paper. Some results concerning nonparametric estimation of the period in the purely cyclic case $a = 0$ can be found in Helmers and Mangku (2003) and Bebbington and Zitikis (2004). We intend to pursue the general case $a > 0$ elsewhere.

Note that the estimator in (23) can be viewed as a special case (take $K = \frac{1}{2}\mathbf{I}_{[-1,1]}(\cdot)$) of a general kernel-type estimator, which is given by

$$\hat{\lambda}_{c,n,K}(s) := \frac{1}{\ln\left(\frac{|W_n|}{\hat{\tau}_n}\right)} \sum_{k=1}^{\infty} \frac{1}{h_n k} \int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) X(\mathrm{d}x) - \hat{a}_n \left(s + \frac{|W_n|}{\ln\left(\frac{|W_n|}{\hat{\tau}_n}\right)} \right), \quad (24)$$

where $K : \mathbf{R} \to \mathbf{R}$ is a function, called kernel, satisfying assumptions: K is a probability density function, bounded, and has support in $[-1, 1]$. However, in this paper we focus on the uniform kernel-type estimator in (23). Similarly as in Helmers et al. (2003, 2005), one can obtain similar results for the estimator given in (24).

Note also that (24) reduces to (1.10) of Helmers et al. (2003), if the weight $1/k$ is replaced by 1 and the second term $\hat{a}_n(s + |W_n|/(\ln(|W_n|/\hat{\tau}_n)))$ is deleted, because in Helmers et al. (2003) $\lambda(s) = \lambda_c(s)$ or in other words $a = 0$.

In the following theorem, we will show that the consistency result in Theorem 1 remains valid, provided the rate of consistency of $\hat{\tau}_n$ to τ is sufficiently fast.

Theorem 3 *Suppose that the assumptions of Theorem 1 are satisfied. If, in addition, for any $\delta > 0$ we have*

$$\mathbf{P}\left(\frac{|W_n|^2}{h_n \ln |W_n|} \left|\hat{\tau}_n - \tau\right| > \delta\right) = o(1) \quad (25)$$

as $n \to \infty$, then

$$\hat{\lambda}_{c,n}(s) \xrightarrow{p} \lambda_c(s) \quad (26)$$

as $n \to \infty$. In other words, $\hat{\lambda}_{c,n}(s)$ is a consistent estimator of $\lambda_c(s)$.

Condition (25) on the estimator $\hat{\tau}_n$ is equivalent to assuming that $|W_n|^2 (h_n \ln |W_n|)^{-1} |\hat{\tau}_n - \tau| \xrightarrow{p} 0$, while condition (2.2) of Helmers et al. (2003) which yields (26) in the purely cyclic case ($a = 0$) considered in Helmers et al. (2003), requires only $(|W_n|h_n^{-1})|\hat{\tau}_n - \tau| \xrightarrow{p} 0$, as $n \to \infty$. An extra factor $|W_n|/(\ln |W_n|)$ is needed here due to the presence of linear trend in our model (1). From the proof of Theorem 3 it appears that (25) is a minimal assumption required to ensure that $\hat{\lambda}_{c,n}(s)$ is a consistent estimator of $\lambda_c(s)$.

In order to be able to derive asymptotic approximations to respectively bias and variance of the estimator of $\lambda_c(s)$ under weak assumptions on the estimator $\hat{\tau}_n$ of the period, it is required to modify our estimator of $\lambda_c(s)$ slightly as follows:

$$\hat{\lambda}^{\diamond}_{c,n}(s) := \mathbf{I}\left(\hat{\lambda}_{c,n}(s) \le D_n\right)\hat{\lambda}_{c,n}(s) + \mathbf{I}\left(\hat{\lambda}_{c,n}(s) > D_n\right) D_n \tag{27}$$

where the nonrandom D_n will approach infinity when $n \to \infty$. The truncation level D_n in the definition of $\hat{\lambda}^{\diamond}_{c,n}(s)$ is needed to avoid accumulation of errors due to estimation of τ in estimating $\lambda_c(s)$, so that statements (29) and (31) would still hold true. The D_n's were introduced in Helmers et al. (2005) for similar purposes in the purely cyclic case, when $a = 0$ in (1).

In the following two theorems, it is shown that one can estimate τ by $\hat{\tau}_n$ without affecting the statistical properties of our estimate $\tilde{\lambda}_{c,n}(s)$ of $\lambda_c(s)$, i.e. its asymptotic bias, variance, and MSE, given in Theorem 2, provided the rate of consistency of $\hat{\tau}_n$ to τ is sufficiently fast.

Theorem 4 *Suppose that the intensity function λ satisfies (1) and is locally integrable. Furthermore, let the bandwidth h_n satisfy (6) and (18), and the sequence D_n be such that, for some $c > 0$ and $\epsilon > 0$, the bound $D_n \ge c(h_n)^{-\epsilon}$ holds for all sufficiently large n. If, in addition, for any $\delta > 0$ we have*

$$\mathbf{P}\left(\frac{|W_n|^2}{h_n^3 \ln |W_n|}\left|\hat{\tau}_n - \tau\right| > \delta\right) = o\left(\frac{h_n^2}{D_n}\right) \tag{28}$$

as $n \to \infty$ and λ_c has finite second derivative λ_c'' at s, then

$$\mathbf{E}\hat{\lambda}^{\diamond}_{c,n}(s) = \lambda_c(s) + \frac{\lambda_c''(s)}{6}h_n^2 + o(h_n^2) \tag{29}$$

as $n \to \infty$.

Theorem 5 *Suppose that the intensity function λ satisfies (1) and is locally integrable. Furthermore, let the bandwidth h_n satisfy (6) and (15), and the sequence D_n be such that, for some $c > 0$ and $\epsilon > 0$, the bound $D_n \ge c(h_n \ln |W_n|)^{\epsilon}$ holds for all sufficiently large n. If, in addition, for any $\delta > 0$ we have*

$$\mathbf{P}\left(\frac{|W_n|^2}{h_n^{1/2}(\ln |W_n|)^{1/2}}\left|\hat{\tau}_n - \tau\right| > \delta\right) = o\left(\frac{1}{D_n^2 h_n \ln |W_n|}\right) \tag{30}$$

as $n \to \infty$, then

$$\mathrm{Var}\left(\hat{\lambda}^{\diamond}_{c,n}(s)\right) = \frac{a\tau}{2h_n \ln(|W_n|/\tau)} + o\left(\frac{1}{h_n \ln|W_n|}\right) \tag{31}$$

as $n \to \infty$, provided s is a Lebesgue point of λ_c.

By Theorems 4 and 5 (i.e. (29) and (31)), we can compute the MSE of $\hat{\lambda}^{\diamond}_{c,n}(s)$ as follows:

$$\mathrm{MSE}\left(\hat{\lambda}^{\diamond}_{c,n}(s)\right) = \frac{a\tau}{2h_n \ln(|W_n|/\tau)} + \frac{(\lambda_c''(s))^2}{36}h_n^4 + o\left(\frac{1}{h_n \ln|W_n|} + h_n^4\right) \tag{32}$$

as $n \to \infty$. Now, we consider the r.h.s. of (32). By minimizing the sum of the first and second term (the leading term for the variance and the squared bias), we get the optimal choice of h_n, which is given by

$$h_n = \left[\frac{9a\tau}{2(\lambda_c''(s))^2}\right]^{\frac{1}{5}} (\ln(|W_n|/\tau))^{-\frac{1}{5}}. \tag{33}$$

With this choice of h_n, the optimal rate of decrease of $\mathrm{MSE}(\hat{\lambda}^{\diamond}_{c,n}(s))$ is of order $\mathcal{O}((\ln|W_n|)^{-4/5})$ as $n \to \infty$.

Remark 1 (*continued*) Recall that in Remark 1 we consider the intensity function in (1) with as replaced by as^b, for some known $b \in (0, 1)$. For the case τ is unknown, $\lambda_c(s)$ can be estimated by $\hat{\lambda}_{c,n,b}(s)$, that is the estimator in (20) with τ is replaced by its estimator $\hat{\tau}_n$. In order to have that $\hat{\lambda}_{c,n,b}(s)$ is a consistent estimator of $\lambda_c(s)$, it is required that the estimator $\hat{\tau}_n$ of τ has to satisfy the condition: for any $\delta > 0$,

$$\mathbf{P}\left(|W_n|^{1+b} h_n^{-1} \left|\hat{\tau}_n - \tau\right| > \delta\right) = o(1)$$

holds true as $n \to \infty$, that is condition (25) with $\ln|W_n|$ is replaced by $|W_n|^{1-b}$. Note that, if $b = 0$ (the case that the intensity function is purely cyclic), this condition reduces to condition (2.2) of Helmers et al. (2003). Similarly, in order to be able to derive asymptotic approximations to respectively bias and variance of $\hat{\lambda}_{c,n,b}(s)$, it is required that $\hat{\tau}_n$ has to satisfy respectively conditions (28) and (30) with $\ln|W_n|$ in both these conditions are replaced by $|W_n|^{1-b}$. Note also that, for the case $b = 0$, these two conditions reduce to respectively conditions (2.4) and (3.3) of Helmers et al. (2005).

To conclude this section, we remark that Helmers and Zitikis (1999) also consider a uniform kernel-type estimator for $\lambda(s)$ in the case where λ is a parametric function of spatial location. These authors focus their attention to the case that X is a Poisson process on $[0, \infty)$ with intensity function

$$\lambda(s) = \exp\left\{\alpha + \beta s + \gamma s^2 + K_1 \sin(\omega_0 s) + K_2 \cos(\omega_0 s)\right\},$$

$s > 0$, where $\alpha, \beta, \gamma, K_1$, and K_2 are unknown parameters, and ω_0 is a known 'frequency'. Helmers and Zitikis (1999) obtain L_2-convergence of their estimator, whenever (3) holds.

The nonparametric maximum likelihood estimator of λ_c, the intensity function of a (purely) cyclic Poisson process X (i.e. $a = 0$ in (1)), with known period τ, was investigated by Dorogovtsev and Kukush (1996) and Kukush and Mishura (1999). To do this, these authors assume that $\lambda_c|[0, \tau)$, the restriction of λ_c to $[0, \tau)$, belongs to a Sobolev space of functions on $[0, \tau)$. An algorithm for the computation of a nonparametric MLE is also given. The paper by Dorogovtsev and Kukush (1996) restricts attention to the case that X is Poisson, while in Kukush and Mishura (1999) X may consists of three components: a drift, a diffusion and a cyclic Poisson process with known period τ. The result obtained in Dorogovtsev and Kukush (1996) was extended to the case that X is a Poisson random field in Kukush and Stepanishcheva (2002).

2 Simulations

For the simulations, we consider the intensity function

$$\lambda(s) = \lambda_c(s) + as = A \exp\left\{\rho \cos\left(\frac{2\pi s}{\tau} + \phi\right)\right\} + as,$$

that is (1), where λ_c is the intensity function discussed in Vere-Jones (1982). We chose $A = 2$, $\rho = 1$, $\tau = 5$ and $\phi = 0$. With this choice of the parameters, we have

$$\lambda(s) = 2 \exp\left\{\cos\left(\frac{2\pi s}{\tau}\right)\right\} + as. \tag{34}$$

The cyclic part $\lambda_c(s)$ of (34) achieves its maximum $2e = 5.4366$ at $s = 5k$ and its minimum $2e^{-1} = 0.7358$ at $s = 2.5 + 5k$, for any integer k. Since λ_c is periodic with period τ, the problem of estimating λ_c at a given $s \in [0, \infty)$ can be reduced to the problem of estimating λ_c at a given $s \in [0, \tau)$. In our simulations we will consider three values of s, namely $s = 2.6$ (a small value of $\lambda_c(s)$), $s = 4.0$ (a moderate value of $\lambda_c(s)$) and $s = 4.9$ (a large value of $\lambda_c(s)$). In each of the examples presented below, we only investigate the performance of an estimator of $\lambda_c(s)$ in the case the period τ is known. We use $W_n = [0, 1000]$ and take the 'optimal' choice for the bandwidth h_n.

Example 1 (*the purely cyclic case* $a = 0$) In this example we consider the purely cyclic Poisson process, that is the Poisson process with intensity function given in (34) in the case we know that $a = 0$. This model is studied extensively in Helmers et al. (2003, 2005). To estimate $\lambda_c(s)$ in the purely cyclic (PC) case, we can use the estimator given by (1.3) of Helmers et al. (2005). In this example, we use this estimator with kernel $K = \frac{1}{2}\mathbf{I}([-1, 1])$, that is

$$\tilde{\lambda}_{c,n,PC}(s) := \frac{\tau}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{X([s + k\tau - h_n, s + k\tau + h_n] \cap W_n)}{2h_n}.$$

Asymptotic approximations to the variance and bias of this estimator are given respectively by (cf. Theorems 2.2 and 3.2 of Helmers et al. (2005))

$$\mathrm{Var}\left(\tilde{\lambda}_{c,n,PC}(s)\right) = \frac{\lambda_c(s)\tau}{2h_n|W_n|} + o\left(\frac{1}{h_n|W_n|}\right), \tag{35}$$

and

$$\mathrm{Bias}\left(\tilde{\lambda}_{c,n,PC}(s)\right) = \frac{\lambda_c''(s)}{6}h_n^2 + o(h_n^2), \tag{36}$$

as $n \to \infty$. The optimal choice for the bandwidth h_n is given by (cf. (3.6) of Helmers et al. 2005)

$$h_n = \left[9\lambda_c(s)\tau/(2(\lambda_c''(s))^2)\right]^{1/5} (|W_n|)^{-1/5}. \tag{37}$$

(i) For $s = 2.6$, we have $\lambda_c(s) = 0.7416$ and $\lambda_c''(s) = 1.1802$. By (37), with $|W_n| = 1000$ and $\tau = 5$, we obtain the (optimal) choice of bandwidth $h_n = 0.4128$. By (35) and (36), we obtain the numerical values of the asymptotic approximations to respectively the variance and the bias of $\tilde{\lambda}_{c,n,PC}(s)$: $\mathrm{Var}(\tilde{\lambda}_{c,n,PC}(s)) = 0.0045$ and $\mathrm{Bias}(\tilde{\lambda}_{c,n,PC}(s)) = 0.0335$. From the simulation, using $M = 10^4$ independent realizations of the process X observed in the $W_n = [0, 1000]$, we obtain respectively $\hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n,PC}(s)) = 0.0047$ and $\hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n,PC}(s)) = 0.0303$, where $\hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n,PC}(s))$ is the sample variance $\frac{1}{M-1}\sum_{i=1}^{M}(\tilde{\lambda}_{c,n,PC,i}(s) - \frac{1}{M}\sum_{j=1}^{M}\tilde{\lambda}_{c,n,PC,j}(s))^2$ and $\hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n,PC}(s))$ is the sample mean $M^{-1}\sum_{j=1}^{M}\tilde{\lambda}_{c,n,PC,j}(s)$ minus $\lambda_c(s)$. Summarizing, we have $\mathrm{Var}(\tilde{\lambda}_{c,n,PC}(s)) - \hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n,PC}(s)) = 0.0045 - 0.0047 = -0.0002$ and $\mathrm{Bias}(\tilde{\lambda}_{c,n,PC}(s)) - \hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n,PC}(s)) = 0.0335 - 0.0303 = 0.0032$.

(ii) For $s = 4.0$, we have $\lambda_c(s) = 2.7242$ and $\lambda_c''(s) = 2.5617$. By (37), we obtain $h_n = 0.3927$. By (35) and (36) and from the simulation ($M = 10^4$) we obtain $\mathrm{Var}(\tilde{\lambda}_{c,n,PC}(s)) - \hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n,PC}(s)) = 0.0173 - 0.0178 = -0.0005$ and $\mathrm{Bias}(\tilde{\lambda}_{c,n,PC}(s)) - \hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n,PC}(s)) = 0.0658 - 0.0472 = 0.0186$.

(iii) For $s = 4.9$, we have $\lambda_c(s) = 5.3939$ and $\lambda_c''(s) = -8.3167$. By (37), we obtain $h_n = 0.2811$. By (35) and (36) and from the simulation ($M = 10^4$) we obtain
$\mathrm{Var}(\tilde{\lambda}_{c,n,PC}(s)) - \hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n,PC}(s)) = 0.0480 - 0.0462 = 0.0018$ and
$\mathrm{Bias}(\tilde{\lambda}_{c,n,PC}(s)) - \hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n,PC}(s)) = -0.1095 - (-0.1404) = 0.0309$.

In this example we find that the asymptotic approximations to respectively the variance and the bias of the estimator proposed in Helmers et al. (2005) are relatively close to the numerical values obtained in the simulation. So we can conclude that the first order asymptotics derived in Helmers et al. (2005) for the purely cyclic case works well in approximating the variance and the bias in finite samples.

Example 2 (*cyclic in the presence of linear trend, $a = 0.05$*) In this example we study the performance of the estimator $\tilde{\lambda}_{c,n}(s)$ in (5), in the case that the intensity function $\lambda(s)$ is given by (34) with $a = 0.05$.

(i) Recall that for $s = 2.6$, we have $\lambda_c(s) = 0.7416$ and $\lambda_c''(s) = 1.1802$. By (33), with $|W_n| = 1000$ and $\tau = 5$, we obtain the 'optimal' $h_n = 0.6865$. By (17), (19) and from the simulation ($M = 10^4$) we obtain respectively
$\mathrm{Var}(\tilde{\lambda}_{c,n}(s)) - \hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n}(s)) = 0.0344 - 0.0750 = -0.0406$ and
$\mathrm{Bias}(\tilde{\lambda}_{c,n}(s)) - \hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n}(s)) = 0.0927 - (-0.8127) = 0.9054$.

(ii) For $s = 4.0$, we have $\lambda_c(s) = 2.7242$ and $\lambda_c''(s) = 2.5617$. By (33), we obtain $h_n = 0.5035$. By (17) and (19) and from the simulation ($M = 10^4$) we obtain
$\mathrm{Var}(\tilde{\lambda}_{c,n}(s)) - \hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n}(s)) = 0.0469 - 0.2255 = -0.1786$ and
$\mathrm{Bias}(\tilde{\lambda}_{c,n}(s)) - \hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n}(s)) = 0.1082 - (-0.6034) = 0.7116$.

(iii) For $s = 4.9$, we have $\lambda_c(s) = 5.3939$ and $\lambda_c''(s) = -8.3167$. By (33), we obtain $h_n = 0.3144$. By (17) and (19) and from the simulation ($M = 10^4$) we obtain
$\mathrm{Var}(\tilde{\lambda}_{c,n}(s)) - \hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n}(s)) = 0.0750 - 0.6012 = -0.5262$ and
$\mathrm{Bias}(\tilde{\lambda}_{c,n}(s)) - \hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n}(s)) = -0.1370 - (-0.5780) = 0.4410$.

In sharp contrast with Example 1, the first order asymptotic approximations to the variance and bias — the ones given by Theorem 2 — fail to predict the variance and bias in finite samples accurately. We also see that the bias of $\tilde{\lambda}_{c,n}(s)$ is unacceptable large (cf. Remark 2).

To improve the accuracy of our approximations for variance and bias we added second order terms to the expansions given in (17) and (19). Expansions (17) and (19) for variance and bias are replaced by the second order approximations (38) and (39) respectively. Expansion (38) improves upon (17) by adding a term of order $h_n^{-1}(\ln|W_n|)^{-2}$ to the original approximation of order $(h_n \ln|W_n|)^{-1}$. On the other hand, the bias expansion (19) without assumption (18) has a remainder term of order $o(h_n^2) + \mathcal{O}((\ln|W_n|)^{-1}))$; assumption (18) in Theorem 2 is in fact meant to eliminate the $\mathcal{O}((\ln|W_n|)^{-1})$ remainder term (cf. paragraph preceding Remark 1). Expansion (39) improves now upon (19) by adding a term of order h_n^4 and also a term of order $(\ln|W_n|)^{-1}$. We restrict attention to the case that τ is known.

Corollary 1 *Suppose that the intensity function λ satisfies (1) and is locally integrable. If $h_n \downarrow 0$ and (15) holds true, then*

$$\mathrm{Var}\left(\tilde{\lambda}_{c,n}(s)\right) = \frac{a\tau}{2h_n \ln(|W_n|/\tau)} + \frac{(\lambda_c(s) + as)(\pi^2/6) + a\tau\gamma}{2h_n(\ln(|W_n|/\tau))^2} + o\left(\frac{1}{h_n(\ln|W_n|)^2}\right) \tag{38}$$

as $n \to \infty$, provided s is a Lebesgue point of λ_c, where $\gamma = 0.577..$ is Euler's *constant. If, in addition,* λ_c *has finite fourth derivative* $\lambda_c^{(\mathrm{iv})}$ *at s then we have*

$$\mathbf{E}\tilde{\lambda}_{c,n}(s) = \lambda_c(s) + \frac{\lambda_c''(s)}{6}h_n^2 - \frac{2\theta - \gamma\lambda_c(s) - (\gamma s + \tau\zeta_n)a}{\ln(|W_n|/\tau)} + \frac{\lambda_c^{(\mathrm{iv})}(s)}{120}h_n^4 + o\left(h_n^4 + \frac{1}{\ln|W_n|}\right) \tag{39}$$

as $n \to \infty$*, where* $\zeta_n = (2h_n)^{-1}\sum_{k=1}^{\infty}\int_{-h_n}^{h_n}\mathbf{I}(x+s+k\tau \in W_n)\mathrm{d}x - (|W_n|/\tau)$ *and* $|\zeta_n| \leq 1$ *for all* $n \geq 1$.

Example 2 (*continued*) Instead of using Theorem 2 we use the second order asymptotic approximations to the variance and bias ((38) and (39) of Corollary 1) to predict the variance and bias in finite samples.

(i) For $s = 2.6$, we obtain
$\mathrm{Var}(\tilde{\lambda}_{c,n}(s)) - \hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n}(s)) = 0.0753 - 0.0750 = 0.0003$ and
$\mathrm{Bias}(\tilde{\lambda}_{c,n}(s)) - \hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n}(s)) = -0.8086 - (-0.8127) = 0.0041$.

(ii) For $s = 4.0$, we obtain
$\mathrm{Var}(\tilde{\lambda}_{c,n}(s)) - \hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n}(s)) = 0.2222 - 0.2255 = -0.0033$ and
$\mathrm{Bias}(\tilde{\lambda}_{c,n}(s)) - \hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n}(s)) = -0.5904 - (-0.6034) = 0.0130$.

(iii) For $s = 4.9$, we obtain
$\mathrm{Var}(\tilde{\lambda}_{c,n}(s)) - \hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n}(s)) = 0.6087 - 0.6012 = 0.0075$ and
$\mathrm{Bias}(\tilde{\lambda}_{c,n}(s)) - \hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n}(s)) = -0.5218 - (-0.5780) = 0.0562$.

We see that the second order approximations (38) and (39) are quite close to the numerical values obtained in the simulation.

Remark 2 **(Bias reduction)**: Note that the bias of $\tilde{\lambda}_{c,n}(s)$ in Example 2 is quite large. However, we can reduce this bias by subtracting and adding respectively estimators of the second and third term on the r.h.s. of (39) into $\tilde{\lambda}_{c,n}(s)$. To indicate that $\tilde{\lambda}_{c,n}(s)$ depends on h_n, let us write $\tilde{\lambda}_{c,n}(s) = \tilde{\lambda}_{c,n,h_n}(s)$. We may define estimators of respectively $\lambda_c''(s)$ and θ as follows:

$$\tilde{\lambda}_{c,n}''(s) := \frac{\tilde{\lambda}_{c,n,h_n'}(s+2h_n') + \tilde{\lambda}_{c,n,h_n'}(s-2h_n') - 2\tilde{\lambda}_{c,n,h_n'}(s)}{4(h_n')^2}$$

where h_n' is a sequence of positive real numbers converging to 0, that is $h_n' \downarrow 0$ as $n \to \infty$, and

$$\hat{\theta}_n := \frac{1}{\ln(|W_n|/\tau)}\sum_{k=1}^{\infty}\frac{1}{k}\frac{X([k\tau,(k+1)\tau]\cap W_n)}{\tau} - \hat{a}_n\left(\frac{\tau}{2} + \frac{|W_n|}{\ln(|W_n|/\tau)}\right).$$

A similar calculation as the one used to compute the expectation of $\tilde{\lambda}_{c,n}(s)$ shows that

$$\mathbf{E}\hat{\theta}_n = \theta - \frac{(2-\gamma)\theta - (\gamma/2+\zeta_n)a\tau}{\ln(|W_n|/\tau)} + o\left(\frac{1}{\ln|W_n|}\right) \tag{40}$$

as $n \to \infty$. Then we obtain a bias corrected estimator of θ as follows:

$$\hat{\theta}_{n,b} := \hat{\theta}_n + \frac{(2-\gamma)\hat{\theta}_n - (\gamma/2+\zeta_n)\tau\hat{a}_n}{\ln(|W_n|/\tau)}. \tag{41}$$

Now, we can define a bias corrected estimator of $\lambda_c(s)$ as

$$\tilde{\lambda}_{c,n,BC}(s) = \tilde{\lambda}_{c,n}(s) - \frac{\tilde{\lambda}''_{c,n}(s)}{6}h_n^2 + \frac{2\hat{\theta}_{n,b} - \gamma\tilde{\lambda}_{c,n}(s) - (\gamma s + \tau\zeta_n)\hat{a}_n}{\ln(|W_n|/\tau)}. \tag{42}$$

Corollary 2 *Suppose that the intensity function λ satisfies (1) and is locally integrable. If $h'_n \downarrow 0$, $h_n = o(h'_n)$ and (15) holds true, then*

$$\begin{aligned}\mathrm{Var}\left(\tilde{\lambda}_{c,n,BC}(s)\right) &= \frac{a\tau}{2h_n\ln(|W_n|/\tau)} + \frac{a\tau h_n^2}{12(h'_n)^3\ln(|W_n|/\tau)} + \frac{a\tau h_n^4}{192(h'_n)^5\ln(|W_n|/\tau)} \\ &\quad + \frac{(\lambda_c(s)+as)(\pi^2/6) - a\tau\gamma}{2h_n(\ln(|W_n|/\tau))^2} + o\left(\frac{1}{h_n(\ln|W_n|)^2}\right)\end{aligned} \tag{43}$$

as $n \to \infty$ provided λ_c is continuous at s. If, in addition, λ_c has finite fourth derivative $\lambda_c^{(\mathrm{iv})}$ in the neighborhood of s then we have

$$\mathbf{E}\tilde{\lambda}_{c,n,BC}(s) = \lambda_c(s) - \frac{\lambda_c^{(\mathrm{iv})}(s)}{12}h_n^2(h'_n)^2 + o\left(h_n^2(h'_n)^2 + \frac{1}{\ln|W_n|}\right) \tag{44}$$

as $n \to \infty$.

Note that we require $h_n = o(h'_n)$ as $n \to \infty$ in order to have the second and third term on the r.h.s. of (43) is of smaller order than its first term.

However, Corollary 2 remains valid when $h'_n = h_n$. In this case, the second and third term on the r.h.s. of (43) are of the same order as the first term, and we obtain

$$\begin{aligned}\mathrm{Var}\left(\tilde{\lambda}_{c,n,BC}(s)\right) &= \frac{113a\tau}{192h_n\ln(|W_n|/\tau)} + \frac{(\lambda_c(s)+as)(\pi^2/6) - a\tau\gamma}{2h_n(\ln(|W_n|/\tau))^2} \\ &\quad + o\left(\frac{1}{h_n(\ln|W_n|)^2}\right)\end{aligned} \tag{45}$$

as $n \to \infty$.

Next we note that the second term on the r.h.s. of (44) is of the same order as the fourth term on the r.h.s. of (39), which directly yields that

$$\mathrm{Bias}(\tilde{\lambda}_{c,n,BC}(s)) = -\frac{3\lambda_c^{(\mathrm{iv})}(s)}{40} h_n^4 + o(h_n^4 + (\ln|W_n|)^{-1}) \tag{46}$$

as $n \to \infty$. We consider this situation in the simulation.

With the aid of (45) and (46) one can obtain the MSE of $\tilde{\lambda}_{c,n,BC}(s)$. Minimizing the MSE of $\tilde{\lambda}_{c,n,BC}(s)$ yields the optimal choice of h_n, which is given by

$$h_n = \left[\frac{200}{9(\lambda_c^{(\mathrm{iv})}(s))^2} \left(\frac{113a\tau}{192} + \frac{(\lambda_c(s) + as)(\pi^2/6) - a\tau\gamma}{2\ln(|W_n|/\tau)} \right) \right]^{\frac{1}{9}} \times (\ln(|W_n|/\tau))^{-\frac{1}{9}}. \tag{47}$$

With this choice of h_n, the optimal rate of decrease of $\mathrm{MSE}(\tilde{\lambda}_{c,n,BC}(s))$ is of order $\mathcal{O}((\ln|W_n|)^{-8/9})$ as $n \to \infty$, slightly faster than the order $\mathcal{O}((\ln|W_n|)^{-4/5})$ obtained with (32) and (33).

Example 3 Here we use $\tilde{\lambda}_{c,n,BC}(s)$ (cf. (42)) to estimate $\lambda_c(s)$. In the simulation we use the 'optimal' choice of h_n given by (47) and take $h'_n = h_n$.

(i) For $s = 2.6$, using (47) and by noting that $\lambda_c^{(\mathrm{iv})}(s) = 3.6831$, we obtain $h_n = 0.7585$. From (45), (46) and the simulation ($M = 10^4$) we obtain $\mathrm{Var}(\tilde{\lambda}_{c,n,BC}(s)) - \hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n,BC}(s)) = 0.0669 - 0.0864 = -0.0195$ and $\mathrm{Bias}(\tilde{\lambda}_{c,n,BC}(s)) - \hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n,BC}(s)) = -0.0914 - (-0.0617) = -0.0297$.

(ii) For $s = 4.0$, using (47) and by noting that $\lambda_c^{(\mathrm{iv})}(s) = -26.3675$, we obtain $h_n = 0.5342$. From (45), (46) and the simulation ($M = 10^4$) we obtain $\mathrm{Var}(\tilde{\lambda}_{c,n,BC}(s)) - \hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n,BC}(s)) = 0.2075 - 0.2376 = -0.0301$ and $\mathrm{Bias}(\tilde{\lambda}_{c,n,BC}(s)) - \hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n,BC}(s)) = 0.1610 - 0.0773 = 0.0837$.

(iii) For $s = 4.9$, using (47) and by noting that $\lambda_c^{(\mathrm{iv})}(s) = 50.9627$, we obtain $h_n = 0.4900$. From (45), (46) and the simulation ($M = 10^4$) we obtain $\mathrm{Var}(\tilde{\lambda}_{c,n,BC}(s)) - \hat{\mathrm{V}}\mathrm{ar}(\tilde{\lambda}_{c,n,BC}(s)) = 0.3886 - 0.4080 = -0.0194$ and $\mathrm{Bias}(\tilde{\lambda}_{c,n,BC}(s)) - \hat{\mathrm{B}}\mathrm{ias}(\tilde{\lambda}_{c,n,BC}(s)) = -0.2203 - (-0.1810) = -0.0393$.

Clearly the bias of $\tilde{\lambda}_{c,n,BC}(s)$ is much smaller than the bias of $\tilde{\lambda}_{c,n}(s)$ (cf. Example 2 (continued)); the bias reduction proposed in (42) works.

A cautionary remark on the range of validity of our results in Example 2 (continued) and Example 3 is in order. The remainder terms in (38), (39), (43) and (44) will depend on the values of the parameters involved, such as $\lambda_c(s)$, $\lambda''_c(s)$, $\lambda_c^{(\mathrm{iv})}(s)$, a, and τ. In order to have a close agreement between the second order approximations and the results of our simulations as in Example 2 (continued) and Example 3, one would need that $\lambda_c(s)$, $|\lambda''_c(s)|h_n^2$, $|\lambda_c^{(\mathrm{iv})}(s)|h_n^4$, and $a\tau$ are relatively small compared with $\ln(|W_n|/\tau)$.

To conclude this section, we remark that, in view of the preceding results, the bias corrected estimator $\tilde{\lambda}_{c,n,BC}(s)$ is to be preferred in practical applications.

3 Proofs of Lemma 1 and Theorems 1 and 2

In this section we prove our main results for the case that the period τ is known.

Proof of Lemma 1 By (4), $\mathbf{E}(\hat{a}_n)$ can be computed as follows:

$$\begin{aligned} \mathbf{E}\left(\hat{a}_n\right) &= \frac{2}{|W_n|^2}\mathbf{E}X(W_n) = \frac{2}{|W_n|^2}\int_{W_n}\lambda(s)\mathrm{d}s \\ &= \frac{2}{|W_n|^2}\int_{W_n}(\lambda_c(s)+as)\mathrm{d}s = \frac{2}{|W_n|^2}\left(\theta|W_n| + \frac{a}{2}|W_n|^2 + \mathcal{O}(1)\right) \\ &= a + \frac{2\theta}{|W_n|} + \mathcal{O}\left(\frac{1}{|W_n|^2}\right) \end{aligned} \tag{48}$$

as $n \to \infty$. The variance of $\hat{a}_n$ is obtained in a similar fashion:

$$\begin{aligned} \mathrm{Var}\left(\hat{a}_n\right) &= \frac{4}{|W_n|^4}\mathrm{Var}\left(X(W_n)\right) = \frac{4}{|W_n|^4}\mathbf{E}X(W_n) = \frac{4}{|W_n|^4}\left(\frac{a}{2}|W_n|^2 + \mathcal{O}(|W_n|)\right) \\ &= \frac{2a}{|W_n|^2} + \mathcal{O}\left(\frac{1}{|W_n|^3}\right) \end{aligned} \tag{49}$$

as $n \to \infty$. This completes the proof of Lemma 1.

We first prove Theorem 2.

Proof of Theorem 2 First we prove (17). Note that

$$\begin{aligned} \mathrm{Var}\left(\tilde{\lambda}_{c,n}(s)\right) &= \mathrm{Var}\left(\bar{\lambda}_{c,n}(s)\right) + \mathrm{Var}\left(\tilde{\lambda}_{c,n}(s) - \bar{\lambda}_{c,n}(s)\right) \\ &\quad + 2\mathrm{Cov}\left(\bar{\lambda}_{c,n}(s), (\tilde{\lambda}_{c,n}(s) - \bar{\lambda}_{c,n}(s))\right). \end{aligned} \tag{50}$$

We will first show that

$$\mathrm{Var}\left(\bar{\lambda}_{c,n}(s)\right) = \frac{a\tau}{2h_n\ln(|W_n|/\tau)} + o\left(\frac{1}{h_n\ln|W_n|}\right) \tag{51}$$

as $n \to \infty$. To establish (51) we argue as follows. By (6), for sufficiently large n, we have that $X([s+k\tau-h_n, s+k\tau+h_n]\cap W_n)$ and $X([s+j\tau-h_n, s+j\tau+h_n]\cap W_n)$ are independent, provided $k \neq j$. Hence, for large n, we have

$$\begin{aligned} &\mathrm{Var}\left(\bar{\lambda}_{c,n}(s)\right) \\ &= \frac{1}{4h_n^2(\ln(|W_n|/\tau))^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\mathrm{Var}\left(X([s+k\tau-h_n, s+k\tau+h_n]\cap W_n)\right) \end{aligned}$$

$$= \frac{1}{4h_n^2(\ln(|W_n|/\tau))^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_{-h_n}^{h_n}(\lambda_c(x+s)+a(x+s))\,\mathbf{I}(x+s+k\tau\in W_n)\mathrm{d}x$$

$$+\frac{a\tau}{4h_n^2(\ln(|W_n|/\tau))^2}\sum_{k=1}^{\infty}\frac{1}{k}\int_{-h_n}^{h_n}\mathbf{I}(x+s+k\tau\in W_n)\mathrm{d}x. \tag{52}$$

Since s is a Lebesgue point of λ_c, a simple calculation shows that the first term on the r.h.s. of (52) is of order $\mathcal{O}(h_n{}^{-1}(\ln|W_n|)^{-2})$, as $n\to\infty$. Clearly

$$\frac{1}{\ln(|W_n|/\tau)}\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{I}(x+s+k\tau\in W_n) = 1+\mathcal{O}\left(\frac{1}{\ln|W_n|}\right) \tag{53}$$

as $n\to\infty$, uniformly in $x\in[-h_n,h_n]$. Using (53), we easily seen that the second term on the r.h.s. of (52) is equal to

$$\frac{a\tau}{4h_n^2\ln(|W_n|/\tau)}\int_{-h_n}^{h_n}\left(\frac{1}{\ln(|W_n|/\tau)}\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{I}(x+s+k\tau\in W_n)\right)\mathrm{d}x$$
$$=\frac{a\tau}{2h_n\ln(|W_n|/\tau)}+\mathcal{O}\left(\frac{1}{h_n(\ln|W_n|)^2}\right) \tag{54}$$

as $n\to\infty$, and (51) follows.

Next we show that the second and third term in (50) are of lower order. By (13), the second term on the r.h.s. of (50) is equal to

$$\left(s+\frac{|W_n|}{\ln(|W_n|/\tau)}\right)^2\operatorname{Var}\left(\hat{a}_n\right)=\frac{2a}{(\ln(|W_n|/\tau))^2}+\mathcal{O}\left(\frac{1}{|W_n|(\ln|W_n|)}\right) \tag{55}$$

as $n\to\infty$. By (54), (55) and the Cauchy–Schwarz inequality, we obtain that the third term on the r.h.s. of (50) is of order $o((h_n\ln|W_n|)^{-1})$ as $n\to\infty$. Combining these results, we obtain (17).

Next we prove (19). Note that

$$\mathbf{E}\tilde{\lambda}_{c,n}(s)=\mathbf{E}\bar{\lambda}_{c,n}(s)-\left(s+\frac{|W_n|}{\ln(|W_n|/\tau)}\right)\mathbf{E}\left(\hat{a}_n-a\right) \tag{56}$$

(cf. (11)) and

$$\mathbf{E}\bar{\lambda}_{c,n}(s)=\frac{1}{\ln\left(\frac{|W_n|}{\tau}\right)}\sum_{k=1}^{\infty}\frac{1}{k}\frac{\mathbf{E}X([s+k\tau-h_n,s+k\tau+h_n]\cap W_n)}{2h_n}$$
$$-a\left(s+\frac{|W_n|}{\ln\left(\frac{|W_n|}{\tau}\right)}\right). \tag{57}$$

The first term on the r.h.s. of (57) is equal to

$$
\begin{aligned}
&\frac{1}{2h_n \ln(|W_n|/\tau)} \sum_{k=1}^{\infty} \frac{1}{k} \int_{-h_n}^{h_n} \lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau \in W_n)\mathrm{d}x \\
&= \frac{1}{2h_n} \int_{-h_n}^{h_n} \lambda_c(x+s) \left(\frac{1}{\ln(|W_n|/\tau)} \sum_{k=1}^{\infty} \frac{1}{k} \mathbf{I}(x+s+k\tau \in W_n) \right) \mathrm{d}x \\
&\quad + \frac{a}{2h_n \ln(|W_n|/\tau)} \sum_{k=1}^{\infty} \frac{1}{k} \int_{-h_n}^{h_n} (x+s+k\tau)\mathbf{I}(x+s+k\tau \in W_n)\mathrm{d}x, \quad (58)
\end{aligned}
$$

where we have used (1) and (2). Since λ_c has finite second derivative λ_c'' at s, by (6) and Taylor's theorem (e.g. see Theorem B.5 of Dudley (1989), p. 413), we have

$$
\begin{aligned}
\frac{1}{2h_n} \int_{-h_n}^{h_n} \lambda_c(x+s)\mathrm{d}x &= \frac{1}{2h_n} \int_{-h_n}^{h_n} \left(\lambda_c(s) + \frac{\lambda_c'(s)}{1!}x + \frac{\lambda_c''(s)}{2!}x^2 + o(x^2) \right) \mathrm{d}x \\
&= \lambda_c(s) + \frac{\lambda_c''(s)}{6} h_n^2 + o(h_n^2) \quad (59)
\end{aligned}
$$

as $n \to \infty$. The second term on the r.h.s. of (58) is easily seen to be equal to

$$
a\left(s + \frac{|W_n|}{\ln(|W_n|/\tau)} \right) + \mathcal{O}\left(\frac{1}{\ln |W_n|} \right) \quad (60)
$$

as $n \to \infty$. Combining (53), (59) and (60), we conclude that

$$
\mathbf{E}\bar{\lambda}_{c,n}(s) = \lambda_c(s) + \frac{\lambda_c''(s)}{6} h_n^2 + o(h_n^2) + \mathcal{O}\left(\frac{1}{\ln |W_n|} \right) \quad (61)
$$

as $n \to \infty$. Using (12), the second term on the r.h.s. of (56) reduces to

$$
-\frac{2\theta}{\ln(|W_n|/\tau)} + \mathcal{O}\left(\frac{1}{|W_n|} \right) = \mathcal{O}\left(\frac{1}{\ln |W_n|} \right) \quad (62)
$$

as $n \to \infty$. By (61), (62) and assumption (18), we obtain (19). This completes the proof of Theorem 2. □

Proof of Theorem 1 By (17) and assumption (15) we obtain

$$
\mathrm{Var}\left(\tilde{\lambda}_{c,n}(s) \right) = o(1) \quad (63)
$$

as $n \to \infty$. We also easily check from the second part of the proof of Theorem 2 that

$$\mathbf{E}\tilde{\lambda}_{c,n}(s) = \lambda_c(s) + o(1) \tag{64}$$

as $n \to \infty$, provided s is a Lebesgue point of λ_c. Together (63) and (64), imply (16). This completes the proof of Theorem 1. □

4 Proofs of Theorems 3, 4 and 5

In this section we give our proofs for the case that the period τ is unknown.

Proof of Theorem 3 Let

$$\hat{\lambda}_{c,n,1}(s) := \frac{1}{\ln\left(\frac{|W_n|}{\tau}\right)} \sum_{k=1}^{\infty} \frac{1}{k} \frac{X([s + k\hat{\tau}_n - h_n, s + k\hat{\tau}_n + h_n] \cap W_n)}{2h_n} - \hat{a}_n \left(s + \frac{|W_n|}{\ln\left(\frac{|W_n|}{\hat{\tau}_n}\right)} \right) \tag{65}$$

and

$$\hat{\lambda}_{c,n,2}(s) := \frac{1}{\ln\left(\frac{|W_n|}{\tau}\right)} \sum_{k=1}^{\infty} \frac{1}{k} \frac{X([s + k\hat{\tau}_n - h_n, s + k\hat{\tau}_n + h_n] \cap W_n)}{2h_n} - \hat{a}_n \left(s + \frac{|W_n|}{\ln\left(\frac{|W_n|}{\tau}\right)} \right). \tag{66}$$

By Theorem 1, to prove this theorem it suffices to check

$$\left(\hat{\lambda}_{c,n}(s) - \hat{\lambda}_{c,n,1}(s)\right) + \left(\hat{\lambda}_{c,n,1}(s) - \hat{\lambda}_{c,n,2}(s)\right) = o_p(1), \tag{67}$$

and

$$\left(\hat{\lambda}_{c,n,2}(s) - \tilde{\lambda}_{c,n}(s)\right) = o_p(1) \tag{68}$$

as $n \to \infty$.

First we prove (68). Recall the notation $B_h(x) := [x - h, x + h]$. Then, (68) holds true, if we can show

$$\frac{1}{\ln\left(\frac{|W_n|}{\tau}\right)} \sum_{k=1}^{\infty} \frac{1}{2h_n k} \left|\left\{X\left(B_{h_n}(s + k\hat{\tau}_n) \cap W_n\right) - X\left(B_{h_n}(s + k\tau) \cap W_n\right)\right\}\right| = o_p(1) \tag{69}$$

as $n \to \infty$. To prove (69), first note that the difference within curly brackets on the l.h.s. of (69) does not exceed

$$X\left(B_{h_n}(s+k\hat{\tau}_n)\Delta B_{h_n}(s+k\tau)\cap W_n\right). \tag{70}$$

Now we notice that

$$B_{h_n-|k(\hat{\tau}_n-\tau)|}(s+k\tau)\subseteq B_{h_n}(s+k\hat{\tau}_n)\subseteq B_{h_n+|k(\hat{\tau}_n-\tau)|}(s+k\tau). \tag{71}$$

This implies that the quantity in (70) does not exceed

$$2X\left(B_{h_n+|k(\hat{\tau}_n-\tau)|}(s+k\tau)\setminus B_{h_n-|k(\hat{\tau}_n-\tau)|}(s+k\tau)\cap W_n\right). \tag{72}$$

Hence, to prove (69), it suffices to show that

$$\frac{1}{\ln\left(\frac{|W_n|}{\tau}\right)}\sum_{k=1}^{\infty}\frac{1}{h_n k}X\left(B_{h_n+|k(\hat{\tau}_n-\tau)|}(s+k\tau)\setminus B_{h_n-|k(\hat{\tau}_n-\tau)|}\right.$$
$$\left.\times\,(s+k\tau)\cap W_n\right)=o_p(1) \tag{73}$$

as $n \to \infty$. To prove (73) we argue as follows. First note that, for any k such that $(s+k\tau)\in W_n$, we have $|k| = \mathcal{O}(|W_n|)$ as $n\to\infty$. Let Λ_n denotes the l.h.s. of (73), and let also $\epsilon > 0$ be any fixed real number. Then, for any fixed $\delta > 0$, we have

$$\mathbf{P}\left(|\Lambda_n|\geq\epsilon\right)\leq\mathbf{P}\left(\{|\Lambda_n|\geq\epsilon\}\cap\left\{|W_n||\hat{\tau}_n-\tau|\leq\delta h_n(\ln(|W_n|\tau^{-1}))|W_n|^{-1}\right\}\right)$$
$$+\mathbf{P}\left(|W_n||\hat{\tau}_n-\tau|>\delta h_n(\ln(|W_n|\tau^{-1}))|W_n|^{-1}\right). \tag{74}$$

By assumption (25), we have that the second term on the r.h.s. of (74) is $o_p(1)$, as $n\to\infty$. Let $\alpha_n := (\ln(|W_n|\tau^{-1}))|W_n|^{-1}$. Then the first term on the r.h.s. of (74), does not exceed $\mathbf{P}(|\bar{\Lambda}_n|\geq\epsilon)$, where

$$\bar{\Lambda}_n=\frac{1}{\ln(|W_n|/\tau)}\sum_{k=1}^{\infty}\frac{1}{h_n k}X\left(B_{h_n+\delta h_n\alpha_n}(s+k\tau)\setminus B_{h_n-\delta h_n\alpha_n}(s+k\tau)\cap W_n\right). \tag{75}$$

Next, by Markov inequality for the first moment, we have that $\mathbf{P}(|\bar{\Lambda}_n|\geq\epsilon)\leq\epsilon^{-1}\mathbf{E}|\bar{\Lambda}_n|$, and $\epsilon^{-1}\mathbf{E}|\bar{\Lambda}_n|$ can also be written as

$$\frac{1}{\epsilon\ln\left(\frac{|W_n|}{\tau}\right)}\sum_{k=1}^{\infty}\frac{1}{h_n k}\int_{B_{(1+\delta\alpha_n)h_n}(0)\setminus B_{(1-\delta\alpha_n)h_n}(0)}\lambda(x+s+k\tau)$$
$$\times\mathbf{I}(x+s+k\tau\in W_n)\mathrm{d}x. \tag{76}$$

Since $\lambda(x+s+k\tau) = \lambda_c(x+s) + a(x+s+k\tau)$, the quantity in (76) can be written as

$$\frac{1}{\epsilon \ln(|W_n|/\tau)} \frac{1}{h_n} \int_{B_{(1+\delta\alpha_n)h_n}(0)\backslash B_{(1-\delta\alpha_n)h_n}(0)} (\lambda_c(x+s) + a(x+s)) \left(L_{n,x}\right) \mathrm{d}x$$
$$+ \frac{a\tau}{\epsilon \ln(|W_n|/\tau)} \frac{1}{h_n} \int_{B_{(1+\delta\alpha_n)h_n}(0)\backslash B_{(1-\delta\alpha_n)h_n}(0)} \left(\sum_{k=1}^{\infty} \mathbf{I}(x+s+k\tau \in W_n)\right) \mathrm{d}x. \quad (77)$$

where $L_{n,x} = \sum_{k=1}^{\infty} k^{-1}\mathbf{I}(x+s+k\tau \in W_n)$. Since $L_{n,x} \sim \ln(|W_n|/\tau)$ as $n \to \infty$ uniformly in x, a simple calculation shows that the first term of (77) is of order $\mathcal{O}(\alpha_n) = \mathcal{O}((\ln|W_n|)|W_n|^{-1})$ as $n \to \infty$. Hence, the proof of Theorem 3 is complete if we can show that the second term of (77) is $o(1)$ as $n \to \infty$. To show this, first note that $\sum_{k=1}^{\infty} \mathbf{I}(s+k\tau+x \in W_n) \leq |W_n|\tau^{-1} + 1 \leq 2|W_n|\tau^{-1}$, for n large enough. Then, the quantity in the last term of (77) does not exceed

$$\frac{2a|W_n|}{\epsilon h_n \ln\left(\frac{|W_n|}{\tau}\right)} \left|B_{(1+\delta\alpha_n)h_n}(0)\backslash B_{(1-\delta\alpha_n)h_n}(0)\right| = \frac{2a|W_n|}{\epsilon h_n \ln\left(\frac{|W_n|}{\tau}\right)} (4\delta\alpha_n h_n) = \frac{8a\delta}{\epsilon}. \quad (78)$$

By taking $\delta = \delta_n \downarrow 0$ as $n \to \infty$, we have that the quantity in (78) converges to zero as $n \to \infty$.

Next we prove (67). The second term on the l.h.s. of (67) is equal to

$$\hat{a}_n|W_n| \left(\frac{1}{\ln(|W_n|/\hat{\tau}_n)} - \frac{1}{\ln(|W_n|/\tau)}\right).$$

A simple calculation shows that

$$\left(\frac{1}{\ln(|W_n|/\hat{\tau}_n)} - \frac{1}{\ln(|W_n|/\tau)}\right) = \mathcal{O}_p\left(\frac{(\hat{\tau}_n - \tau)}{(\ln(|W_n|/\tau))^2}\right) \quad (79)$$

as $n \to \infty$. By assumption (25) and (79), we obtain the second term on the l.h.s. of (67) is $o_p(1)$ as $n \to \infty$. The first term on the l.h.s. of (67) is equal to

$$\left(\frac{1}{\ln(|W_n|/\hat{\tau}_n)} - \frac{1}{\ln(|W_n|/\tau)}\right) \sum_{k=1}^{\infty} \frac{1}{k} \frac{X([s+k\hat{\tau}_n - h_n, s+k\hat{\tau}_n + h_n] \cap W_n)}{2h_n}.$$

Since $\hat{\lambda}_{c,n,2}(s) = \lambda_c(s) + o_p(1)$, we have

$$\sum_{k=1}^{\infty} \frac{1}{k} \frac{X([s+k\hat{\tau}_n - h_n, s+k\hat{\tau}_n + h_n] \cap W_n)}{2h_n} = \mathcal{O}_p(|W_n|), \quad (80)$$

as $n \to \infty$. By assumption (25), (79) and (80), we obtain the first term on the l.h.s. of (67) is $o_p(1)$ as $n \to \infty$. This completes the proof of Theorem 3. □

Proof of Theorem 4 Denote

$$B_n = \left\{ |\hat{\tau}_n - \tau| \leq \frac{\delta h_n^3 \ln |W_n|}{|W_n|^2} \right\}. \tag{81}$$

Then we can write $\hat{\lambda}^{\diamond}_{c,n}(s)$ as follows:

$$\hat{\lambda}^{\diamond}_{c,n}(s) = \mathbf{I}(B_n)\hat{\lambda}_{c,n}(s) - R_{n,1} + R_{n,2} + R_{n,3} + R_{n,4}, \tag{82}$$

where

$$\begin{aligned} R_{n,1} &= \mathbf{I}(B_n)\mathbf{I}\left(\hat{\lambda}_{c,n}(s) > D_n\right)\hat{\lambda}_{c,n}(s) \\ R_{n,2} &= \mathbf{I}(B_n^c)\mathbf{I}\left(\hat{\lambda}_{c,n}(s) \leq D_n\right)\hat{\lambda}_{c,n}(s) \\ R_{n,3} &= \mathbf{I}(B_n)\mathbf{I}\left(\hat{\lambda}_{c,n}(s) > D_n\right)D_n \\ R_{n,4} &= \mathbf{I}(B_n^c)\mathbf{I}\left(\hat{\lambda}_{c,n}(s) > D_n\right)D_n. \end{aligned}$$

We see that $\mathbf{E}(R_{n,2} + R_{n,4}) \leq 2D_n\mathbf{P}(B_n^c)$. By the assumption (28), this quantity is of order $o(h_n^2)$, as $n \to \infty$. While

$$\mathbf{E}(R_{n,3} - R_{n,1}) \leq 2D_n^{-r}\mathbf{E}\mathbf{I}(B_n)\hat{\lambda}^{1+r}_{c,n}(s) \leq 2c^{-r}h^{r\epsilon}\mathbf{E}\mathbf{I}(B_n)\hat{\lambda}^{1+r}_{c,n}(s)$$

for sufficiently large n. The latter inequality is due to the lower bound of D_n. Since, for sufficiently large n we have $B_n \subset A_n$, then by Lemma 2 and choosing sufficiently large r such that $r\epsilon > 2$, we have this term is of order $o(h_n^2)$, as $n \to \infty$. Hence, it remains to show

$$\mathbf{E}\left(\mathbf{I}(B_n)\hat{\lambda}_{c,n}(s)\right) = \lambda_c(s) + \frac{\lambda_c''(s)}{6}h_n^2 + o(h_n^2), \tag{83}$$

as $n \to \infty$. To prove (83) we argue as follows. First we write

$$\begin{aligned} \mathbf{E}\left(\mathbf{I}(B_n)\hat{\lambda}_{c,n}(s)\right) &= \mathbf{E}\mathbf{I}(B_n)\left(\hat{\lambda}_{c,n}(s) - \hat{\lambda}_{c,n,1}(s)\right) + \mathbf{E}\mathbf{I}(B_n)\left(\hat{\lambda}_{c,n,1}(s) - \hat{\lambda}_{c,n,2}(s)\right) \\ &\quad + \mathbf{E}\mathbf{I}(B_n)\left(\hat{\lambda}_{c,n,2}(s) - \tilde{\lambda}_{c,n}(s)\right) + \mathbf{E}\tilde{\lambda}_{c,n}(s) - \mathbf{E}\mathbf{I}(B_n^c)\tilde{\lambda}_{c,n}(s), \end{aligned} \tag{84}$$

where $\tilde{\lambda}_{c,n}(s)$, $\hat{\lambda}_{c,n,1}(s)$ and $\hat{\lambda}_{c,n,2}(s)$ are given respectively by (5), (65) and (66). By Theorem 2, we have $\mathbf{E}\tilde{\lambda}_{c,n}(s) = \lambda_c(s) + (\lambda_c''(s)h_n^2)/6 + o(h_n^2)$, as $n \to \infty$. By Lemma 2 (for the case τ is known), for any positive integer m, we have $\mathbf{E}\tilde{\lambda}^{2m}_{c,n}(s) = \mathcal{O}(1)$, as $n \to \infty$. Then, by assumption (28) and Hölder's inequality, we obtain $\mathbf{E}\mathbf{I}(B_n^c)\tilde{\lambda}_{c,n}(s) = o(h_n^2)$, as $n \to \infty$. A simple calculation using assumption (28), also shows that the first and second term on the r.h.s. of (84) is of order $o(h_n^2)$, as $n \to \infty$. Hence, it remains to check that the third term on the r.h.s. of (84) is of order $o(h_n^2)$, as $n \to \infty$.

This term does not exceed

$$\mathbf{E}\mathbf{I}(B_n)\frac{1}{\ln(|W_n|/\tau)}\sum_{k=1}^{\infty}\frac{1}{2h_nk}\left|\left\{X\left(B_{h_n}(s+k\hat{\tau}_n)\cap W_n\right)-X\left(B_{h_n}(s+k\tau)\cap W_n\right)\right\}\right|. \tag{85}$$

By a similar calculation as in the proof of (69), but with assumption (25) now replaced by assumption (28), we obtain that the quantity in (85) is of order $o(h_n^2)$, as $n\to\infty$. This completes the proof of Theorem 4. □

Proof of Theorem 5 Denote

$$C_n=\left\{|\hat{\tau}_n-\tau|\le\frac{\delta h_n^{1/2}(\ln|W_n|)^{1/2}}{|W_n|^2}\right\}. \tag{86}$$

Then we can write $\hat{\lambda}^{\diamond}_{c,n}(s)$ as follows:

$$\hat{\lambda}^{\diamond}_{c,n}(s)=\mathbf{I}(C_n)\hat{\lambda}_{c,n}(s)-R^*_{n,1}+R^*_{n,2}+R^*_{n,3}+R^*_{n,4}, \tag{87}$$

where $R^*_{n,1}$, $R^*_{n,2}$, $R^*_{n,3}$ and $R^*_{n,4}$ are the same as respectively $R_{n,1}$, $R_{n,2}$, $R_{n,3}$ and $R_{n,4}$ in (82) but with set B_n replaced by set C_n. Then, to prove this theorem it suffices to check that

$$\mathrm{Var}\left(\mathbf{I}(C_n)\hat{\lambda}_{c,n}(s)\right)=\frac{a\tau}{2h_n\ln|W_n|}+o\left(\frac{1}{h_n\ln|W_n|}\right) \tag{88}$$

and

$$\mathbf{E}\left(R^*_{n,1}\right)^2+\mathbf{E}\left(R^*_{n,2}\right)^2+\mathbf{E}\left(R^*_{n,3}\right)^2+\mathbf{E}\left(R^*_{n,4}\right)^2=o\left(\frac{1}{h_n\ln|W_n|}\right) \tag{89}$$

as $n\to\infty$. First, we consider (89). We see that $\mathbf{E}(R^*_{n,2})^2+\mathbf{E}(R^*_{n,4})^2\le 2D_n^2\mathbf{P}(C_n^c)$. By the assumption (30), this quantity is of order $o((h_n\ln|W_n|)^{-1})$, as $n\to\infty$. While

$$\mathbf{E}(R^*_{n,1})^2+\mathbf{E}(R^*_{n,3})^2\le 2D_n^{-r}\mathbf{E}\mathbf{I}(C_n)\hat{\lambda}^{2+r}_{c,n}(s)\le(2\mathbf{E}\mathbf{I}(C_n)\hat{\lambda}^{1+r}_{c,n}(s))/(c^r(h_n\ln|W_n|)^{r\epsilon}),$$

for sufficiently large n. The latter inequality is due to the bound $D_n\ge c(h_n\ln|W_n|)^{\epsilon}$. Since, for sufficiently large n we have $C_n\subset A_n$, then by Lemma 2 and choosing sufficiently large r such that $r\epsilon>1$, we have this term is of order $o((h_n\ln|W_n|)^{-1})$, as $n\to\infty$.

Next, we consider (88). To prove (88) we argue as follows. First we write

$$\begin{aligned}\mathbf{I}(C_n)\hat{\lambda}_{c,n}(s)&=\mathbf{I}(C_n)\left(\hat{\lambda}_{c,n}(s)-\hat{\lambda}_{c,n,1}(s)\right)+\mathbf{I}(C_n)\left(\hat{\lambda}_{c,n,1}(s)-\hat{\lambda}_{c,n,2}(s)\right)\\&\quad+\mathbf{I}(C_n)\left(\hat{\lambda}_{c,n,2}(s)-\tilde{\lambda}_{c,n}(s)\right)+\tilde{\lambda}_{c,n}(s)-\mathbf{I}(C_n^c)\tilde{\lambda}_{c,n}(s),\end{aligned} \tag{90}$$

where $\tilde{\lambda}_{c,n}(s)$, $\hat{\lambda}_{c,n,1}(s)$ and $\hat{\lambda}_{c,n,2}(s)$ are given respectively by (5), (65) and (66). Then, to prove (88), it suffices to check

$$\operatorname{Var}\left(\tilde{\lambda}_{c,n}(s)\right) = \frac{a\tau}{2h_n \ln|W_n|} + o\left(\frac{1}{h_n \ln|W_n|}\right), \tag{91}$$

$$\mathbf{E}\left(\mathbf{I}(C_n)\left(\hat{\lambda}_{c,n,2}(s) - \tilde{\lambda}_{c,n}(s)\right)\right)^2 = o\left(\frac{1}{h_n \ln|W_n|}\right), \tag{92}$$

and

$$\mathbf{E}\left(\mathbf{I}(C_n)\left(\hat{\lambda}_{c,n}(s) - \hat{\lambda}_{c,n,1}(s)\right)\right)^2 + \mathbf{E}\left(\mathbf{I}(C_n)\left(\hat{\lambda}_{c,n,1}(s) - \hat{\lambda}_{c,n,2}(s)\right)\right)^2 + \mathbf{E}\left(\mathbf{I}(C_n^c)\tilde{\lambda}_{c,n}(s)\right)^2 = o\left(\frac{1}{h_n \ln|W_n|}\right) \tag{93}$$

as $n \to \infty$. By Theorem 2, we have (91). A simple calculation using assumption (30) shows that the first and second term on the l.h.s. of (93) is of order $o(h_n^{-1}(\ln|W_n|)^{-1})$ as $n \to \infty$. By Lemma 2 (for the case τ is known), for every positive integer m, we have $\mathbf{E}\tilde{\lambda}_{c,n}^{2m}(s) = \mathcal{O}(1)$, as $n \to \infty$. Then, by assumption (30) and Hölder's inequality, we obtain that the third term on the l.h.s. of (93) is of order $o(h_n^{-1}(\ln|W_n|)^{-1})$ as $n \to \infty$. Hence, it remains to prove (92).

The l.h.s. of (92) does not exceed

$$\frac{1}{\left(\ln\left(\frac{|W_n|}{\tau}\right)\right)^2}\mathbf{E} \times\left(\sum_{k=1}^{\infty}\frac{1}{2h_n k}\left|\left\{X\left(B_{h_n}(s+k\hat{\tau}_n)\cap W_n\right) - X\left(B_{h_n}(s+k\tau)\cap W_n\right)\right\}\right|\mathbf{I}(C_n)\right)^2. \tag{94}$$

By writing the square of the sum in (94) as a double sum $\sum_k \sum_j$ we can, by Fubini's theorem, interchange expectation and summation. Next we split the double sum into two parts, one corresponding to the case of different indices $k \neq j$, and the other one to the case $k = j$. For sufficiently large n, we can now write the quantity in (94) as

$$\begin{aligned} &\frac{1}{\left(\ln\left(\frac{|W_n|}{\tau}\right)\right)^2}\sum_{k=1}^{\infty}\sum_{j\neq k}^{\infty}\frac{1}{4h_n^2k^2}\left(\mathbf{E}\left|\left\{X\left(B_{h_n}(s+k\hat{\tau}_n)\cap W_n\right)\right.\right.\right. \\ &\left.\left.\left.-X\left(B_{h_n}(s+k\tau)\cap W_n\right)\right\}\right|\mathbf{I}(C_n)\right)\times\left(\mathbf{E}\left|\left\{X\left(B_{h_n}(s+j\hat{\tau}_n)\cap W_n\right)\right.\right.\right. \\ &\left.\left.\left.-X\left(B_{h_n}(s+j\tau)\cap W_n\right)\right\}\right|\mathbf{I}(C_n)\right) + \frac{1}{\left(\ln\left(\frac{|W_n|}{\tau}\right)\right)^2}\sum_{k=1}^{\infty}\frac{1}{4h_n^2k^2} \\ &\mathbf{E}\left(\left\{X\left(B_{h_n}(s+k\hat{\tau}_n)\cap W_n\right) - X\left(B_{h_n}(s+k\tau)\cap W_n\right)\right\}\right)^2\mathbf{I}(C_n). \end{aligned} \tag{95}$$

Now we see that, for large n, the second term of (95) does not exceed the first term. Hence, for large n, the quantity in (95) does not exceed

$$2\left(\frac{1}{\ln\left(\frac{|W_n|}{\tau}\right)}\sum_{k=1}^{\infty}\frac{1}{2h_nk}\mathbf{E}\left|\left\{X\left(B_{h_n}(s+k\hat{\tau}_n)\cap W_n\right)\right.\right.\right.$$
$$\left.\left.\left.-X\left(B_{h_n}(s+k\tau)\cap W_n\right)\right\}\right|\mathbf{I}(C_n)\right)^2. \tag{96}$$

Then, by a similar calculation as that in the proof of (69), but with assumption (25) replaced by (30), we obtain that the quantity in (96) is of order $o((h_n \ln |W_n|)^{-1})$, as $n \to \infty$. Hence, we obtain (92). This completes the proof of Theorem 5. □

In the proofs of Theorems 4 and 5 we require the following lemma. Consider the assumption (25) and let

$$A_n = \left\{|\hat{\tau}_n - \tau| \leq \frac{\delta h_n \ln |W_n|}{|W_n|^2}\right\}.$$

Lemma 2 *Suppose that the intensity function λ satisfies (1) and is locally integrable. If, in addition, $h_n \downarrow 0$ and $h_n \ln |W_n| \to \infty$, then for any positive integer m we have*

$$\mathbf{E}\left(\mathbf{I}(A_n)\hat{\lambda}_{c,n}^{2m}(s)\right) = \mathcal{O}(1) \tag{97}$$

as $n \to \infty$, provided s is a Lebesgue point of λ_c.

Proof First we write

$$\begin{aligned}\mathbf{I}(A_n)\hat{\lambda}_{c,n}(s) &= \mathbf{I}(A_n)\left(\hat{\lambda}_{c,n}(s)-\hat{\lambda}_{c,n,1}(s)\right)+\mathbf{I}(A_n)\left(\hat{\lambda}_{c,n,1}(s)-\hat{\lambda}_{c,n,2}(s)\right)\\ &\quad+\mathbf{I}(A_n)\left(\hat{\lambda}_{c,n,2}(s)-\tilde{\lambda}_{c,n}(s)\right)+\mathbf{I}(A_n)\left(\tilde{\lambda}_{c,n}(s)-\mathbf{E}\tilde{\lambda}_{c,n}(s)\right)\\ &\quad+\mathbf{I}(A_n)\mathbf{E}\tilde{\lambda}_{c,n}(s)\end{aligned} \tag{98}$$

where $\tilde{\lambda}_{c,n}(s)$, $\hat{\lambda}_{c,n,1}(s)$ and $\hat{\lambda}_{c,n,2}(s)$ are given respectively by (5), (65) and (66). Then, to prove (97) it suffices to check

$$\begin{aligned}&\mathbf{EI}(A_n)\left(\hat{\lambda}_{c,n}(s)-\hat{\lambda}_{c,n,1}(s)\right)^{2m}+\mathbf{EI}(A_n)\left(\hat{\lambda}_{c,n,1}(s)-\hat{\lambda}_{c,n,2}(s)\right)^{2m}\\ &\quad+\mathbf{EI}(A_n)\left(\hat{\lambda}_{c,n,2}(s)-\tilde{\lambda}_{c,n}(s)\right)^{2m}+\mathbf{E}\left(\tilde{\lambda}_{c,n}(s)-\mathbf{E}\tilde{\lambda}_{c,n}(s)\right)^{2m}\\ &\quad+\left(\mathbf{E}\tilde{\lambda}_{c,n}(s)\right)^{2m}=\mathcal{O}(1)\end{aligned} \tag{99}$$

as $n \to \infty$. Now we see that the leading term on the l.h.s. of (99) is $(\mathbf{E}\tilde{\lambda}_{c,n}(s))^{2m}$. Because of (64), we can write

$$\left(\mathbf{E}\tilde{\lambda}_{c,n}(s)\right)^{2m} = (\lambda_c(s) + o(1))^{2m} = (\lambda_c(s))^{2m} + o(1), \tag{100}$$

which is $\mathcal{O}(1)$ as $n \to \infty$. From the proof of (92), but with the set C_n replaced by A_n, we see for the case $m = 1$ that the first term on the l.h.s. of (99) is $o(1)$ as $n \to \infty$. This argument can be extended to the case $m > 1$, and we conclude that this term is asymptotically bounded. Note that we did not require this lemma in the proof of (92). For the case $m = 1$, the second term on the l.h.s. of (99) is nothing but the variance of $\tilde{\lambda}_{c,n}(s)$. By (63), we see that this term is $o(1)$ as $n \to \infty$. The argument in the proof of (63) can also be extended to the case $m > 1$, to conclude that this term is asymptotically bounded. This completes the proof of Lemma 2. □

5 Proofs of Corollaries 1 and 2

In this section we derive second order terms for bias and variance of $\tilde{\lambda}_{c,n}(s)$ (cf. Corollary 1). Similar results for a bias corrected estimator $\tilde{\lambda}_{c,n,BC}(s)$ (cf. Corollary 2) are also established.

Proof of Corollary 1 First we check (38). To do this we follow the argument given in (50) – (52) and note that $\sum_{k=1}^{\infty} k^{-2}\mathbf{I}(x + s + k\tau \in W_n) = \pi^2/6 + o(1)$ as $n \to \infty$ uniformly in $x \in [-h_n, h_n]$. This directly yields that the first term on the r.h.s. of (52) is equal to

$$\frac{(\lambda_c(s) + as)(\pi^2/6)}{2h_n(\ln(|W_n|/\tau))^2} + o\left(\frac{1}{h_n(\ln |W_n|)^2}\right) \tag{101}$$

as $n \to \infty$.

The second term on the r.h.s. of (52) is easily seen to be equal to

$$\frac{a\tau}{2h_n \ln(|W_n|/\tau)} + \frac{a\tau\gamma}{2h_n(\ln(|W_n|/\tau))^2} + o\left(\frac{1}{h_n(\ln |W_n|)^2}\right) \tag{102}$$

as $n \to \infty$, where we have used the well-known fact that

$$\sum_{k=1}^{\infty} k^{-1}\mathbf{I}(x + s + k\tau \in W_n) = \ln(|W_n|/\tau) + \gamma + o(1) \tag{103}$$

as $n \to \infty$ uniformly in $x \in [-h_n, h_n]$. Finally we note that simple computations show that the second and third term on the r.h.s. of (50) are of lower order. This completes the proof of (38).

Next we prove (39). Since λ_c has finite fourth derivative $\lambda_c^{(\mathrm{iv})}$ at s, by (6) and Taylor's theorem (e.g. see Theorem B.5 of Dudley 1989, p. 413), we have

$$\frac{1}{2h_n}\int_{-h_n}^{h_n}\lambda_c(x+s)\mathrm{d}x = \lambda_c(s) + \frac{\lambda_c''(s)}{6}h_n^2 + \frac{\lambda_c^{(\mathrm{iv})}(s)}{120}h_n^4 + o(h_n^4) \tag{104}$$

as $n \to \infty$. In view of (103), (56), (57) and (104), we easily check that the first term on the r.h.s. of (58) is equal to

$$\lambda_c(s) + \frac{\lambda_c''(s)}{6}h_n^2 + \frac{\lambda_c^{(\mathrm{iv})}(s)}{120}h_n^4 + \frac{\lambda_c(s)\gamma}{\ln(|W_n|/\tau)} + o\left(h_n^4 + \frac{1}{\ln|W_n|}\right) \tag{105}$$

as $n \to \infty$.

The second term on the r.h.s. of (58) is equal to

$$\begin{aligned}&\frac{a}{2h_n\ln(|W_n|/\tau)}\sum_{k=1}^{\infty}\frac{1}{k}\int_{-h_n}^{h_n}(x+s)\mathbf{I}(x+s+k\tau\in W_n)\mathrm{d}x\\&\quad+\frac{a\tau}{2h_n\ln(|W_n|/\tau)}\sum_{k=1}^{\infty}\int_{-h_n}^{h_n}\mathbf{I}(x+s+k\tau\in W_n)dx.\end{aligned} \tag{106}$$

Using (103), it easily verified that the first term of (106) reduces to

$$as + \frac{as\gamma}{\ln(|W_n|/\tau)} + o\left(\frac{1}{\ln|W_n|}\right) \tag{107}$$

as $n \to \infty$. With $\zeta_n = (2h_n)^{-1}\sum_{k=1}^{\infty}\int_{-h_n}^{h_n}\mathbf{I}(x+s+k\tau\in W_n)dx - |W_n|/\tau$ (cf. also Corollary 1) we easily see that the second term of (106) can be written as

$$\frac{a|W_n|}{\ln(|W_n|/\tau)} + \frac{a\tau\zeta_n}{\ln(|W_n|/\tau)}. \tag{108}$$

The sum of the first term in (107) and the first term in (108) cancels with the second term on the r.h.s. of (57). Note also that the l.h.s. of (62) and the line preceding it yields the term $-2\theta/(\ln(|W_n|/\tau))$ appearing in (39). Combining this with (105), the second term of (107) and the second term of (108), we obtain the r.h.s. of (39). This completes the proof of Corollary 1. □

Proof of Corollary 2 First we prove (43). Note first that the bias corrected estimator $\tilde{\lambda}_{c,n,BC}(s)$ given in (42) can be written as

$$\left(1 - \frac{\gamma}{\ln(|W_n|/\tau)}\right)\tilde{\lambda}_{c,n}(s) - \frac{\left(\tilde{\lambda}_{c,n,h_n'}(s+2h_n') + \tilde{\lambda}_{c,n,h_n'}(s-2h_n') - 2\tilde{\lambda}_{c,n,h_n'}(s)\right) h_n^2}{24(h_n')^2}$$
$$+ \frac{2\hat{\theta}_{n,b} - (\gamma s + \tau\zeta_n)\hat{a}_n}{\ln(|W_n|/\tau)}. \tag{109}$$

A simple calculation using (41) shows that $\mathrm{Var}(\hat{\theta}_{n,b}) = a(\ln(|W_n|/\tau))^{-1} + \mathcal{O}((\ln|W_n|)^{-2})$ as $n \to \infty$. This, together with (13), (38) and Cauchy–Schwarz, easily implies that both the variance of the third term in (109) and the covariances of this term with the other two terms are of negligible order $o(h_n^{-1}(\ln|W_n|)^{-2})$, as $n \to \infty$.

It remains to show that the variance of the sum of the first two terms in (109) is equal to the r.h.s. of (43). To verify this, let $A_{n,h_n}(s)$ and $-B_n(s)$ denote respectively the first and second term on the r.h.s. of (5), in other words we write $\tilde{\lambda}_{c,n}(s) = \tilde{\lambda}_{c,n,h_n}(s) = A_{n,h_n}(s) - B_n(s)$. Then, simple algebra shows that the sum of the first and second term in (109) can be written as

$$\left(1 - \frac{\gamma}{\ln(|W_n|/\tau)}\right) A_{n,h_n}(s) - \frac{A_{n,h_n'}(s+2h_n)\, h_n^2}{24(h_n')^2} - \frac{A_{n,h_n'}(s-2h_n)\, h_n^2}{24(h_n')^2}$$
$$+ \frac{A_{n,h_n'}(s)\, h_n^2}{12(h_n')^2} - \left(1 - \frac{\gamma}{\ln(|W_n|/\tau)}\right) B_n(s). \tag{110}$$

From the proof of Corollary 1, we can infer that $\mathrm{Var}(A_{n,h_n}(s))$ is equal to the r.h.s. of (38). Similarly, using $h_n = o(h_n')$ as $n \to \infty$ and the fact λ_c is continuous at s, we also see that $\mathrm{Var}(A_{n,h_n'}(s+2h_n'))$, $\mathrm{Var}(A_{n,h_n'}(s-2h_n'))$ and $\mathrm{Var}(A_{n,h_n'}(s))$ are all equal to

$$\frac{a\tau}{2h_n' \ln(|W_n|/\tau)} + o\left(\frac{1}{h_n(\ln|W_n|)^2}\right) \tag{111}$$

as $n \to \infty$. Next we show that the variance of the last term in (110) and the covariances of this term with all the other terms in (110) are of negligible order $o(h_n^{-1}(\ln|W_n|)^{-2})$, as $n \to \infty$. To verify this, note that from the proof of Theorem 1 we can infer that $\mathrm{Var}(B_n(s))$ is of order $o(h_n^{-1}(\ln|W_n|)^{-2})$ as $n \to \infty$ (cf. (55)). From the proof of Corollary 1 we know that $\mathrm{Cov}(A_{n,h_n}(s), B_n(s))$ is of order $o(h_n^{-1}(\ln|W_n|)^{-2})$ as $n \to \infty$. By a similar argument we also see that $\mathrm{Cov}(A_{n,h_n'}(s+2h_n'), B_n(s))$, $\mathrm{Cov}(A_{n,h_n'}(s-2h_n'), B_n(s))$ and $\mathrm{Cov}(A_{n,h_n'}(s), B_n(s))$ are all of order $o(h_n^{-1}(\ln|W_n|)^{-2})$, as $n \to \infty$. Hence the variance of the last term in (110) and the covariances of this term with the other terms in (110) are indeed of negligible order $o(h_n^{-1}(\ln|W_n|)^{-2})$, as $n \to \infty$.

We can conclude that it suffices now to show that the variance of the sum of the first four terms in (110) is equal to the r.h.s. of (43). To prove this, we argue as follows: Since h_n is of lower order than h_n', for n sufficiently large, we easily check that, $A_{n,h_n}(s)$ and $A_{n,h_n'}(s+2h_n')$, $A_{n,h_n}(s)$ and $A_{n,h_n'}(s-2h_n')$, $A_{n,h_n'}(s)$ and $A_{n,h_n'}(s+2h_n')$, $A_{n,h_n'}(s)$ and $A_{n,h_n'}(s-2h_n')$ and also $A_{n,h_n'}(s+2h_n')$ and $A_{n,h_n'}(s-2h_n')$ are all independent. The same argument is also valid when $h_n = h_n'$ (cf. (45)). A simple calculation shows that $\mathrm{Cov}(A_{n,h_n}(s), A_{n,h_n'}(s))$ is equal to the quantity in (111). Hence, the variance of the sum of the first four terms in (110) is equal to the sum of variances of the first four terms in (110) plus two times the covariance of its first and fourth term. By (38), we see that variance of the first term in (110) is equal to the sum of the first and fourth term on the r.h.s. of (43). Note that the $+a\tau\gamma$ coefficient in the second term on the r.h.s. of (38) is replaced by $-a\tau\gamma$ in the fourth term on the r.h.s. of (43). This is because the variance of the first term in (110) is equal to the r.h.s. of (38) multiplied by $\left(1-\frac{\gamma}{\ln(|W_n|/\tau)}\right)^2$. From the fact that $\mathrm{Cov}(A_{n,h_n}(s), A_{n,h_n'}(s))$ is equal to the quantity in (111), it easily seen that two times covariance of the first and fourth term in (110) is equal to the second term on the r.h.s. of (43). By (111) and the line preceding it, and by noting that $\frac{1}{24^2}+\frac{1}{24^2}+\frac{1}{12^2}=\frac{1}{96}$, the sum of the variances of the second, third and fourth terms in (110) is equal to the third term on the r.h.s. of (43). This completes the proof of (43).

Next we prove (44). Since λ_c has finite fourth derivative $\lambda_c^{(\mathrm{iv})}$ in the neighborhood of s and $h_n' \downarrow 0$ as $n \to \infty$, a simple calculation using (39) and Taylor's theorem (e.g. see Theorem B.5 of Dudley 1989, p. 413), shows that

$$\mathbf{E}\tilde{\lambda}_{c,n}''(s) = \lambda_c''(s) + \frac{\lambda_c^{(\mathrm{iv})}(s)}{2}(h_n')^2 + o\left((h_n')^2 + \frac{1}{(h_n')^2 \ln|W_n|}\right) \tag{112}$$

as $n \to \infty$. A simple calculation using (12) and (40) shows that $\mathbf{E}\hat{\theta}_{n,b} = \theta + o((\ln|W_n|)^{-1})$ as $n \to \infty$. Replacing $\tilde{\lambda}_{c,n}(s)$, $\hat{a}_n$, $\tilde{\lambda}_{c,n}''(s)$ and $\hat{\theta}_{n,b}$ on the r.h.s. of (42) by respectively $\mathbf{E}\tilde{\lambda}_{c,n}(s)$ (cf. (39)), $\mathbf{E}\hat{a}_n$ (cf. (12)), $\mathbf{E}\tilde{\lambda}_{c,n}''(s)$ (cf. (112)) and $\mathbf{E}\hat{\theta}_{n,b}$, we directly obtain (44). The same argument is also valid when $h_n' = h_n$ (cf. (46)). This completes the proof of Corollary 2. □

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