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Semi-self-decomposable distributions on $\mathbf{Z}_+$

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Abstract We present a notion of semi-self-decomposability for distributions with support in $\mathbf{Z}_+$. We show that discrete semi-self-decomposable distributions are infinitely divisible and are characterized by the absolute monotonicity of a specific function. The class of discrete semi-self-decomposable distributions is shown to contain the discrete semistable distributions and the discrete geometric semistable distributions. We identify a proper subclass of semi-self-decomposable distributions that arise as weak limits of subsequences of binomially thinned sums of independent $\mathbf{Z}_+$-valued random variables. Multiple semi-self-decomposability on $\mathbf{Z}_+$ is also discussed.

Keywords Discrete distributions · Infinite divisibility · Semistability · Poisson mixtures · Probability generating functions · Weak convergence

1 Introduction

The notion of semi-self-decomposability and the related concept of semistability have been the object of growing interest in recent years. Semi-self-decomposable and semistable distributions derive their importance from the fact that they arise as solutions to central limit-type problems. We cite the articles by Maejima and Naito (1998), Maejima et al. (1999), Maejima (2001), Huillet et al. (2001),Meerschaert and Scheffler (2001), and Becker-Kern and Scheffler (2005). These concepts are also closely connected with Lévy processes,

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in particular semistable processes and semi-self-similar processes (see Choi 1994, Maejima and Sato 1999, and Sato 1999).

A distribution on the real line is said to be semi-self-decomposable of order $\alpha \in (0,1)$ if its characteristic function $f(t)$ satisfies

$$f(t) = f(\alpha t) f_\alpha(t) \quad (t \in \mathbf{R}), \tag{1}$$

for some infinitely divisible characteristic function $f_\alpha(t)$.

Maejima and Naito (1998) showed that semi-self-decomposable distributions are infinitely divisible and that their Lévy measures are characterized by the representation

$$\nu(E) = \int_E \mathrm{d}M_1(x) \quad \text{and} \quad \nu(-E) = \int_E \mathrm{d}M_2(x),$$

for every $E \in \mathcal{B}((0,\infty))$ (the tribe of Borel sets in $(0,\infty)$), where for $i = 1,2$, $M_i(x)$ is right-continuous and non-decreasing on $(0,\infty)$, $\lim_{x\to\infty} M_i(x) = 0$,

$$M_i(\alpha x) - M_i(\alpha(x+y)) \leq M_i(x) - M_i(x+y) \leq 0,$$

and

$$0 < \int_0^\infty (1 \wedge x^2)\,\mathrm{d}M_1(x) = \int_0^\infty (1 \wedge x^2)\,\mathrm{d}M_2(x) < \infty.$$

Importantly, Maejima and Naito (1998) showed that the class of semi-self-decomposable distributions coincides with the class of weak limits of subsequences of normalized sums of independent random variables.

Steutel and van Harn (1979) introduced the binomial thinning operation $\odot$ which they defined as follows:

$$\alpha \odot X = \sum_{i=1}^{X} X_i, \tag{2}$$

where $\alpha \in (0,1)$, X is a $\mathbf{Z}_+$-valued random variable, here $\mathbf{Z}_+ := \{0,1,2,\cdots\}$, and $\{X_i\}$ is a sequence of independent identically distributed (iid) Bernoulli(α) random variables independent of X. The authors used the operation $\odot$ to introduce the concepts of discrete self-decomposability and stability. A $\mathbf{Z}_+$-valued random variable X is said to have a discrete self-decomposable distribution if for every $\alpha \in (0,1)$,

$$X \stackrel{d}{=} \alpha \odot X' + X_\alpha, \tag{3}$$

where X' and X_α are $\mathbf{Z}_+$-valued, $\alpha \odot X'$ and X_α are independent, and X' has the same distribution as X. Similarly to their continuous counterparts, discrete self-decomposable distributions are infinitely divisible, unimodal, and can be characterized via canonical representations of their probability generating functions (pgfs). The class of discrete self-decomposable distributions contains the discrete stable as well as the discrete geometric stable distributions. We refer to the recent monograph by Steutel and van Harn (2004) for details and for more on the topic of discrete self-decomposability.

The purpose of this paper is to present a notion of semi-self-decomposability for distributions with support in $\mathbf{Z}_+$ that generalizes the concept of discrete self-decomposability. Our approach parallels that of Maejima and Naito (1998) in the continuous case. In Sect. 2, we give a definition of discrete semi-self-decomposability that is analogous to (1). We obtain several properties, including characterization results, of discrete semi-self-decomposable distributions. Notably, we establish that these distributions possess the property of infinite divisibility and that they are characterized by the absolute monotonicity of a specific function. We show that the class of discrete semi-self-decomposable distributions contains the discrete semistable distributions and the discrete geometric semistable distributions. We construct some examples and offer a counterexample that shows that, in general, discrete semi-self-decomposable distributions do not possess the property of unimodality. In Sect. 3, we establish a connection between semi-self-decomposability on $\mathbf{R}_+$ and discrete semi-self-decomposability by way of Poisson mixtures and we proceed to identify a proper subclass of semi-self-decomposable distributions that arise as weak limits of subsequences of binomially thinned sums of independent $\mathbf{Z}_+$-valued random variables. A related limit theorem that leads to discrete semistability is also given. Finally, multiple semi-self-decomposability on $\mathbf{Z}_+$ is discussed in Sect. 4.

We recall a useful characterization of infinite divisibility for distributions on $\mathbf{Z}_+$ (see Theorem II.4.2 in Steutel and van Harn 2004).

Theorem 1 *A distribution $(p_n, n \geq 0)$ on $\mathbf{Z}_+$ with pgf $P(z)$, $0 < P(0) < 1$, is infinitely divisible if and only if the function $R(z) = P'(z)/P(z)$ is absolutely monotone on $[0, 1)$, with power series expansion*

$$R(z) = \sum_{n=0}^{\infty} r_n z^n \quad (z \in [0, 1)), \tag{4}$$

where $r_n \geq 0$ for every $n \geq 0$ and, necessarily, $\sum_{n=0}^{\infty} r_n/(n+1) < \infty$.

Following Steutel and van Harn (2004), we will refer to the function $R(z)$ (respectively, the sequence $(r_n, n \geq 0)$) in Theorem 1 as the R-function (respectively, the canonical sequence) of $(p_n, n \geq 0)$, or that of its pgf $P(z)$.

2 Discrete semi-self-decomposability

Definition 1 A nondegenerate distribution $(p_n, n \geq 0)$ on $\mathbf{Z}_+$ is said to be discrete semi-self-decomposable of order $\alpha \in (0, 1)$ if its pgf $P(z)$ satisfies for all $|z| \leq 1$

$$P(z) = P(1-\alpha+\alpha z)P_\alpha(z), \tag{5}$$

where $P_\alpha(z)$ is the pgf of an infinitely divisible distribution.

Let X be a $\mathbf{Z}_+$-valued random variable with pgf $P(z)$. Noting that $P(1-\alpha+\alpha z)$ is the pgf of $\alpha \odot X$ of (2), it follows that X has a discrete semi-self-decomposable distribution of order $\alpha \in (0,1)$ if and only if it admits the representation (3) with the further property that X_α has an infinitely divisible distribution.

Discrete semi-self-decomposability implies infinite divisibility.

Theorem 2 *A discrete semi-self-decomposable distribution of order $\alpha \in (0,1)$ is infinitely divisible.*

Proof Let $P(z)$ be the pgf of a discrete semi-self-decomposable distribution of order $\alpha \in (0,1)$. By (5) and an induction argument, we have for all $k \geq 1$,

$$P(z) = P(1-\alpha^k+\alpha^k z)\prod_{i=0}^{k-1} P_\alpha(1-\alpha^i+\alpha^i z) \quad (|z| \leq 1), \tag{6}$$

for some infinitely divisible pgf $P_\alpha(z)$. By Proposition II.6.1 in Steutel and van Harn (2004), $P_\alpha(1-\alpha^i+\alpha^i z)$ is infinitely divisible for every $i \geq 0$. Closure under convolution of infinite divisibility implies that $\prod_{i=0}^{k-1} P_\alpha(1-\alpha^i+\alpha^i z)$ is an infinitely divisible pgf. Moreover, we have by (6)

$$P(z) = \lim_{k\to\infty}\prod_{i=0}^{k} P_\alpha(1-\alpha^i+\alpha^i z) \quad (|z| \leq 1).$$

Since the class of infinitely divisible discrete distributions is closed under weak convergence, we conclude that $P(z)$ is infinitely divisible. □

Next, we obtain some useful characterizations of discrete semi-self-decomposability.

Theorem 3 *Let $(p_n, n \geq 0)$ be a distribution on $\mathbf{Z}_+$ with pgf $P(z)$ and let $\alpha \in (0,1)$. The following assertions are equivalent.*

(i) *$(p_n, n \geq 0)$ is discrete semi-self-decomposable of order $\alpha \in (0,1)$.*
(ii) *$(p_n, n \geq 0)$ is infinitely divisible and the function*

$$R_\alpha(z) = R(z) - \alpha R(1-\alpha+\alpha z), \tag{7}$$

(where $R(z)$ is the R-function of $P(z)$) is absolutely monotone on $[0,1)$.

(iii) $(p_n, n \geq 0)$ *is infinitely divisible and its canonical sequence* $(r_n, n \geq 0)$ *satisfies for every* $n \geq 0$,

$$r_n - \alpha^{n+1} \sum_{j=0}^{\infty} \binom{j+n}{n} (1-\alpha)^j r_{j+n} \geq 0. \tag{8}$$

Proof Assume that (i) holds. By Theorem 2, $(p_n, n \geq 0)$ is infinitely divisible. Therefore, we have by (5) $\ln P(z) = \ln P(1-\alpha+\alpha z) + \ln P_\alpha(z)$. This implies that the R-functions $R(z)$ and $R_\alpha(z)$ of $P(z)$ and $P_\alpha(z)$, respectively, are related by Eq. (7) and hence (ii) follows by Theorem 1. The converse ((ii)⇒(i)) is deduced straightforwardly from Theorem 1 applied to $P_\alpha(z)$ and its R-function $R_\alpha(z)$ of (7). To establish ((ii)⇔(iii)), we note that if $(r_n, n \geq 0)$ is the canonical sequence of $P(z)$, then by (4) and (7)

$$R_\alpha(z) = \sum_{n=0}^{\infty} r_n z^n - \sum_{n=0}^{\infty} r_n (1-\alpha+\alpha z)^n = \sum_{n=0}^{\infty} \left(r_n - \sum_{k=n}^{\infty} \binom{k}{n} \alpha^{n+1} (1-\alpha)^{k-n} r_k \right) z^n,$$

for any $z \in [0, 1)$. Therefore, $R_\alpha(z)$ is absolutely monotone if and only if (8) holds. □

Corollary 1 *The support of a discrete semi-self-decomposable distribution* $(p_n, n \geq 0)$ *of order* $\alpha \in (0, 1)$ *is equal to* $\mathbf{Z}_+$. *i.e.,* $p_n > 0$ *for every* $n \geq 0$.

Proof By Theorem 2, $(p_n, n \geq 0)$ is infinitely divisible in the discrete sense, i.e., the factor in the k-fold convolution ($k \geq 1$) is $\mathbf{Z}_+$-valued. It follows that $p_0 > 0$ (see Sect. II.1 in Steutel and van Harn 2004). Let $(r_n, n \geq 0)$ be the canonical sequence of $(p_n, n \geq 0)$. If $r_0 = 0$, then by (8) $r_n = 0$ for every $n \geq 1$. This implies that $(p_n, n \geq 0)$ is a degenerate distribution, which is a contradiction. Therefore, $r_0 > 0$ and hence $p_1 > 0$ (as $p_1 = r_0 p_0$). The conclusion follows by Corollary II.8.3 in Steutel and van Harn (2004). □

The following result gives some characterizations of discrete self-decomposability.

Proposition 1 *Let* $(p_n, n \geq 0)$ *be a distribution on* $\mathbf{Z}_+$. *The following assertions are equivalent.*

(i) $(p_n, n \geq 0)$ *is discrete self-decomposable.*
(ii) $(p_n, n \geq 0)$ *is infinitely divisible and its canonical sequence* $(r_n, n \geq 0)$ *is nonincreasing.*
(iii) $(p_n, n \geq 0)$ *is discrete semi-self-decomposable of order* α *for every* $\alpha \in (0, 1)$.

Proof (i)⇒(ii) follows from Theorem V.4.13 in Steutel and van Harn (2004). To show (ii)⇒(iii), we note that if $(r_n, n \geq 0)$ is nonincreasing, then for every

$\alpha \in (0,1)$ and $n \geq 0$

$$r_n - \alpha^{n+1} \sum_{j=0}^{\infty} \binom{j+n}{n} (1-\alpha)^j r_{j+n} \geq r_n \left[1 - \alpha^{n+1} \sum_{j=0}^{\infty} \binom{j+n}{n} (1-\alpha)^j \right] = 0.$$

Therefore, (8) holds and (iii) follows by Theorem 3. (iii)$\Rightarrow$(i) is true by definition. $\square$

We next present an example.

Let $\lambda > 0$ and $\theta \in [0,1)$. We define the function $P_{\lambda,\theta}(z)$ by

$$P_{\lambda,\theta}(z) = \exp\left\{ -\lambda \frac{1-z}{1-\theta z} \right\}. \tag{9}$$

$P_{\lambda,\theta}(z)$ is the pgf of the compound Poisson (and therefore infinitely divisible) distribution $(p_n(\lambda,\theta), n \geq 0)$ given by

$$p_n(\lambda,\theta) = \begin{cases} \mathrm{e}^{-\lambda} & \textit{if } n = 0 \\ \sum_{i=1}^{n} \mathrm{e}^{-\lambda} \frac{\lambda^i}{i!} \binom{n-1}{i-1} (1-\theta)^i \theta^{n-i} & \textit{if } n \geq 1. \end{cases} \tag{10}$$

The canonical sequence $(r_n(\lambda,\theta), n \geq 0)$ of $(p_n(\lambda,\theta), n \geq 0)$ is easily shown to be $r_n(\lambda,\theta) = \lambda(1-\theta)(n+1)\theta^n$ $(n \geq 0)$. Letting $\alpha \in (0,1)$, we obtain by straightforward calculations

$$\sum_{j=0}^{\infty} \binom{j+n}{n} (1-\alpha)^j r_{j+n}(\lambda,\theta) = (1-(1-\alpha)\theta)^{-n-2} r_n(\lambda,\theta).$$

Since $\frac{1-(1-\alpha)\theta}{\alpha} \geq 1$, the inequality (8) holds if and only if $\left(\frac{1-(1-\alpha)\theta}{\alpha}\right)^2 \geq \frac{1}{\alpha}$. Therefore, by Theorem 3, $(p_n(\lambda,\theta), n \geq 0)$ is semi-self-decomposable of order α if and only if

$$\theta \in [0, (1+\sqrt{\alpha})^{-1}]. \tag{11}$$

Moreover, if $\theta \in [0, 1/2]$, then (11) holds for every $\alpha \in (0,1)$ and therefore, $(p_n(\lambda,\theta), n \geq 0)$ is discrete self-decomposable. If $\theta \in (1/2, (1+\sqrt{\alpha})^{-1}]$, then $r_0(\lambda,\theta) < r_1(\lambda,\theta)$. Hence, $(r_n(\lambda,\theta), n \geq 0)$ is not nonincreasing, implying (by Proposition 1) that $(p_n(\lambda,\theta), n \geq 0)$ is discrete semi-self-decomposable of order α, but not discrete self-decomposable.

The next results identify some classes of discrete semi-self-decomposable distributions.

We recall (see Satheesh and Nair 2002, Bouzar 2004) that a nondegenerate distribution on $\mathbf{Z}_+$ is said to be discrete semistable with exponent $\gamma \in (0,1]$ and order $\alpha \in (0, 1)$ if its pgf $P(z)$ satisfies for all $|z| \le 1$, $P(z) \neq 0$ and

$$\ln P(1 - \alpha + \alpha z) = \alpha^{\gamma} \ln P(z). \tag{12}$$

Proposition 2 *A discrete semistable distribution with exponent* $\gamma \in (0, 1]$ *and order* $\alpha \in (0, 1)$*, and pgf* $P(z)$*, is discrete semi-self-decomposable of order* α*. In this case, the pgf* $P_\alpha(z)$ *of (5) has the form*

$$P_\alpha(z) = [P(z)]^c \quad (z \in [0, 1]), \tag{13}$$

for some $c \in (0, 1)$*. Conversely, a discrete semi-self-decomposable distribution of order* $\alpha \in (0, 1)$ *such that the pgf* $P_\alpha(z)$ *of (5) satisfies (13) for some* $c \in (0, 1)$*, is discrete semistable with some exponent* $\gamma \in (0, 1]$ *and order* α*.*

Proof The R-function of a discrete semistable distribution with exponent $\gamma \in (0, 1]$ and order $\alpha \in (0, 1)$ is absolutely monotone and has the form (Bouzar 2004, Proposition 2.4)

$$R(z) = (1 - z)^{\gamma - 1} r_1(|\ln(1 - z)|) \quad (z \in [0, 1)), \tag{14}$$

where $r_1(x)$, defined over $[0, \infty)$, is periodic with period $-\ln\alpha$. It follows that

$$R_\alpha(z) = R(z) - \alpha R(1 - \alpha + \alpha z) = (1 - \alpha^{\gamma})(1 - z)^{\gamma - 1} r_1(|\ln(1 - z)|) \quad (z \in [0, 1)),$$

or, $R_\alpha(z) = (1 - \alpha^{\gamma}) R(z)$. Therefore, $R_\alpha(z)$ is absolutely monotone on $[0, 1)$ and $P_\alpha(z)$ has the form (13) with $c = 1 - \alpha^{\gamma}$. Semi-self-decomposability follows by Theorem 3. Conversely, by combining (5) and (13) we obtain $P(1 - \alpha + \alpha z) = [P(z)]^{1-c}$, $z \in [0, 1]$. Letting $\gamma = \ln(1 - c)/\ln\alpha$, it follows that (12) holds for every $z \in [0, 1]$, and hence, by analytic continuation, for all $|z| \le 1$. Lemma 2.1 in Bouzar (2004) insures $\gamma \in (0, 1]$. □

A nondegenerate distribution on $\mathbf{Z}_+$ is said to be discrete geometric semistable with exponent $\gamma \in (0, 1]$ and order $\alpha \in (0, 1)$ (Bouzar 2004) if its pgf $P(z)$ satisfies

$$P(1 - \alpha + \alpha z) = \frac{P(z)}{\alpha^{\gamma} + (1 - \alpha^{\gamma}) P(z)} \quad (|z| \le 1). \tag{15}$$

Proposition 3 *A discrete geometric semistable distribution with exponent* $\gamma \in (0, 1]$ *and order* $\alpha \in (0, 1)$*, and pgf* $P(z)$*, is discrete semi-self-decomposable of order* α*. In this case, the pgf* $P_\alpha(z)$ *of (5) has the form*

$$P_\alpha(z) = 1 - c + cP(z) \quad (z \in [0, 1]). \tag{16}$$

for some $c \in (0, 1)$*. Conversely, a discrete semi-self-decomposable distribution of order* $\alpha \in (0, 1)$ *such that the pgf* $P_\alpha(z)$ *of (5) satisfies (16) for some* $c \in (0, 1)$*, is discrete geometric semistable with some exponent* $\gamma \in (0, 1]$ *and order* α*.*

Proof Let $P(z)$ be the pgf of a discrete geometric semistable distribution with exponent $\gamma \in (0,1]$ and order $\gamma \in (0,1)$. It follows by (15) that

$$P_\alpha(z) = \frac{P(z)}{P(1-\alpha+\alpha z)} = \alpha^\gamma + (1-\alpha^\gamma)P(z). \tag{17}$$

Therefore, (16) holds with $c = 1-\alpha^\gamma$. Combining (15) and (17) leads to

$$R_\alpha(z) = R(z) - \alpha R(1-\alpha+\alpha z) = \frac{P'_\alpha(z)}{P_\alpha(z)} = (1-\alpha^\gamma)P(1-\alpha+\alpha z)R(z) \quad (z \in [0,1)),$$

where $R(z)$ is the R-function of $P(z)$. The absolute monotonicity of $R(z)$ and $P(1-\alpha+\alpha z)$ over $[0,1)$ implies that of $R_\alpha(z)$, and semi-self-decomposability follows by Theorem 3. Conversely, we have by (5) and (16) $P(1-\alpha+\alpha z) = P(z)/(c+(1-c)P(z))$, $z \in [0,1]$. Letting $\gamma = \ln(1-c)/\ln\alpha$, we deduce (15) holds for every $z \in [0,1]$ and hence, by analytic continuation, for all $|z| \le 1$. Corollary 3.1 in Bouzar (2004) insures $\gamma \in (0,1]$. □

The pgf $P(z)$ of a discrete geometric semistable distribution with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$ admits the representation (Corollary 3.2 in Bouzar 2004)

$$P(z) = (1-\ln H(z))^{-1}, \tag{18}$$

where $H(z)$ is the pgf of a discrete semistable distribution with the same exponent and the same order. In view of (18), any discrete geometric semistable distribution is an exponential compounding of discrete semistable distributions. This suggests an extension of the first part of Proposition 3 to a wider class of discrete compound distributions.

Proposition 4 *Let $\phi(\tau)$ be the Laplace-Stieltjes (LST) of a self-decomposable distribution on $\mathbf{R}_+$ and let $H(z)$ be the pgf of a discrete semistable distribution with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$. Then the (discrete) compound distribution with pgf*

$$P(z) = \phi(-\ln H(z)), \quad |z| \le 1, \tag{19}$$

is discrete semi-self-decomposable of order α.

Proof We refer to Steutel and van Harn (2004), Sect. II.3, for a discussion on compound distributions of the type (19). By self-decomposability, we have for every $\tau \ge 0$

$$\phi(\tau) = \phi(\alpha^\gamma \tau)\phi_{\alpha^\gamma}(\tau), \tag{20}$$

for some infinitely divisible LST $\phi_{\alpha^\gamma}(\tau)$ (see Corollary V.2.7 in Steutel and van Harn 2004). Combining (12) (applied to $H(z)$), (19), and (20) yields

$$P(z) = \phi(-\alpha^\gamma \ln H(z))\phi_{\alpha^\gamma}(-\ln H(z)) = \phi(-\ln H(1-\alpha+\alpha z))\phi_{\alpha^\gamma}(-\ln H(z)).$$

Therefore, (5) holds for $P(z)$, with $P_\alpha(z) = \phi_{\alpha^\gamma}(-\ln H(z))$. By Proposition II.3.5 in Steutel and van Harn (2004), $P_\alpha(z) = \phi_{\alpha^\gamma}(-\ln H(z))$ is an infinitely divisible pgf. □

The gamma distribution with LST $\phi(\tau) = \left(\frac{\lambda}{\lambda+\tau}\right)^r$, for some $\lambda > 0$ and $r > 0$, is an example of a self-decomposable distribution on $\mathbf{R}_+$ from which one can generate discrete semi-self-decomposable distributions of the type (19).

Self-decomposable distributions are unimodal as shown by Yamazato (1978) (see also Sato 1999) for the continuous case and by Steutel and van Harn (1979) for the discrete case. This property does not hold in general for discrete semi-self-decomposable distributions. As a counterexample, we consider the distribution $(p_n(\lambda,\theta), n \geq 0)$ of (10). Let $\theta = 0.9$, $\lambda = 5$, and $\alpha = 0.01$. Since $\theta \in (1/2, (1+\sqrt(\alpha))^{-1}]$, it follows that $(p_n(\lambda,\theta), n \geq 0)$ is discrete semi-self-decomposable of order α (but not discrete self-decomposable). Using the computer algebra system MAPLE, calculations show that $(p_n(\lambda,\theta), n \geq 0)$ has a mode at $n = 0$ and at $n = 35$.

3 Limit theorems

In this section, we identify a proper subclass of discrete semi-self-decomposable distributions that arise as weak limits of subsequences of binomially thinned sums of $\mathbf{Z}_+$-valued independent random variables.

If $(N_\lambda(t), t \geq 0)$ is a Poisson process of rate $\lambda > 0$ and X is an $\mathbf{R}_+$-valued random variable independent of $(N_\lambda(t), t \geq 0)$, then the $\mathbf{Z}_+$-valued random variable $N_\lambda(X)$ is called a Poisson mixture with mixing random variable X. Its pgf $P_\lambda(z)$ is given by

$$P_\lambda(z) = \phi(\lambda(1-z)) \quad (z \in [0,1]), \tag{21}$$

where $\phi(\tau)$ is the LST of X.

The following result is a useful characterization of semi-self-decomposability on $\mathbf{R}_+$ in terms of discrete semi-self-decomposability.

Lemma 1 *An $\mathbf{R}_+$-valued random variable X has a semi-self-decomposable distribution of order $\alpha \in (0,1)$ if and only if for every $\lambda > 0$, the Poisson mixture $N_\lambda(X)$ is discrete semi-self-decomposable of order α.*

Proof Let $\phi(\tau)$ be the LST of X. Assume X has a semi-self-decomposable distribution of order $\alpha \in (0,1)$ and let $\lambda > 0$. By definition, there exists an infinitely divisible LST $\phi_\alpha(\tau)$ such that $\phi(\tau) = \phi(\alpha\tau)\phi_\alpha(\tau)$, $\tau \geq 0$. This implies, along with (21), that

$$P_\lambda(z) = \phi(\alpha\lambda(1-z))\phi_\alpha(\lambda(1-z)) = P_\lambda(1-\alpha+\alpha z)\phi_\alpha(\lambda(1-z)),$$

for every $z \in [0,1]$. By Theorem VI.6.4 in Steutel and van Harn (2004), $\phi_\alpha(\lambda(1-z))$ is an infinitely divisible distribution on $\mathbf{Z}_+$. Therefore, $N_\lambda(X)$ is discrete semi-self-decomposable of order α. Conversely, assume that for every $\lambda > 0$, $N_\lambda(X)$ is discrete semi-self-decomposable of order α. Then by (5) and (21),

$$\phi(\lambda(1-z)) = \phi(\alpha\lambda(1-z))P_{\lambda,\alpha}(z) \quad (z \in [0,1)),$$

for some infinitely divisible pgf $P_{\lambda,\alpha}(z)$. Note that as an infinitely divisible pgf (Theorem 2), $\phi(\lambda(1-z)) \neq 0$ for all $\lambda > 0$ and $z \in [0,1]$. Therefore, $\phi(\tau) \neq 0$ for all $\tau \geq 0$. Letting $\phi_\alpha(\tau) = \phi(\tau)/\phi(\alpha\tau)$, $\tau \geq 0$, we have for all $\lambda > 0$, $\phi_\alpha(\lambda(1-z)) = P_{\lambda,\alpha}(z)$. This implies that $\phi_\alpha(\lambda(1-z))$ is an infinitely divisible pgf. By Theorem 6.4 and Proposition VI.6.5 in Steutel and van Harn (2004), $\phi_\alpha(\tau)$ must then be an infinitely divisible LST. □

The main result of the section follows.

Theorem 4 *Let $(X_n, n \geq 1)$ be a sequence of independent $\mathbf{Z}_+$-valued random variables and $\alpha \in (0,1)$. Let $(c_n, n \geq 1)$ be an increasing sequence of real numbers such that $c_n \geq 1$ and $c_n \uparrow \infty$ and let $(k_n, n \geq 1)$ be a strictly increasing sequence in $\mathbf{Z}_+$ such that $k_n \uparrow \infty$. Further, assume*

(i) $c_n^{-1} \odot \sum_{i=1}^{k_n} X_i$ *converges weakly to a $\mathbf{Z}_+$-valued random variable X;*

(ii) $\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = \alpha$;

(iii) $\lim_{n\to\infty} \max_{1\leq i\leq k_n} P(c_n^{-1} \odot X_i \geq \epsilon) = 0$, *for every $\epsilon > 0$.*

Then X has a discrete semi-self-decomposable distribution of order α. Moreover, X admits a Poisson mixture representation with a semi-self-decomposable mixing distribution on $\mathbf{R}_+$ of order α.

Conversely, if a $\mathbf{Z}_+$-valued random variable X admits a Poisson mixture representation with a semi-self-decomposable mixing distribution of order $\alpha \in (0,1)$, then there exist sequences $(X_n, n \geq 1)$, $(c_n, n \geq 1)$ and $(k_n, n \geq 1)$, as defined above, that satisfy (i)–(iii).

Proof Let $Q_i(z)$ $(i \geq 1)$ and $P(z)$ be the pgf's of X_i and X, respectively. By (i) and Theorem 8.4 in van Harn et al. (1982) (applied to the semi-group of pgf's $F_t(z) = 1 - e^{-t} + e^{-t}z$, $t \geq 0$), the $\mathbf{R}_+$-valued sequence $(c_n^{-1}\sum_{i=1}^{k_n} X_i, n \geq 1)$ converges weakly to an $\mathbf{R}_+$-valued random variable Y whose LST $\phi(\tau)$ $(\tau \geq 0)$ is related to $P(z)$ via the equation $P(z) = \phi((1-z))$, $z \in [0,1]$. Therefore, $X \stackrel{d}{=} N_1(Y)$, where $(N_1(t), t \geq 0)$ is a Poisson process of rate one. It is easily seen (due to the discreteness of the X_i's and the definition of the operator $\odot$) that (iii) is equivalent to

$$\lim_{n\to\infty} \min_{1\le i\le k_n} Q_i(1-c_n^{-1}) = 1.$$

Letting $\phi_i(\tau)$ denote the LST of X_i ($i \ge 1$) and noting that $\phi_i(c_n^{-1}) = Q_i(e^{-c_n^{-1}}) \ge Q_i(1-c_n^{-1})$, we deduce that $\lim_{n\to\infty} \min_{1\le i\le k_n} \phi_i(c_n^{-1}) = 1$. This in turn leads to (see the discussion following Definition 8.1 in van Harn et al. 1982)

$$\lim_{n\to\infty} \min_{1\le i\le k_n} \phi_i(\tau/c_n) = 1 \quad (\tau \ge 0),$$

or, equivalently,

$$\lim_{n\to\infty} \max_{1\le i\le k_n} P(c_n^{-1}X_i \ge \epsilon) = 0 \quad (\epsilon > 0). \tag{22}$$

The weak convergence of $(c_n^{-1}\sum_{i=1}^{k_n} X_i, n \ge 1)$ to Y, combined with condition (ii) above and (22) imply, by Theorem 2.1(i) in Maejima and Naito (1998), that Y has a semi-self-decomposable distribution of order α. We conclude by Lemma 1 that $P(z)$ is the pgf of a discrete semi-self-decomposable distribution of order α. Conversely, assume that $X \overset{d}{=} N_\lambda(Y)$ for some $\lambda > 0$ and some $\mathbf{R}_+$-valued random variable Y with a semi-self-decomposable distribution of order α. By Theorem 2.1(ii) in Maejima and Naito (1998) (note that no constant term is needed in the normalized sums in their result, as shown in the proof of the theorem), there exists a sequence $(Y_n, n \ge 1)$ of independent $\mathbf{R}_+$-valued random variables such that $(c_n^{-1}\sum_{i=1}^{k_n} Y_i, n \ge 1)$ converges weakly to Y, where $c_n > 0$ and $c_n \uparrow \infty$, $k_n, \in \mathbf{Z}_+$, $k_n \uparrow \infty$, $(c_n, n \ge 1)$ satisfies (ii) and $\lim_{n\to\infty} \max_{1\le i\le k_n} P(c_n^{-1}Y_i \ge \epsilon) = 0$, $\epsilon > 0$. Letting $\phi_i(\tau)$ and $\phi(\tau)$ be the LST's of Y_i and Y respectively, it follows that

$$\lim_{n\to\infty} \prod_{i=1}^{k_n} \phi_i\left(\frac{\lambda}{c_n}(1-z)\right) = \phi(\lambda(1-z)) \quad (z \in [0,1]), \tag{23}$$

and (referring again to the discussion following Definition 8.1 in van Harn et al. 1982)

$$\lim_{n\to\infty} \min_{1\le i\le k_n} \phi_i(\lambda/c_n) = 1. \tag{24}$$

Define $X_i \overset{d}{=} N_\lambda(Y_i)$ for $i \ge 1$. It is easily seen from (23) and (24) that (i)–(iii) are satisfied by $(X_n, n \ge 1)$, $(c_n, n \ge 1)$ and $(k_n, n \ge 1)$ (we have assumed without loss of generality that $c_n \ge 1$). $\square$

Unlike the case of continuous distributions (see Maejima and Naito 1998), there are discrete semi-self-decomposable distributions that do not arise as limiting distributions under the assumptions of Theorem 4. As a counterexample, we consider the scaled Sibuya distribution of Christoph and Schreiber (2000). Its pgf is given by

$$P(z) = 1 - c(1-z)^a \quad (z \in [0,1]),$$

for some $c \in (0,1]$ and $a \in (0,1]$. We assume further $0 < c \leq \frac{1-a}{1+a}$. This insures, by Theorem 1 in Christoph and Schreiber (2000), the discrete self-decomposability of the scaled Sibuya distribution. Since $P(z)$ cannot be the pgf of a Poisson mixture, as it fails to satisfy $P(z) \geq 0$ for all $z \in (-\infty, 1]$, it follows that the scaled Sibuya distribution (with $0 < c \leq \frac{1-a}{1+a}$) cannot arise as a limiting distribution under the assumptions (i)–(iii) of Theorem 4.

If $(X_n, n \geq 1)$ in Theorem 4 is a sequence of iid random variables, then, as shown next, the limiting distribution is discrete semistable. We note that in this case condition (iii) is necessarily satisfied.

Theorem 5 *Let $(X_n, n \geq 1)$ be a sequence of $\mathbf{Z}_+$-valued iid random variables and $\alpha \in (0,1)$. Let $(c_n, n \geq 1)$ be an increasing sequence of real numbers such that $c_n \geq 1$ and $c_n \uparrow \infty$ and let $(k_n, n \geq 1)$ be a strictly increasing sequence in $\mathbf{Z}_+$ such that $k_n \uparrow \infty$. Further, assume*

(i) $c_n^{-1} \odot \sum_{i=1}^{k_n} X_i$ *converges weakly to a $\mathbf{Z}_+$-valued random variable X;*

(ii) $\lim_{n\to\infty} \dfrac{c_n}{c_{n+1}} = \alpha.$

Then X has a discrete semistable distribution with some exponent $\gamma \in (0,1]$ and order α. Moreover, X admits a Poisson mixture representation with a semistable mixing distribution on $\mathbf{R}_+$ with exponent γ and order α. The sequence $(k_n, n \geq 1)$ necessarily satisfies

$$\lim_{n\to\infty} \frac{k_n}{k_{n+1}} = \gamma. \tag{25}$$

Conversely, if a $\mathbf{Z}_+$-valued random variable X has a discrete semistable distribution with exponent $\gamma \in (0,1]$ and order α, then there exist sequences $(X_n, n \geq 1)$, $(c_n, n \geq 1)$ and $(k_n, n \geq 1)$, as in (25), that satisfy (i)–(ii) above.

Proof The proof of the direct statement is similar to that of Theorem 4. The details are omitted. We simply note that Theorem 2.1 in Maejima (2001) (instead of Theorem 2.1 in Maejima and Naito 1998) and Proposition 5.2 in Bouzar (2004) (instead of Lemma 1) are needed. The converse is due to Bouzar (2004) (Proposition 4.1). □

The following characterization of discrete semistability is a direct consequence of Theorem 5. The proof is omitted.

Corollary 2 *A $\mathbf{Z}_+$-valued random variable X has a discrete semistable distribution with exponent $\gamma \in (0, 1]$ and order $\alpha \in (0, 1)$ if and only if X admits a Poisson mixture representation with a semistable mixing distribution on $\mathbf{R}_+$ with exponent γ and order α.*

Theorems 4 and 5 are extensions of results obtained by van Harn et al. (1982) for discrete self-decomposable and discrete stable distributions. In the latter case the statements of the theorems are expectedly stronger. The weak convergence [condition (i)] is assumed to occur along the entire sequence of sums (i.e., $k_n = n$), condition (ii) is not needed, and condition (iii) (in Theorem 4) is assumed to hold for $k_n = n$. The limiting distribution will have a Poisson mixture representation with a self-decomposable (stable) mixing distribution on $\mathbf{R}_+$.

Remark The Poisson mixture representations in Theorems 4 and 5 are not unique. In fact, if a $\mathbf{Z}_+$-valued random variable X can be written as $X \stackrel{d}{=} N_{\lambda_0}(Y)$ for some $\lambda_0 > 0$ and some $\mathbf{R}_+$-valued random variable Y with a semi-self-decomposable distribution, then for any $\lambda > 0$, $X \stackrel{d}{=} N_\lambda((\lambda_0/\lambda)Y)$. It is easy to see that $(\lambda_0/\lambda)Y$ has a semi-self-decomposable distribution with the exponent and order of Y. The same argument holds for semistability.

4 Multiple semi-self-decomposability on $\mathbf{Z}_+$

Maejima and Naito (1998) constructed and studied a sequence of nested subclasses of semi-self-decomposable distributions on $\mathbf{R}^d$ that contain the semistable distributions (see also Maejima et al. 1999). In this section we establish similar results for discrete semi-self-decomposable distributions. Our results generalize those obtained by Berg and Forst (1983) for discrete self-decomposability.

We start out by defining several classes of pgf's.

PGF will denote the collection of all pgf's $P(z)$ such that $P(0) < 1$ and $I(Z_+)$ will denote the subset of PGF that consists of all the pgf's of infinitely divisible distributions on $\mathbf{Z}_+$. For $\alpha \in (0, 1)$, the sequence $((DL)_n(\alpha), n \geq 0)$ of subsets of PGF is defined inductively as follows:

$$(DL)_0(\alpha) = \left\{P(\cdot) \in \text{PGF} \mid P(\cdot) \text{ satisfies (5) with } P_\alpha(\cdot) \in I(\mathbf{Z}_+)\right\},$$

and for $n \geq 0$,

$$(DL)_{n+1}(\alpha) = \left\{P(\cdot) \in (DL)_n(\alpha) | \; P(\cdot) \text{ satisfies (5) with } P_\alpha(\cdot) \in (DL)_n(\alpha))\right\}.$$

A distribution on $\mathbf{Z}_+$ whose pgf belongs to $(DL)_n(\alpha)$ for some $n \geq 0$ is said to be n-times discrete semi-self-decomposable of order α.

The subclass $(DL)_\infty(\alpha)$ is defined by

$$(DL)_\infty(\alpha) = \bigcap_{n=0}^{\infty}(DL)_n(\alpha).$$

For $\alpha \in (0,1)$, we have (by construction)

$$I(\mathbf{Z}_+) \supset (DL)_0(\alpha) \supset \cdots \supset (DL)_n(\alpha) \supset \cdots \supset (DL)_\infty(\alpha).$$

Note that if DL denotes the subset of PGF that consists of the pgf's of discrete self-decomposable distributions, then by Proposition 1,

$$DL = \bigcap_{0<\alpha<1}(DL)_0(\alpha).$$

Theorem 6 *Let* $\alpha \in (0,1)$ *and* $n \geq 0$. *A pgf* $P(\cdot)$ *belongs to* $(DL)_n(\alpha)$ *if and only if* $P(\cdot)$ *belongs to* $I(\mathbf{Z}_+)$ *and for every* $1 \leq j \leq n+1$,

$$R_\alpha^{(j)}(z) = \sum_{i=0}^{j}(-1)^i\binom{j}{i}\alpha^i R(1-\alpha^i+\alpha^i z) \tag{26}$$

is absolutely monotone on $[0,1)$ *[here* $R(\cdot)$ *is the R-function of* $P(\cdot)$*].*

Proof It is easily shown by induction that for every $n \geq 0$, $P(\cdot) \in (DL)_n(\alpha)$ if and only if the functions $P_\alpha^{(j)}(\cdot)$, $0 \leq j \leq n$, defined by the recurrence relation

$$\begin{cases} P_\alpha^{(0)}(z) &= P_\alpha(z) \\ P_\alpha^{(j-1)}(z) &= P_\alpha^{(j-1)}(1-\alpha+\alpha z)P_\alpha^{(j)}(z) \quad 1 \leq j \leq n, \end{cases} \tag{27}$$

satisfy $P_\alpha^{(j)}(\cdot) \in I(\mathbf{Z}_+)$ for all $0 \leq j \leq n$. The conclusion will follow (from Theorem 3) if we show that $R_\alpha^{(j)}(\cdot)$ of (26) is the R-function of $P_\alpha^{(j-1)}(\cdot)$ for every $1 \leq j \leq n+1$. The statement is true for $j=1$. Assume the statement holds for $1 \leq j \leq n$. Then by (27), and the argument used in the proof of Theorem 3 ((i)$\Leftrightarrow$(ii)), the R-function $S(\cdot)$ of $P_\alpha^{(j)}(\cdot)$ satisfies

$$S(z) = R_\alpha^{(j)}(z) - \alpha R_\alpha^{(j)}(1-\alpha+\alpha z) \quad z \in [0,1).$$

Therefore, by the induction hypothesis and (26),

$$\begin{aligned} S(z) &= \sum_{i=0}^{j}(-1)^i\binom{j}{i}\alpha^i R(1-\alpha^i+\alpha^i z) + \sum_{i=1}^{j+1}(-1)^i\binom{j}{i-1}\alpha^i R(1-\alpha^i+\alpha^i z) \\ &= \sum_{i=0}^{j+1}(-1)^i\binom{j+1}{i}\alpha^i R(1-\alpha^i+\alpha^i z), \end{aligned}$$

where the second equation is derived by way of the identity $\binom{j+1}{i} = \binom{j}{i} + \binom{j}{i-1}$. Therefore, $S(z) = R_\alpha^{(j+1)}(z)$, $z \in [0,1)$. □

Next, we characterize the subclass $(DL)_\infty(\alpha)$ in terms of an invariance property.

Theorem 7 *Let $\alpha \in (0,1)$ and $P(\cdot) \in PGF$. Then $P(\cdot) \in (DL)_\infty(\alpha)$ if and only if (5) holds for some $P_\alpha(\cdot) \in (DL)_\infty(\alpha)$. Moreover, if H is a subclass of $I(\mathbf{Z}_+)$ such that for any $P(\cdot) \in H$, (5) holds for some $P_\alpha(\cdot) \in H$, then $H \subset (DL)_\infty(\alpha)$.*

Proof By definition, $P(\cdot) \in (DL)_\infty(\alpha)$ implies that for every $n \geq 1$, $P(\cdot) \in (DL)_n(\alpha)$ and $P_\alpha(\cdot) \in (DL)_{n-1}(\alpha)$. Therefore, $P_\alpha(\cdot) \in (DL)_\infty(\alpha)$. Conversely, assume (5) holds for some $P_\alpha(\cdot) \in (DL)_\infty(\alpha)$. It follows that $P(\cdot) \in (DL)_0(\alpha)$ and, since $P_\alpha(\cdot) \in (DL)_0(\alpha)$, $P(\cdot) \in (DL)_1(\alpha)$. A simple induction leads to $P(\cdot) \in (DL)_n(\alpha)$ for all $n \geq 1$. An induction argument is again used to prove the second part of the proposition. If $P(\cdot) \in H$, then $P_\alpha(\cdot) \in I(\mathbf{Z}_+)$ (since $H \subset I(\mathbf{Z}_+)$). We thus have $H \subset (DL)_0(\alpha)$. Assume that for $n \geq 0$, $H \subset (DL)_n(\alpha)$. Then for any $P(\cdot) \in H$, $P_\alpha(\cdot) \in (DL)_n(\alpha)$, which implies $P(\cdot) \in (DL)_{n+1}(\alpha)$. □

We denote by $SS(\gamma, \alpha)$ the subset of PGF that consists of all the pgf's of discrete semistable distributions with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$.

Corollary 3 *Let $\alpha \in (0,1)$. Then*

$$\bigcup_{0<\gamma\leq 1} SS(\gamma,\alpha) \subset (DL)_\infty(\alpha).$$

Proof Let $P(\cdot) \in SS(\gamma,\alpha)$ for some $\gamma \in (0,1]$. We show that $P(\cdot) \in (DL)_n(\alpha)$ for every $n \geq 0$. By (14), the R-function of $P(\cdot)$ is absolutely monotone on $[0,1)$ and can be written as $R(z) = (1-z)^{\gamma-1} r_1(|\ln(1-z)|)$ where $r_1(\cdot)$, defined over $[0,\infty)$, is periodic with period $-\ln\alpha$. By (26), we have for $n \geq 0$ and $1 \leq j \leq n+1$,

$$R_\alpha^{(j)}(z) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} \alpha^{\gamma i} (1-z)^{\gamma-1} r_1(|\ln(1-z)| - i\ln\alpha) \quad (z \in [0,1)).$$

It follows by the binomial formula and the fact that $r_1(\cdot)$ is periodic that

$$R_\alpha^{(j)}(z) = (1-\alpha^\gamma)^j (1-z)^{\gamma-1} r_1(|\ln(1-z)|) = (1-\alpha^\gamma)^j R(z),$$

for every $z \in [0,1)$. Therefore, $R^{(j)}(\cdot)$ is absolutely monotone on $[0,1)$ and, by Theorem 6, $P(\cdot) \in (DL)_n(\alpha)$. □

By Eq. (6), we have for every $k \geq 1$,

$$(DL)_0(\alpha) \subset (DL)_0(\alpha^k). \tag{28}$$

Since $\alpha^k < \alpha$, (28) suggests the claim $(DL)_0(\alpha_1) \subset (DL)_0(\alpha_2)$ for all $0 < \alpha_1 < \alpha_2 < 1$. A counterexample by way of Poisson mixtures shows the claim to be false. We start out with an infinitely divisible distribution μ on $\mathbf{R}_+$ with Lévy measure

$$L(dy) = \sum_{n=0}^{\infty} g(2^n)\, \delta_{2^n}(dy),$$

where $g(\cdot)$ is a monotone decreasing function on $\mathbf{R}_+$ and $\delta_a(dy)$ is the Dirac point-mass measure (at a) on the σ-algebra of Borel sets in $\mathbf{R}_+$. Using the exact same argument as in Maejima and Naito (1998), Example 4.1, it is shown that μ is semi-self-decomposable of order 1/2, but that μ is not semi-self-decomposable of order 1/3. Therefore, by Lemma 1, there exists a Poisson mixture distribution on $\mathbf{Z}_+$ (with mixing distribution μ) such that its pgf $P(\cdot)$ satisfies $P(\cdot) \in (DL)_0(1/2)$ and $P(\cdot) \notin (DL)_0(1/3)$.

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