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Multiple comparisons of several homoscedastic multivariate populations

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Abstract The limiting joint distribution of correlated Hotelling's T^2 statistics associated with multiple comparisons with a control in multivariate one-way layout model is a multivariate central nonsingular chi-square distribution with one-factorial correlation matrix, which has the distribution function expressed in a closed form as an integral of a product of noncentral chi-square distribution functions with respect to a central chi-square density function. For pairwise comparisons, it is a multivariate central singular chi-square distribution whose distribution function is generally intricate. To overcome the complexity of the (exact or asymptotic) distribution theory of $T^2_{\max}$-type statistics appeared in simultaneous confidence intervals of mean vectors, improved Bonferroni-type inequalities are applied to construct asymptotically conservative simultaneous confidence intervals for pairwise comparisons as well as comparisons with a control.

Keywords Multiple comparisons · Bonferroni-type inequality · Maximum of correlated Hotelling's T^2 statistics · Multivariate central nonsingular or singular chi-square distribution

1 Introduction

Given $q \geq 2$ levels, let $\boldsymbol{X}_i^{(a)} = (X_{1i}^{(a)}, \ldots, X_{pi}^{(a)})'$ be the i-th observation on the a-th level and assume the linear model (one-way layout model)

$$\boldsymbol{X}_i^{(a)} = \boldsymbol{\theta}^{(a)} + \boldsymbol{U}_i^{(a)}, \quad a = 1, \ldots, q\,;\; i = 1, \ldots, N_a, \tag{1}$$

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where $\boldsymbol{U}_i^{(a)}$s are (unobservable) independent $p \times 1$ random vectors with mean zero vector and positive definite covariance matrix Σ, which is assumed to be unknown. The total number of such vectors is $\sum_{a=1}^{q} N_a = N$ (say). In the model (1), the least squares estimates of the $\boldsymbol{\theta}^{(a)}$'s are given by the sample mean vector $\overline{\boldsymbol{X}}^{(a)} = N_a^{-1} \sum_{i=1}^{N_a} \boldsymbol{X}_i^{(a)}$, $a = 1, \ldots, q$. Due to the equality of covariance matrices, a usual unbiased estimate of Σ is given by the pooled sample covariance matrix

$$S_{\mathrm{pool,X}} = \frac{1}{N-q} \sum_{a=1}^{q} (N_a - 1)\, S_X^{(a)},$$

where $S_X^{(a)} = (N_a - 1)^{-1} \sum_{i=1}^{N_a} (\boldsymbol{X}_i^{(a)} - \overline{\boldsymbol{X}}^{(a)})(\boldsymbol{X}_i^{(a)} - \overline{\boldsymbol{X}}^{(a)})'$, $a = 1, \ldots, q$.

We consider the problem of constructing simultaneous confidence intervals among mean vectors. We focus on the multivariate case $p > 1$ and deal with (I) comparisons with a control when the q-th level is regarded as a control and (II) pairwise comparisons. Roy and Bose (1953; (4.3.1)) and Siotani (1960; (12) and (13)) gave exact $100(1-\alpha)\%$ simultaneous confidence intervals of the form

$$\begin{aligned} &\boldsymbol{\ell}'(\boldsymbol{\theta}^{(a)} - \boldsymbol{\theta}^{(q)}) \in \boldsymbol{\ell}'\left(\overline{\boldsymbol{X}}^{(a)} - \overline{\boldsymbol{X}}^{(q)}\right) \pm \left\{N_{aq}^{-1} t_{\mathrm{max,I}}^2(\alpha)(\boldsymbol{\ell}' S_{\mathrm{pool},X}\boldsymbol{\ell})\right\}^{1/2} \\ &\text{for all } \boldsymbol{\ell} \in \mathbf{R}^p - \{\mathbf{0}\}, \quad a = 1, \ldots, q-1 \end{aligned} \tag{2.I}$$

and

$$\begin{aligned} &\boldsymbol{\ell}'(\boldsymbol{\theta}^{(a)} - \boldsymbol{\theta}^{(b)}) \in \boldsymbol{\ell}'\left(\overline{\boldsymbol{X}}^{(a)} - \overline{\boldsymbol{X}}^{(b)}\right) \pm \left\{N_{ab}^{-1} t_{\mathrm{max,II}}^2(\alpha)(\boldsymbol{\ell}' S_{\mathrm{pool},X}\boldsymbol{\ell})\right\}^{1/2} \\ &\text{for all } \boldsymbol{\ell} \in \mathbf{R}^p - \{\mathbf{0}\}, \quad a, b = 1, \ldots, q; \quad a < b, \end{aligned} \tag{2.II}$$

where $N_{ab} = N_a N_b/(N_a + N_b)$. Here, $t_{\mathrm{max,I}}^2(\alpha)$ and $t_{\mathrm{max,II}}^2(\alpha)$ are, respectively, the upper $100\alpha\%$ point of maximum of correlated Hotelling's T^2 statistics

$$T_{\mathrm{max,I}}^2 = \max_{a=1,\ldots,q-1} (T_{aq}^2) \quad \text{and} \quad T_{\mathrm{max,II}}^2 = \max_{1 \le a < b \le q} (T_{ab}^2), \tag{3}$$

where

$$T_{ab}^2 = N_{ab}\left(\overline{\boldsymbol{U}}^{(a)} - \overline{\boldsymbol{U}}^{(b)}\right)' S_{\mathrm{pool},U}^{-1}\left(\overline{\boldsymbol{U}}^{(a)} - \overline{\boldsymbol{U}}^{(b)}\right), \quad a, b = 1, \ldots, q; \quad a \neq b. \tag{4}$$

Obviously, we have $T_{ab}^2 = T_{ba}^2$ for $a \neq b$.

To implement (2.I) and (2.II) practically, even in the normal populations, one encounters the difficulty of computing exact two percentiles $t_{\mathrm{max,I}}^2(\alpha)$ and $t_{\mathrm{max,II}}^2(\alpha)$, for which one requires an expression for the joint distribution of $(T_{ab}^2)_{a,b=1,\ldots,q;a<b}$ or its marginal distribution of $(T_{aq}^2)_{a=1,\ldots,q-1}$. For any pair $(a, b) \in \{(a, b) : a, b = 1, \ldots, q; a < b\} \equiv J$, let $\boldsymbol{\lambda}_{ab} \in \mathbf{R}^q$ be a column vector with $\{N_b/(N_a+N_b)\}^{1/2}$ at the a-th position, $-\{N_a/(N_a + N_b)\}^{1/2}$ at the b-th position and zero at other position. By virtue of the multivariate sampling theory (e.g. Anderson 2003; p. 77), we know only that given a subset $\{(a_i, b_i) : i = 1, \ldots, K\} \subset J$ for some $K = 1, \ldots, q(q-1)/2$, the distribution of $(T^2_{a_i b_i}/(N-q))_{i=1,\ldots,K}$ under normality is characterized as the distribution of $(\boldsymbol{Z}'_{a_i b_i} W^{-1} \boldsymbol{Z}_{a_i b_i})_{i=1,\ldots,K}$ with $\boldsymbol{Z}_{ab} = (\boldsymbol{\lambda}'_{ab} \otimes I_p)\boldsymbol{U}$, where W is distributed as Wishart distribution $W_p(I_p, N-q)$, independent of $\boldsymbol{U} \sim N_{pq}(\boldsymbol{0}, I_{pq})$. A better characterization for the distribution of $(\boldsymbol{Z}'_{a_i b_i} W^{-1} \boldsymbol{Z}_{a_i b_i})_{i=1,\ldots,K}$ is not available at present for either $p > 1$ or $K > 1$. Hence, the exact percentiles $t^2_{\max,\mathrm{I}}(\alpha)$ and $t^2_{\max,\mathrm{II}}(\alpha)$, equivalently, the distribution functions of the maximum statistics (3), given by

$$P(T^2_{\max,\mathrm{I}} \le x) = P(T^2_{aq} \le x, a = 1, \ldots, q-1) \tag{5.I}$$

and

$$P(T^2_{\max,\mathrm{II}} \le x) = P(T^2_{ab} \le x, a, b = 1, \ldots, q; a < b), \tag{5.II}$$

are not generally computable. It may be noted that letting Λ_K be a $K \times q$ matrix whose i-th row is $\boldsymbol{\lambda}'_{a_i b_i}$, $(\boldsymbol{Z}'_{a_1 b_1}, \ldots, \boldsymbol{Z}'_{a_K b_K})' = (\Lambda_K \otimes I_p)\boldsymbol{U}$ is distributed as a nonsingular or singular normal $N_{Kp}(\boldsymbol{0}, \Lambda_K \Lambda'_K \otimes I_p)$ according as the rank of Λ_K is equal to K or less than K (hence the case $K > q$ is singular).

Historically, many authors discussed the distribution of $(\boldsymbol{Z}'_{a_i b_i} \boldsymbol{Z}_{a_i b_i})_{i=1,\ldots,K}$, where $(\boldsymbol{Z}'_{a_1 b_1}, \ldots, \boldsymbol{Z}'_{a_K b_K})' \sim N_{Kp}(\boldsymbol{0}, \Omega_K \otimes I_p)$. Provided that Ω_K $(K \times K)$ is the correlation matrix, the distribution of $(\boldsymbol{Z}'_{a_i b_i} \boldsymbol{Z}_{a_i b_i})_{i=1,\ldots,K}$ is referred to as a K-variate central nonsingular or singular chi-square distribution with p degrees of freedom according as Ω_K is nonsingular or singular (e.g. Krishnaiah 1965). See also Krishnamoorthy and Parthasarathy (1951) and Royen (1991b) on a multivariate Gamma-type distribution for the nonsingular case (apart from some special cases, the multivariate Gamma-type distribution is complicated to handle numerically). We say that the $K \times K$ correlation matrix Ω_K has a product (or one-factorial) structure if $\Omega_K = D_K + \boldsymbol{\rho}_K \boldsymbol{\rho}'_K$, where $\boldsymbol{\rho}_K$ is a $K \times 1$ column vector whose i-th element is $\rho_{a_i b_i} \in (-1, 1)$ and D_K is a $K \times K$ diagonal matrix whose i-th diagonal is $1 - \rho^2_{a_i b_i}$. The following formula due to Royen (1984, 1991a) is available:

$$\begin{aligned} &P\left(\boldsymbol{Z}'_{a_i b_i} \boldsymbol{Z}_{a_i b_i} \le x_{a_i b_i},\ i = 1, \ldots, K | \Omega_K = D_K + \boldsymbol{\rho}_K \boldsymbol{\rho}'_K\right) \\ &\quad = \int_0^\infty g_p(s) \prod_{i=1}^K G_p\left(\frac{x_{a_i b_i}}{1-\rho^2_{a_i b_i}}, \frac{\rho^2_{a_i b_i} s}{1-\rho^2_{a_i b_i}}\right) \mathrm{d}s, \end{aligned} \tag{6}$$

where $g_p(x)$ is the density function of the central chi-square distribution with p degrees of freedom, and $G_p(x, \omega^2)$ denotes the distribution function of the noncentral chi-square distribution with p degrees of freedom and noncentrality parameter ω^2 (e.g. Anderson (2003; p. 82));

$$G_p(x, \omega^2) = \sum_{\ell=0}^\infty \frac{1}{\ell!} \left(\frac{\omega^2}{2}\right)^\ell \exp\left(-\frac{\omega^2}{2}\right) G_{p+2\ell}(x).$$

Fujikoshi and Seo (1999) attempted to derive an asymptotic expansion up to order ν^{-2} of $P(\mathbf{Z}_a' S^{-1} \mathbf{Z}_a \le x_a, a = 1, \ldots, K)$, where νS is distributed as Wishart distribution $W_p(\Sigma, \nu)$, independent of $(\mathbf{Z}_1', \ldots, \mathbf{Z}_K')' \sim N_{Kp}(\mathbf{0}, \Gamma \otimes \Sigma)$. In view of their derivation, their matrix Γ must be nonsingular, although they did not mention it. But, their results were restricted to a special case of $\Gamma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ with $K = 2$ (see also Siotani 1959) or $\Gamma = I_K$ (this case is referred to as being quasi-independent). Under elliptical populations with equal sample sizes $N_1 = \cdots = N_q = N_0$ (say) or unequal sample sizes $N_1, \ldots, N_q$ satisfying $\max_{a=2,\ldots,q} N_a \le N_1 = N_0$ (say), Seo (2002) and Okamoto (2005) gave an asymptotic expansion up to order N_0^{-1} of the joint probability $P(T_{ab}^2 > x, T_{cd}^2 > x)$ for $(a, b), (c, d) \in J$ satisfying $(a, b) \ne (c, d)$, which was extended by Kakizawa (2006) to a situation where the underlying population distributions satisfy $\boldsymbol{U}^{(a)} \overset{d}{=} -\boldsymbol{U}^{(a)}$, $a = 1, \ldots, q$. Recently, Kakizawa (2005) derived an asymptotic expansion up to order N^{-1} for the joint distribution of quasi-independent Hotelling's T^2 statistics $N_a \overline{\boldsymbol{U}}^{(a)'} S_{\mathrm{pool},U}^{-1} \overline{\boldsymbol{U}}^{(a)}$, $a = 1, \ldots, q$ under nonnormality.

2 Background

2.1 Improved Bonferroni inequality

An easy idea to overcome the distributional complexity of the statistics (3) is based on the Bonferroni inequality, which provides upper and lower bounds for the probability of the union of a sequence of events $A_1, \ldots, A_m$;

$$\sum_{i=1}^{m} P(A_i) - \sum_{1 \le i_1 < i_2 \le m} P(A_{i_1} \cap A_{i_2}) \le P\left(\bigcup_{i=1}^{m} A_i\right) \le \sum_{i=1}^{m} P(A_i). \tag{7}$$

If A_i is, for each $i = 1, \ldots, q-1$, the event that Hotelling's T^2 statistic T_{iq}^2 is greater than a certain positive value x, we have from (7) with $m = q - 1$

$$1 - \sum_{a=1}^{q-1} \overline{P}_{aq}(x) + \sum_{1 \le a < b \le q-1} \overline{P}_{aq:bq}(x) \ge P(T_{\max,\mathrm{I}}^2 \le x) \ge 1 - \sum_{a=1}^{q-1} \overline{P}_{aq}(x), \tag{8}$$

where

$$\overline{P}_{aq}(x) = P(T_{aq}^2 > x) \quad \text{and} \quad \overline{P}_{aq:bq}(x) = P\left(T_{aq}^2 > x, T_{bq}^2 > x\right).$$

Suppose that the exact or asymptotic formulae on $\overline{P}_{aq}(x)$'s are available. Then, replacing the critical value $t_{\max,\mathrm{I}}^2(\alpha)$ in (2.I) with the solution $t_{\mathrm{I},0}^2$ of

$$\sum_{a=1}^{q-1} \overline{P}_{aq}(x) = \alpha, \tag{9}$$

we have conservative or asymptotically conservative $100(1-\alpha)\%$ simultaneous confidence intervals for comparisons with a control (the conservative property follows from the lower bound in (8)). It is obvious that equating other lower bound $L(x)$ for $P(T^2_{\max,\mathrm{I}} \le x)$ to $1-\alpha$ yields also conservative simultaneous confidence intervals. The sharper the bound $L(x)$ is, the closer to $1-\alpha$ the coverage probability of the resulting simultaneous confidence intervals is. We remark that equating the left hand side of (8) to $1-\alpha$, that is, solving the equation $\sum_{a=1}^{q-1} \overline{P}_{aq}(x) - \sum_{1\le a<b\le q-1} \overline{P}_{aq:bq}(x) = \alpha$, is useless for $q > 3$, since the coverage probability in this case is less than or equal to $1-\alpha$. Instead, the upper bound in (8) or another improved upper bound $U(x)$ for $P(T^2_{\max,\mathrm{I}} \le x)$ should be used to estimate the accuracy of any approximation to the true critical value $t^2_{\max,\mathrm{I}}(\alpha)$.

On the other hand, the modified second approximation that Siotani's (1959, 1960, 1964) proposed is a creative but ad-hoc approximation method in such a way that the critical value $t^2_{\mathrm{I},S}$ is calculated as the solution of the equation

$$\sum_{a=1}^{q-1} \overline{P}_{aq}(x) = \alpha + \sum_{1\le a<b\le q-1} \overline{P}_{aq:bq}(t^2_{\mathrm{I},0}), \tag{10}$$

where $t^2_{\mathrm{I},0}$ is the solution of (9). Unfortunately, there will be no theoretically support whether the coverage probability of Siotani's simultaneous confidence intervals with the critical value $t^2_{\mathrm{I},S}$ is larger than or equal to $1-\alpha$ for the case $q > 3$. Seo and Siotani (1993) gave an extensive simulation study in order to examine the accuracy of their proposal $t^2_{\mathrm{I},S}$ (see also Seo and Siotani 1992, Seo 2002, Okamoto 2005 for pairwise comparisons).

The aim of this paper is to construct simultaneous confidence intervals (2.I) and (2.II) with some critical values in place of the exact percentiles $t^2_{\max,\mathrm{I}}(\alpha)$ and $t^2_{\max,\mathrm{II}}(\alpha)$ of $T^2_{\max,\mathrm{I}}$ and $T^2_{\max,\mathrm{II}}$, respectively. We will develop an easily computable procedure by applying improved Bonferroni-type lower bounds for (5.I) and (5.II), which is essential for the conservative property of the resulting procedure. We drop some multiplicative probability inequalities (e.g. Hochberg and Tamhane 1987; Appendix 2) from consideration, since they are valid only under somewhat restrictive distributional assumptions. To the author's knowledge, the Bonferroni-type inequalities for the probability of the union of a sequence of events $A_1, \ldots, A_m$ ($m \in \mathbf{N}$) have been proposed in the literature by introducing a general framework based on the concepts of graphs. A graph is a pair $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \mathcal{V}(G)$ is the set of vertices and $\mathcal{E} = \mathcal{E}(G)$ is the set of edges, each described as an unordered pair of vertices. A weighted graph is a graph which has a value associated with each edge (given m vertices $v_1, \ldots, v_m$ representing events $A_1, \ldots, A_m$, respectively, the probability $P(A_{i_1} \cap A_{i_2}) = P(A_{i_2} \cap A_{i_1})$, $i_1 \ne i_2$, is regarded as the weight of edge $(i_1, i_2) = (i_2, i_1)$ between vertices v_{i_1} and v_{i_2}). In a complete graph, each pair of vertices is assumed to be joined by an edge. We now consider the complete graph on m vertices, denoted by $K_m = (\mathcal{V}(K_m), \mathcal{E}(K_m))$, where $\mathcal{V}(K_m) = \{v_1, \ldots, v_m\}$ and $\mathcal{E}(K_m) = \{(i_1, i_2), i_1, i_2 = 1, \ldots, m; i_1 < i_2\}$. A spanning tree $\mathcal{T}$ of K_m is a connected subgraph of K_m satisfying $\mathcal{V}(\mathcal{T}) = \mathcal{V}(K_m)$, $\mathcal{E}(\mathcal{T}) \subset \mathcal{E}(K_m)$ and $|\mathcal{E}(\mathcal{T})| = m - 1$. The Hunter-Worsley inequality, which was established independently by Hunter (1976) and Worsley (1982), provides a more general bound

$$P\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} P(A_i) - \sum_{(i_1,i_2)\in\mathcal{E}(\mathcal{T})} P\left(A_{i_1} \cap A_{i_2}\right) \tag{11}$$

for any spanning tree $\mathcal{T}$ of K_m (there are different $M = m^{m-2}$ spanning trees $\mathcal{T}^1, \ldots, \mathcal{T}^M$ according to a theorem of A. Cayley), where $\sum_{(i_1,i_2)\in\mathcal{E}(\mathcal{T})}$ denotes summation over all edges $(i_1, i_2) \in \mathcal{E}(\mathcal{T})$. Remarkably, the inequality

$$P\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} P(A_i) - \max_{\mathcal{T}\in\{\mathcal{T}\}} \sum_{(i_1,i_2)\in\mathcal{E}(\mathcal{T})} P\left(A_{i_1} \cap A_{i_2}\right)$$

for any subset $\{\mathcal{T}\} \subset \{\mathcal{T}^1, \ldots, \mathcal{T}^M\}$ is a class of sharper upper bounds than (7), and the sharpest upper bound in this class is given by

$$\begin{aligned} P\left(\bigcup_{i=1}^{m} A_i\right) &\leq \sum_{i=1}^{m} P(A_i) - \max_{\mathcal{T}\in\{\mathcal{T}^1,\ldots,\mathcal{T}^M\}} \sum_{(i_1,i_2)\in\mathcal{E}(\mathcal{T})} P\left(A_{i_1} \cap A_{i_2}\right) \\ &= \sum_{i=1}^{m} P(A_i) - \sum_{(i_1,i_2)\in\mathcal{E}(\mathcal{T}^*)} P\left(A_{i_1} \bigcap A_{i_2}\right), \end{aligned} \tag{12}$$

where $\mathcal{T}^*$ is the maximal spanning tree, having the property

$$\sum_{(i_1,i_2)\in\mathcal{E}(\mathcal{T}^*)} P\left(A_{i_1} \cap A_{i_2}\right) \geq \sum_{(i_1,i_2)\in\mathcal{E}(\mathcal{T})} P\left(A_{i_1} \cap A_{i_2}\right)$$

for $\mathcal{T} = \mathcal{T}^1, \ldots, \mathcal{T}^M$.

It is worth describing the following two subclasses of spanning trees $\mathcal{T}$ of K_m. One subclass is a class of m trees $\mathcal{T}(i') = (\{v_1, \ldots, v_m\}, \mathcal{E}_{i'})$, where $\mathcal{E}_{i'} = \{(i, i'), i = 1, \ldots, m;\ i \neq i'\}$ for $i' = 1, \ldots, m$. The other subclass consists of $m!/2$ trees $\mathcal{T}(i_1 \ldots i_m) = (\{v_1, \ldots, v_m\}, \mathcal{P}_{i_1\ldots i_m})$, where $\mathcal{P}_{i_1\ldots i_m} = \{(i_1, i_2), (i_2, i_3), \ldots, (i_{m-1}, i_m)\}$ is an elementary path of length $m - 1$, with $i_1, \ldots, i_m$ being a permutation of the first m positive integers but $i_1 < i_m$.

The Hunter-Worsley inequality (11), with $\mathcal{T} = \mathcal{T}(i')$ for some $i' = 1, \ldots, m$, implies not only Kounias' (1968; (2)) inequality

$$P\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} P(A_i) - \sum_{\substack{i=1 \\ i\neq i'}}^{m} P\left(A_i \cap A_{i'}\right) \tag{13}$$

but also two variants

$$P\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} P(A_i) - \max_{i'=1,\ldots,m} \sum_{\substack{i=1\\ i\neq i'}}^{m} P\left(A_i \cap A_{i'}\right) \tag{14}$$

$$\leq \sum_{i=1}^{m} P(A_i) - \frac{2}{m} \sum_{1\leq i_1<i_2\leq m} P\left(A_{i_1} \cap A_{i_2}\right) \tag{15}$$

by minimizing Kounias' inequality (13) over $i' = 1, \ldots, m$ and by noting

$$\max_{i'=1,\ldots,m} \sum_{\substack{i=1\\ i\neq i'}}^{m} P\left(A_i \bigcap A_{i'}\right) \geq \frac{1}{m} \sum_{i'=1}^{m} \sum_{\substack{i=1\\ i\neq i'}}^{m} P\left(A_i \cap A_{i'}\right)$$

$$= \frac{2}{m} \sum_{1\leq i_1<i_2\leq m} P\left(A_{i_1} \cap A_{i_2}\right).$$

Interestingly, the worse bound (15) than (14) (hence (12)) is optimal among all the bounds of the type $c_1 \sum_{i=1}^{m} P(A_i) + c_2 \sum_{1\leq i_1<i_2\leq m} P(A_{i_1} \cap A_{i_2})$, where c_1 and c_2 are real numbers (e.g. Galambos and Simonelli 1996; p. 20). Further, another upper bound

$$P\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} P(A_i) - \sum_{(i,i')\in\mathcal{P}_{i_1\ldots i_m}} P\left(A_i \cap A_{i'}\right) \tag{16}$$

due to Worsley (1982; Corollary 1) and Efron (1997; the two-point formula) is a special case of the Hunter-Worsley inequality (11) with $\mathcal{T} = \mathcal{T}(i_1, \ldots, i_m)$. One may minimize (16) over all similar $m!/2$ bounds. Some other Bonferroni-type inequalities are found in Galambos and Simonelli (1996).

Remark 1 In the 1990s, Naiman and Wynn (1992, 1997) introduced the topological concept of an abstract tube in order to get improved Bonferroni's inequalities for a finite family of events. Dohmen (2003) recently provided some elementary alternative proofs (not using abstract tubes) and gave a new Bonferroni-type inequality which is determined by a chordal graph. By varying such a graph, several known inequalities have been illustrated in a unified way.

2.2 Contents

These Bonferroni-type bounds are applied for the distribution functions (5.I) and (5.II) at the beginning of Sect. 3, which require the knowledge of the joint distribution of pairs of (T^2_{ab}, T^2_{cd}), $(a, b), (c, d) \in J$; $(a, b) \neq (c, d)$. As we saw previously, no simple exact formula for $\overline{P}_{ab:cd}(x) = P(T^2_{ab} > x, T^2_{cd} > x)$ is known. Since the resulting large sample procedure (P1) based on the limits of $\overline{P}^{\infty}_{ab}(x) = \lim_{N\to\infty} \overline{P}_{ab}(x)$ and

$\overline{P}^{\infty}_{ab:cd}(x) = \lim_{N\to\infty} \overline{P}_{ab:cd}(x)$ requires a root finding algorithm to solve some complicated nonlinear equations, we propose, in Sect. 4, a procedure (P2) as a modification of Siotani's (1959, 1960, 1964) procedure, which is relatively easy to implement. Unlike Siotani, our procedures are shown to guarantee the designated simultaneous confidence level conservatively. Section 5 reports numerical values using our large sample procedures (P1) and (P2). Higher-order variants of (P1) and (P2) are briefly discussed in Sect. 6. Finally, concluding remarks are given in Sect. 7.

2.3 Assumptions and notation

We set down the following assumptions on the N vectors $\boldsymbol{U}_i^{(a)}$'s:

- (A_1) $\boldsymbol{U}_i^{(a)}$ and $\boldsymbol{U}_j^{(b)}$ are independent if either $a \neq b$ or $i \neq j$;
- (A_2) for every fixed a, the random vectors $\boldsymbol{U}_1^{(a)}, \dots, \boldsymbol{U}_{N_a}^{(a)}$ are identically distributed as a distribution of $\boldsymbol{U}^{(a)} = (U_1^{(a)}, \dots, U_p^{(a)})'$ with mean zero vector, positive definite covariance matrix Σ and $E[||\boldsymbol{U}^{(a)}||^4] < \infty$;
- (A_3) all N_a's are large, in such a way that the total number N of observations goes to infinity, while the ratio $N_a/N = \eta_N^{(a)}$ (say) converges to $\eta_a > 0$, $a = 1, \dots, q$, where $\sum_{a=1}^q \eta_a = 1$.

Throughout this paper, $G_\nu(x)$ is the distribution function of the central chi-square distribution with ν degrees of freedom, whose density function and upper $100\alpha\%$ point are $g_\nu(x)$ and $\chi^2_{\nu,\alpha}$, respectively, and let $\overline{G}_\nu(x) = 1 - G_\nu(x)$. For simplicity, we write

$$\alpha_{\mathrm{I}} = \frac{\alpha}{q-1} \quad \text{and} \quad \alpha_{\mathrm{II}} = \frac{\alpha}{q(q-1)/2},$$

and we use the notation

$$\overline{P}_{ab}(x) = P(T^2_{ab} > x) \quad \text{and} \quad \overline{P}_{ab:cd}(x) = P(T^2_{ab} > x, T^2_{cd} > x)$$

for any $a, b, c, d \in \{1, \dots, q\}$ satisfying $a \neq b$, $c \neq d$ and $\{a, b\} \neq \{c, d\}$. We sometimes adopt the lexicographically order $(a, b) < (c, d)$ iff (i) $a < c$ or (ii) $a = c$ and $b < d$.

3 Improved Bonferroni procedures

A given confidence level $1 - \alpha$, the simultaneous confidence intervals

$$\boldsymbol{\ell}'\left(\overline{\boldsymbol{X}}^{(a)} - \overline{\boldsymbol{X}}^{(q)}\right) \pm \left\{N_{aq}^{-1} t_{\mathrm{I}}^2 \left(\boldsymbol{\ell}' S_{\mathrm{pool},X} \boldsymbol{\ell}\right)\right\}^{1/2}, \quad \boldsymbol{\ell} \in \mathbf{R}^p - \{\mathbf{0}\}, \;\; a = 1, \dots, q-1 \tag{17.I}$$

for comparisons with a control are conservative, if the critical value t_{I}^2 satisfies

$$P\left(T^2_{\max,\mathrm{I}} \leq t_{\mathrm{I}}^2\right) \geq 1 - \alpha \quad \text{equivalently} \quad t^2_{\max,\mathrm{I}}(\alpha) \leq t_{\mathrm{I}}^2.$$

Similarly, the simultaneous confidence intervals

$$\boldsymbol{\ell}'\left(\overline{\boldsymbol{X}}^{(a)}-\overline{\boldsymbol{X}}^{(b)}\right)\pm\left\{N_{ab}^{-1}t_{\mathrm{II}}^{2}(\boldsymbol{\ell}'S_{\mathrm{pool},X}\boldsymbol{\ell})\right\}^{1/2},\quad \boldsymbol{\ell}\in\mathbf{R}^{p}-\{\mathbf{0}\},\ \ (a,b)\in J \quad (17.\mathrm{II})$$

for pairwise comparisons are conservative, if the critical value t_{II}^2 satisfies

$$P\left(T_{\max,\mathrm{II}}^{2}\leq t_{\mathrm{II}}^{2}\right)\geq 1-\alpha \quad \text{equivalently} \quad t_{\max,\mathrm{II}}^{2}(\alpha)\leq t_{\mathrm{II}}^{2}.$$

Furthermore, if the critical value t_{I}^2 or t_{II}^2 satisfies

$$\lim_{N\to\infty}P\left(T_{\max,\mathrm{I}}^{2}\leq t_{\mathrm{I}}^{2}\right)\geq 1-\alpha \quad \text{or} \quad \lim_{N\to\infty}P\left(T_{\max,\mathrm{II}}^{2}\leq t_{\mathrm{II}}^{2}\right)\geq 1-\alpha,$$

the corresponding simultaneous confidence intervals (17.I) or (17.II) are then referred to as being asymptotically conservative.

The usual Bonferroni procedure is a consequence of the inequality (7); $P(\cup_{i=1}^{m}A_i) \leq \sum_{i=1}^{m}P(A_i)$, which implies

$$P\left(T_{\max,\mathrm{I}}^{2}\leq x\right)\geq 1-\sum_{a=1}^{q-1}\overline{P}_{aq}(x) \quad (18.\mathrm{I})$$

for comparisons with a control and

$$P\left(T_{\max,\mathrm{II}}^{2}\leq x\right)\geq 1-\sum_{1\leq a<b\leq q}\overline{P}_{ab}(x) \quad (18.\mathrm{II})$$

for pairwise comparisons. The Bonferroni upper bounds $t_{\mathrm{I},0}^2(\alpha)$ and $t_{\mathrm{II},0}^2(\alpha)$ (say) on the true critical values $t_{\max,\mathrm{I}}^2(\alpha)$ and $t_{\max,\mathrm{II}}^2(\alpha)$ are thus found by equating the lower bounds (18.I) and (18.II) to $1-\alpha$, respectively. That is, $t_{\mathrm{I},0}^2(\alpha)$ is the solution of

$$\sum_{a=1}^{q-1}\overline{P}_{aq}(x)=\alpha \quad (19.\mathrm{I})$$

and $t_{\mathrm{II},0}^2(\alpha)$ is the solution of

$$\sum_{1\leq a<b\leq q}\overline{P}_{ab}(x)=\alpha. \quad (19.\mathrm{II})$$

It is important to discuss progressively less conservative upper bounds on $t_{\max,\mathrm{I}}^2(\alpha)$ and $t_{\max,\mathrm{II}}^2(\alpha)$. The fundamental idea behind is to apply sharper lower bounds than (18.I) and (18.II). From among many improvements of the inequality $P(\cup_{i=1}^{m}A_i) \leq \sum_{i=1}^{m}P(A_i)$, we choose (12) and (15).

(I) If A_i is, for each $i = 1, \ldots, q-1$, the event that Hotelling's T^2 statistic T_{iq}^2 is greater than a certain positive value x, it then follows from (12) and (15) with $m = q-1$ that

$$P\left(T_{\max,\mathrm{I}}^2 \le x\right) \ge 1 - \sum_{a=1}^{q-1} \overline{P}_{aq}(x) + \sum_{(a,b)\in\mathcal{E}(\mathcal{T}_{\mathrm{I}}^*)} \overline{P}_{aq:bq}(x) \tag{20.I}$$

$$\ge 1 - \sum_{a=1}^{q-1} \overline{P}_{aq}(x) + \frac{2}{q-1} \sum_{1\le a<b\le q-1} \overline{P}_{aq:bq}(x), \tag{21.I}$$

where $\mathcal{T}_{\mathrm{I}}^*$ corresponds to the maximal spanning tree of the complete graph on the vertex set $\mathcal{V}_{\mathrm{I}} = \{v_a : a = 1, \ldots, q-1\}$.

Remark 2 If $\overline{P}_{aq}(x) = \overline{P}_{1q}(x)$ and $\overline{P}_{aq:bq}(x) = \overline{P}_{1q:2q}(x)$ for all $a, b = 1, \ldots, q-1$; $a < b$, (20.I) and (21.I) are reduced to the same lower bound

$$P\left(T_{\max,\mathrm{I}}^2 \le x\right) \ge 1 - (q-1)\overline{P}_{1q}(x) + (q-2)\overline{P}_{1q:2q}(x).$$

This is the case where the equality of sample sizes $N_1 = \cdots = N_{q-1}$ and normality are assumed (normality is unnecessary for the large sample case).

(II) Suppose that A_{ab} is, for each $a, b = 1, \ldots, q$; $a < b$, the event that Hotelling's T^2 statistic T_{ab}^2 is greater than a certain positive value x. Then, we have from (12) and (15) with $m = q(q-1)/2$

$$P\left(T_{\max,\mathrm{II}}^2 \le x\right) \ge 1 - \sum_{1\le a<b\le q} \overline{P}_{ab}(x) + \sum_{(i_{ab},i_{cd})\in\mathcal{E}(\mathcal{T}_{\mathrm{II}}^*)} \overline{P}_{ab:cd}(x) \tag{20.II}$$

$$\ge 1 - \sum_{1\le a<b\le q} \overline{P}_{ab}(x) + \frac{2}{q(q-1)/2} \sum_{(ab)(cd)} \overline{P}_{ab:cd}(x), \tag{21.II}$$

where $\mathcal{T}_{\mathrm{II}}^*$ corresponds to the maximal spanning tree of the complete graph on the vertex set $\mathcal{V}_{\mathrm{II}} = \{v_{i_{ab}} : (a,b) \in J\}$ with $i_{ab} = (2q-a)(a-1)/2 + b - a$. Here, $\sum_{(ab)(cd)}$ denotes summations over all $(a,b), (c,d) \in J$ satisfying $(a,b) < (c,d)$. More precisely, $\sum_{(ab)(cd)} \overline{P}_{ab:cd}(x)$ is the sum of

$$P_1(x) = \sum_{1\le a<b<c<d\le q} \left\{\overline{P}_{ab:cd}(x) + \overline{P}_{ac:bd}(x) + \overline{P}_{ad:bc}(x)\right\}$$

and

$$P_2(x) = \sum_{1\le a<b<c\le q} \left\{\overline{P}_{ab:ac}(x) + \overline{P}_{ab:bc}(x) + \overline{P}_{ac:bc}(x)\right\},$$

where $P_1(x)$ and $P_2(x)$ consist of $q(q-1)(q-2)(q-3)/8$ and $q(q-1)(q-2)/2$ terms, respectively.

Remark 3 If the equality of sample sizes $N_1 = \cdots = N_q$ and normality are assumed (normality is unnecessary for the large sample case), the following three statements hold:

(i) $\overline{P}_{ab}(x) = \overline{P}_{12}(x)$ for all $(a, b) \in J$,
(ii) $\overline{P}_{ab:cd}(x) = \overline{P}_{13:23}(x)$ for all (a, b), $(c, d) \in J$ such that (a, b) and (c, d) have exactly one common index, and
(iii) $\overline{P}_{ab:cd}(x) = \overline{P}_{12:34}(x)$ for all (a, b), $(c, d) \in J$ such that a, b, c, d are all different.

Then, letting $\mathcal{E}_1(\mathcal{T}_{\mathrm{II}}) = \{(i_{ab}, i_{cd}) \in \mathcal{E}(\mathcal{T}_{\mathrm{II}}) : a, b, c, d$ are all different$\}$ and $\mathcal{E}_2(\mathcal{T}_{\mathrm{II}}) = \mathcal{E}(\mathcal{T}_{\mathrm{II}}) - \mathcal{E}_1(\mathcal{T}_{\mathrm{II}})$, we have

$$\sum_{(i_{ab}, i_{cd}) \in \mathcal{E}(\mathcal{T}_{\mathrm{II}})} \overline{P}_{ab:cd}(x) = |\mathcal{E}_1(\mathcal{T}_{\mathrm{II}})| \overline{P}_{12:34}(x) + |\mathcal{E}_2(\mathcal{T}_{\mathrm{II}})| \overline{P}_{13:23}(x)$$

for any spanning tree $\mathcal{T}_{\mathrm{II}}$ of the complete graph on the vertex set $\mathcal{V}_{\mathrm{II}}$, where $|\mathcal{E}_1(\mathcal{T}_{\mathrm{II}})| + |\mathcal{E}_2(\mathcal{T}_{\mathrm{II}})| = q(q-1)/2 - 1$. There exists a spanning tree $\mathcal{T}_{\mathrm{II}}$ satisfying $|\mathcal{E}_1(\mathcal{T}_{\mathrm{II}})| = 0$. Actually, we have an elementary path $\{\mathcal{V}_{\mathrm{II}}, \mathcal{P}^1 \cup \mathcal{P}^2 \cup \mathcal{P}^3\}$, where

$$\begin{aligned}
\mathcal{P}^1 &= \left\{(i_{ab}, i_{a,b+1}) : 1 \le a < b \le q - 1\right\}, \\
\mathcal{P}^2 &= \left\{(i_{aq}, i_{a+1,q}) : a \text{ is odd and } 1 \le a \le q - 2\right\}, \\
\mathcal{P}^3 &= \left\{(i_{a,a+1}, i_{a+1,a+2}) : a \text{ is even and } 2 \le a \le q - 2\right\}.
\end{aligned}$$

The following lemma is useful to check the existence of a solution of $B(x) = 1 - \alpha$, where $B(x)$ is any lower bound for $P(T^2_{\max,\mathrm{I}} \le x)$ or $P(T^2_{\max,\mathrm{II}} \le x)$.

Lemma 1 *Suppose that $U(x) \ge I(x) > L(x)$ for all $x \in (0, \infty) \equiv \mathbf{R}^+$, where $I(x)$ is a continuous function in $x \in \mathbf{R}^+$, and $U(x)$ and $L(x)$ are strictly increasing functions in $x \in \mathbf{R}^+$. Assume that $U(x) = c$ has a solution $x_* > 0$ and that $L(x) = c$ has a solution $x_{**} > 0$($x_{**} > x_*$ holds automatically). Then, $I(x) = c$ has at least one solution in $[x_*, x_{**})$ and $I(x) \ne c$ for all $x \in \mathbf{R}^+ - [x_*, x_{**})$. In addition, assume that $U(x) \ge I_2(x) \ge I_1(x) > L(x)$ for all $x \in \mathbf{R}^+$, where $I_1(x)$ and $I_2(x)$ are continuous functions in $x \in \mathbf{R}^+$. Then, $I_2(x) \ne c$ for all $x \in \mathbf{R}^+ - [x_*, x_{**})$ and $I_2(x) = c$ has at least one solution in $[x_*, x_1]$, where $x_1 \in [x_*, x_{**})$ is a solution of $I_1(x) = c$.*

Proof By virtue of the intermediate value theorem, the equation $I(x) = c$ has at least one solution in $[x_*, x_{**})$. It is easy to show that the equation $I(x) = c$ has no solution in $\mathbf{R}^+ - [x_*, x_{**})$. In particular, suppose that $x_0 > 0$ is a solution of $I(x) = c$. Then, the inequality $U(x_0) \ge I(x_0) = c = U(x_*)$ implies $x_* \le x_0$, because of the monotonicity of $U(x)$. Similarly, the inequality $L(x_{**}) = c = I(x_0) > L(x_0)$ implies $x_0 < x_{**}$, because of the monotonicity of $L(x)$. □

3.1 Small sample procedure under normality

If the population distribution is assumed to be normal $\boldsymbol{U}^{(a)} \sim N_p(\mathbf{0}, \Sigma), a = 1, \ldots, q$, it is well-known that $(n/p)[T_{ab}^2/(N-q)]$, $(a, b) \in J$, is identically distributed as a central F distribution $F_{p,n}$ with (p, n) degrees of freedom (e.g. Anderson 2003; pp. 176); $P(T_{ab}^2 > x) = 1 - P(F_{p,n} \le \overline{x})$, where $n = N - q - p + 1$ and $\overline{x} = (n/p)[x/(N-q)]$. It follows from (19.I) and (19.II) that the usual Bonferroni upper bounds $t_{\mathrm{I},0}^2(\alpha)$ and $t_{\mathrm{II},0}^2(\alpha)$ are exactly found to be

$$\frac{(N-q)p}{n} f_{p,n}(\alpha_{\mathrm{I}}) \quad \text{and} \quad \frac{(N-q)p}{n} f_{p,n}(\alpha_{\mathrm{II}}),$$

respectively, where $f_{\nu_1\nu_2}(\alpha)$ is the upper $100\alpha\%$ point of $F_{\nu_1\nu_2}$. As we saw in Introduction, no simple formula for the joint probability of $P(T_{ab}^2 > x, T_{cd}^2 > x)$, $(a, b), (c, d) \in J$; $(a, b) < (c, d)$, is available. Therefore, we can not apply the Bonferroni-type inequalities (20.I) and (21.I) (or (20.II) and (21.II)) directly, except for the univariate case $p = 1$, in which we will have procedures based on a bivariate central nonsingular F distribution with $(1, N-q)$ degrees of freedom (equivalently a bivariate central nonsingular t^2 distribution with $N-q$ degrees of freedom). We do not discuss these topics any longer in this paper.

3.2 Large sample procedure (P1) under general distributions

We next consider the large sample but possibly multivariate nonnormal case. Compared with the notation of $\boldsymbol{\lambda}_{ab} \in \mathbf{R}^q$ given in Introduction, we define $\widetilde{\boldsymbol{\lambda}}_{ab} = \lim_{N\to\infty} \boldsymbol{\lambda}_{ab}$, which is a column vector in $\mathbf{R}^q$ with $\{\eta_b/(\eta_a+\eta_b)\}^{1/2}$ at the a-th position, $-\{\eta_a/(\eta_a+\eta_b)\}^{1/2}$ at the b-th position and zero at other position. Whatever the underlying p-variate distributions $\boldsymbol{U}^{(a)}$, $a = 1, \ldots, q$, the central limit theorem and Slutsky's theorem tell us that $(T_{a_ib_i}^2)_{i=1,\ldots,K}$ is asymptotically distributed as $(\chi_{a_ib_i}^2)_{i=1,\ldots,K}$ with $\chi_{ab}^2 = \boldsymbol{Z}_{ab}'\boldsymbol{Z}_{ab}$, where $(\boldsymbol{Z}_{ab})_{a,b=1,\ldots,q;a<b} \sim N_{pq(q-1)/2}(\mathbf{0}, \widetilde{P}_{\mathrm{II}} \otimes I_p)$. Here, $\widetilde{P}_{\mathrm{II}}$ is a $q(q-1)/2 \times q(q-1)/2$ correlation matrix whose (i_{ab}, i_{cd})th element is $\widetilde{\boldsymbol{\lambda}}_{ab}'\widetilde{\boldsymbol{\lambda}}_{cd}$, where $(a, b), (c, d) \in J$. More precisely,

$$\widetilde{\boldsymbol{\lambda}}_{ab}'\widetilde{\boldsymbol{\lambda}}_{cd} = \begin{cases} 1, & (a,b) = (c,d) \\ 0, & \text{if } a, b, c, d \text{ are all different} \\ \eta_{a_2a_3,a_1}, & (a,b,c,d) = (a_1,a_2,a_1,a_3) \\ -\eta_{a_1a_3,a_2}, & (a,b,c,d) = (a_1,a_2,a_2,a_3) \\ \eta_{a_1a_2,a_3}, & (a,b,c,d) = (a_1,a_3,a_2,a_3) \end{cases} \tag{22}$$

for $a_1, a_2, a_3 \in \{1, \ldots, q\}$; $a_1 < a_2 < a_3$, where

$$\eta_{b_1b_2,b_3} = \left(\frac{\eta_{b_1}}{\eta_{b_1}+\eta_{b_3}}\right)^{1/2}\left(\frac{\eta_{b_2}}{\eta_{b_2}+\eta_{b_3}}\right)^{1/2} \in (0, 1).$$

Therefore,

$$\lim_{N\to\infty} P(T^2_{\max,\mathrm{I}} \le x) = P(\chi^2_{\max,\mathrm{I}} \le x) \tag{23.I}$$

and

$$\lim_{N\to\infty} P(T^2_{\max,\mathrm{II}} \le x) = P(\chi^2_{\max,\mathrm{II}} \le x), \tag{23.II}$$

where

$$\chi^2_{\max,\mathrm{I}} = \max_{a=1,\ldots,q-1}(\chi^2_{aq}) \quad \text{and} \quad \chi^2_{\max,\mathrm{II}} = \max_{1\le a<b\le q}(\chi^2_{ab}).$$

Then, the upper $100\alpha\%$ points of $\chi^2_{\max,\mathrm{I}}$ and $\chi^2_{\max,\mathrm{II}}$, denoted by $\chi^2_{\max,\mathrm{I}}(\alpha)$ and $\chi^2_{\max,\mathrm{II}}(\alpha)$, yield asymptotically $100(1-\alpha)\%$ simultaneous confidence intervals for comparisons with a control and for pairwise comparisons, respectively. We notice that $\chi^2_{\max,\mathrm{I}}$ is the maximum of correlated chi-square distributions with p degrees of freedom, whose correlation matrix $\widetilde{P}_{\mathrm{I}} = (\widetilde{\boldsymbol{\lambda}}'_{aq}\widetilde{\boldsymbol{\lambda}}_{bq})_{a,b=1,\ldots,q-1}$ has a product structure $\widetilde{P}_{\mathrm{I}} = \mathrm{diag}(1-\rho_1^2,\ldots,1-\rho_{q-1}^2) + \boldsymbol{\rho}\boldsymbol{\rho}'$, where $\boldsymbol{\rho} = (\rho_1,\ldots,\rho_{q-1})'$ with $\rho_a = \{\eta_a/(\eta_a+\eta_q)\}^{1/2}$. In view of (6), the critical value $\chi^2_{\max,\mathrm{I}}(\alpha)$ is computable as a solution of

$$\int_0^\infty \frac{s^{p/2-1}\mathrm{e}^{-s}}{\Gamma(p/2)} \prod_{a=1}^{q-1} G_p\left(\frac{x}{1-\rho_a^2}, \frac{2\rho_a^2 s}{1-\rho_a^2}\right) \mathrm{d}s = 1-\alpha. \tag{24}$$

This kind of the integral may be evaluated by means of the Gauss-Laguerre quadrature formula. However, a simple formula on a multivariate singular Gamma-type distribution with $p > 1$ (hence the distribution of $\chi^2_{\max,\mathrm{II}}$) is not available. It is important to discuss a conservative critical point for $\chi^2_{\max,\mathrm{II}}(\alpha)$.

Now, T^2_{ab}, $(a,b) \in J$, is asymptotically distributed as the central chi-square distribution with p degrees of freedom, so that

$$\lim_{N\to\infty} P(T^2_{ab} > x) = P(\chi^2_{ab} > x) = \overline{G}_p(x). \tag{25}$$

On the other hand, the limiting distribution of (T^2_{ab}, T^2_{cd}), $(a,b), (c,d) \in J$; $(a,b) < (c,d)$, is the bivariate central nonsingular chi-square distribution $(\chi^2_{ab}, \chi^2_{cd})' = (\boldsymbol{Z}'_{ab}\boldsymbol{Z}_{ab}, \boldsymbol{Z}'_{cd}\boldsymbol{Z}_{cd})'$, where

$$\begin{pmatrix} \boldsymbol{Z}_{ab} \\ \boldsymbol{Z}_{cd} \end{pmatrix} \sim N_{2p}\left(\boldsymbol{0}, \begin{pmatrix} 1 & \widetilde{\boldsymbol{\lambda}}'_{ab}\widetilde{\boldsymbol{\lambda}}_{cd} \\ \widetilde{\boldsymbol{\lambda}}'_{ab}\widetilde{\boldsymbol{\lambda}}_{cd} & 1 \end{pmatrix} \otimes I_p\right).$$

So, if a, b, c, d are all different, we have

$$\lim_{N\to\infty} P(T^2_{ab} > x, T^2_{cd} > x) = P(\chi^2_{ab} > x)P(\chi^2_{cd} > x) = \{\overline{G}_p(x)\}^2. \tag{26}$$

We know from Siotani et al. (1985; p. 259) that the bivariate density of $(\sum_{i=1}^{p} Y_{i1}^2, \sum_{i=1}^{p} Y_{i2}^2)'$ when

$$\mathrm{vec}[(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_p)] \sim N_{2p}\left(\mathbf{0},\ I_p \otimes \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) \quad \text{with } \boldsymbol{Y}_i = (Y_{i1}, Y_{i2})'$$

is given by

$$\begin{aligned} f_p(y_1, y_2; \rho) &= f_p(y_1, y_2; -\rho) \\ &= (1-\rho^2)^{p/2} \sum_{\ell=0}^{\infty} \frac{\Gamma(p/2+\ell)}{\ell!\Gamma(p/2)} \rho^{2\ell} \\ &\quad \times \frac{(y_1 y_2)^{(p+2\ell)/2-1}}{[2^{(p+2\ell)/2}\Gamma((p+2\ell)/2)(1-\rho^2)^{(p+2\ell)/2}]^2} \exp\left\{-\frac{y_1+y_2}{2(1-\rho^2)}\right\} \end{aligned}$$

for $\rho \in (-1, 1)$, whose probability integral is

$$\begin{aligned} &\int_{x_1^L}^{x_1^U} \int_{x_2^L}^{x_2^U} f_p(y_1, y_2; |\rho|)\, \mathrm{d}y_1 \mathrm{d}y_2 \\ &= (1-\rho^2)^{p/2} \sum_{\ell=0}^{\infty} \frac{\Gamma(p/2+\ell)}{\ell!\Gamma(p/2)} \rho^{2\ell} \prod_{i=1}^{2} \left\{ G_{p+2\ell}\left(\frac{x_i^U}{1-\rho^2}\right) - G_{p+2\ell}\left(\frac{x_i^L}{1-\rho^2}\right) \right\}. \end{aligned}$$

We have another integral expression using the formula (6) with $K = 2$:

$$\begin{aligned} &\int_{x_1^L}^{x_1^U} \int_{x_2^L}^{x_2^U} f_p(y_1, y_2; |\rho|)\, \mathrm{d}y_1 \mathrm{d}y_2 \\ &= \left\{ \int_0^{x_1^U} \int_0^{x_2^U} - \int_0^{x_1^L} \int_0^{x_2^U} - \int_0^{x_1^U} \int_0^{x_2^L} + \int_0^{x_1^L} \int_0^{x_2^L} \right\} f_p(y_1, y_2; |\rho|)\, \mathrm{d}y_1 \mathrm{d}y_2 \\ &= \int_0^{\infty} g_p(s) \prod_{i=1}^{2} \left\{ G_p\left(\frac{x_i^U}{1-|\rho|}, \frac{|\rho| s}{1-|\rho|}\right) - G_p\left(\frac{x_i^L}{1-|\rho|}, \frac{|\rho| s}{1-|\rho|}\right) \right\} \mathrm{d}s, \end{aligned}$$

by noting

$$\begin{pmatrix} 1 & |\rho| \\ |\rho| & 1 \end{pmatrix} = \begin{pmatrix} 1-|\rho| & 0 \\ 0 & 1-|\rho| \end{pmatrix} + |\rho| \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1, 1).$$

Remark 4 (i) Such a product (or one-factorial) structure of 2×2 correlation matrix is not unique. In fact, the decomposition

$$\begin{pmatrix} 1 & |\rho| \\ |\rho| & 1 \end{pmatrix} = \begin{pmatrix} 1-\rho_1^2 & 0 \\ 0 & 1-\rho_2^2 \end{pmatrix} + \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} (\rho_1, \rho_2)$$

is valid, provided that $|\rho| = \rho_1\rho_2$. Consequently, there are infinitely many ways for the bivariate central chi-square distribution

$$\int_{x_1^L}^{x_1^U}\int_{x_2^L}^{x_2^U} f_p(y_1, y_2; |\rho|)\,\mathrm{d}y_1\mathrm{d}y_2 = \int_0^\infty g_p(s)\prod_{i=1}^{2}\left\{G_p\left(\frac{x_i^U}{1-\rho_i^2}, \frac{\rho_i^2 s}{1-\rho_i^2}\right) - G_p\left(\frac{x_i^L}{1-\rho_i^2}, \frac{\rho_i^2 s}{1-\rho_i^2}\right)\right\}\mathrm{d}s.$$

(ii) It may be noted from Kimball (1951) that

$$\int_{x_1^L}^{\infty}\int_{x_2^L}^{\infty} f_p(y_1, y_2; |\rho|)\,\mathrm{d}y_1\mathrm{d}y_2 > \overline{G}_p(x_1^L)\overline{G}_p(x_2^L),$$

since $1 - G_p(\cdot, \omega^2)$ is a strictly increasing function of $\omega^2 > 0$.

Anyway, if a, b, c, d have exactly one common index as in (22), we have

$$\widetilde{\rho}_{ab:cd} \equiv |\widetilde{\boldsymbol{\lambda}}'_{ab}\widetilde{\boldsymbol{\lambda}}_{cd}| = \begin{cases} \eta_{a_2a_3,a_1}, & (a,b,c,d) = (a_1,a_2,a_1,a_3) \\ \eta_{a_1a_3,a_2}, & (a,b,c,d) = (a_1,a_2,a_2,a_3) \\ \eta_{a_1a_2,a_3}, & (a,b,c,d) = (a_1,a_3,a_2,a_3) \end{cases}$$

and

$$\lim_{N\to\infty} P(T_{ab}^2 > x, T_{cd}^2 > x) = P(\chi_{ab}^2 > x, \chi_{cd}^2 > x) = S_{ab:cd}(x) > \{\overline{G}_p(x)\}^2, \quad (27)$$

where

$$\begin{aligned} S_{ab:cd}(x) &= (1-\widetilde{\rho}_{ab:cd}^2)^{p/2}\sum_{\ell=0}^{\infty}\frac{\Gamma(p/2+\ell)}{\ell!\Gamma(p/2)}(\widetilde{\rho}_{ab:cd}^2)^{\ell}\left\{\overline{G}_{p+2\ell}\left(\frac{x}{1-\widetilde{\rho}_{ab:cd}^2}\right)\right\}^2 \\ &= \int_0^\infty g_p(s)\left\{1 - G_p\left(\frac{x}{1-\widetilde{\rho}_{ab:cd}}, \frac{\widetilde{\rho}_{ab:cd}\,s}{1-\widetilde{\rho}_{ab:cd}}\right)\right\}^2\mathrm{d}s \\ &= 1 - 2G_p(x) + \int_0^\infty g_p(s)\left\{G_p\left(\frac{x}{1-\widetilde{\rho}_{ab:cd}}, \frac{\widetilde{\rho}_{ab:cd}\,s}{1-\widetilde{\rho}_{ab:cd}}\right)\right\}^2\mathrm{d}s. \end{aligned}$$

Therefore, we summarize (26) and (27), as follows:

$$\begin{aligned} &\lim_{N\to\infty} P(T_{ab}^2 > x, T_{cd}^2 > x) \\ &= \{\overline{G}_p(x)\}^2 + \mathcal{I}_{ab:cd}(x), \quad (a,b),(c,d) \in J;\ (a,b) < (c,d), \end{aligned} \quad (28)$$

where

$$\mathcal{I}_{ab:cd}(x) = \begin{cases} 0, & \text{if } a, b, c, d \text{ are all different} \\ I_{a_2a_3,a_1}(x), & (a, b, c, d) = (a_1, a_2, a_1, a_3) \\ I_{a_1a_3,a_2}(x), & (a, b, c, d) = (a_1, a_2, a_2, a_3) \\ I_{a_1a_2,a_3}(x), & (a, b, c, d) = (a_1, a_3, a_2, a_3) \end{cases}$$

for $a_1, a_2, a_3 \in \{1, \ldots, q\}$; $a_1 < a_2 < a_3$, with

$$I_{b_1b_2,b_3}(x) = \int_0^\infty \frac{s^{p/2-1}\mathrm{e}^{-s}}{\Gamma(p/2)} \left\{ G_p\left(\frac{x}{1-\eta_{b_1b_2,b_3}}, \frac{2\eta_{b_1b_2,b_3}s}{1-\eta_{b_1b_2,b_3}} \right) \right\}^2 \mathrm{d}s - \{G_p(x)\}^2 > 0.$$

We now define

$$B^\infty_{\mathrm{I},1}(x) = B^\infty_{\mathrm{I}}(x) + \frac{2}{q-1} \sum_{1\le a<b\le q-1} I_{ab,q}(x), \tag{29.I}$$

$$B^\infty_{\mathrm{I},2}(x) = B^\infty_{\mathrm{I}}(x) + \sum_{(a,b)\in\mathcal{E}(\mathcal{T}_{\mathrm{I}}^*)} I_{ab,q}(x) \tag{30.I}$$

with

$$B^\infty_{\mathrm{I}}(x) = 1 - (q-1)\overline{G}_p(x) + (q-2)\{\overline{G}_p(x)\}^2,$$

and

$$B^\infty_{\mathrm{II},1}(x) = B^\infty_{\mathrm{II}}(x) + \frac{2}{q(q-1)/2} \sum_{1\le a<b<c\le q} \left\{ I_{bc,a}(x) + I_{ac,b}(x) + I_{ab,c}(x) \right\}, \tag{29.II}$$

$$B^\infty_{\mathrm{II},2}(x) = B^\infty_{\mathrm{II}}(x) + \sum_{(i_{ab},i_{cd})\in\mathcal{E}(\mathcal{T}_{\mathrm{II}}^*)} \mathcal{I}_{ab:cd}(x) \tag{30.II}$$

with

$$B^\infty_{\mathrm{II}}(x) = 1 - \frac{q(q-1)}{2}\overline{G}_p(x) + \frac{(q+1)(q-2)}{2}\{\overline{G}_p(x)\}^2.$$

Combining the improved Bonferroni inequalities (20.I) and (21.I) (or (20.II) and (21.II)) with (23.I) (or (23.II)), (25) and (28), we obtain the following inequalities

$$P(\chi^2_{\max,\mathrm{I}} \le x) \ge B^{\infty}_{\mathrm{I},2}(x) \ge B^{\infty}_{\mathrm{I},1}(x) > 1-(q-1)\overline{G}_p(x) \tag{31.I}$$

for comparisons with a control (we notice that $B^{\infty}_{\mathrm{I},1}(x) = P(\chi^2_{\max,\mathrm{I}} \le x)$ holds exactly for the exceptional case $q=3$), and

$$P(\chi^2_{\max,\mathrm{II}} \le x) \ge B^{\infty}_{\mathrm{II},2}(x) \ge B^{\infty}_{\mathrm{II},1}(x) > 1-\frac{q(q-1)}{2}\overline{G}_p(x) \tag{31.II}$$

for pairwise comparisons.

It is not difficult to see that by equating the right hand sides of (31.I) and (31.II) to $1-\alpha$, the usual Bonferroni upper bounds on $\chi^2_{\max,\mathrm{I}}(\alpha)$ and $\chi^2_{\max,\mathrm{II}}(\alpha)$ are found to be $\chi^2_{p,\alpha_{\mathrm{I}}}$ and $\chi^2_{p,\alpha_{\mathrm{II}}}$, respectively. The lower bounds $B^{\infty}_{\mathrm{I},i}(x)$ and $B^{\infty}_{\mathrm{II},i}(x)$ with two options $i=1,2$, given in (31.I) and (31.II), can be also applied to get improved upper bounds

$$\chi^2_{\max,\mathrm{I}}(\alpha) \le t^2_{\mathrm{I},2}(\alpha) \le t^2_{\mathrm{I},1}(\alpha) < \chi^2_{p,\alpha_{\mathrm{I}}} \tag{32.I}$$

and

$$\chi^2_{\max,\mathrm{II}}(\alpha) \le t^2_{\mathrm{II},2}(\alpha) \le t^2_{\mathrm{II},1}(\alpha) < \chi^2_{p,\alpha_{\mathrm{II}}}, \tag{32.II}$$

by using Lemma 1. That is,
(P1). For $i=1,2$, find a solution of $B^{\infty}_{\mathrm{I},i}(x) = 1-\alpha$, denoted by $t^2_{\mathrm{I},i}(\alpha)$, for comparisons with a control, and find a solution of $B^{\infty}_{\mathrm{II},i}(x) = 1-\alpha$, denoted by $t^2_{\mathrm{II},i}(\alpha)$, for pairwise comparisons.

Remark 5 (I) If $\eta_1 = \cdots = \eta_{q-1} = \eta_0$ (say), both $B^{\infty}_{\mathrm{I},1}(x) = 1-\alpha$ and $B^{\infty}_{\mathrm{I},2}(x) = 1-\alpha$ are then reduced to the same equation

$$1-(q-1)\overline{G}_p(x) + (q-2)[\{\overline{G}_p(x)\}^2 + I_{[2]}(x;\rho^2)] = 1-\alpha \quad \text{with } \rho^2 = \frac{\eta_0}{\eta_0+\eta_q}$$

(see also Remark 2), where

$$I_{[2]}(x;\rho^2) = \int_0^{\infty} \frac{s^{p/2-1}\mathrm{e}^{-s}}{\Gamma(p/2)} \left\{G_p\left(\frac{x}{1-\rho^2}, \frac{2\rho^2 s}{1-\rho^2}\right)\right\}^2 \mathrm{d}s - \{G_p(x)\}^2 > 0.$$

(II) If $\eta_1 = \cdots = \eta_q = 1/q$, the formulae $B^{\infty}_{\mathrm{II},i}(x)$, $i=1,2$, are then simplified to

$$B^{\infty}_{\mathrm{II},1}(x) = 1-\frac{q(q-1)}{2}\overline{G}_p(x) + \frac{(q+1)(q-2)}{2}\{\overline{G}_p(x)\}^2 + 2(q-2)I_{[2]}(x;1/2)$$

and

$$B^{\infty}_{\mathrm{II},2}(x) = 1-\frac{q(q-1)}{2}\overline{G}_p(x) + \frac{(q+1)(q-2)}{2}[\{\overline{G}_p(x)\}^2 + I_{[2]}(x;1/2)]$$

(see also Remark 3), where $B^{\infty}_{\mathrm{II},1}(x) = B^{\infty}_{\mathrm{II},2}(x)$ iff $q=3$.

Remark 6 The numerical computation of the infinite series is implemented by truncating the infinite series to a finite series and then has the problem of estimating the truncation error. Fortunately, our consideration is based on the lower bound, hence (29.I) and (30.I) (or (29.II) and (30.II)) have some variants by using the truncated series for $S_{ab:cd}(x)$, as follows:

$$\begin{aligned}
S_{ab:cd}(x) &= (1-\widetilde{\rho}^2_{ab:cd})^{p/2}\sum_{\ell=0}^{\infty}\frac{\Gamma(p/2+\ell)}{\ell!\Gamma(p/2)}(\widetilde{\rho}^2_{ab:cd})^{\ell}\left\{\overline{G}_{p+2\ell}\left(\frac{x}{1-\widetilde{\rho}^2_{ab:cd}}\right)\right\}^2\\
&> (1-\widetilde{\rho}^2_{ab:cd})^{p/2}\sum_{\ell=0}^{L}\frac{\Gamma(p/2+\ell)}{\ell!\Gamma(p/2)}(\widetilde{\rho}^2_{ab:cd})^{\ell}\left\{\overline{G}_{p+2\ell}\left(\frac{x}{1-\widetilde{\rho}^2_{ab:cd}}\right)\right\}^2\\
&= S^L_{ab:cd}(x) \quad \text{(say) for an integer } L \ge 0.
\end{aligned}$$

4 Large sample procedure (P2) under general distributions

The procedure (P1) enable us to choose the critical values satisfying (32.I) and (32.II), which imply that whatever the underlying population distribution, we have asymptotically conservative $100(1-\alpha)\%$ simultaneous confidence intervals among mean vectors. Apart from some special cases (see Remark 5), the equations $B^{\infty}_{\mathrm{I},2}(x) = 1-\alpha$ and $B^{\infty}_{\mathrm{II},2}(x) = 1-\alpha$ are extremely complicated to handle numerically, since they require a root finding numerical algorithm, together with the maximization over all spanning trees. So, it is important to develop an easily computable method.

The idea is simple, as follows: We have from (20.I)

$$\begin{aligned}
P(T^2_{\max,\mathrm{I}} \le x) &\ge 1-\sum_{a=1}^{q-1}\overline{P}_{aq}(x) + \sum_{(a,b)\in\mathcal{E}(\mathcal{T}^*_{\mathrm{I}})}\overline{P}_{aq:bq}(x \vee t^2_{\mathrm{I},0})\\
&\ge 1-\sum_{a=1}^{q-1}\overline{P}_{aq}(x) + \frac{2}{q-1}\sum_{1\le a<b\le q-1}\overline{P}_{aq:bq}(x \vee t^2_{\mathrm{I},0}),
\end{aligned}$$

where $x \vee y = \max(x, y)$ and $t^2_{\mathrm{I},0}$ is the solution of (19.I). Hence, if the exact or asymptotic formulae on $\overline{P}_{aq}(x)$ and $\overline{P}_{aq:bq}(x)$ are available, by equating the above lower bounds to $1-\alpha$ (the existence of a solution in $(t^2_{\max,\mathrm{I}}(\alpha), t^2_{\mathrm{I},0})$ is guaranteed by Lemma 1), we can get simultaneous confidence intervals (2.I) with other critical value, which is the solution of the equation

$$\sum_{a=1}^{q-1}\overline{P}_{aq}(x) = \alpha + \frac{2}{q-1}\sum_{1\le a<b\le q-1}\overline{P}_{aq:bq}(t^2_{\mathrm{I},0})$$

or

$$\sum_{a=1}^{q-1} \overline{P}_{aq}(x) = \alpha + \sum_{(a,b)\in\mathcal{E}(\mathcal{T}_{\mathrm{I}}^*)} \overline{P}_{aq:bq}(t_{\mathrm{I},0}^2).$$

Such a method turns out to be a modification of Siotani's (1959, 1960, 1964) procedure, except for the case $q = 3$ (note that for $q = 3$, both equations coincides with Siotani's proposal (10)).

Similarly, we have from (20.II)

$$\begin{aligned} P(T_{\max,\mathrm{II}}^2 \le x) &\ge 1 - \sum_{1\le a<b\le q} \overline{P}_{ab}(x) + \sum_{(i_{ab},i_{cd})\in\mathcal{E}(\mathcal{T}_{\mathrm{II}}^*)} \overline{P}_{ab:cd}(x \vee t_{\mathrm{II},0}^2) \\ &\ge 1 - \sum_{1\le a<b\le q} \overline{P}_{ab}(x) + \frac{2}{q(q-1)/2} \sum_{(ab)(cd)} \overline{P}_{ab:cd}(x \vee t_{\mathrm{II},0}^2), \end{aligned}$$

where $t_{\mathrm{II},0}^2$ is the solution of (19.II). Equating the above lower bounds to $1-\alpha$ (the existence of a solution in $(t_{\max,\mathrm{II}}^2(\alpha), t_{\mathrm{II},0}^2)$ is guaranteed by Lemma 1), the critical value for pairwise comparisons is proposed as the solution of the equation

$$\sum_{1\le a<b\le q} \overline{P}_{ab}(x) = \alpha + \frac{2}{q(q-1)/2} \sum_{(ab)(cd)} \overline{P}_{ab:cd}(t_{\mathrm{II},0}^2)$$

or

$$\sum_{1\le a<b\le q} \overline{P}_{ab}(x) = \alpha + \sum_{(i_{ab},i_{cd})\in\mathcal{E}(\mathcal{T}_{\mathrm{II}}^*)} \overline{P}_{ab:cd}(t_{\mathrm{II},0}^2).$$

It may be noted that Siotani's (1959) procedure for pairwise comparisons was to use the solution of the equation

$$\sum_{1\le a<b\le q} \overline{P}_{ab}(x) = \alpha + \sum_{(ab)(cd)} \overline{P}_{ab:cd}(t_{\mathrm{II},0}^2).$$

In this way, we propose an easily computable and conservative procedure (P2) for the large sample but possibly nonnormal case, as follows:

(P2) and Siotani's original procedure. Compute the critical values $\chi^2_{p,\alpha_{\mathrm{I},i}}$ and $\chi^2_{p,\alpha_{\mathrm{II},i}}$ with

$$\alpha_{\mathrm{I},i} = \frac{\alpha + A_{\mathrm{I},i}^{\infty}}{q-1} \quad \text{and} \quad \alpha_{\mathrm{II},i} = \frac{\alpha + A_{\mathrm{II},i}^{\infty}}{q(q-1)/2}$$

for three options $i = 1, 2$ and S, where

$$A^{\infty}_{\mathrm{I},S} = \frac{(q-1)(q-2)}{2}\,\alpha_{\mathrm{I}}^2 + \sum_{1\le a<b\le q-1} I_{ab,q}(\chi^2_{p,\alpha_{\mathrm{I}}}), \tag{33.I}$$

$$A^{\infty}_{\mathrm{I},1} = (q-2)\alpha_{\mathrm{I}}^2 + \frac{2}{q-1}\sum_{1\le a<b\le q-1} I_{ab,q}(\chi^2_{p,\alpha_{\mathrm{I}}}), \tag{34.I}$$

$$A^{\infty}_{\mathrm{I},2} = (q-2)\alpha_{\mathrm{I}}^2 + \sum_{(a,b)\in\mathcal{E}(\mathcal{T}_{\mathrm{I}}^*)} I_{ab,q}(\chi^2_{p,\alpha_{\mathrm{I}}}) \tag{35.I}$$

and

$$\begin{aligned} A^{\infty}_{\mathrm{II},S} &= \frac{(q+1)q(q-1)(q-2)}{8}\,\alpha_{\mathrm{II}}^2 \\ &\quad + \sum_{1\le a<b<c\le q} \{I_{bc,a}(\chi^2_{p,\alpha_{\mathrm{II}}}) + I_{ac,b}(\chi^2_{p,\alpha_{\mathrm{II}}}) + I_{ab,c}(\chi^2_{p,\alpha_{\mathrm{II}}})\}, \end{aligned} \tag{33.II}$$

$$\begin{aligned} A^{\infty}_{\mathrm{II},1} &= \frac{(q+1)(q-2)}{2}\,\alpha_{\mathrm{II}}^2 \\ &\quad + \frac{2}{q(q-1)/2}\sum_{1\le a<b<c\le q} \{I_{bc,a}(\chi^2_{p,\alpha_{\mathrm{II}}}) + I_{ac,b}(\chi^2_{p,\alpha_{\mathrm{II}}}) + I_{ab,c}(\chi^2_{p,\alpha_{\mathrm{II}}})\}, \end{aligned} \tag{34.II}$$

$$A^{\infty}_{\mathrm{II},2} = \frac{(q+1)(q-2)}{2}\,\alpha_{\mathrm{II}}^2 + \sum_{(i_{ab},i_{cd})\in\mathcal{E}(\mathcal{T}_{\mathrm{II}}^*)} \mathcal{I}_{ab:cd}(\chi^2_{p,\alpha_{\mathrm{II}}}). \tag{35.II}$$

Apart from special cases as in Remark 5, the minimal spanning tree algorithm due to Kruskal (see Hunter 1976) that we used here as the maximization is needed once for the sums $\sum_{(a,b)\in\mathcal{E}(\mathcal{T}_{\mathrm{I}}^*)} I_{ab,q}(\chi^2_{p,\alpha_{\mathrm{I}}})$ and $\sum_{(i_{ab},i_{cd})\in\mathcal{E}(\mathcal{T}_{\mathrm{II}}^*)} \mathcal{I}_{ab:cd}(\chi^2_{p,\alpha_{\mathrm{II}}})$, defined in terms of the maximal spanning tree $\mathcal{T}_{\mathrm{I}}^*$ and $\mathcal{T}_{\mathrm{II}}^*$, respectively.

The proposed critical values with two options $i = 1, 2$ satisfy

$$\chi^2_{\max,\mathrm{I}}(\alpha) < \chi^2_{p,\alpha_{\mathrm{I},2}} \le \chi^2_{p,\alpha_{\mathrm{I},1}} < \chi^2_{p,\alpha_{\mathrm{I}}}$$

and

$$\chi^2_{\max,\mathrm{II}}(\alpha) < \chi^2_{p,\alpha_{\mathrm{II},2}} \le \chi^2_{p,\alpha_{\mathrm{II},1}} < \chi^2_{p,\alpha_{\mathrm{II}}}.$$

However, (P2) is worse than (P1), that is,

$$\chi^2_{\max,\mathrm{I}}(\alpha) \le t^2_{\mathrm{I},i}(\alpha) < \chi^2_{p,\alpha_{\mathrm{I},i}} \quad \text{and} \quad \chi^2_{\max,\mathrm{II}}(\alpha) \le t^2_{\mathrm{II},i}(\alpha) < \chi^2_{p,\alpha_{\mathrm{II},i}}$$

for $i = 1, 2$.

5 Comparison of (P1),(P2) and Siotani's procedure

5.1 Comparisons with a control

We now compare the proposed procedures (P1) and (P2) with option $i = 1$ and Siotani's procedure to the exact percentile $\chi^2_{\max,\mathrm{I}}(\alpha)$, being a solution of (24). For simplicity, we assume that the correlation matrix $\widetilde{P}_{\mathrm{I}}$ is equicorrelated. That is, $\widetilde{P}_{\mathrm{I}} = (1-\rho^2)I_{q-1} + \rho^2\mathbf{1}\mathbf{1}'$, where $\mathbf{1} = (1, \ldots, 1)' \in \mathbf{R}^{q-1}$ and $\rho^2 \in (0, 1)$, which corresponds to the large sample case of $N_a \to \infty, a = 1, \ldots, q$, in such a way that the total number N of observations goes to infinity, while the ratio vector $(N_1/N, \ldots, N_q/N)$ converges to $(\eta_0, \ldots, \eta_0, \eta_q)$ satisfying $(q-1)\eta_0 + \eta_q = 1$ and $\eta_0/(\eta_0 + \eta_q) = \rho^2$. Thus, we have $\rho^2 = 1/2$ iff $N_1 \approx N_2 \approx \cdots \approx N_q$. A double precision Fortran program that yields $\chi^2_{\max,\mathrm{I}}(\alpha)$, $t^2_{\mathrm{I},1}(\alpha)$, $\chi^2_{p,\alpha_{\mathrm{I},1}}$, $\chi^2_{p,\alpha_{\mathrm{I},S}}$ for inputted values of (p, q, α, ρ^2) was written. We notice $t^2_{\mathrm{I},1}(\alpha) = t^2_{\mathrm{I},2}(\alpha)$ and $\chi^2_{p,\alpha_{\mathrm{I},1}} = \chi^2_{p,\alpha_{\mathrm{I},2}}$ for the present equicorrelated case.

For comparative purposes, it is convenient to determine the ratios of the square of the length of simultaneous confidence intervals (17.I) using $\chi^2_{\max,\mathrm{I}}(\alpha)$ (the limiting procedure) to those for (P1), (P2), Siotani's procedure and the usual Bonferroni procedure; $R_{1,\alpha} = t^2_{\mathrm{I},1}(\alpha)/\chi^2_{\max,\mathrm{I}}(\alpha)$, $R_{2,\alpha} = \chi^2_{p,\alpha_{\mathrm{I},1}}/\chi^2_{\max,\mathrm{I}}(\alpha)$, $R_{S,\alpha} = \chi^2_{p,\alpha_{\mathrm{I},S}}/\chi^2_{\max,\mathrm{I}}(\alpha)$ and $R_{0,\alpha} = \chi^2_{p,\alpha_{\mathrm{I}}}/\chi^2_{\max,\mathrm{I}}(\alpha)$. These ratios, except for $R_{S,\alpha}$, are theoretically supported to be greater than 1, which means that their procedures are asymptotically conservative.

From Tables 1, 2, 3, the improved Bonferroni procedure is more efficient than the usual Bonferroni procedure. Also, the following observations can be made about the efficiencies of several procedures to the limiting procedure:

A. Both (P1) and (P2) are very efficient for small ρ^2 values. In fact, even the usual Bonferroni procedure with $\chi^2_{p,\alpha_{\mathrm{I}}}$ has a good performance.
B. The efficiencies of both (P1) and (P2) deteriorate somewhat for the larger ρ^2 values, especially for large q values.
C. Remarkably, Siotani's procedure is a good approximation for $\chi^2_{\max,\mathrm{I}}(\alpha)$. But, Table 3 shows that his critical value $\chi^2_{p,\alpha_{\mathrm{I},S}}$ can be less than $\chi^2_{\max,\mathrm{I}}(\alpha)$. In other words, his procedure is sometimes liberal.

Table 1 Critical values for $p = 3, q = 3, \alpha = 0.05$, where $\chi^2_{p,\alpha_{\mathrm{I}}} = 9.348$

ρ^2	$\chi^2_{\max,\mathrm{I}}(\alpha) = t^2_{\mathrm{I},1}(\alpha)$ $(R_{0,\alpha})$	$\chi^2_{p,\alpha_{\mathrm{I},1}} = \chi^2_{p,\alpha_{\mathrm{I},S}}$ $(R_{2,\alpha} = R_{S,\alpha})$
0.125	9.315(1.004)	9.316(1.000)
0.250	9.298(1.005)	9.300(1.000)
0.333	9.278(1.008)	9.281(1.000)
0.500	9.210(1.015)	9.221(1.001)
0.667	9.081(1.029)	9.118(1.004)
0.750	8.978(1.041)	9.043(1.007)
0.875	8.726(1.071)	8.880(1.018)

Table 2 Critical values for $p = 3, q = 6, \alpha = 0.05$, where $\chi^2_{p,\alpha_{\mathrm{I}}} = 11.345$

ρ^2	$\chi^2_{\max,\mathrm{I}}(\alpha)$ $(R_{0,\alpha})$	$t^2_{\mathrm{I},1}(\alpha)$ $(R_{1,\alpha})$	$\chi^2_{p,\alpha_{\mathrm{I},1}}$ $(R_{2,\alpha})$	$\chi^2_{p,\alpha_{\mathrm{I},S}}$ $(R_{S,\alpha})$
0.125	11.288(1.005)	11.322(1.003)	11.323(1.003)	11.290(1.000)
0.250	11.245(1.009)	11.304(1.005)	11.306(1.005)	11.248(1.000)
0.333	11.195(1.013)	11.282(1.008)	11.285(1.008)	11.199(1.000)
0.500	11.019(1.030)	11.198(1.016)	11.212(1.017)	11.026(1.001)
0.667	10.689(1.061)	11.014(1.030)	11.070(1.036)	10.712(1.002)
0.750	10.430(1.088)	10.848(1.040)	10.959(1.051)	10.484(1.005)
0.875	9.818(1.155)	10.384(1.058)	10.712(1.091)	10.011(1.020)

Table 3 Critical values for $p = 3, q = 9, \alpha = 0.05$, where $\chi^2_{p,\alpha_{\mathrm{I}}} = 12.359$

ρ^2	$\chi^2_{\max,\mathrm{I}}(\alpha)$ $(R_{0,\alpha})$	$t^2_{\mathrm{I},1}(\alpha)$ $(R_{1,\alpha})$	$\chi^2_{p,\alpha_{\mathrm{I},1}}$ $(R_{2,\alpha})$	$\chi^2_{p,\alpha_{\mathrm{I},S}}$ $(R_{S,\alpha})$
0.125	12.294(1.005)	12.343(1.004)	12.343(1.004)	12.296(1.000)
0.250	12.237(1.010)	12.328(1.007)	12.328(1.007)	12.239(1.000)
0.333	12.169(1.016)	12.308(1.011)	12.310(1.012)	12.170(1.000)
0.500	11.929(1.036)	12.230(1.025)	12.240(1.026)	11.918(0.999)
0.667	11.482(1.076)	12.046(1.049)	12.096(1.053)	11.455(0.998)
0.750	11.136(1.110)	11.872(1.066)	11.979(1.076)	11.124(0.999)
0.875	10.334(1.196)	11.361(1.099)	11.711(1.133)	10.464(1.013)

5.2 Pairwise comparisons

This subsection contains a brief comparison of the proposed procedures (P1) and (P2) with option $i = 2$ and Siotani's procedure, together with the usual Bonferroni procedure. We now consider the large sample case of $N_a \to \infty$, $a = 1, \ldots, q$, in such a way that the total number N of observations goes to infinity, while the ratio N_a/N converges to $1/q$, $a = 1, \ldots, q$. Contrast with the percentile $\chi^2_{\max,\mathrm{I}}(\alpha)$ in Sect. 5.1, no exact tabulation of the percentile $\chi^2_{\max,\mathrm{II}}(\alpha)$ is available, for which the large simulations were done by Royen (1984). Only for the case $p = 1$, the exact value of $\{\chi^2_{\max,\mathrm{II}}(\alpha)\}^{1/2}$ is computable (e.g. Hsu (1996; Tables E.1 with $k = q$ and $\nu = \infty$)). A double precision Fortran program that yields $\chi^2_{p,\alpha_{\mathrm{II},2}}$, $t^2_{\mathrm{II},2}(\alpha)$, $\chi^2_{p,\alpha_{\mathrm{II},S}}$ for inputted values of (p, q, α) was written. We notice $t^2_{\mathrm{II},1}(\alpha) \geq t^2_{\mathrm{II},2}(\alpha)$ and $\chi^2_{p,\alpha_{\mathrm{II},1}} \geq \chi^2_{p,\alpha_{\mathrm{II},2}}$, where the equality holds iff $q = 3$.

From Table 4, the improved Bonferroni procedure is more efficient than the usual Bonferroni procedure, while the difference $\chi^2_{p,\alpha_{\mathrm{II}}} - t^2_{\mathrm{II},2}(\alpha)$ decreases as q increases. Such a tendency also tells us that the improved Bonferroni bounds do not work well. The values $\chi^2_{p,\alpha_{\mathrm{II},S}}$ based on Siotani's (1959) proposal are, by chance, close to the values that Royen (1984; Table 2b with $\nu = p$ and $m = q$) gave by simulation. But, the

Table 4 Critical values for $p = 3, \alpha = 0.05$

q	$\chi^2_{p,\alpha_{\mathrm{II}}}$	$\chi^2_{p,\alpha_{\mathrm{II},2}}$	$t^2_{\mathrm{II},2}(\alpha)$	$\chi^2_{p,\alpha_{\mathrm{II},S}}$	Royen
3	10.236	10.096	10.081	10.029	9.97
4	11.739	11.611	11.599	11.435	11.36
5	12.838	12.727	12.718	12.474	12.41
6	13.706	13.609	13.603	13.300	13.25
7	14.424	14.338	14.333	13.986	13.94
8	15.037	14.959	14.955	14.572	14.53

reason why Siotani's original procedure has a good performance is not yet explained theoretically.

6 Discussion

In principle, the formulae $\overline{P}^{\infty}_{ab}(x) = \overline{G}_p(x)$ and $\overline{P}^{\infty}_{ab:cd}(x) = \{\overline{G}_p(x)\}^2 + \mathcal{I}_{ab:cd}(x)$ we used in this paper may be replaced by

$$\overline{P}^{\mathrm{AE1}}_{ab}(x) = \overline{G}_p(x) + \frac{1}{N}\overline{P}^{[1]}_{ab}(x), \quad (a,b) \in J$$

and

$$\overline{P}^{\mathrm{AE1}}_{ab:cd}(x) = \overline{P}^{\infty}_{ab:cd}(x) + \frac{1}{N}\overline{P}^{[1]}_{ab:cd}(x), \quad (a,b),(c,d) \in J; \ \ (a,b) < (c,d),$$

where

$$\overline{P}^{[1]}_{ab}(x) = \lim_{N\to\infty} N[\overline{P}_{ab}(x) - \overline{G}_p(x)]$$

and

$$\overline{P}^{[1]}_{ab:cd}(x) = \lim_{N\to\infty} N[\overline{P}_{ab:cd}(x) - \overline{P}^{\infty}_{ab:cd}(x)]$$

(see Seo 2002, Okamoto and Seo, Okamoto 2005, Kakizawa 2006). Then, more accurate variants for (P1) and (P2) are possible. For example, (29.I) and (29.II) for the case of option $i = 1$ are replaced by

$$B^{\mathrm{AE1}}_{\mathrm{I},1}(x) = B^{\infty}_{\mathrm{I},1}(x) + \frac{1}{N}B^{[1]}_{\mathrm{I},1}(x) \quad \text{and} \quad B^{\mathrm{AE1}}_{\mathrm{II},1}(x) = B^{\infty}_{\mathrm{II},1}(x) + \frac{1}{N}B^{[1]}_{\mathrm{II},1}(x),$$

where

$$B^{[1]}_{\mathrm{I},1}(x) = 1 - \sum_{a=1}^{q-1}\overline{P}^{[1]}_{aq}(x) + \frac{2}{q-1}\sum_{1\le a<b\le q-1}\overline{P}^{[1]}_{aq:bq}(x)$$

and

$$B_{\mathrm{II},1}^{[1]}(x) = 1 - \sum_{1 \le a < b \le q} \overline{P}_{ab}^{[1]}(x) + \frac{2}{q(q-1)/2} \sum_{(ab)(cd)} \overline{P}_{ab:cd}^{[1]}(x).$$

With K = I, II, we can see that a solution of $B_{\mathrm{K},1}^{\mathrm{AE1}}(x) = 1 - \alpha$ is expanded as

$$t_{\mathrm{K},1}^2(\alpha) - \frac{1}{N} \frac{B_{\mathrm{K},1}^{[1]}\{t_{\mathrm{K},1}^2(\alpha)\}}{\frac{\mathrm{d}}{\mathrm{d}x} B_{\mathrm{K},1}^{\infty}\{t_{\mathrm{K},1}^2(\alpha)\}} + \mathrm{o}(N^{-1}), \tag{36}$$

where $t_{\mathrm{K},1}^2$ is a solution of $B_{\mathrm{K},1}^{\infty}(x) = 1 - \alpha$ (see (P1)). To assure the positiveness, we may use the formula

$$t_{\mathrm{K},1}^2(\alpha) \left[1 - \frac{1}{2N} \frac{B_{\mathrm{K},1}^{[1]}\{t_{\mathrm{K},1}^2(\alpha)\}}{t_{\mathrm{K},1}^2(\alpha) \frac{d}{dx} B_{\mathrm{K},1}^{\infty}\{t_{\mathrm{K},1}^2(\alpha)\}} \right]^2$$

in place of (36) without the remainder term. Further, (34.I) and (34.II) for the case of option $i = 1$ are replaced by

$$A_{\mathrm{I},1}^{\mathrm{AE1}} = \frac{2}{q-1} \sum_{1 \le a < b \le q-1} \overline{P}_{aq:bq}^{\mathrm{AE1}} \left\{ CF_{\mathrm{I}}(\chi_{p,\alpha_{\mathrm{I}}}^2) \right\}$$

and

$$A_{\mathrm{II},1}^{\mathrm{AE1}} = \frac{2}{q(q-1)/2} \sum_{(ab)(cd)} \overline{P}_{ab:cd}^{\mathrm{AE1}} \left\{ CF_{\mathrm{II}}(\chi_{p,\alpha_{\mathrm{II}}}^2) \right\},$$

and using them, (P2) is now modified as $CF_{\mathrm{K}}(\chi_{p,\alpha'_{\mathrm{K},1}}^2)$, K = I, II, where

$$\alpha'_{\mathrm{I},1} = \frac{\alpha + A_{\mathrm{I},1}^{\mathrm{AE1}}}{q-1} \quad \text{and} \quad \alpha'_{\mathrm{II},1} = \frac{\alpha + A_{\mathrm{II},1}^{\mathrm{AE1}}}{q(q-1)/2},$$

with $CF_{\mathrm{I}}(\cdot)$ and $CF_{\mathrm{II}}(\cdot)$ being the Cornish-Fisher type polynomial (see Seo 2002, Okamoto 2005, Kakizawa 2006).

7 Concluding remarks

The aim of this paper was to construct asymptotically conservative $100(1-\alpha)\%$ simultaneous confidence intervals among mean vectors for cases of comparisons with a control and pairwise comparisons. This was accomplished by following two procedures (P1) and (P2) presented in Sects. 3 and 4. We emphasize that although we focused on the Hunter-Worsley inequalities based on the spanning tree of the complete graph, any Bonferroni-type lower bounds are applicable. It may be true that Siotani's (1959, 1960, 1964) original procedure gives very close approximation to the percentile of $\chi^2_{\max}$-type statistic, but its mathematical property is not yet clarified theoretically (as shown numerically in Sect. 5.1, it may be sometimes liberal).

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