Ann Inst Stat Math (2008) 60:801–812
DOI 10.1007/s10463-007-0125-5

On a simple strategy weakly forcing the strong law of large numbers in the bounded forecasting game

Masayuki Kumon · Akimichi Takemura

Received: 2 August 2006 / Revised: 6 December 2006 / Published online: 9 March 2007

Abstract In the framework of the game-theoretic probability of Shafer and Vovk it is of basic importance to construct an explicit strategy weakly forcing the strong law of large numbers in the bounded forecasting game. We present a simple finite-memory strategy based on the past average of Reality's moves, which weakly forces the strong law of large numbers with the convergence rate of $O(\sqrt{\log n/n})$. Our proof is very simple compared to a corresponding measure-theoretic result of Azuma (*The Tôhoku Mathematical Journal*, *19*, 357–367, 1967) on bounded martingale differences and this illustrates effectiveness of game-theoretic approach. We also discuss one-sided protocols and extension of results to linear protocols in general dimension.

Keywords Azuma-Hoeffding-Bennett inequality · Capital process · Game-theoretic probability · Large deviation

1 Introduction

The book by Shafer and Vovk (2001) established the whole new field of game-theoretic probability and finance. Their framework provides an attractive alternative foundation of probability theory. Compared to the conventional measure

M. Kumon
Risk Analysis Research Center, Institute of Statistical Mathematics,
4-6-7 Minami-azabu, Minato-ku, Tokyo 106-8569, Japan
e-mail: kumon@ism.ac.jp

A. Takemura (✉)
Graduate School of Information Science and Technology, University of Tokyo,
7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, Japan
e-mail: takemura@stat.t.u-tokyo.ac.jp

theoretic probability, the game theoretic probability treats the sets of measure zero in a very explicit way when proving various probabilistic laws, such as the strong law of large numbers. In a game-theoretic proof, we can explicitly describe the behavior of the paths on a set of measure zero, whereas in measure-theoretic proofs the sets of measure zero are often simply ignored. This feature of game-theoretic probability is well illustrated in the explicit construction of Skeptic's strategy forcing SLLN in Chap. 3 of Shafer and Vovk (2001).

However the strategy given in Chap. 3 of Shafer and Vovk (2001), which we call a mixture ϵ-strategy in this paper, is not yet satisfactory, in the sense that it requires combination of infinite number of "accounts" and it needs to keep all the past moves of Reality in memory. In fact in Sect. 3.5 of their book, Shafer and Vovk pose the question of required memory for strategies forcing strong law of large numbers (SLLN). In a forthcoming paper we will study the problem of various ways of mixing ϵ-strategies in a somewhat more general form than in Chap. 3 of Shafer and Vovk (2001).

In this paper we prove that a very simple single strategy, based only on the past average of Reality's moves is weakly forcing SLLN. Furthermore it weakly forces SLLN with the convergence rate of $O(\sqrt{\log n/n})$. In this sense, our result is a substantial improvement over the mixture ϵ-strategy of Shafer and Vovk. Since ϵ-strategies are used as essential building blocks for the "defensive forecasting" (Vovk et al. 2005b) the performance of defensive forecasting might be improved by incorporating our simple strategy.

Our thinking was very much influenced by the detailed analysis by Takeuchi (2004b) and Chap. 5 of Takeuchi (2004a) of the optimum strategy of Skeptic in the games, which are favorable for Skeptic. We should also mention that the intuition behind our strategy is already discussed several times throughout the book by Shafer and Vovk (see e.g. Sect. 5.2). Our contribution is in proving that the strategy based on the past average of Reality's moves is actually weakly forcing SLLN.

In this paper we only consider weakly forcing by a strategy. A strategy weakly forcing an event E can be transformed to a strategy forcing E as in Lemma 3.1 of Shafer and Vovk (2001). We do not present anything new for this step of the argument. An extension of the present paper to unbounded games is presented in Kumon et al. (2006).

The organization of this paper is as follows. In Sect. 2 we formulate the bounded forecasting game and motivate the strategy based on the past average of Reality's moves as an approximately optimum ϵ-strategy. In Sect. 3 we prove that our strategy is weakly forcing SLLN with the convergence rate of $O(\sqrt{\log n/n})$. In Sect. 4 we consider the one-sided protocol and prove that the one-sided version of our strategy weakly forces the one-sided SLLN with the same order. In Sect. 5 we treat a multivariate extension to linear protocols.

2 Approximately optimum single ϵ-strategy for the bounded forecasting game

Consider the bounded forecasting game in Sect. 3.2 of Shafer and Vovk (2001).

Bounded Forecasting Game
Protocol:
$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \dots$:
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in [-1, 1]$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.
END FOR

For a fixed ϵ, $|\epsilon| < 1$, the ϵ-strategy sets $M_n = \epsilon \mathcal{K}_{n-1}$. Under this strategy Skeptic's capital process $\mathcal{K}_n$ is written as $\mathcal{K}_n = \prod_{i=1}^n (1 + \epsilon x_i)$ or

$$\log \mathcal{K}_n = \sum_{i=1}^n \log(1 + \epsilon x_i).$$

For sufficiently small $|\epsilon|$, $\log \mathcal{K}_n$ is approximated as

$$\log \mathcal{K}_n \simeq \epsilon \sum_{i=1}^n x_i - \frac{1}{2}\epsilon^2 \sum_{i=1}^n x_i^2.$$

The right-hand side is maximized by taking

$$\epsilon = \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2}.$$

In particular in the fair-coin game, where x_n is restricted as $x_n = \pm 1$, approximately optimum ϵ is given as

$$\epsilon = \bar{x}_n = \frac{1}{n}\sum_{i=1}^n x_i.$$

Actually, as shown by Takeuchi (2004b), it is easy to check that $\epsilon = \bar{x}_n$ exactly maximizes $\prod_{i=1}^n (1 + \epsilon x_i)$ for the case of the fair-coin game. Recently Kumon et al. (2005) give a detailed analysis of Bayesian strategies for the biased-coin games, which include the strategy $\epsilon = \bar{x}_n$ as a special case.

Of course, the above approximately optimum ϵ is chosen in hindsight, i.e., we can choose optimum ϵ after seeing the moves $x_1, \dots, x_n$. However it suggests choosing M_n based on the past average $\bar{x}_{n-1}$ of Reality's moves. Therefore consider a strategy $\mathcal{P} = \mathcal{P}^c$

$$M_n = c\bar{x}_{n-1}\mathcal{K}_{n-1}. \tag{1}$$

In the next section we prove that for $0 < c \leq 1/2$ this strategy is weakly forcing SLLN. The restriction $0 < c \leq 1/2$ is just for convenience for the proof and in Kumon et al. (2005) we consider $c = 1$ for biased-coin games.

Compared to a single fixed ϵ-strategy $M_n = \epsilon\mathcal{K}_{n-1}$ or the mixture ϵ-strategy in Chap. 3 of Shafer and Vovk (2001), letting $\epsilon = c\bar{x}_{n-1}$ depend on $\bar{x}_{n-1}$ seems to be reasonable from the viewpoint of effectiveness of Skeptic's strategy. The basic reason is that as $\bar{x}_{n-1}$ deviates more from the origin, Skeptic should try to exploit this bias in Reality's moves by betting a larger amount. Clearly this reasoning is shaky because for each round Skeptic has to move first and Reality can decide her move after seeing Skeptic's move. However in the next section we show that the strategy in (1) is indeed weakly forcing SLLN with the convergence rate of $O(\sqrt{\log n/n})$.

3 Weakly forcing SLLN by past averages

In this section we prove the following result.

Theorem 1 *In the bounded forecasting game, if Skeptic uses the strategy* (1) *with* $0 < c \le 1/2$, *then* $\limsup_n \mathcal{K}_n = \infty$ *for each path* $\xi = x_1 x_2 \cdots$ *of Reality's moves such that*

$$\limsup_n \frac{\sqrt{n}|\bar{x}_n|}{\sqrt{\log n}} > 1. \tag{2}$$

This theorem states that the strategy (1) weakly forces that $\bar{x}_n$ converge to 0 with the convergence rate of $O(\sqrt{\log n/n})$. Therefore it is much stronger than the mixture ϵ-strategy in Chap. 3 of Shafer and Vovk (2001), which only forces convergence to 0. A corresponding measure theoretic result was stated in Theorem 1 of Azuma (1967) as discussed in Remark 2 at the end of this section. The rest of this section is devoted to a proof of Theorem 1.

By comparing $1, 1/2, 1/3, \ldots,$ and the integral of $1/x$ we have at first

$$\log(n+1) = \int_1^{n+1} \frac{1}{x} \mathrm{d}x \le 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \le 1 + \int_1^n \frac{1}{x} \mathrm{d}x = 1 + \log n.$$

Next with $s_n = x_1 + \cdots + x_n$ we have the relation

$$s_{n-1}x_n = (x_1 + \cdots + x_{n-1})x_n = \frac{1}{2}(s_n^2 - s_{n-1}^2 - x_n^2). \tag{3}$$

Now by denoting $c_i = c/(i-1)$, the capital process $\mathcal{K}_n = \mathcal{K}_n^{\mathcal{P}}$ of (1) is written with the convention $c_1 = 0,\ s_0 = 0$ as

$$\mathcal{K}_n = \prod_{i=1}^n (1 + c_{i-1}s_{i-1}x_i).$$

As in Chap. 3 of Shafer and Vovk (2001) we use

$$\log(1+t) \ge t - t^2, \quad |t| \le 1/2.$$

Then for $0 < c \leq 1/2$ we have

$$\begin{aligned}\log \mathcal{K}_n &= \sum_{i=1}^{n} \log(1 + c_i s_{i-1} x_i) \\ &\geq \sum_{i=1}^{n} c_i s_{i-1} x_i - \sum_{i=1}^{n} c_i^2 s_{i-1}^2 x_i^2 \\ &\geq \sum_{i=1}^{n} c_i s_{i-1} x_i - \sum_{i=1}^{n} c_i^2 s_{i-1}^2.\end{aligned}$$

By substituting (3) into the right-hand side, we can further bound $\log \mathcal{K}_n$ from below as

$$\begin{aligned}\log \mathcal{K}_n &\geq \frac{1}{2}\sum_{i=1}^{n} c_i(s_i^2 - s_{i-1}^2 - x_i^2) - \sum_{i=1}^{n} c_i^2 s_{i-1}^2 \\ &= \frac{1}{2}\sum_{i=1}^{n} c_i(s_i^2 - s_{i-1}^2 - 2c_i s_{i-1}^2) - \frac{1}{2}\sum_{i=1}^{n} c_i x_i^2 \\ &\geq \frac{1}{2}\sum_{i=1}^{n} c_i(s_i^2 - (1 + 2c_i)s_{i-1}^2) - \frac{1}{2}\sum_{i=1}^{n} c_i \\ &= \frac{1}{2}\sum_{i=1}^{n-1}[c_i - c_{i+1}(1 + 2c_{i+1})]s_i^2 + \frac{c_n}{2}s_n^2 - \frac{1}{2}\sum_{i=1}^{n} c_i.\end{aligned}$$

In the right-hand side for $2 \leq i \leq n-1$ we have

$$c_i - c_{i+1}(1 + 2c_{i+1}) = \frac{c(i - 2c(i-1))}{i^2(i-1)} > 0$$

if $0 < c \leq 1/2$, and

$$\sum_{i=1}^{n} c_i < c(1 + \log(n-1)).$$

Thus $\log \mathcal{K}_n$ is bounded from below as

$$\begin{aligned}\log \mathcal{K}_n &\geq -\frac{1}{2}c(1 + 2c)s_1^2 + \frac{c}{2(n-1)}s_n^2 - \frac{c}{2}(1 + \log(n-1)) \\ &\geq c\frac{n}{2}\bar{x}_n^2 - \frac{c}{2}(3 + \log n) \\ &= \frac{c}{2}\log n\left(\frac{n\bar{x}_n^2}{\log n} - 1\right) - \frac{3}{2}c.\end{aligned}$$

Now if $\limsup_n \sqrt{n}|\bar{x}_n|/\sqrt{\log n} > 1$, then $\limsup_n \log \mathcal{K}_n = +\infty$ because $\log n \uparrow \infty$. This proves the theorem.

Remark 1 As noted before, the same line of the proof will establish the result for $c \leq 1 - \epsilon$ for any $\epsilon > 0$. It can be obtained by using

$$\log(1+t) \geq t - \frac{1}{2}t^2 - \frac{1}{3}\left(\frac{|t|}{1+t}\right)^3, \quad |t| < 1,$$

which is also given in Chap. 15 of Shafer and Vovk (2001). Then the resulting third order terms decrease $\log \mathcal{K}_n$ at most with the magnitude $n|\bar{x}_n|^3$, which is dominated by the second order term $n\bar{x}_n^2$.

Remark 2 In the framework of the conventional measure theoretic probability, a strong law of large numbers analogous to Theorem 3.1 can be proved using Azuma-Hoeffding-Bennett inequality (Appendix A.7 of Vovk et al. 2005a, Sect. 2.4 of Dembo and Zeitouni 1998, Azuma 1967, Hoeffding 1963, Bennett 1962). Let $X_1, X_2, \ldots$ be a sequence of martingale differences such that $|X_n| \leq 1, \forall n$. Then for any $\epsilon > 0$

$$P(|\bar{X}_n| \geq \epsilon) \leq 2\exp(-n\epsilon^2/2).$$

Fix an arbitrary $\alpha > 1/2$. Then for any $\epsilon > 0$

$$\sum_n P(|\bar{X}_n| \geq \epsilon(\log n)^\alpha/\sqrt{n}) \leq \sum_n \exp\left(-\frac{\epsilon^2}{2}(\log n)^{2\alpha}\right) < \infty.$$

Therefore by Borel-Cantelli $\sqrt{n}|\bar{X}_n|/(\log n)^\alpha \to 0$ almost surely. Actually Theorem 1 of Azuma (1967) states the following stronger result

$$\limsup_{n\to\infty} \frac{\sqrt{n}\bar{x}_n}{\sqrt{\log n}} \leq \sqrt{2} \qquad \text{a.s.} \tag{4}$$

Although our Theorem 1 is better in the constant factor of $\sqrt{2}$, Azuma's result (4) and our result (2) are virtually the same. However we want to emphasize that our game theoretic proof requires much less mathematical background than the measure theoretic proof. Also see the factor of $\sqrt{3/2}$ in the one-sided version of our result in Theorem 2 below.

We also add a comment on the relation between "forcing" in the game theoretic framework and "a.s." in the measure theoretic probability. If $x_1, x_2, \ldots$ is regarded as a martingale difference sequence in a probabilistic model and if there exist a Skeptic's strategy for which $\limsup \mathcal{K}_n = \infty$ for any of the path $\xi = x_1x_2\cdots$ belonging to a set A, then $P(\xi \in A) = 0$.

4 One-sided protocol

In this section we consider a one-sided bounded forecasting game where M_n is restricted to be nonnegative ($M_n \geq 0$), i.e. Skeptic is only allowed to buy tickets. We also consider the restriction $M_n \leq 0$. In Chap. 3 of Shafer and Vovk, weak forcing of SLLN is proved by combining positive and negative one-sided strategies, whereas in the previous section we proved that a single strategy $\mathcal{P} = \mathcal{P}^c$ weakly forces SLLN. Therefore it is of interest to investigate whether a one-sided version of our strategy weakly forces a one-sided SLLN. We adopt the same notations as Sect. 5 of Kumon et al. (2005), where one-sided protocols for biased-coin games are studied.

For the positive one-sided case consider the strategy $\mathcal{P}^+$ with

$$M_n = c\bar{x}_{n-1}^+ \mathcal{K}_{n-1}, \quad \bar{x}_{n-1}^+ = \max(\bar{x}_{n-1}, 0).$$

Similarly we consider a negative one-sided strategy $\mathcal{P}^-$ with $M_n = -c\bar{x}_{n-1}^- \mathcal{K}_{n-1}$, $\bar{x}_{n-1}^- = \max(-\bar{x}_{n-1}, 0)$.

For these protocols we have the following theorem.

Theorem 2 *If Skeptic uses the strategy $\mathcal{P}^+$ with $0 < c \leq 1/2$, then $\limsup_n \mathcal{K}_n = \infty$ for each path $\xi = x_1 x_2 \cdots$ of Reality's moves such that*

$$\limsup_n \frac{\sqrt{n}\bar{x}_n}{\sqrt{\log n}} > \sqrt{\frac{3}{2}}.$$

Similarly if Skeptic uses the strategy $\mathcal{P}^-$ with $0 < c \leq 1/2$, then $\limsup_n \mathcal{K}_n = \infty$ for each path $\xi = x_1 x_2 \ldots$ of Reality's moves such that $\liminf_n \sqrt{n}\bar{x}_n / \sqrt{\log n} < -\sqrt{3/2}$.

The rest of this section is devoted to a proof of this theorem for $\mathcal{P}^+$. If the $\bar{x}_n$ are eventually all nonnegative, then the behavior of the capital process $\mathcal{K}_n^{\mathcal{P}}$ and $\mathcal{K}_n^{\mathcal{P}^+}$ are asymptotically equivalent except for a constant factor reflecting some initial segment of Reality's path ξ. Then the theorem follows from Theorem 1. On the other hand if the $\bar{x}_n$ are eventually all negative, then $\mathcal{K}_n^{\mathcal{P}^+}$ stays constant and Theorem 2 holds trivially. Therefore we only need to consider the case that the $\bar{x}_n$ change sign infinitely often. Note that at time n when the $\bar{x}_n$ change the sign, the overshoot is bounded as

$$|\bar{x}_n| \leq 1/n.$$

We consider capital process after a sufficiently large time n_0 such that $\bar{x}_{n_0} \simeq 0$, and proceed to divide the sequence $\{\bar{x}_n\}$ into the following two types of blocks. For $n_0 \leq k \leq l-1$, consider a block $\{k, \ldots, l-1\}$. We call it a *nonnegative block* if

$$\bar{x}_{k-1} < 0, \quad \bar{x}_k \geq 0, \quad \bar{x}_{k+1} \geq 0, \ldots, \bar{x}_{l-1} \geq 0, \quad \bar{x}_l < 0.$$

Similarly we call it a *negative block* if

$$\bar{x}_{k-1} \geq 0, \quad \bar{x}_k < 0, \quad \bar{x}_{k+1} < 0, \ldots, \bar{x}_{l-1} < 0, \quad \bar{x}_l \geq 0.$$

By definition, negative and nonnegative blocks are alternating.

For a nonnegative block

$$\mathcal{K}_l^{\mathcal{P}+} = \mathcal{K}_k^{\mathcal{P}+} \prod_{i=k+1}^{l} (1 + c_i s_{i-1} x_i) \tag{5}$$

whereas for a negative block $\mathcal{K}_l^{\mathcal{P}+} = \mathcal{K}_k^{\mathcal{P}+}$. Taking the logarithm of (5) we can bound it from below as

$$\begin{aligned}
\log \mathcal{K}_l^{\mathcal{P}+} - \log \mathcal{K}_k^{\mathcal{P}+} &= \sum_{i=k+1}^{l} \log(1 + c_i s_{i-1} x_i) \\
&\geq \frac{1}{2} \sum_{i=k+1}^{l-1} [c_i - c_{i+1}(1 + 2c_{i+1})] s_i^2 + \frac{c_l}{2} s_l^2 \\
&\quad - \frac{c_{k+1}(1 + 2c_{k+1})}{2} s_k^2 - \frac{1}{2} \sum_{i=k+1}^{l} c_i \\
&\geq -\frac{c_{k+1}(1 + 2c_{k+1})}{2} s_k^2 - \frac{c}{2} \left(\frac{1}{k} + \log \frac{l}{k} \right).
\end{aligned}$$

In the above, we used the approximation formula

$$\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n} \leq \int_m^n \frac{\mathrm{d}x}{x} + \frac{1}{m} = \log \frac{n}{m} + \frac{1}{m}.$$

Thus we obtain

$$\log \mathcal{K}_l^{\mathcal{P}+} - \log \mathcal{K}_k^{\mathcal{P}+} \geq -\frac{c}{k} - \frac{c}{2} \log \frac{l}{k} + O(k^{-2}). \tag{6}$$

Now starting at n_0, we consider adding the right-hand side of (6) for nonnegative blocks and $0 = \log \mathcal{K}_l^{\mathcal{P}+} - \log \mathcal{K}_k^{\mathcal{P}+}$ for negative blocks. Then after passing sufficiently many blocks, $\log \mathcal{K}_l^{\mathcal{P}+} - \log \mathcal{K}_k^{\mathcal{P}+}$ behave as (6) during half number of the entire blocks. Therefore at the beginning n_k of the last nonnegative block, we have

$$\log \mathcal{K}_{n_k}^{\mathcal{P}+} - \log \mathcal{K}_{n_0}^{\mathcal{P}+} \geq -\frac{c}{2} \log \frac{n_k}{n_0} - \frac{c}{4} \log \frac{n_k}{n_0} + O(1) = -\frac{3c}{4} \log \frac{n_k}{n_0} + O(1). \tag{7}$$

To finish the proof of Theorem 2, let n be in a middle of the last nonnegative block $\{n_k, \dots n_{l-1}\}$. Then as above, we have

$$\log \mathcal{K}_n^{\mathcal{P}+} - \log \mathcal{K}_{n_k}^{\mathcal{P}+} \geq \frac{cn}{2}\bar{x}_n^2 - \frac{c}{n_k} - \frac{c}{2}\log\frac{n}{n_k}. \tag{8}$$

Adding (7) and (8) we obtain

$$\log \mathcal{K}_n^{\mathcal{P}+} - \log \mathcal{K}_{n_0}^{\mathcal{P}+} \geq \frac{cn}{2}\bar{x}_n^2 - \frac{3c}{4}\log n + \frac{3c}{4}\log n_0 + O(1).$$

Thus we can bound $\log \mathcal{K}_n^{\mathcal{P}+}$ from below as

$$\log \mathcal{K}_n^{\mathcal{P}+} \geq \frac{cn}{2}\bar{x}_n^2 - \frac{3c}{4}\log n + O(1) = \frac{c}{2}\log n\left(\frac{n\bar{x}_n^2}{\log n} - \frac{3}{2}\right) + O(1).$$

This completes the proof of Theorem 2.

5 Multivariate linear protocol

In this section we generalize Theorem 1 to multivariate linear protocols. See Sect. 3 of Vovk et al. (2005c) and Sect. 6 of Takemura and Suzuki (2005) for discussions of linear protocols. Since the following generalization works for any dimension, including the case of infinite dimension, we assume that Skeptic and Reality choose elements from a Hilbert space H. The inner product of $x, y \in H$ is denoted by $x \cdot y$ and the norm of $x \in H$ is denoted by $\|x\| = (x \cdot x)^{1/2}$. Actually we do not specifically use properties of infinite dimensional space and readers may just think of H as a finite dimensional Euclidean space $\mathbb{R}^m$. For example the spectral resolution below corresponds to the spectral decomposition of a nonnegative definite matrix.

Let $\mathcal{X} \subset H$ denote the move space of Reality, and assume that $\mathcal{X}$ is bounded. Then by rescaling we can say without loss of generality that $\mathcal{X}$ is contained in the unit ball

$$\mathcal{X} \subset \{x \in H \mid \|x\| \leq 1\}.$$

In this case the closed convex hull $\overline{\mathrm{co}}(\mathcal{X})$ of $\mathcal{X}$ is contained in the unit ball. As in Takemura and Suzuki (2005) we also assume that the origin 0 belongs to $\overline{\mathrm{co}}(\mathcal{X})$. Note also that the average $\bar{x}_n$ of Reality's moves always belongs to $\overline{\mathrm{co}}(\mathcal{X})$ and hence $\|\bar{x}_n\| \leq 1$. In order to be clear, we write out the multivariate linear protocol.

Bounded Linear Game in General Dimension
Protocol:
$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \dots$:

Skeptic announces $M_n \in H$.
Reality announces $x_n \in \mathcal{X}$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n \cdot x_n$.
END FOR

As a natural multivariate generalization of the strategy $\mathcal{P}^c$ given by (1), we consider the strategy $\mathcal{P} = \mathcal{P}^A$

$$M_n = A\bar{x}_{n-1}\mathcal{K}_{n-1}, \tag{9}$$

where A is a self-adjoint operator in H. Then A has the spectral resolution

$$A = \int_{-\infty}^{\infty} \lambda E(\mathrm{d}\lambda), \tag{10}$$

where E denotes the real spectral measure of A, or the resolution of the identity corresponding to A. Let $\sigma(A)$ denote the spectrum of A (i.e. the support of E) and let

$$c_0 = \inf\{\lambda \mid \lambda \in \sigma(A)\}, \quad c_1 = \sup\{\lambda \mid \lambda \in \sigma(A)\}.$$

In the finite dimensional case, c_0 and c_1 correspond to the smallest and the largest eigenvalues of the matrix A, respectively.

Now we have the following generalization of Theorem 1.

Theorem 3 *In the bounded linear protocol game in general dimension, if Skeptic uses the strategy* (9) *with* $0 < c_0 \le c_1 \le 1/2$, *then* $\limsup_n \mathcal{K}_n = \infty$ *for each path* $\xi = x_1 x_2 \cdots$ *of Reality's moves such that*

$$\limsup_n \frac{\sqrt{n}\|\bar{x}_n\|}{\sqrt{\log n}} > \sqrt{\frac{c_1}{c_0}}.$$

Proof In the expression

$$\mathcal{K}_n = \prod_{i=1}^{n}(1 + A\bar{x}_{i-1} \cdot x_i), \tag{11}$$

we have

$$A\bar{x}_{i-1} \cdot x_i = \int_{c_0}^{c_1} \lambda(E(\mathrm{d}\lambda)\bar{x}_{i-1} \cdot x_i) = \bar{y}_{i-1} \cdot y_i, \tag{12}$$

with

$$\bar{y}_{i-1} = \int_{c_0}^{c_1} \sqrt{\lambda}E(\mathrm{d}\lambda)\bar{x}_{i-1}, \quad y_i = \int_{c_0}^{c_1} \sqrt{\lambda}E(\mathrm{d}\lambda)x_i.$$

By the Schwarz's inequality,

$$|A\bar{x}_{i-1} \cdot x_i| \le \|\bar{y}_{i-1}\| \|y_i\| \le c_1 \|\bar{x}_{i-1}\| \|x_i\| \le \frac{1}{2}.$$

Hence as in Sect. 3, we can bound $\log \mathcal{K}_n$ from below as

$$\log \mathcal{K}_n \ge \sum_{i=1}^{n} d_i t_{i-1} \cdot y_i - c_1 \sum_{i=1}^{n} d_i^2 \|t_{i-1}\|^2, \tag{13}$$

where $t_{i-1} = i\bar{y}_{i-1},\ d_i = 1/(i-1)$. Then by using the relation

$$t_{n-1} \cdot y_n = \frac{1}{2}(\|t_n\|^2 - \|t_{n-1}\|^2 - \|y_n\|^2),$$

we can further bound $\log \mathcal{K}_n$ from below as follows.

$$\begin{aligned}
\log \mathcal{K}_n &\ge \frac{1}{2} \sum_{i=1}^{n-1} (d_i - d_{i+1}(1 + 2c_1 d_{i+1})) \|t_i\|^2 + \frac{d_n}{2} \|t_n\|^2 - \frac{c_1}{2} \sum_{i=1}^{n} d_i \\
&\ge \frac{1}{2(n-1)} \|t_n\|^2 - \frac{c_1}{2}(3 + \log(n-1)) \\
&\ge \frac{n}{2} \|\bar{y}_n\|^2 - \frac{c_1}{2} \log n - \frac{3}{2} c_1 \\
&= \frac{c_1}{2} \log n \left(\frac{n\|\bar{y}_n\|^2}{c_1 \log n} - 1 \right) - \frac{3}{2} c_1.
\end{aligned} \tag{14}$$

It follows that if $\limsup_n \sqrt{n}\|\bar{y}_n\|/\sqrt{c_1 \log n} > 1$, then $\limsup_n \log \mathcal{K}_n = +\infty$. Now the theorem follows from $c_0\|\bar{x}_n\|^2 \le \|\bar{y}_n\|^2$. $\square$

Note that

$$c_0 \|\bar{x}_n\|^2 \le \|\bar{y}_n\|^2 \le c_1 \|\bar{x}_n\|^2$$

and the equalities hold if and only if A is a scalar multiplication operator $A = \int_{-\infty}^{\infty} cE(\mathrm{d}\lambda)$.

Remark 3 Suppose that $\{A_m\}$ is a sequence of positive definite degenerate self-adjoint operators with finite dimensional ranges $R_{A_m}(H) \subsetneq R_{A_{m+1}}(H) \cdots$, and with the supports (ranges of eigenvalues) $\sigma(A_m) \subsetneq \sigma(A_{m+1}) \cdots \subset (0, 1/2]$. Also suppose that A_∞ is a compact operator with infinite dimensional range $R_{A_\infty}(H) \subset H$, and A_∞ is obtained in the limit $A_m \to A_\infty,\ m \to \infty$. Then $c_0(A_m) \to c_0(A_\infty) = 0,\ m \to \infty$, so that in Theorem 3, $\sqrt{c_1(A_m)/c_0(A_m)} \to \sqrt{c_1(A_\infty)/c_0(A_\infty)} = \infty$, implying that the strategy $\mathcal{P}^{A_\infty}$ cannot weakly force SLLN with any rate. This phenomenon reflects one feature that the dimension of Skeptic's move space is related to the effective weak forcing of his strategy. It is a subject we will treat in the forthcoming paper.

References

Azuma, K. (1967). Weighted sums of certain dependent random variables. *The Tôhoku Mathematical Journal*, *19*, 357–367.

Bennett, G. (1962). Probability inequalities for the sum of independent random variables. *Journal of American Statistical Association*, *57*, 33–45.

Dembo, A., Zeitouni, O. (1998). *Large deviations techniques and applications* (2nd ed.). New York: Springer.

Hoeffding, W. (1963). Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, *58*, 13–30.

Kumon, M., Takemura, A., Takeuchi, K. (2005). Capital process and optimality properties of Bayesian Skeptic in the fair and biased coin games. arxiv:math.ST/0510662 (Submitted).

Kumon, M., Takemura, A., Takeuchi, K. (2006). Game-theoretic versions of strong law of large numbers for unbounded variables. arxiv:math.PR/0603184 (Submitted).

Shafer, G., Vovk, V. (2001). *Probability and finance: it's only a game!* New York: Wiley.

Takemura, A., Suzuki, T. (2005). Game theoretic derivation of discrete distributions and discrete pricing formulas. arxiv:math.PR/0509367. *Journal of the Japan Statistical Society* (in press).

Takeuchi, K. (2004a). *Kake no suuri to kinyu kogaku* (Mathematics of betting and financial engineering) (in Japanese). Tokyo: Saiensusha.

Takeuchi, K. (2004b). On strategies in favourable betting games. Unpublished manuscript.

Vovk, V., Gammerman, A., Shafer, G. (2005a). *Algorithmic learning in a random world*. New York: Springer.

Vovk, V., Takemura, A., Shafer, G. (2005b). Defensive forecasting. In R. G. Cowell, Z. Ghahramani, (Eds.). *Proceedings of the tenth international workshop on artificial intelligence and statistics* (pp. 365–372) (available electronically at http://www.gatsby.ucl.ac.uk/aistats/).

Vovk, V., Nouretdinov, I., Takemura, A., Shafer, G. (2005c). Defensive forecasting for linear protocols. In S. Jain, H. U. Simon, E. Tomita, (Eds.). *Proceedings of the sixteenth international conference on algorithmic learning theory*, LNAI 3734, Berlin: Springer. pp. 459–473.