Ann Inst Stat Math (2008) 60:605–626
DOI 10.1007/s10463-007-0115-7

Nonparametric inference for sequential k-out-of-n systems

Eric Beutner

Received: 1 February 2006 / Revised: 4 August 2006 / Published online: 16 March 2007

Abstract The k-out-of-n model is commonly used in reliability theory. In this model the failure of any component of the system does not influence the components still at work. Sequential k-out-of-n systems have been introduced as an extension of k-out-of-n systems where the failure of some component of the system may influence the remaining ones. We consider nonparametric estimation of the cumulative hazard function, the reliability function and the quantile function of sequential k-out-of-n systems. Furthermore, nonparametric hypothesis testing for sequential k-out-of-n-systems is examined. We make use of counting processes to show strong consistency and weak convergence of the estimators and to derive the asymptotic distribution of the test statistics.

Keywords Sequential k-out-of-n systems · Nonparametric estimation · Nonparametric hypothesis testing · Nelson–Aalen estimator · Martingale methods

1 Introduction

Consider a system consisting of n components. The system is functioning as long as k, $1 \leq k \leq n$, components are functioning, and it fails if $n - k + 1$ or more components fail. Such systems are called k-out-of-n system. Particular cases are parallel and series systems corresponding to $k = 1$ and $k = n$, respectively. It is often assumed that the failure times T_i, $1 \leq i \leq n$, of the n components are iid random variables. Assuming that the failure times T_i, $1 \leq i \leq n$, are iid random variables means that the failure of any component of the system does not affect

E. Beutner (✉)
Institute of Statistics, RWTH Aachen University, Aachen 52056, Germany
e-mail: beutner@stochastik.rwth-aachen.de

the remaining components. This model may be inadequate in various practical situations. For example, the failure of the engine of an airplane will increase the load put on the remaining two or three engines so that their lifetimes tend to be shorter.

In this context, sequential k-out-of-n models have been introduced by Kamps (1995) as an extension of the k-out-of-n model based on iid random variables (see also Hollander and Peña 1995). They are flexible in the sense that, after the failure of some component, the distribution of the residual lifetime of the remaining components may change, i.e., the underlying failure rate of the remaining components is adjusted according to the number of preceding failures. The failure times of a sequential k-out-of-n system are called sequential order statistics and will be denoted by $X_1^*, \ldots, X_{nk+1}^*$. Thus the life length of a sequential k-out-of-n system is described by X_{n-k+1}^*, $1 \leq k \leq n$.

General accounts of theoretical developments and applications concerning sequential order statistics are given by Cramer and Kamps (2001b, 2003). Statistical inference for sequential order statistics was mainly concerned with parametric models (cf. Cramer and Kamps 1996, 2001a) including exponential and Weibull distributions. Here, we focus on nonparametric statistical methods.

In Sect. 2 we describe sequential order statistics and introduce related point processes. Section 3 introduces the estimators of the cumulative hazard rate function Λ, the reliability function $R = 1 - F$ and the quantile function F^{-1}. Here F denotes the distribution function. Strong consistency and weak convergence of the estimators is established. Two nonparametric hypothesis tests related to sequential k-out-of-n systems are examined in Sect. 4. Finally, in Sect. 5 we present a simulation study illustrating the behavior of the estimator of Λ.

2 Description of the model

A definition of sequential order statistics with a view to the motivation given in the introduction can be found in Cramer and Kamps (1996). As shown in Cramer and Kamps (2003) they can also be defined as follows.

Definition 1 Let $F_1, \ldots, F_n$ be distribution functions with $F_1^{-1}(1) \leq \cdots \leq F_n^{-1}(1)$, and let $V_1, \ldots, V_n$ be independent random variables with $V_r \sim \text{Beta}(n-r+1, 1)$, $1 \leq r \leq n$.
Then the random variables

$$X_r^* = F_r^{-1}(1 - V_r R_r(X_{r-1}^*)), \quad 1 \leq r \leq n, \quad X_0^* = -\infty,$$

are called sequential order statistics (based on $F_1, \ldots, F_n$), where R_r denotes the reliability function $1 - F_r$.

Assumption 1 In the following we restrict ourselves to a particular choice of the distribution functions $F_1, \ldots, F_n$, namely

$$F_i(t) = 1 - (1 - F(t))^{\alpha_i} \tag{1}$$

for positive real numbers $\alpha_1, \ldots, \alpha_n$.

Taking $\alpha_1 = \cdots = \alpha_n = 1$ it is easily seen that common k-out-of-n models are contained in the sequential k-out-of-n model under the assumption (1).

Remark 1 The restriction to the choice $F_i(t) = 1 - (1 - F(t))^{\alpha_i}$, $1 \le i \le n$, has two advantages. The first advantage is that, in this case, the model of sequential order statistics coincides with the model of generalized order statistics in the distributional theoretical sense. The model of generalized order statistics contains for example order statistics and progressively type-II censored order statistics (for nonparametric estimation with progressively type-II censored order statistics see Bordes 2004; Guilbaud 2004). The second advantage is that the model uncertainty reduces to the parameters $\alpha_1, \ldots, \alpha_n$ and the distribution function F.

In what follows we consider the case where the parameter vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is known and only the distribution function F is unknown and the case where both are unknown.

Assumption 2 In the case where the parameter vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and the distribution function F are unknown a condition is needed in order for the model to be identifiable; therefore we assume that $\alpha_1 = 1$ to make the model identifiable.

We aim at estimating the cumulative hazard function Λ, the reliability function $R = 1 - F$ and the quantile function F^{-1} of the distribution function F based on the observation of m independent copies of a sequential k-out-of-n system. For notational convenience we suppose that we observe a sequential 1-out-of-n system. As can be seen in the rest of the paper we can choose $k \in \{1, \ldots, n\}$ arbitrarily without changing anything in the proof given below.
When estimating Λ, R and F^{-1} based on the observation of m independent copies of a sequential k-out-of-n system two problems arise. The first problem is that, in contrast to k-out-of-n models based on iid random variables, the failure times of the components of a sequential k-out-of-n system are no longer iid random variables. The second problem is that, if we have $\alpha_i \neq 1$ for all $i \in \{1, \ldots, n\}$, none of the random variables X_i^*, $1 \le i \le n$, is the minimum of a sample having distribution function F. In order to construct estimators for Λ, R and F^{-1} it is helpful to recognize that the hazard rate function λ_i of F_i is given by

$$\lambda_i(t) = \alpha_i \lambda(t),$$

where λ is the hazard rate function of F. Nonparametric estimation of Λ is often based on the processes N and Y where $N(t)$ represents the number of observed failures by time t and $Y(t)$ represents the number of the risk set at time t. Given the close connection between the hazard rate functions of F_i and F it is reasonable to expect that nonparametric estimation for sequential order statistics can be based on these processes or on adequate modifications of these processes.

Define the counting process N based on sequential order statistics by

$$N(t) = \sum_{i=1}^{n} I_{\{X_i^* \le t\}}$$

and let $\mathcal{F}_t^N = \sigma(\{N_s : 0 \le s \le t; \alpha_i, 1 \le i \le n\})$ the natural filtration generated by N and α_i. Here and in the following $I_{\{\cdot\}}$ denotes the indicator function.

Lemma 1 *The stochastic intensity $\tilde{\lambda}$ of the counting process N based on sequential order statistics is given by*

$$\tilde{\lambda}(t) = \boldsymbol{\alpha}\check{\mathbf{Y}}'(t) \cdot \frac{f(t)}{1 - F(t)}$$

where $\check{\mathbf{Y}}(t) = (nI_{\{X_0^* < t \le X_1^*\}}, (n-1)I_{\{X_1^* < t \le X_2^*\}}, \ldots, I_{\{X_{n-1}^* < t \le X_n^*\}})$, $X_0^* = 0$, *and* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

Proof From Karr (1991, p. 70) we conclude that N admits a stochastic intensity which is given by

$$\tilde{\lambda}_t(\omega) = \frac{f_{i-1}(t - X_i^*(\omega))}{\int_{t-X_{i-1}^*(\omega)}^{\infty} f_{i-1}(u)\mathrm{d}u}, \quad t \in (X_{i-1}^*(\omega), X_i^*(\omega)]. \tag{2}$$

From Kamps (1995, p. 27) we obtain for $\hat{t} > 0$

$$\begin{aligned} P(X_i^* - X_{i-1}^* > \hat{t} | X_{i-1}^* = s) &= \left(\frac{1 - (1 - (1 - F(\hat{t} + s))^{\alpha_i})}{1 - (1 - (1 - F(s))^{\alpha_i})}\right)^{n-i+1} \\ &= \left(\frac{1 - F(\hat{t} + s)}{1 - F(s)}\right)^{(n-i+1)\alpha_i}. \end{aligned}$$

Hence the density f_{i-1} of the conditional distribution is given by

$$f_{i-1}(\hat{t}) = (n - i + 1)\alpha_i \frac{(1 - F(\hat{t} + s))^{(n-i+1)\alpha_i - 1}}{(1 - F(s))^{(n-i+1)\alpha_i}} f(\hat{t}).$$

This implies for $t \in [s, \infty)$

$$f(t - X_{i-1}^*) = (n - i + 1)\alpha_i \frac{(1 - F(t))^{(n-i+1)\alpha_i - 1}}{(1 - F(s))^{(n-i+1)\alpha_i}} f(t). \tag{3}$$

Finally for $t \in [s, \infty)$,

$$\int_{t-X_{i-1}^*(\omega)}^{\infty} f_{i-1}(u)\mathrm{d}u = P(X_i^* - X_{i-1}^* > t - X_{i-1}^* | X_{i-1}^* = s)$$
$$= \left(\frac{1-F(t)}{1-F(s)}\right)^{(n-i+1)\alpha_i}. \tag{4}$$

From (2) to (4) we conclude that

$$\tilde{\lambda}_t(\omega) = (n-i+1)\alpha_i \frac{f(t)}{1-F(t)}, \qquad t \in (X_{i-1}^*(\omega), X_i^*(\omega)].$$

Now the assertion follows.

From the preceding Lemma it follows immediately that

$$M(t) = N(t) - \int_0^t \lambda(s) Y(s) \mathrm{d}s, \tag{5}$$

where $\lambda(s) = \frac{f(s)}{1-F(s)}$ and $Y(s) \equiv \boldsymbol{\alpha} \check{\mathbf{Y}}'(s)$, is a martingale.

Remark 2 Notice that, in general, the process Y is not monotonically decreasing in t. If we consider the common k-out-of-n model then it is obvious that $Y(t)$ is monotonically decreasing in t since then $Y(t) = \sum_{i=1}^n I_{\{T_{i:n} \geq t\}}$.

In the next section we define the estimators of Λ, R and F^{-1} and examine their asymptotic behavior.

3 Asymptotic results for the estimators

Suppose that the parameter vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is known and that we observe m independent copies of the processes N and Y on a finite interval $[0, T]$ to estimate Λ, R and F^{-1}. Let N_m, $\check{\mathbf{Y}}_m$, Y_m, and M_m be the sum of the first m copies of N, $\check{\mathbf{Y}}$, Y, and M. We denote $\frac{N_m}{m}$ by $\bar{N}_m$ and $\frac{Y_m}{m}$ by $\bar{Y}_m$. Since M_m is a martingale, an obvious estimator for $\int_0^t \lambda(s) I_{\{Y_m(s)>0\}} \mathrm{d}s$ and hence for $\Lambda(t) = \int_0^t \lambda(s) \mathrm{d}s$ is given by

$$\hat{\Lambda}_m(t) = \int_0^t \frac{J_m(s)}{\boldsymbol{\alpha} \check{\mathbf{Y}}'_m(s)} \mathrm{d}N_m(s), \tag{6}$$

where $J_m(s) = I_{\{Y_m(s)>0\}} = I_{\{\boldsymbol{\alpha}\check{\mathbf{Y}}'_m(s)>0\}}$. Our estimator $\hat{\Lambda}_m$ does not coincide exactly with the Nelson–Aalen estimator but as it is very close we refer to it as Nelson–Aalen estimator.

To obtain an estimator for Λ in the more general case where the parameter vector $\boldsymbol{\alpha} = (1, \alpha_2, \ldots, \alpha_n)$ is unknown we proceed as follows (for a detailed description see Kvam and Peña 2005). First, fix a parameter vector $\bar{\boldsymbol{\alpha}}$ to obtain an estimator for Λ from (6). Then this estimator is plugged into the likelihood process which is maximized with respect to $\alpha_2, \ldots, \alpha_n$ to obtain the estimator $\hat{\boldsymbol{\alpha}} = (1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$. The estimator $\hat{\Lambda}(\cdot, \hat{\boldsymbol{\alpha}})$ of Λ is then obtained by plugging $\hat{\boldsymbol{\alpha}}$ in (6).

Remark 3 For the asymptotic properties of the estimator $\hat{\boldsymbol{\alpha}}$ see Kvam and Peña (2005).

To facilitate the presentation we introduce the following notations. By $X^*_{r,i}$, $1 \leq r \leq n$, $1 \leq i \leq m$, we denote the r-th sequential order statistic for the i-th observation.

$$\begin{aligned}
\boldsymbol{\delta}_i(t) &= (I_{\{X^*_{0,i}<t\leq X^*_{1,i}\}}, I_{\{X^*_{1,i}<t\leq X^*_{2,i}\}}, \ldots, I_{\{X^*_{n-1,i}<t\leq X^*_{n,i}\}}), \quad i = 1, \ldots, m \\
\mathbf{e}(t) &= (nE[I_{\{X^*_0<t\leq X^*_1\}}], (n-1)E[I_{\{X^*_1<t\leq X^*_2\}}], \ldots, E[I_{\{X^*_{n-1}<t\leq X^*_n\}}]), \\
\boldsymbol{\varepsilon}(t, \boldsymbol{\alpha}) &= \frac{1}{E[Y(t)]} \cdot \boldsymbol{\alpha} \star \mathbf{e}(t, \boldsymbol{\alpha}) \\
\boldsymbol{\gamma} &= (n, \ldots, 1) \star \boldsymbol{\alpha}.
\end{aligned}$$

Here and in the following, $\star$ represents component-by-component multiplication. Given a vector $\boldsymbol{\zeta}$ we denote by $\mathbf{D}(\zeta)$ a diagonal matrix with diagonal elements $\boldsymbol{\zeta}$. Later on we have to distinguish between the mth observation of $\check{\mathbf{Y}}$ and the sum of m copies of $\check{\mathbf{Y}}$ we let for $i = 1, \ldots, m$

$$\underline{\check{\mathbf{Y}}}_i(t) = (nI_{\{X^*_{0,i}<t\leq X^*_{1,i}\}}, (n-1)I_{\{X^*_{1,i}<t\leq X^*_{2,i}\}}, \ldots, I_{\{X^*_{n-1,i}<t\leq X^*_{n,i}\}}).$$

As $R(t) = \prod_{s\leq t}(1 - \mathrm{d}\Lambda(s))$ it is natural to use

$$\hat{R}_m(t) = \prod_{s\leq t}(1 - \mathrm{d}\hat{\Lambda}_m(s)) \tag{7}$$

as an estimator for R in the case where only F is unknown and

$$\hat{R}_m(t, \hat{\alpha}) = \prod_{s\leq t}(1 - \mathrm{d}\hat{\Lambda}_m(s, \hat{\alpha})) \tag{8}$$

as an estimator for R in the case where α and F are unknown. Here $\prod_{s\leq t}$ denotes the product integral (see Gill and Johansen 1990). The estimator (8) is called Kaplan–Meier estimator.

Remark 4 Notice that, in contrast to the usual Nelson–Aalen estimator, $\hat{\Lambda}_m$ and $\hat{\Lambda}(\cdot, \hat{\boldsymbol{\alpha}})$ can have jumps greater than 1, in the case that the number of components that are still at work and the parameters α_i (or the estimator) of these components are small. In that case, (8) can be negative. One way to circumvent this problem is to exclude this observations form the definition of $\hat{R}_m$. Since the proportion of excluded observations to the total number of observations is getting smaller and smaller as m increases this does not affect the asymptotic behavior of $\hat{R}_m$.

Assumption 3 For the rest of the paper we assume that $F(T) < 1$.

This assumption enables us to show that $E[Y(t)]$ is uniformly bounded from below by a constant $c > 0$ on $[0, T]$. This fact allows us to prove uniform consistency of the estimators.

Lemma 2 *Under the Assumption 3 we have for all* $0 \leq t \leq T$

$$E[Y(t)] \geq \gamma_1 \cdot \tilde{F}\big(1 - F(T)\big)$$

where $\tilde{F}$ denotes the distribution function of a beta$(\gamma_1, 1)$ *random variable.*

Proof Using the Definition 1, Cramer and Kamps (2003) (cf. Theorem 3.1) showed that under the assumption (1) sequential order statistics can be represented as

$$X_i^* = R^{-1}\left(\prod_{j=1}^{i} B_j\right) \tag{9}$$

where B_j, $1 \leq j \leq m$, are independent beta distributed random variables with parameters γ_j and 1. Hence,

$$\begin{aligned}
E[Y(t)] &= E\left[\sum_{i=1}^{m} \gamma_i I_{\{X_{i-1}^* < t \leq X_i^*\}}\right] \\
&\geq \gamma_1 P(X_1^* \geq t) \\
&= \gamma_1 P(R^{-1}(B_1) \geq t) \\
&= \gamma_1 P(F^{-1}(1 - B_1) \geq t) \\
&\geq \gamma_1 P(1 - F(T) \geq B_1) \\
&= \gamma_1 \tilde{F}\big(1 - F(T)\big) \geq c > 0.
\end{aligned}$$

Before proving uniform consistency of $\hat{\Lambda}_m$ to Λ on $[0, T]$ we establish a Lemma which will be helpful when showing uniform consistency. By $||.||_0^T$ we will denote the supremum norm on $[0, T]$.

Lemma 3 *Let $\bar{Y}_m(t)$ be as above. Under the Assumption 3 with probability* 1, *$\bar{Y}_m(t)$ converges to $E[Y(t)]$ uniformly on $[0, T]$ as $m \to \infty$.*

Proof Notice that $Y(t)$ can be written as

$$Y(t) = \sum_{i=1}^{n} (\gamma_i - \gamma_{i+1}) I_{\{X_i^* \geq t\}} \tag{10}$$

where $\gamma_{n+1} = 0$. The processes $Y_1(t) = \sum_{i=1}^{n} \gamma_i I_{\{X_i^* \geq t\}}$ and $Y_2(t) = \sum_{i=1}^{n} \gamma_{i+1} I_{\{X_i^* \geq t\}}$ are both monotonically decreasing in t. Let Y_{1m} and Y_{2m} be the sum of the first m copies of Y_1 and Y_2 and denote $\frac{Y_{1m}}{m}$ by $\bar{Y}_{1m}$ and $\frac{Y_{2m}}{m}$ by $\bar{Y}_{2m}$. By the Glivenko–Cantelli theorem

$$||\bar{Y}_{1m} - E[Y_1(t)]||_0^T \to 0 \quad \text{and} \quad ||\bar{Y}_{2m} - E[Y_2(t)]||_0^T \to 0$$

with probability 1 as $m \to \infty$. Hence, with probability 1, $\bar{Y}_m(t)$ converges to $E[Y(t)]$ uniformly on $[0, T]$ as $m \to \infty$.

Theorem 1 *Under the Assumption 3 we have* $||\hat{\Lambda}_m(t) - \Lambda(t)||_0^T \to 0$ *with probability 1 as* $m \to \infty$.

Proof Taking expectations in (5) it can be seen that $\Lambda(t) = \int_0^t \frac{1}{E[Y(s)]} \mathrm{d}E[N(s)]$. By Lemma 4.1 of Dorado et al. (1997) for m sufficiently large

$$\begin{aligned} &||\hat{\Lambda}_m(t) - \int_0^t \frac{1}{E[Y(s)]} \mathrm{d}E[N(s)]||_0^T \\ &\quad \leq \frac{||\bar{Y}_m(t) - E[Y(t)]||_0^T \big(E[N(T)] + ||\bar{N}_m(t) - E[N(t)]||_0^T\big)}{||E[Y(t)]||_0^T \big(\mid ||E[Y(t)]||_0^T - ||E[Y(t)] - \bar{Y}_m(t)||_0^T \mid \big)} \\ &\qquad + \frac{2}{||E[Y(t)]||_0^T} ||\bar{N}_m(t) - E[N(t)]||_0^T. \end{aligned}$$

From Lemma 2 we conclude that $||E[Y(t)]||_0^T \geq \gamma_1 \tilde{F}\big(1 - F(T)\big) \geq c > 0$. Since $N(t)$ is monotonically increasing in t the Glivenko–Cantelli theorem implies $||\bar{N}_m(t) - E[N(t)]||_0^T \to 0$ with probability 1. According to Lemma 3 $||\bar{Y}_m(t) - E[Y(t)]||_0^T \to 0$ with probability 1. Therefore,

$$||\hat{\Lambda}_m(t) - \Lambda(t)||_0^T \to 0 \ \text{ w.p.1 } \ \text{as} \ \ m \to \infty.$$

Uniform consistency of $\hat{R}_m$ now follows from Theorem 1 and the continuity of the product integral as shown in the next result.

Theorem 2 *Under the Assumption 3 we have* $||\hat{R}_m(t) - R(t)||_0^T \to 0$ *with probability* 1 *as* $m \to \infty$.

Proof Recall that $R(t) = 1 - F(t) = \prod_{s \leq t} (1 - \mathrm{d}\Lambda(s))$. According to Theorem 7 of Gill and Johansen (1990), the convergence of $|| \prod_{s \leq t} (1 - \mathrm{d}\hat{\Lambda}_m(s)) - \prod_{s \leq t} (1 - \mathrm{d}\Lambda(s))||_0^T \to 0$ holds w.p.1 if

$$||\hat{\Lambda}_m(t) - \Lambda(t)||_0^T \to 0 \text{ w.p.1 as } m \to \infty \tag{11}$$

and

$$\limsup ||\hat{\Lambda}_m(t)||_0^T < \infty \text{ w.p.1 as } m \to \infty. \tag{12}$$

Condition (11) is fulfilled according to Theorem 1 and condition (12) holds since $\hat{\Lambda}_m(t)$ is increasing in t for each m, $\hat{\Lambda}_m(T) \to \Lambda(T)$ a.s. and $\Lambda(T) < \infty$ under the Assumption 3.

We now present the main results concerning the asymptotic distribution of the estimators $\hat{\Lambda}_m$, $\hat{\Lambda}(\cdot, \hat{\alpha})$, $\hat{R}_m$, $\hat{R}_m(\cdot, \hat{\alpha})$ and $\hat{F}_m^{-1}$ where $\hat{F}_m^{-1}$ is the estimator of F^{-1} defined by

$$\hat{F}_m^{-1}(p) = \inf\{x : \hat{R}_m(x) \leq 1 - p\}. \tag{13}$$

We will denote by $D[0, T]$ the cadlag functions on $[0, T]$ and by $\Rightarrow$ weak convergence. The following theorem is shown in Kvam and Peña (2005) and will be used to show weak convergence of F_m^{-1} and to establish the asymptotic distribution of the test statistics in Sect. 4.

Theorem 3 *Let W be a zero mean Gaussian process with independent increments and variance function $v(t) = \mathrm{Var}[W(t)] = \int_0^t \frac{1}{E[Y(s)]}\lambda(s)\mathrm{d}s$. Then under the Assumption 3 we obtain*

(a) (i) $\sqrt{m}(\hat{\Lambda}_m(t) - \Lambda(t)) \Rightarrow W(t)$ *on* $D[0, T]$.

(ii) $P(\sup_{0\leq t\leq T} |\hat{v}_m(t) - v(t)| > \epsilon) \to 0$ *as* $m \to \infty$, *for all* $\epsilon > 0$ *with* $\hat{v}_m$ *defined by* $\hat{v}_m(t) = m\int_0^t \frac{1}{Y_m^2(s)}\mathrm{d}N_m(s)$.

(iii) $\sqrt{m}(\hat{R}_m(t) - R(t)) \Rightarrow -R \cdot W(t)$ *on* $D[0, T]$.

(b)(iv) $\sqrt{m}((\hat{\Lambda}_m(t, \hat{\alpha}) - \Lambda(t)) \Rightarrow W_1(t)$,

(v) $\sqrt{m}(\hat{R}_m(t, \hat{\alpha}) - R(t)) \Rightarrow -R \cdot W_1(t)$

where W_1 is a Gaussian process with variance function

$$v_1(t) = v(t) + \left(\int_0^t \varepsilon(s)\lambda(s)\mathrm{d}s\right) \Psi(T, \boldsymbol{\alpha})^{-1} \left(\int_0^t \varepsilon(s)\lambda(s)\mathrm{d}s\right)'$$

where

$$\Psi(t, \boldsymbol{\alpha})^{-1} = \int_0^t \left[\mathbf{D}(\varepsilon(s, \alpha)) - \varepsilon(s, \alpha)'\varepsilon(s, \alpha)\right] E[Y(s)]\lambda(s)\mathrm{d}s.$$

Proof See Kvam and Peña (2005), Theorem 1 and Corollary 1.

Given the above results we can easily establish the following theorem for the estimator $\hat{F}_m^{-1}$ of the quantile function.

Theorem 4 *Under the Assumption 3 for $0 < p < F^{-1}(T)$:*

(a) (i) $\hat{F}_m^{-1}(p) \to F^{-1}(p)$ *w.p.1 as* $m \to \infty$.

(ii) *Provided that* $f(F^{-1}(p)) > 0$ *we have* $\sqrt{m}(\hat{F}_m^{-1}(p) - F^{-1}(p)) \Rightarrow N\left(0, \frac{(1-p)^2 v(F^{-1}(p))}{f^2(F^{-1}(p))}\right)$ *where* v *is defined in Theorem 3.*

(b) *Moreover, if* f *is continuous and positive on* $[t_1-\epsilon, t_2+\epsilon]$ *for* $0 < t_1 < t_2 < T$ *and* $\epsilon > 0$ *we have that*

(iii) $\sqrt{m}(\hat{F}_m^{-1}(\cdot) - F^{-1}(\cdot)) \Rightarrow \frac{(1-(\cdot))W \circ F^{-1}(\cdot)}{f(F^{-1}(\cdot))}$ *on* $D[p_1, p_2]$ *where* W *is the Gaussian process defined in Theorem 3 and* $p_i = F(t_i)$, $i = 1, 2$.

(iv) *For* $p_1 \leq p \leq p_2$ *we have that* $\frac{(1-p)^2 v(F^{-1}(p))}{f^2(F^{-1}(p))} = \frac{v(F^{-1}(p))}{\lambda^2(F^{-1}(p))}$ *may be consistently estimated by* $\hat{F}_m^{-1}$, $\hat{v}_m$ *and* $\hat{\lambda}_m$ *where* $\hat{\lambda}_m(t) = \frac{1}{b_m}\int_0^T K(\frac{t-s}{b_m})\mathrm{d}\hat{\Lambda}_m(s)$ *is an estimator of* λ *with bandwidth* $b_m \to 0$ *as* $m \to \infty$ *and* K *denotes a kernel function of bounded variation.*

Proof (i) According to Theorem 1 we have for all $\epsilon > 0$ that $\hat{F}_m^{-1}(p) = \inf\{x : \hat{R}_m(x) \leq 1 - p\} \leq F^{-1}(p + \epsilon)$ and $\hat{F}_m^{-1}(p) \geq F^{-1}(p - \epsilon)$ with probability 1 as $m \to \infty$. Hence $\hat{F}_m^{-1}(p)$ converges to $F^{-1}(p)$ with probability 1 as $m \to \infty$.
(ii) Follows from Theorem 3 (iii), the functional delta method and the fact that the function ϕ from the space of distribution functions to the real line defined by $\phi(G) = \inf\{x : G(p) \geq p\}$ is compactly differentiable at F with derivative $\mathrm{d}\phi(F) \cdot h = -\frac{h(F^{-1}(p))}{f(F^{-1}(p))}$ (cf. Andersen et al. 1993 Theorem II 8.4).
(iii) Follows with the arguments given in part (ii) and Theorem 1 of Doss and Gill (1992).
(iv) Notice that

$$
\begin{aligned}
P(|\hat{v}_m(\hat{F}_m^{-1}(p)) - v(F^{-1}(p))| > \epsilon) &\leq P\left(|\hat{v}_m(\hat{F}_m^{-1}(p)) - v(\hat{F}_m^{-1}(p))| > \frac{\epsilon}{2}\right) \\
&\quad + P\left(|v(\hat{F}_m^{-1}(p)) - v(F^{-1}(p))| > \frac{\epsilon}{2}\right) \\
&\leq P\left(\hat{F}_m^{-1}(p) \notin [t_1 - \delta, t_2 + \delta]\right) \\
&\quad + P\left(\sup_{0 \leq t \leq T} |\hat{v}_m(t) - v(t)| > \frac{\epsilon}{2}\right) \\
&\quad + P\left(|v(\hat{F}_m^{-1}(p)) - v(F^{-1}(p))| > \frac{\epsilon}{2}\right).
\end{aligned}
$$

By part (i) and Theorem 3 (ii) the last sum converges to zero as $m \to \infty$. Applying the above inequality to $P(|\hat{\lambda}_m(\hat{F}_m^{-1}(p)) - \lambda(F^{-1}(p))| > \epsilon)$ and using the uniform convergence of $\hat{\lambda}_m$ to λ on $[t_1 - \delta, t_2 + \delta]$ (cf. Theorem IV.2.2 of Andersen et al. 1993) and part (i) we obtain $P(|\hat{\lambda}_m(\hat{F}_m^{-1}(p)) - \lambda(F^{-1}(p))| > \epsilon) \to 0$ for all $\epsilon > 0$ as $m \to \infty$. Now the assertion follows.

Remark 5 In their paper, Kvam and Peña (2005) considered an equally load sharing model which leads to the same mathematical structure as in (6). They assumed due to the interpretation of their model that the unknown parameters α_i, $2 \leq i \leq k$, fulfil

$$\alpha_1 = 1, \text{ and } \alpha_2 < \cdots < \alpha_k. \tag{14}$$

As they did not use this ordering in their proof it can be directly applied to our estimators.

4 Nonparametric hypothesis testing

In the first subsection we consider nonparametric hypothesis testing for sequential 1-out-of-n systems in the case where only the distribution function F is unknown. The asymptotic distribution of the test statistics in the case, where the parameter vector $\boldsymbol{\alpha} = (1, \alpha_2, \dots, \alpha_n)$ and the distribution function F are unknown, is examined in the second subsection.

4.1 Nonparametric hypothesis testing for known parameter vector α

First, we consider one-sample tests for the hypothesis that the hazard rate function λ of a given sequential 1-out-of-n system equals a known hazard rate function λ_0. Afterwards we examine two-sample tests for the hypothesis that the hazard rate functions λ_1 and λ_2 of two sequential 1-out-of-n_i, $i = 1, 2$, systems coincide. Their parameters $\alpha_{11}, \dots, \alpha_{1n_1}$ and $\alpha_{21}, \dots, \alpha_{2n_2}$ may be different or equal.
Our test statistics for the one-sample tests will be based on the stochastic processes

$$Z_m(t) = \int_0^t K_m(s) \mathrm{d}\hat{\Lambda}_m(s) - \int_0^t K_m(s)\lambda_0(s)\mathrm{d}s \tag{15}$$

where $K_m(t) = Y_m(t)R_0^\rho(t)$, $0 \le \rho \le 1$. Here R_0^ρ is the reliability function under the hypothesis $\lambda = \lambda_0$.

Remark 6 The choice of $\rho = 0$ leads to the log rank statistic and for $0 < \rho \le 1$ we arrive at the family of test statistics suggested by Harrington and Fleming (1982).

Theorem 5 *Let W be a zero mean Gaussian process with independent increments and variance function $v^1(t) = \mathrm{Var}[W(t)] = \int_0^t E[Y(s)]R_0^{2\rho}(s)\lambda_0(s)\mathrm{d}s$. Then under the Assumption 3 and under $H_0 : \lambda = \lambda_0$ we have*

(i) $\frac{1}{\sqrt{m}} Z_m \Rightarrow W$ *on* $D[0, T]$ *as* $m \to \infty$, *and*
(ii) *for all* $\epsilon > 0$ *we have* $P(\sup_{0 \le t \le T} |\hat{v}_m^1(t) - v^1(t)| > \epsilon) \to 0$ *as* $m \to \infty$ *where* $\hat{v}_m^1(t) = \frac{1}{m} \int_0^t \frac{(K_m(s))^2}{Y_m(s)} \mathrm{d}\hat{\Lambda}_m(s)$
for every $K_m(t) = Y_m(t)R_0^\rho(t)$ *where* $0 \le \rho \le 1$.

Proof Notice that $Z_m(t)$ is equal to $\int_0^t \frac{K_m(s)}{Y_m(s)} \mathrm{d}M_m(s)$, hence a martingale. In order to apply Rebolledo's Theorem (see Andersen et al. 1993 Theorem II. 5.1) to the martingale $\frac{1}{\sqrt{m}} Z_m$ we have to show that for all $t \in [0, T]$ and all $\epsilon > 0$

(a) $\left\langle \frac{1}{\sqrt{m}} \int_0^t \frac{K_m(s)}{Y_m(s)} \mathrm{d}M_m(s) \right\rangle \xrightarrow{P} v^1(t)$ as $m \to \infty$, and

(b) $\left\langle \frac{1}{\sqrt{m}} \int_0^t \frac{K_m(s)}{Y_m(s)} I_{\{|\frac{K_m(s)}{\sqrt{m}Y_m(s)}>\epsilon|\}} \mathrm{d}M_m(s) \right\rangle \xrightarrow{P} 0 \quad \text{as } m \to \infty.$

Condition (a) is fulfilled since

$$\left\langle \frac{1}{\sqrt{m}} \int_0^t \frac{K_m(s)}{Y_m(s)} \mathrm{d}M_m(s) \right\rangle = \frac{1}{m} \int_0^t \frac{(K_m(s))^2}{Y_m(s)} \lambda_0(s)\mathrm{d}s$$
$$= \int_0^t R_0^{2\rho}(s) \frac{Y_m(s)}{m} \lambda_0(s)\mathrm{d}s$$

and the last term converges to $\int_0^t E[Y(s)]R_0^{2\rho}(s)\lambda_0(s)\mathrm{d}s$, P.a.s, $\forall t \in [0, T]$ by Lemma 3 and the boundedness of R_0^ρ for all $0 \leq \rho \leq 1$.
To show that condition (b) is satisfied notice that

$$\left\langle \frac{1}{\sqrt{m}} \int_0^t \frac{K_m(s)}{Y_m(s)} I_{\{|\frac{K_m(s)}{\sqrt{m}Y_m(s)}>\epsilon|\}} \mathrm{d}M_m(s) \right\rangle = \int_0^t R_0^{2\rho}(s) \frac{Y_m(s)}{m} I_{\{|\frac{K_m(s)}{\sqrt{m}Y_m(s)}>\epsilon|\}} \lambda_0(s)\mathrm{d}s$$
$$= \int_0^t R_0^{2\rho}(s) \frac{Y_m(s)}{m} I_{\{|\frac{R_0^\rho(s)}{\sqrt{m}}>\epsilon|\}} \lambda_0(s)\mathrm{d}s$$

and the last term is equal to 0 for m sufficiently large. Hence, condition (b) is satisfied and the assertion follows.
(ii) We have that

$$\frac{1}{m} \int_0^t \frac{(K_m(s))^2}{Y_m(s)} \mathrm{d}\hat{\Lambda}_m(s) = \frac{1}{m} \int_0^t \frac{(K_m(s))^2}{(Y_m(s))^2} \mathrm{d}M_m(s)$$
$$+ \frac{1}{m} \int_0^t \frac{(K_m(s))^2}{Y_m(s)} \lambda_0(s)\mathrm{d}s. \tag{16}$$

Applying Lenglart's inequality to the first term on the right hand side in (16), which is a martingale, we get for any $\delta, \eta > 0$

$$P\left(\sup_{0 \leq t \leq T} |\frac{1}{m} \int_0^t \frac{(K_m(s))^2}{(Y_m(s))^2} \mathrm{d}M_m(s)| > \eta \right)$$
$$\leq \frac{\delta}{\eta^2} + P\left(\int_0^T \frac{(R_0^\rho(s))^4}{m} \frac{Y_m(s)}{m} \lambda_0(s)\mathrm{d}s > \delta \right).$$

From Lemma 3 we obtain $P\left(\sup_{0 \leq t \leq T} |\frac{1}{m} \int_0^t \frac{(K_m(s))^2}{(Y_m(s))^2} \mathrm{d}M_m(s)| > \eta \right) \to 0$ for all $\eta > 0$.

The uniform convergence of the second term on the right hand side in (16) to ν^1 follows also from Lemma 3.

For the construction of the two-sample tests let, for $i = 1, 2, N_{m_i}, Y_{m_i}$ and M_{m_i} be the sum of the first m_i copies of the sequential 1-out-of-n_i system, and denote by $\hat{\Lambda}_{m_i}$ the estimators of the cumulative hazard functions. The test statistics for the two-sample tests we will based on the processes

$$\begin{aligned}\tilde{Z}_{m_1,m_2}(t) &= \int_0^t L_{m_1,m_2}(s)\mathrm{d}\hat{\Lambda}_{m_1}(s) - \int_0^t L_{m_1,m_2}(s)\mathrm{d}\hat{\Lambda}_{m_2}(s) \\ &= \int_0^t \frac{L_{m_1,m_2}(s)}{Y_{m_1}(s)}\mathrm{d}M_{m_1}(s) - \int_0^t \frac{L_{m_1,m_2}(s)}{Y_{m_2}(s)}\mathrm{d}M_{m_2}(s) \\ &\quad + \int_0^t L_{m_1,m_2}(s)(\lambda_1(s) - \lambda_2(s))\mathrm{d}s \end{aligned} \tag{17}$$

where $L_{m_1,m_2}(s) = \tilde{K}_m(s)\frac{Y_{m_1}(s)\cdot Y_{m_2}(s)}{Y_{m_1}(s)+Y_{m_2}(s)}$. In the following we consider weight functions of the type $\tilde{K}_m(s) = (\hat{R}_m(s-))^\rho(1 - \hat{R}_m(s-))^\delta$ where $0 \le \rho, \delta \le 1$. Here and in the following the subscript $_m$ denotes the obvious estimators or quantities in the pooled sample. It follows from (17) that $\tilde{Z}_{m_1,m_2}$ is a martingale under the hypothesis $H_0 : \lambda_1 = \lambda_2$.

Theorem 6 *Let W be a zero mean Gaussian process with independent increments and variance function*

$$v^2(t) = \mathrm{Var}[W(t)] = \int_0^t \frac{E[Y_1(s)]E[Y_2(s)]}{E[Y_1(s) + Y_2(s)]}(R(s-))^{2\rho}(1 - R(s-))^{2\delta}\lambda_1(s)\mathrm{d}s.$$

Here R denotes the reliability function under the hypothesis $\lambda_1 = \lambda_2$. Then under the Assumption 3 and under $H_0 : \lambda_1 = \lambda_2$ we have

(i) $\sqrt{\frac{m_1+m_2}{m_1 m_2}}\tilde{Z}_{m_1,m_2}(t) \Rightarrow W(t)$ *on* $D[0, T]$ *as* $m_1, m_2 \to \infty$, *and*
(ii) *for all* $\epsilon > 0$ $P(\sup_{0\le t\le T} |\hat{v}_m^2(t) - v^2(t)| > \epsilon) \to 0$ *as* $m_1, m_2 \to \infty$ *where*

$$\hat{v}_m^2(t) = \frac{m_1 + m_2}{m_1 m_2}\int_0^t \frac{(\tilde{K}_m(s))^2 Y_{m_1} Y_{m_2}}{Y_m(s)}\mathrm{d}\hat{\Lambda}_m(s)$$

for every $\tilde{K}_m(t) = R_m^\rho(t-)(1 - R_m(t-))^\delta$ *where* $0 \le \delta, \rho \le 1$.

Proof (i) In order to show the result we apply Rebolledo's theorem to the martingale $\sqrt{\frac{m_1+m_2}{m_1 m_2}}\tilde{Z}_{m_1 m_2}$. It is easily seen from (17) that

$$\left\langle \sqrt{\frac{m_1 + m_2}{m_1 m_2}}\tilde{Z}_{m_1 m_2}(t)\right\rangle = \int_0^t (\tilde{K}_m(s))^2 \frac{Y_{m_1}(s)Y_{m_2}(s)}{m_1 m_2}\frac{1}{\frac{Y_m}{m_1+m_2}}\lambda_1(s)\mathrm{d}s.$$

From Lemma 3 and Theorem 2 we conclude that $\left\langle \sqrt{\frac{m_1+m_2}{m_1 m_2}} \tilde{Z}_{m_1 m_2}(t) \right\rangle \xrightarrow{P} v^2(t)$. The only jumps in the martingale $\sqrt{\frac{m_1+m_2}{m_1 m_2}} \tilde{Z}_{m_1 m_2}$ are due to $\tilde{K}_m$. Thus,

$$\left\langle \tilde{Z}_{m_1,m_2}(t) I_{\{\tilde{K}_m(s)\sqrt{\frac{m_1+m_2}{m_1 m_2}}>\epsilon\}} \right\rangle = \int_0^t (\tilde{K}_m(s))^2 \frac{Y_{m_1}(s) Y_{m_2}(s)}{m_1 m_2} \frac{1}{\frac{Y_m}{m_1+m_2}} \times \lambda_1(s) I_{\left\{\tilde{K}_m(s)\sqrt{\frac{m_1+m_2}{m_1 m_2}}>\epsilon\right\}} \mathrm{d}s.$$

As $\tilde{K}_m(s)$ is bounded from above by 1, the result follows now by Rebolledo's theorem.
(ii) Since this can be shown in the same way as in Theorem 5 (ii) the proof is omitted.

4.2 Nonparametric hypothesis testing for unknown parameter vector α and unknown distribution function F

The one-sample tests in the case where α and F are unknown will be based on the stochastic processes

$$U_m(t) = \int_0^t H_m(s) \mathrm{d}\hat{\Lambda}_m(s, \hat{\alpha}) - \int_0^t H_m(s) \lambda_0(s) \mathrm{d}s \tag{18}$$

where $H_m(s) = \hat{\boldsymbol{\alpha}} \check{\mathbf{Y}}'_m(s) R_0^{\rho}(s)$, $0 \le \rho \le 1$. Here R_0 denotes again the reliability function under the hypothesis $\lambda = \lambda_0$.

Remark 7 In the case where only F is unknown the test statistics (18) coincide with the test statistics (15) if we replace $\hat{\alpha}$ by α.

Theorem 7 *Let W be a zero mean Gaussian process with independent increments and variance function*

$$\bar{v}^1(t) = \mathrm{Var}[W(t)] = \int_0^t E[Y(s)] R_0^{2\rho}(s) \lambda_0(s) \mathrm{d}s + \left(\int_0^t e(s) R_0^{\rho}(s) \lambda_0(s) \mathrm{d}s \right) \mathbf{D}(\boldsymbol{\alpha}) \Psi^{-1}(T, \boldsymbol{\alpha}) \mathbf{D}(\boldsymbol{\alpha}) \left(\int_0^t e(s) R_0^{\rho}(s) \lambda_0(s) \mathrm{d}s \right)'.$$

Then under the Assumption 3 and under $H_0 : \lambda = \lambda_0$ we have

$$\frac{1}{\sqrt{m}} U_m \Rightarrow W \text{ on } D[0, T] \text{ as } m \to \infty.$$

Proof We have

$$\begin{aligned}\frac{1}{\sqrt{m}}U_m(t) &= \frac{1}{\sqrt{m}}\int_0^t R_0^{\rho}(s)\mathrm{d}N_m(s)\\ &\quad -\frac{1}{\sqrt{m}}\left(\int_0^t(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})\check{\mathbf{Y}}'_m(s)R_0^{\rho}(s)\lambda_0(s)\mathrm{d}s+\int_0^t\boldsymbol{\alpha}\check{\mathbf{Y}}'_m(s)R_0^{\rho}(s)\lambda_0(s)\mathrm{d}s\right)\\ &= \frac{1}{\sqrt{m}}Z_m(t)-\frac{1}{\sqrt{m}}\int_0^t(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})\check{\mathbf{Y}}'_m(s)R_0^{\rho}(s)\lambda_0(s)\mathrm{d}s \qquad (19)\end{aligned}$$

where Z_m is defined as in Subsect. 4.1. Hence, according to Theorem 5

$$\frac{1}{\sqrt{m}}Z_m \Rightarrow W_1 \text{ on } D[0,T] \text{ as } m\to\infty$$

where W_1 denotes the Gaussian process of Theorem 5.

From Kvam and Peña (2005) (cf. Theorem 1) it follows that the second term in (19) can be written as

$$\mathbf{D}(\boldsymbol{\alpha})\left(\frac{1}{m}D_m(T,\boldsymbol{\alpha})\right)^{-1}\frac{1}{\sqrt{m}}G_m(T,\boldsymbol{\alpha})\int_0^t\frac{1}{m}\check{\mathbf{Y}}'_m(s)R_0^{\rho}(s)\lambda_0(s)ds+o_P(1) \qquad (20)$$

where the matrix-valued process $\left(\frac{1}{m}D_m(\cdot,\boldsymbol{\alpha})\right)^{-1}\xrightarrow{P}\Psi^{-1}(\cdot,\boldsymbol{\alpha})$ and $\frac{1}{\sqrt{m}}G_m(t,\alpha)$ is given by

$$\frac{1}{\sqrt{m}}\sum_{i=1}^m\int_0^t\left[\boldsymbol{\delta}_i(s)-\frac{\boldsymbol{\alpha}\star\check{\mathbf{Y}}_m(s)}{\boldsymbol{\alpha}\check{\mathbf{Y}}'_m(s)}\right]\mathrm{d}M_i(s)$$

and converges to W_2 on $D[0,T]$. Here W_2 denotes a Gaussian process with independent increments and variance function $v^2(t)=\Psi(t,\alpha)$. Moreover $\int_0^t\frac{1}{m}\check{\mathbf{Y}}'_m(s)R_0^{\rho}(s)\lambda_0(s)ds\xrightarrow{P}\int_0^t\mathbf{e}(s)R_0^{\rho}(s)\lambda_0(s)ds$. Finally, the covariance process between Z_m and G_m is

$$\begin{aligned}\langle Z_m(t),G_m(t,\alpha)\rangle &= \sum_{i=1}^m\int_0^t\frac{K_m(s)}{Y_m(s)}\left[\boldsymbol{\delta}_i(s)-\frac{\boldsymbol{\alpha}\star\check{\mathbf{Y}}_m(s)}{\boldsymbol{\alpha}\check{\mathbf{Y}}'_m(s)}\right]\lambda_0(s)\boldsymbol{\alpha}\underline{\check{\mathbf{Y}}}'_i(s)\mathrm{d}s\\ &= 0\end{aligned}$$

since $\boldsymbol{\delta}_i(s)\boldsymbol{\alpha}\underline{\check{\mathbf{Y}}}'_i(s)=\boldsymbol{\alpha}\star\underline{\check{\mathbf{Y}}}_i(s)$, $\sum_{i=1}^m\boldsymbol{\alpha}\star\underline{\check{\mathbf{Y}}}_i(s)=\boldsymbol{\alpha}\star\check{\mathbf{Y}}_m(s)$, and $\sum_{i=1}^m\boldsymbol{\alpha}\underline{\check{\mathbf{Y}}}'_i(s)=\boldsymbol{\alpha}\check{\mathbf{Y}}'_m(s)$. This completes the proof.

Our two-sample test statistics will be based on the processes

$$\tilde{U}_{m_1,m_2}(t) = \int_0^t \tilde{H}_m(s) \frac{\hat{\boldsymbol{\alpha}}_1 \check{\mathbf{Y}}'_{m_1}(s) \cdot \hat{\boldsymbol{\alpha}}_2 \check{\mathbf{Y}}'_{m_2}(s)}{\hat{\boldsymbol{\alpha}}_1 \check{\mathbf{Y}}'_{m_1}(s) + \hat{\boldsymbol{\alpha}}_2 \check{\mathbf{Y}}'_{m_2}(s)} \mathrm{d}\hat{\Lambda}_{m_1}(s, \hat{\boldsymbol{\alpha}}_1) - \int_0^t \tilde{H}_m(s) \frac{\hat{\boldsymbol{\alpha}}_1 \check{\mathbf{Y}}'_{m_1}(s) \cdot \hat{\boldsymbol{\alpha}}_2 \check{\mathbf{Y}}'_{m_2}(s)}{\hat{\boldsymbol{\alpha}}_1 \check{\mathbf{Y}}'_{m_1}(s) + \hat{\boldsymbol{\alpha}}_2 \check{\mathbf{Y}}'_{m_2}(s)} \mathrm{d}\hat{\Lambda}_{m_2}(s, \hat{\boldsymbol{\alpha}}_2) \tag{21}$$

where $\tilde{H}_m$ is defined as $R_m^{\rho}(t-, (\hat{\boldsymbol{\alpha}}_1, \hat{\boldsymbol{\alpha}}_2))(1 - R_m(t-, (\hat{\boldsymbol{\alpha}}_1, \hat{\boldsymbol{\alpha}}_2))^{\delta}$ where $0 \leq \delta, \rho \leq 1$. Here and in the following, the subscript m as well as $(\hat{\boldsymbol{\alpha}}_1, \hat{\boldsymbol{\alpha}}_2)$ stands for the estimators in the pooled sample obtained by combining the Nelson–Aalen estimators of the first and the second sequential 1-out-of-n system. The subscripts 1 and 2 will be used to distinguish between the quantities belonging to the first and second system, respectively. In the following we let $m_1 = m_2 = m$.

Remark 8 It is easily seen that if the parameter vectors $\boldsymbol{\alpha}_1 = (\alpha_{11}, \ldots, \alpha_{1n_1})$ and $\boldsymbol{\alpha}_2 = (\alpha_{21}, \ldots, \alpha_{2n_2})$ are known and if $\hat{\boldsymbol{\alpha}}_1$ and $\hat{\boldsymbol{\alpha}}_2$ are replaced by $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$, respectively, then the test statistics (21) coincide with the test statistics (17).

Theorem 8 *Let W be a zero mean Gaussian process with independent increments and variance function*

$$\bar{v}^2(t) = \mathrm{Var}[W(t)] = \int_0^t \frac{E[Y_1(s)]E[Y_2(s)]}{E[Y_1(s) + Y_2(s)]} (R(s-))^{2\rho}(1 - R(s-))^{2\delta} \lambda_1(s)\mathrm{d}s + \frac{1}{2} b(t, (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)) D((\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)) \tilde{\Psi}(T, (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)) \times D((\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)) b'(t, (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2))$$

where

$$\tilde{\Psi}(t, (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)) = \begin{pmatrix} \Psi_1^{-1}(t, \boldsymbol{\alpha}_1) & \mathbf{0} \\ \mathbf{0} & \Psi_2^{-1}(t, \boldsymbol{\alpha}_2) \end{pmatrix}$$

with $\mathbf{0}$ *denoting the zero matrix and*

$$b(t, (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)) = \int_0^t (R(s-))^{\rho}(1 - R(s-))^{\delta}(-\boldsymbol{\alpha}_2 \mathbf{e}'_2(s)\mathbf{e}_1(s), \boldsymbol{\alpha}_1 \mathbf{e}'_1(s)\mathbf{e}_2(s)) \times \frac{1}{(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)(\mathbf{e}_1(s), \mathbf{e}_2(s))'} \lambda_1(s)\mathrm{d}s$$

As before, R denotes the reliability function under the hypothesis $\lambda_1 = \lambda_2$. Then under the Assumption 3 and under $H_0 : \lambda_1 = \lambda_2$ we have

$$\sqrt{\frac{2m}{m^2}} \tilde{U}_{m_1,m_2}(t) \Rightarrow W(t) \text{ on } D[0, T] \text{ as } m \to \infty$$

Proof Notice that

$$\sqrt{\frac{2m}{m^2}}\tilde{U}_{m_1,m_2}(t) = \sqrt{\frac{2m}{m^2}}\int_0^t \tilde{H}_m(s)\frac{\hat{\boldsymbol{\alpha}}_2\check{\mathbf{Y}}'_{m_2}(s)}{\hat{\boldsymbol{\alpha}}_1\check{\mathbf{Y}}'_{m_1}(s)+\hat{\boldsymbol{\alpha}}_2\check{\mathbf{Y}}'_{m_2}(s)}\mathrm{d}N_{m_1}(s) - \sqrt{\frac{2m}{m^2}}\int_0^t \tilde{H}_m(s)\frac{\hat{\boldsymbol{\alpha}}_1\check{\mathbf{Y}}'_{m_1}(s)}{\hat{\boldsymbol{\alpha}}_1\check{\mathbf{Y}}'_{m_1}(s)+\hat{\boldsymbol{\alpha}}_2\check{\mathbf{Y}}'_{m_2}(s)}\mathrm{d}N_{m_2}(s). \quad (22)$$

A first order Taylor expansion of $\frac{\hat{\boldsymbol{\alpha}}_2\check{\mathbf{Y}}'_{m_2}}{\hat{\boldsymbol{\alpha}}_1\check{\mathbf{Y}}'_{m_1}+\hat{\boldsymbol{\alpha}}_2\check{\mathbf{Y}}'_{m_2}}$ and $\frac{\hat{\boldsymbol{\alpha}}_1\check{\mathbf{Y}}'_{m_1}}{\hat{\boldsymbol{\alpha}}_1\check{\mathbf{Y}}'_{m_1}+\hat{\boldsymbol{\alpha}}_2\check{\mathbf{Y}}'_{m_2}}$ around $(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)$ leads that (22) is equal to

$$\begin{aligned}
&\sqrt{\frac{2m}{m^2}}\left(\int_0^t \tilde{H}_m(s)\frac{Y_{m_2}(s)}{Y_{m_1}(s)+Y_{m_2}(s)}\mathrm{d}N_{m_1}(s) - \int_0^t \tilde{H}_m(s)\frac{Y_{m_1}(s)}{Y_{m_1}(s)+Y_{m_2}(s)}\mathrm{d}N_{m_2}(s)\right)\\
&+\sqrt{\frac{2m}{m^2}}\int_0^t \tilde{H}_m(s)((\hat{\boldsymbol{\alpha}}_1,\hat{\boldsymbol{\alpha}}_2)-(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2))(-\boldsymbol{\zeta}_2\check{\mathbf{Y}}'_{m_2}(s)\check{\mathbf{Y}}_{m_1}(s),\boldsymbol{\zeta}_1\check{\mathbf{Y}}'_{m_1}(s)\check{\mathbf{Y}}_{m_2}(s))'\\
&\times\frac{1}{(\boldsymbol{\zeta}_1\check{\mathbf{Y}}'_{m_1}(s)+\boldsymbol{\zeta}_2\check{\mathbf{Y}}'_{m_2}(s))^2}\mathrm{d}N_{m_1}(s)\\
&-\sqrt{\frac{2m}{m^2}}\int_0^t \tilde{H}_m(s)((\hat{\boldsymbol{\alpha}}_1,\hat{\boldsymbol{\alpha}}_2)-(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2))(\boldsymbol{\zeta}_2\check{\mathbf{Y}}'_{m_2}(s)\check{\mathbf{Y}}_{m_1}(s),-\boldsymbol{\zeta}_1\check{\mathbf{Y}}'_{m_1}(s)\check{\mathbf{Y}}_{m_2}(s))'\\
&\times\frac{1}{(\boldsymbol{\zeta}_1\check{\mathbf{Y}}'_{m_1}(s)+\boldsymbol{\zeta}_2\check{\mathbf{Y}}'_{m_2}(s))^2}\mathrm{d}N_{m_2}(s) \qquad (23)
\end{aligned}$$

where $(\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)=(1,\zeta_{12},\ldots,\zeta_{1n_1},1,\zeta_{22},\ldots,\zeta_{2n_2})$ lies in the line segment connecting $(\hat{\boldsymbol{\alpha}}_1,\hat{\boldsymbol{\alpha}}_2)$ and $(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)$.

The first line in (23) equals (17) except that $\tilde{K}_m$ is replaced by $\tilde{H}_m$. Hence, we conclude as in Theorem 6 that under the hypothesis

$$\begin{aligned}
&\sqrt{\frac{2m}{m^2}}\left(\int_0^t \tilde{H}_m(s)\frac{Y_{m_2}(s)}{Y_{m_1}(s)+Y_{m_2}(s)}\mathrm{d}N_{m_1}(s)\right.\\
&\qquad\left.-\int_0^t \tilde{H}_m(s)\frac{Y_{m_1}(s)}{Y_{m_1}(s)+Y_{m_2}(s)}\mathrm{d}N_{m_2}(s)\right)\\
&\Rightarrow W_1(t) \text{ on } D[0,T],
\end{aligned}$$

where W_1 is the Gaussian process from Theorem 6. Noticing that $(\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)=(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)+o_p(1)$ the second and the third line in (23) equal asymptotically

$$\sqrt{\frac{2m}{m^2}}\int_0^t \tilde{H}_m(s)((\hat{\boldsymbol{\alpha}}_1,\hat{\boldsymbol{\alpha}}_2)-(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2))(-\boldsymbol{\zeta}_2\check{\mathbf{Y}}'_{m_2}(s)\check{\mathbf{Y}}_{m_1}(s),\boldsymbol{\zeta}_1\check{\mathbf{Y}}'_{m_1}(s)\check{\mathbf{Y}}_{m_2}(s))'$$
$$\times\frac{1}{\boldsymbol{\zeta}_1\check{\mathbf{Y}}'_{m_1}(s)+\boldsymbol{\zeta}_2\check{\mathbf{Y}}'_{m_2}(s)}\lambda_1(s)$$
$$=\sqrt{\frac{m}{2}}((\hat{\boldsymbol{\alpha}}_1,\hat{\boldsymbol{\alpha}}_2)-(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2))\int_0^t \tilde{H}_m(s)\frac{1}{m^2}(-\boldsymbol{\alpha}_2\check{\mathbf{Y}}'_{m_2}(s)\check{\mathbf{Y}}_{m_1}(s),\boldsymbol{\alpha}_1\check{\mathbf{Y}}'_{m_1}(s)\check{\mathbf{Y}}_{m_2}(s))$$
$$\times\frac{1}{\frac{\boldsymbol{\alpha}_1\check{\mathbf{Y}}'_{m_1}(s)+\boldsymbol{\alpha}_2\check{\mathbf{Y}}'_{m_2}(s)}{2m}}\lambda_1(s)+o_p(1)$$
$$=\sqrt{\frac{m}{2}}((\hat{\boldsymbol{\alpha}}_1,\hat{\boldsymbol{\alpha}}_2)-(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2))\int_0^t \tilde{H}_m(s)\frac{1}{m^2}(-\boldsymbol{\alpha}_2\check{\mathbf{Y}}'_{m_2}(s)\check{\mathbf{Y}}_{m_1}(s),\boldsymbol{\alpha}_1\check{\mathbf{Y}}'_{m_1}(s)\check{\mathbf{Y}}_{m_2}(s))$$
$$\times\frac{1}{\frac{Y_{m_1}(s)+Y_{m_2}(s)}{2m}}\lambda_1(s)+o_p(1).$$

As in the proof of Theorem 7 it follows that

$$\sqrt{m}((\hat{\boldsymbol{\alpha}}_1,\hat{\boldsymbol{\alpha}}_2)-(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2))=\Bigg(\mathbf{D}(\boldsymbol{\alpha}_1)\left(\frac{1}{m}D_{m_1}(T,\boldsymbol{\alpha}_1)\right)^{-1}\frac{1}{\sqrt{m}}G_{m_1}(T,\boldsymbol{\alpha}_1),$$
$$\mathbf{D}(\boldsymbol{\alpha}_2)\left(\frac{1}{m}D_{m_2}(T,\boldsymbol{\alpha}_2)\right)^{-1}\frac{1}{\sqrt{m}}G_{m_2}(T,\boldsymbol{\alpha}_2)\Bigg)+o_p(1)$$

where $(\frac{1}{\sqrt{m}}G_{m_1}(\cdot,\boldsymbol{\alpha}_1),\frac{1}{\sqrt{m}}G_{m_2}(\cdot,\boldsymbol{\alpha}_2))$ converges to a Gaussian process with independent increments and covariance function

$$\tilde{\Psi}(t,(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)).$$

The structure of the covariance function follows from Theorem 7 and the independence of $\hat{\boldsymbol{\alpha}}_1$ and $\hat{\boldsymbol{\alpha}}_2$.
Moreover, we have

$$\int_0^t \tilde{H}_m(s)\frac{1}{m^2}(-\boldsymbol{\alpha}_2\check{\mathbf{Y}}'_{m_2}(s)\check{\mathbf{Y}}_{m_1}(s),\boldsymbol{\alpha}_1\check{\mathbf{Y}}'_{m_1}(s)\check{\mathbf{Y}}_{m_2}(s))'$$
$$\times\frac{1}{\frac{Y_{m_1}(s)+Y_{m_2}(s)}{2m}}\lambda_1(s)$$
$$\xrightarrow{P}\int_0^t (R(s-))^{\rho}(1-R(s-))^{\delta}(-\boldsymbol{\alpha}_2\mathbf{e}'_2(s)\mathbf{e}_1(s),\boldsymbol{\alpha}_1\mathbf{e}'_1(s)\mathbf{e}_2(s))$$
$$\times\frac{1}{(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)(\mathbf{e}_1(s),\mathbf{e}_2(s))'}\lambda_1(s)\mathrm{d}s.$$

Using the independence of the processes $N_{m_1}, Y_{m_1}, \check{\mathbf{Y}}_{m_1}$ and $N_{m_2}, Y_{m_2}, \check{\mathbf{Y}}_{m_2}$ it follows as in Theorem 7 that the covariance process between

$$\begin{aligned}&\int_0^t \tilde{H}_m(s)\frac{Y_{m_2}(s)}{Y_{m_1}(s)+Y_{m_2}(s)}\mathrm{d}N_{m_1}(s) - \int_0^t \tilde{H}_m(s)\frac{Y_{m_1}(s)}{Y_{m_1}(s)+Y_{m_2}(s)}\mathrm{d}N_{m_2}(s)\\ &\quad= \int_0^t \tilde{H}_m(s)\frac{Y_{m_2}(s)}{Y_{m_1}(s)+Y_{m_2}(s)}\mathrm{d}M_{m_1}(s) - \int_0^t \tilde{H}_m(s)\frac{Y_{m_1}(s)}{Y_{m_1}(s)+Y_{m_2}(s)}\mathrm{d}M_{m_2}(s)\end{aligned}$$

and

$$\left(\frac{1}{\sqrt{m}}G_{m_1}(\cdot,\boldsymbol{\alpha}_1), \frac{1}{\sqrt{m}}G_{m_2}(\cdot,\boldsymbol{\alpha}_2)\right)$$

is zero.

5 Simulation study

In this section, we present three simulation studies for the Nelson–Aalen estimator if the true parameter vector $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ is *assumed* to be known. Note that the Nelson–Aalen estimator (6) depends on the failure times and the assumed parameter vector $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n)$. To distinguish between the assumed and the true parameter vector they are denoted by $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n)$ and $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$, respectively. The first simulation study shows the behavior of the Nelson–Aalen estimator if the true parameter vector $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ is correctly specified. The second and the third simulation study concern the case where the true parameter vector $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ is misspecified.
The simulations were carried out using R. Notice that (9) provides an easy formula for the simulation of n sequential order statistics with parameters $\alpha_1,\ldots,\alpha_n$ and distribution function F. First, we simulate n independent beta random variables where $B_i \sim Beta((n-i+1)\alpha_i, 1)$, $1 \le i \le n$. Then we calculate the n products B_1,$B_1B_2,\ldots,\prod_{i=1}^n B_i$. The i-th sequential order statistics X_i^*, $1 \le i \le n$, is then obtained by setting $X_i^* = F^{-1}(1-\prod_{j=1}^i B_j)$. Finally, we use the simulated failure times to compute the Nelson–Aalen estimator according to (6).
By $W(a,b)$ we denote a Weibull distribution with density

$$f(x) = \frac{a}{b}\left(\frac{x}{b}\right)^{a-1} e^{-(\frac{x}{b})^a}, \quad 0 < x < \infty$$

where $a, b > 0$.

Figure 1 shows the Nelson–Aalen estimator for the cumulative hazard function of a $W(2,1.5)$ distribution. The calculation of the Nelson–Aalen estimator was based on the failure times of 20 simulations of a sequential 1-out-of-10 system with parameter vector $\boldsymbol{\alpha} = (1, 1.65, 1.90, 2.20, 2.60, 3.10, 3.55, 4.10, 4.40, 5.25)$, distribution function $F = W(2,1.5)$, and assumed parameter vector $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$. It can be seen from Fig. 1 that the Nelson–Aalen estimator seems to perform well for correctly specified $\boldsymbol{\alpha}$.

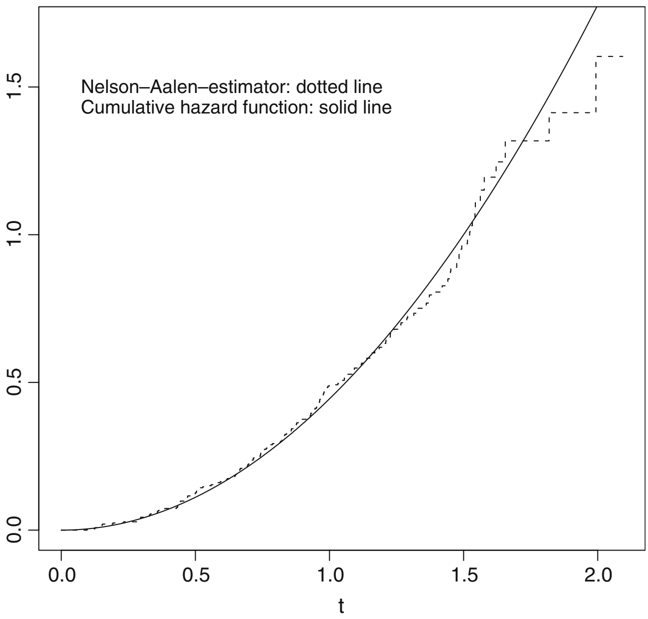

Fig. 1 Nelson–Aalen estimator of the cumulative hazard function of a $W(2, 1.5)$ distribution based on 20 observations of a 1-out-of-10 system with correctly specified α

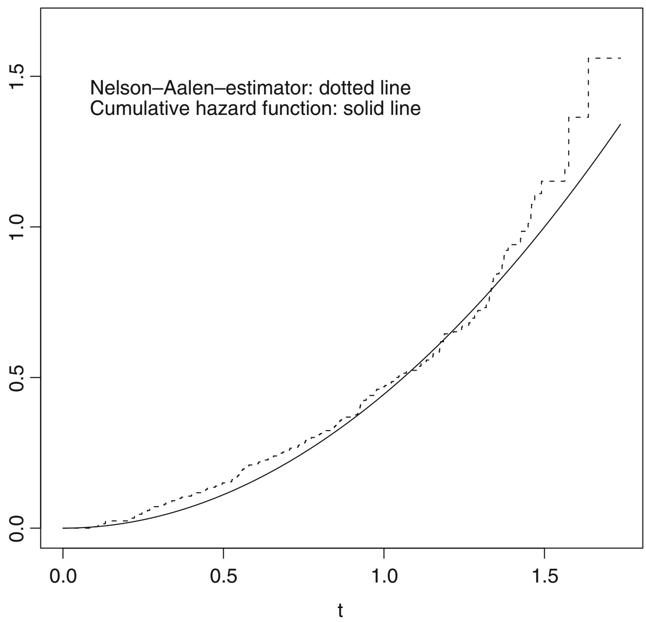

Fig. 2 Nelson–Aalen estimator of the cumulative hazard function of a $W(2, 1.5)$ distribution based on 20 observations of a 1-out-of-10 system with slightly misspecified α

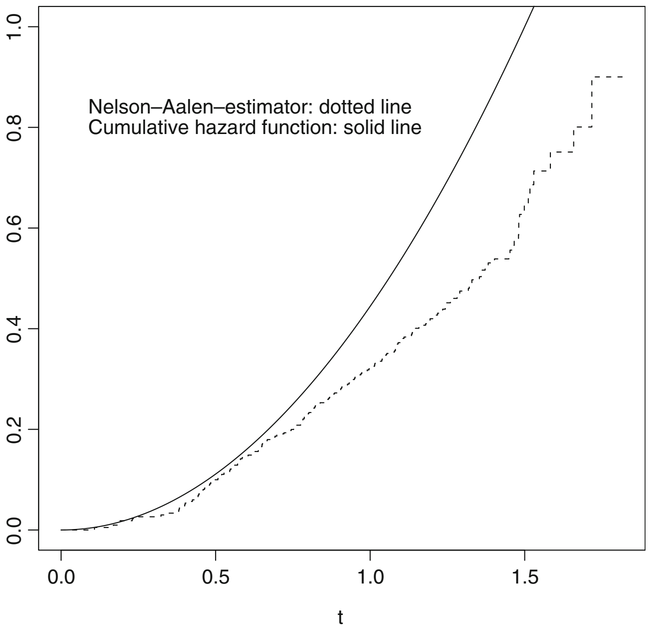

Fig. 3 Nelson–Aalen estimator of the cumulative hazard function of a $W(2, 1.5)$ distribution based on 20 observations of a 1-out-of-10 system with strongly misspecified α

For the calculation of the Nelson–Aalen estimator in Figures 2 and 3 we used the failure times of 20 simulations of a sequential 1-out-of-10 system based on $\boldsymbol{\alpha} = (1, 1.65, 1.90, 2.20, 2.60, 3.10, 3.55, 4.10, 4.40, 5.25)$, $F = W(2, 1.5)$ and assumed parameter vector $\tilde{\boldsymbol{\alpha}} = (1, 1.55, 1.95, 2.30, 2.75, 3.25, 3.60, 4.25, 4.50, 5.10)$ and $\tilde{\boldsymbol{\alpha}} = (1, 2.45, 2.60, 3.30, 4.65, 5.30, 6.60, 7.95, 8.25, 10.05)$, respectively. Figure 2 indicates that a slight misspecification of $\boldsymbol{\alpha}$ has only a slight impact on the Nelson–Aalen estimator, and Fig. 3 indicates that a strong misspecification has a strong impact on the behavior of the Nelson–Aalen estimator.

Acknowledgments The author wish to thank the referee for his careful reading, valuable comments and criticisms.

References

Andersen, P. K., Borgan, O., Gill, R. D., Keidling, N. (1993). *Statistical models based on counting processes*. New York: Springer

Bordes, L. (2004). Nonparametric estimation under progressive censoring. *Journal of Statistical Planning and Inference*, *119*, 171–189

Cramer, E., Kamps, U. (1996). Sequential order statistics and k-out-of-n systems with sequentially adjusted failure rates. *Annals of the Institute of Statistical Mathematics*, *48*(3), 535–549

Cramer, E., Kamps, U. (2001a). Estimation with sequential order statistics from exponential distributions. *Annals of the Institute of Statistical Mathematics*, *53*(2), 307–324

Cramer, E., Kamps, U. (2001b). Sequential k-out-of-n systems. In N. Balakrishnan, & C.R. Rao (Eds.), *Handbook of Statistics,* Vol. 20 (pp. 301–372). Amsterdam: Elsevier

Cramer, E., Kamps, U. (2003). Marginal distributions of sequential and generalized order statistics. *Metrika*, *58*(3), 293–310

Dorado, C., Hollander, M., Sethuraman, J. (1997). Nonparametric estimation for a general repair model. *The Annals of Statistics*, *25*(3), 1140–1160

Doss, H., Gill, R. D. (1992). An elementary approach to weak convergence for quantile processes, with applications to censored survival data. *Journal of the American Statistical Association*, *100*, 869–877

Gill, R. D., Johansen, S. (1990). A survey of product-integration with a view toward application in survival analysis. *The Annals of Statistics*, *18*(4), 1501–1555

Guilbaud, O. (2004). Exact non-parametric confidence, prediction and tolerance intervals with progressive type-II censoring. *Scandinavian Journal of Statistics*, *31*(2), 265–281

Harrington, D. P., Fleming, T. R. (1982). A class of rank test procedures for censored survival data. *Biometrika*, *69*, 133–143

Hollander, M., Peña, E. A., (1995). Dynamic reliability models with conditional proportional hazards. *Lifetime Data Analysis*, *1* (4), 377–401

Kamps, U. (1995). *A Concept of Generalized Order Statistics*. Stuttgart: Teubner

Karr, A. F. (1991). *Point Processes and their Statistical Inference*. New York: Marcel Dekker

Kvam, P. H., Peña, E. A. (2005). Estimating load-sharing properties in a dynamic reliability system. *Journal of the American Statistical Association*, *100*, 262–272