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The semi-Sibuya distribution

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Abstract In this note we propose several characterizations of the Sibuya distribution. A related distribution we call the semi-Sibuya distribution is introduced and studied.

Keywords The Sibuya distribution · Probability generating functions · Discrete semistability

1 Introduction

Sibuya (1979) introduced the discrete distribution with probability generating distribution (pgf)

$$P(z) = 1 - (1 - z)^{\gamma}, \tag{1}$$

for some parameter $\gamma \in (0, 1]$. This distribution is known as the Sibuya distribution with exponent γ. A random variable (rv) with a Sibuya distribution can be represented as one plus a variable with a special generalized hypergeometric distribution (see Sibuya 1979). Devroye (1993) offered a similar distributional representation of the Sibuya distribution by way of a Poisson mixture. Christoph and Schreiber (2000) introduced and studied a more general distribution called the scaled Sibuya distribution with exponent $\gamma \in (0, 1]$ and scale parameter $\lambda \in (0, 1]$, and referred to as the Sibuya(γ, λ) distribution.

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Its pgf is

$$P(z) = 1 - \lambda(1 - z)^{\gamma}. \tag{2}$$

The scaled Sibuya(γ, λ) distribution results from a mixture of a Sibuya$(\gamma, 1)$ distribution and a distribution concentrated at zero; with weighing factors λ and $1 - \lambda$.

In this note, we show that the scaled Sibuya distribution arises as the solution of some functional equations. We also introduce a related discrete distribution we call the semi-Sibuya distribution. We establish several properties of the semi-Sibuya distribution, including characterizations in terms of its pgf. Results that relate the class of Sibuya distributions with the classes of semi-Sibuya distributions are also given.

We recall the definition of the binomial thinning operation $\odot$

$$\alpha \odot X = \sum_{i=1}^{X} X_i, \tag{3}$$

where X is a $\mathbf{Z}_+$-valued rv, $\alpha \in (0, 1)$, and $\{X_i\}$ is a sequence of independent identically distributed (iid) Bernoulli(α) rv's independent of X (see Steutel and van Harn, 2004).

2 The semi-Sibuya distribution

We show that the scaled Sibuya distribution arises as a solution of some functional equations. This results in several new characterizations of the distribution.

Theorem 1 *Let X be a $\mathbf{Z}_+$-valued rv with pgf $P(z)$ and with $P(0) < 1$. The following assertions are equivalent.*

(i) For every integer $n \geq 1$, there exists $c_n \in (0, 1]$ (with $c_1 = 1$) such that

$$P(1 - c_n + c_n z) = 1 - \frac{1}{n} + \frac{1}{n} P(z) \quad (0 \leq z \leq 1). \tag{4}$$

(ii) There exists $\gamma > 0$ such that for every real number $x \geq 1$,

$$P(1 - x^{-1/\gamma} + x^{-1/\gamma} z) = 1 - \frac{1}{x} + \frac{1}{x} P(z) \quad (0 \leq z \leq 1). \tag{5}$$

(iii) There exists $\gamma > 0$ such that for every $\alpha \in (0, 1)$,

$$P(1 - \alpha + \alpha z) = 1 - \alpha^{\gamma} + \alpha^{\gamma} P(z) \quad (0 \leq z \leq 1). \tag{6}$$

(iv) There exists $\gamma > 0$ such that for every $n \geq 1$,

$$n^{-1/\gamma} \odot X \stackrel{d}{=} I_n X_n \tag{7}$$

where I_n is Bernoulli$(1/n)$, $X_n \stackrel{d}{=} X$ and I_n and X_n are independent

(v) X has a scaled Sibuya(γ, λ) distribution for some $\lambda, \gamma \in (0, 1]$.

Proof (i)⇒(ii): we use an argument due to Steutel and van Harn (2004) in their proof of a similar result on discrete stability (Theorem V. 5.1, p 263). By (4), $(c_n, n \geq 1)$ is nonincreasing. Let $c(x) = c_x$ for x integer, $x \geq 1$. Moreover, again by (4), we have for all integers $x, y \geq 1$,

$$c(xy) = c(x)c(y). \tag{8}$$

The function $c(x)$, as well as Eqs (4) and (8), can be shown to extend to all rationals $x \geq 1$ by letting $c(x) = c_n/c_k$ for $x = n/k, n \geq k$. A limiting argument leads in turn to the extension of $c(x)$, (4), and (8) to all reals $x \geq 1$ (see the reference above for details). By (8) and the fact that $c(x) \leq 1$, we have $c(x) = x^{-1/\gamma}$ for some $\gamma > 0$. Thus (5) holds. (ii)⇒(iii) is immediate by letting $x = \alpha^{-\gamma}$ in (5). Assuming (iii) and setting $z = 0$ and then $\alpha = 1 - z$ for $z \in (0, 1)$ in Eq. (6) yields $P(z) = 1 - \lambda(1 - z)^{\gamma}$ for all $z \in [0, 1]$, where $\lambda = 1 - P(0)$. Since $P(z)$ is a pgf, $P''(0) \geq 0$ implies $\gamma \in (0, 1]$ and hence (v̇) holds. The representation (7) follows from (2) and (3) for every $n \geq 1$, and thus (v̇)⇒(iv). Finally, if (iv) holds, then (4) results from (7) by letting $c_n = n^{-1/\gamma}$. □

It is of interest to study the solution of the functional Eq. (6) if the latter is restricted to hold for a single value of $\alpha \in (0, 1)$ (or, equivalently, when (5) is restricted to hold for a single value of $x > 1$). This leads to the following definition.

Definition 1 *A nondegenerate distribution on* $\mathbf{Z}_+$ *is said to be semi-Sibuya if its pgf* $P(z)$ *satisfies the functional Eq. (6) for some* $\gamma > 0$ *and* $\alpha \in (0, 1)$. *We will refer to* γ *(respectively* α*) as the exponent (respectively, the order) of the distribution.*

It follows easily from the definition above that a semi-Sibuya distribution with exponent $\gamma = 1$ is necessarily a Bernoulli distribution.

A few additional properties are listed below without proof.

Proposition 1 (i) *If there exists a semi-Sibuya distribution with exponent* $\gamma > 0$ *and order* $\alpha \in (0, 1)$, *then, necessarily,* $0 < \gamma \leq 1$. *In addition, if this distribution has finite mean, then* $\gamma = 1$.

(ii) *If* $(p_n, n \geq 0)$ *is a semi-Sibuya distribution with exponent* $\gamma \in (0, 1)$ *and order* $\alpha \in (0, 1)$, *then* $p_n > 0$ *for every* $n \geq 1$.

(iii) *A* $\mathbf{Z}_+$*-valued rv* X *has a semi-Sibuya distribution with exponent* $\gamma \in (0, 1)$ *and order* $\alpha \in (0, 1)$ *if and only if it satisfies the equation* $\alpha \odot X \stackrel{d}{=} IX$, *where* I *is a Bernoulli*(α^{γ}) *rv independent of* X.

An example of a semi-Sibuya distribution is presented next.

Let $\alpha, \gamma \in (0,1)$ and $\beta \in (0,1]$. We define

$$P(z) = 1 - \beta c \int_0^\infty (1 - e^{-(1-z)x}) x^{-1-\gamma} \left| \sin \frac{2\pi \ln x}{-\ln \alpha} \right| \mathrm{d}x \quad (z \in [0,1]), \tag{9}$$

where $c = \left(\int_0^\infty (1 - e^{-x}) x^{-1-\gamma} \left| \sin \frac{2\pi \ln x}{-\ln \alpha} \right| dx \right)^{-1}$. $P(z)$ is the pgf of $(p_n, n \geq 0)$ given by

$$p_0 = 1 - \beta \quad \text{and} \quad p_n = \frac{\beta c}{n!} \int_0^\infty x^{n-1-\gamma} e^{-x} \left| \sin \frac{2\pi \ln x}{-\ln \alpha} \right| \mathrm{d}x \quad (n \geq 1).$$

A simple change of variable argument leads to

$$P(1-\alpha+\alpha z) = 1 - \alpha^\gamma \beta c \int_0^\infty (1 - e^{-(1-z)x}) x^{-1-\gamma} \left| \sin \frac{2\pi \ln x}{-\ln \alpha} \right| \mathrm{d}x = 1 - \alpha^\gamma + \alpha^\gamma P(z).$$

Therefore, $(p_n, n \geq 0)$ is semi-Sibuya with exponent γ and order α.

Note that example (9) can be extended by replacing $\left| \sin \frac{2\pi \ln x}{-\ln \alpha} \right|$ with $\psi(\ln x)$ where $\psi(x)$ is a continuous bounded nonnegative and periodic function (with period $-\ln \alpha$).

Semi-Sibuya distributions can be characterized via their pgf's, as the next result shows.

Theorem 2 *A distribution on $\mathbf{Z}_+$ is semi-Sibuya with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$ if and only if its pgf $P(z)$ admits the representation*

$$P(z) = 1 - (1-z)^\gamma h(z) \quad (0 \leq z < 1), \tag{10}$$

where $h(\cdot)$ satisfies $h(1-\alpha+\alpha z) = h(z)$ for any $z \in [0,1)$, or, equivalently,

$$P(z) = 1 - (1-z)^\gamma g(|\ln(1-z)|) \quad (0 \leq z < 1), \tag{11}$$

where $g(\cdot)$, defined over $[0,\infty)$, is a periodic function with period $-\ln \alpha$.

Proof Clearly, if (10) holds for some $\gamma \in (0,1]$ and $\alpha \in (0,1)$, then $P(z)$ satisfies (6) for all $z \in [0,1)$, and hence, the distribution is semi-Sibuya. Conversely, if (6) holds for $P(z)$ for some $\gamma \in (0,1]$ and $\alpha \in (0,1)$, then $h(z) = (1-z)^{-\gamma}(1-P(z))$ satisfies

$$\begin{aligned} h(1-\alpha+\alpha z) &= (\alpha(1-z))^{-\gamma}(1 - P(1-\alpha+\alpha z)) \\ &= \alpha^{-\gamma}(1-z)^{-\gamma}\alpha^\gamma(1-P(z)) = h(z) \end{aligned}$$

for all for $z \in [0,1)$, which implies (10). We conclude by showing that (10) and (11) are equivalent. If (10) holds, define $g(\tau) = h(1-e^{-\tau})$ for $\tau \geq 0$. Then $g(|\ln(1-z)|) = h(z)$ for any $z \in [0,1)$. Moreover, $g(\tau - \ln \alpha) = h(1-\alpha e^{-\tau}) =$

$h(1-\alpha+\alpha(1-e^{-\tau})) = h(1-e^{-\tau}) = g(\tau)$, which implies that $g(\cdot)$ is periodic with period $-\ln\alpha$ and thus (11) is proven. If (11) holds, define $h(z) = g(|\ln(1-z)|)$ for $z \in [0,1)$. Then $h(1-\alpha+\alpha z) = g(-\ln\alpha + |\ln(1-z)|) = g(|\ln(1-z)|) = h(z)$, implying (10). □

Simple calculations show that the example of a semi-Sibuya distribution with pgf (9) admits the representation (10) with

$$h(z) = \beta c \int_0^\infty (1-e^{-x})x^{-1-\gamma}\left|\sin\frac{2\pi \ln(x/(1-z))}{-\ln\alpha}\right| \mathrm{d}x \quad (z \in [0,1)),$$

and the representation (11) with

$$g(\tau) = \beta c \int_0^\infty (1-e^{-x})x^{-1-\gamma}\left|\sin\frac{2\pi(\tau + \ln x)}{-\ln\alpha}\right| \mathrm{d}x \quad (\tau \geq 0).$$

We recall (Bouzar 2004) that a nondegenerate distribution on $\mathbf{Z}_+$ is said to be discrete semistable with exponent γ, γ necessarily in $(0,1]$, and order $\alpha \in (0,1)$ if its pgf $H(z)$ satisfies for all $|z| \leq 1$, $H(z) \neq 0$ and

$$\ln H(1-\alpha+\alpha z) = \alpha^\gamma \ln H(z). \tag{12}$$

Proposition 2 *A function $P(z)$ over $[0,1]$ is the pgf of a semi-Sibuya distribution with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$ if and only if*

$$P(z) = 1 + c\ln\ H(z) \quad (0 \leq z \leq 1), \tag{13}$$

where $H(z)$ is the pgf of a discrete semistable distribution with exponent γ and order α, and $0 < c \leq -1/\ln H(0)$.

Proof Assume $P(z)$ is the pgf of a semi-Sibuya distribution with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$ and let $H(z) = \exp(P(z)-1)$. $H(z)$ is a pgf (of a compound Poisson distribution) which satisfies (12). Therefore, (13) holds with $c = 1$. Conversely, if $H(z)$ is the pgf of a semistable distribution with exponent γ and order α, then the infinite divisibility of $H(z)$ (Bouzar 2004) and the condition $0 < c \leq -1/\ln H(0)$ imply that $P(z)$ of (13) is a pgf. It follows easily from (12) that (6) holds for $P(z)$. □

We denote by Sb the class of scaled Sibuya distributions and by $SSb(\alpha)$, $\alpha \in (0,1)$, the class of semi-Sibuya distributions of order α. It is easily seen that

$$Sb = \bigcap_{0<\alpha<1} SSb(\alpha). \tag{14}$$

An additional assumption leads to the following stronger result.

Theorem 3 *If* α_1 *and* α_2 *in* $(0,1)$ *are such that* $\ln\alpha_1/\ln\alpha_2$ *is irrational, then*

$$Sb = SSb(\alpha_1)\bigcap SSb(\alpha_2). \tag{15}$$

Proof Let $P(z)$ be the pgf of a semi-Sibuya distribution. Note by (11) that $P(z)$ is defined for all $z \leq 1$. Let A_P the set of all $\alpha \in (0,\infty)$ for which (6) holds for some $\gamma > 0$. Since P is the pgf of a semi-Sibuya distribution, $A_P \cap (0,1) \neq \emptyset$. We prove that A_P is a closed multiplicative subgroup of $(0,\infty)$. Clearly, $1 \in A_P$. If α and α' belong to A_P with respective exponents $\gamma > 0$ and $\gamma' > 0$, then $P(1-\alpha\alpha'+\alpha\alpha' z) = 1-\alpha^{\gamma}\alpha'^{\gamma'}+\alpha^{\gamma}\alpha'^{\gamma'}P(z)$. Therefore, $\alpha\alpha' \in A_P$ with exponent $\ln(\alpha^{\gamma}\alpha'^{\gamma'})/\ln(\alpha\alpha')$. Let $\alpha \in A_P$ with exponent $\gamma > 0$. Solving for $P(z)$ in (6) yields $P(z) = 1-\alpha^{-\gamma}+\alpha^{-\gamma}P(1-\alpha+\alpha z)$, which implies that $P(1-\alpha^{-1}+\alpha^{-1}z) = 1-\alpha^{-\gamma}+\alpha^{-\gamma}P(z)$. Therefore, $\alpha^{-1} \in A_P$ (with exponent γ). To show that A_P is closed in $(0,\infty)$, let $(\alpha_n, n \geq 1)$ in A_P such that $\lim_{n\to\infty}\alpha_n = \alpha$ for some $\alpha > 0$. Let γ_n be the exponent corresponding to α_n through (6) and let $z_0 \in [0,1)$ be such that $P(z_0) < 1$. It follows by (6) applied to (α_n, γ_n) that

$$\gamma = \lim_{n\to\infty}\gamma_n = \frac{1}{\ln\alpha}\ln\frac{1-P(1-\alpha+\alpha z_0)}{1-P(z_0)} > 0$$

and that $\alpha \in A_P$ with exponent γ. Let $a_0 = \sup A_P \cap (0,1)$. Using the same argument as the one in the proof of Theorem 13.11, p. 73, in Sato (1999), it can be shown that if $a_0 = 1$, then $A_P = (0,\infty)$, and if $a_0 < 1$, then $A_P = \{a_0^n : n \in \mathbf{Z}\}$. To establish (15) we only need to show that $SSb(\alpha_1) \cap SSb(\alpha_2) \subset Sb$. Let $P(z)$ be the pgf of a distribution in $SSb(\alpha_1) \cap SSb(\alpha_2)$. Since $\ln\alpha_1/\ln\alpha_2$ is irrational, α_1 and α_2 cannot be written as powers with some common base $a_0 > 0$ and integer exponents. It follows by the first part of the proof that $A_P = (0,\infty)$. Therefore, $P(z)$ satisfies (6) for every $\alpha \in (0,1)$. The conclusion follows by (14). □

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References

Bouzar, N. (2004). Discrete semi-stable distributions. *Annals of the Institute of Statistical Mathematics*, *56*, 497–510.

Christoph, G., Schreiber, K. (2000). Scaled Sibuya distribution and discrete self-decomposability. *Statistics and Probability Letters*, *48*, 181–187.

Devroye, L. (1993) A triptych of discrete distributions related to the stable law. *Statistics and Probability Letters 18*, 349–351.

Sato, K. (1999) *Lévy Processes and infinitely divisible distributions*. Cambridge: Cambridge University Press.

Sibuya, M. (1979). Generalized hypergeometric digamma and trigamma distributions. *Annals of the Institute of Statistical Mathematics*, *31*, 373–390.

Steutel, F. W., van Harn, K. (2004). *Infinite divisibility of probability distributions on the real line*. New York: Marcel Dekker.