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Asymptotic normality of a covariance estimator for nonsynchronously observed diffusion processes

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Abstract We consider the problem of estimating the covariance of two diffusion-type processes when they are observed only at discrete times in a nonsynchronous manner. In our previous work in 2003, we proposed a new estimator which is free of any 'synchronization' processing of the original data and showed that it is consistent for the true covariance of the processes as the observation interval shrinks to zero; Hayashi and Yoshida (*Bernoulli, 11*, 359–379, 2005). This paper is its sequel. Specifically, it establishes *asymptotic normality* of the estimator in a general nonsynchronous sampling scheme.

Keywords Diffusions · Discrete-time observations · High-frequency data · Nonsynchronicity · Quadratic variation · Realized volatility

1 Introduction

1.1 Background

Consider the case when two continuous diffusion processes are observed only at discrete times in a *nonsynchronous* manner. We are interested in estimating the

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covariance of the two processes accurately in such a situation. This type of problem arises typically in *high-frequency finance*. A popular approach for this is to compute

$$V_{\pi(m)} := \sum_{i=1}^{m} \left(P_{t_i}^1 - P_{t_{i-1}}^1\right)\left(P_{t_i}^2 - P_{t_{i-1}}^2\right), \tag{1}$$

which is often called the *realized covariance* (estimator) in the literature; see Andersen et al. (2001), for instance. Here, P^1 and P^2 are continuous semimartingales representing log-prices, $0 = t_0 < t_1 < \cdots < t_m = T$ are grid points for measuring their respective changes with the mesh size $\pi(m) := \max_{1\le i\le m} |t_i - t_{i-1}|$, where T is a given time to evaluate the quantity. The popularity of the estimator comes from its consistency, i.e., as $\pi(m) \to 0$, one has $V_{\pi(m)} \to \langle P^1, P^2\rangle_T$ in probability, not to mention from its ease of implementation. $\langle , \rangle$ denotes the quadratic covariation. For practical convenience it is standard to take equal spacing, i.e., $t_i - t_{i-1} = T/m$ $(=: h)$, $i \ge 1$.

Actual transaction data are recorded at irregular times in a nonsynchronous manner. This fact requires one who adopts (1) to 'synchronize' the original multivariate time series *a priori*; choose a common interval length h first, then impute missing observations by some interpolation scheme such as previous-tick interpolation or linear interpolation (Dacorogna et al. 2001). Inevitably, the value of V_h depends heavily on the choice of h as well as an interpolation method adopted. By and large, most of the existing approaches rely on the 'synchronization' of the original data, hence, suffer 'synchronization' bias (e.g., Hayashi and Yoshida 2005b).[1]

In the preceding work, we proposed a new procedure which is free of 'synchronization' hence of any bias due to it. In the case of diffusion-type processes with independent random observation times, they showed that their estimator is consistent for the underlying (deterministic) covariation as the size of observation intervals goes to zero (in our 2003 paper, now Hayashi and Yoshida 2005b), which is *not* in general possessed by the realized covariance estimator subject to nonsynchronicity of observations.

This paper extends our previous work. Specifically, it demonstrates *asymptotic normality* of the proposed estimator in a general nonsynchronous sampling scheme of multivariate diffusion-type processes, as the observation interval shrinks to zero. Central limit theories for the realized volatility/covariance and related estimators have been discussed in the statistics literature for a long time (e.g., Dacunha-Castelle and Florens-Zmirou 1986); however, nonsynchronicity has rarely been taken into account. At the best of our knowledge, this paper is the first one in the literature to show asymptotic normality in the case of nonsynchronous sampling.

[1] In the univariate case, volatility estimation problems in the presence of measurement error, or *market microstructure* noise, have been actively studied recently (e.g., Zhang et al. 2005). Recognizing that nonsynchronicity is a fundamental, salient feature for the multivariate case yet has been rarely addressed, we focus on it in this paper without the existence of additive microstructure noises taken into account. It is deferred for future research.

1.2 The estimator (review)

Let $T \in (0, \infty)$ be an arbitrary terminal time for observation. On a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, suppose that P^l follows the one-dimensional continuous Itô process

$$\mathrm{d}P_t^l = \mu_t^l \mathrm{d}t + \sigma_t^l \mathrm{d}W_t^l, \quad P_0^l = p_0^l, \quad 0 \le t \le T, \ l = 1, 2, \tag{2}$$

where W^l, $l = 1, 2$, are Wiener processes with $d\left\langle W^1, W^2\right\rangle_t = \rho_t \mathrm{d}t$, $\rho_\cdot \in (-1, 1)$ is an unknown, deterministic and measurable function of t, p_0^l is a constant,[2] $\mu_\cdot^l$ is a progressively measurable (possibly unknown) function, and $\sigma_\cdot^l$ is a deterministic and bounded (possibly unknown), measurable function of t.

Let $\Pi^1 := (I^i)_{i=1,2,\ldots}$ and $\Pi^2 := (J^i)_{i=1,2,\ldots}$ be random intervals reading from left to right, each of which partitions $(0, T]$. Let $T^{1,i} := \inf I^{i+1}$ represent the ith observation time of P^1, and $T^{2,i} := \inf J^{i+1}$ that of P^2. Let n be the index that controls the (random) size of Π^1 and Π^2; see the Poisson sampling case below. The length of an interval I is denoted by $|I|$. We assume (temporarily) that the sampling intervals $\Pi := (\Pi^1, \Pi^2)$ satisfy the following:

Condition (C0): *(i)* (I^i) *and* (J^i) *are independent of* P^1 *and* P^2*; (ii) As* $n \to \infty$*,* $E\left[\max_i \left|I^i\right| \vee \max_j |J^j|\right] = o(1)$.

Remark (ii) is equivalent to either of the conditions: (ii′) $\max_i \left|I^i\right| \vee \max_j |J^j| \to 0$ in probability as $n \to \infty$; (iii) $\sum_i \left|I^i\right|^2 + \sum_j \left|J^j\right|^2 \to 0$ in probability as $n \to \infty$. Moreover, for (ii) to hold it is sufficient that (iv) $P\left[\max_i \left|I^i\right| \vee \max_j |J^j| > n^{-q}\right] = o(1)$ for some $q > 0$.

Remark The independence condition (i), which may be too restrictive in financial applications, is removed in a subsequent paper (Hayashi and Yoshida 2006).

Example (Synchronous sampling scheme): Notice that there is *no* assumption for dependency between (I^i) and (J^i). In particular, any perfectly synchronous sampling scheme (deterministic or stochastic) with $I^i = J^i$, for every i, is covered by the framework so far as $\left(I^i\right)$ satisfies (C0). See Sect. 3.1.

Example (*Poisson* random sampling scheme): Let N^1 and N^2 be Poisson processes with intensity $\lambda^l := np^l$, $p^l \in (0, \infty)$, $n \in \mathbb{N}$, $l = 1, 2$. If $\widetilde{T}^{l,i}$ is the ith arrival time of the lth Poisson process with $\widetilde{T}^{l,0} := 0$, $l = 1, 2$, then we construct $\Pi^1 := (I^i)_{i=1,2,\ldots}$ and $\Pi^2 := (J^i)_{i=1,2,\ldots}$, by setting $I^i := (\widetilde{T}^{1,i-1}, \widetilde{T}^{1,i}] \cap (0, T]$ and $J^i := (\widetilde{T}^{2,i-1}, \widetilde{T}^{2,i}] \cap (0, T]$. In this case, $E\left[N_T^l\right] = \lambda^l T = np^l T$, i.e., the mean partition size of Π^l is proportional to n. This Poisson random sampling scheme is covered so far as (C0)(i) is satisfied (note: (C0)(ii) is implied). See Sect. 3.2.

The parameter of interest is the (deterministic) covariation of P^1 and P^2,

$$\left\langle P^1, P^2\right\rangle_T = \int_0^T \sigma_t^1 \sigma_t^2 \rho_t \mathrm{d}t =: \theta.$$

[2] In Hayashi and Yoshida (2005b), the authors assumed $p_0^l > 0$ (note: there they used the symbol p^l instead of p_0^l). However, the positivity was simply unnecessary.

In finance, θ is the *integrated* covariance (over $[0, T]$) of the logarithmic prices P^1 and P^2 of two securities. It is an essential quantity to be measured for risk management purposes; T is set to 1 (day), for instance.

Previously, the authors proposed an estimator for θ of the following form, which is based only on the observations of P^1 and P^2, and the times they were recorded at.

Definition 1 (Cumulative covariance estimator):

$$U_n := \sum_{i,j} \left(P^1_{T^{1,i}} - P^1_{T^{1,i-1}}\right) \left(P^2_{T^{2,j}} - P^2_{T^{2,j-1}}\right) 1_{\{I^i \cap J^j \neq \emptyset\}}. \tag{3}$$

That is, the product of any pair of increments $(P^1_{T^{1,i}} - P^1_{T^{1,i-1}})$ and $(P^2_{T^{2,j}} - P^2_{T^{2,j-1}})$ will make a contribution to the sum only when the respective observation intervals I^i and J^j are overlapping. Observe that U_n utilizes the information regarding not only process changes but also the observation times–through the indicator functions—at which they were recorded. The fact contrasts with the realized covariance estimator (1), which discards the observation time information through synchronization. Besides, it should be noted that the there is no serious increase in computational load; the number of summation required in (3) is essentially of the same order as that of the realized covariance regardless of its appearance as a double sum in i and j. Specifically, the number equals to the number of grids in Π^1 plus that in Π^2, minus the number of grids that are common in both Π^1 and Π^2 (i.e., synchronous observation time points).

Theorem 1 (Hayashi and Yoshida 2005b) *Suppose* (C0) *holds.*

(1) *If* $\sup_{0\le t\le T} \left|\mu^l_t\right| \in L^4$, *$l = 1, 2$, then $U_n \to \theta$ in L^2 as $n \to \infty$.*

(2) *If* $\sup_{0\le t\le T} \left|\mu^l_t\right| < \infty$ *almost surely, $l = 1, 2$, then U_n is consistent for θ, i.e., $U_n \to \theta$ in probability as $n \to \infty$.*

(C0) alone is insufficient to derive asymptotic normality of the proposed estimator U_n. Indeed, we will overwrite (C0) with a stronger set of conditions (C1)–(C4) in the next section.

2 Asymptotic normality

We are going to demonstrate *asymptotic normality* of our estimator in a general non-synchronous sampling scheme of multivariate diffusion processes, as the observation interval shrinks to zero.

We maintain the same set-up as stated in the previous section. For exploring asymptotic normality of U_n we need to elaborate conditions not only on Π but also on the underlying processes. Let $r_n := \max_{1\le i<\infty} \left|I^i\right| \vee \max_{1\le j<\infty} \left|J^j\right|$, the largest interval size.

Condition (C1): (I^i) *and* (J^i) *are independent of* P^1 *and* P^2.

We define (signed) measures by, for each $I \in \mathcal{B}_{[0,T]}$, where $\mathcal{B}_{[0,T]}$ is the Borel σ-field on $[0, T]$,

$$v(I) := v^0(I) := \int_I \sigma^1_t \sigma^2_t \rho_t \mathrm{d}t; \; v^l(I) := \int_I \left(\sigma^l_t\right)^2 \mathrm{d}t, \quad l = 1, 2.$$

Condition (C2): *There exist a sequence of positive numbers* $(b_n) \subset (0, 1)$ *and some constant* $c \in (0, \infty)$ *such that, as* $n \to \infty$, $b_n \to 0$ *and*

$$b_n^{-1} \left\{ \sum_{i,j} v^1\left(I^i\right) v^2\left(J^j\right) 1_{\{I^i \cap J^j \neq \emptyset\}} + \sum_i v\left(I^i\right)^2 + \sum_j v\left(J^j\right)^2 - \sum_{i,j} v\left(I^i \cap J^j\right)^2 \right\} \xrightarrow{P} c. \tag{4}$$

This condition postulates the (asymptotic) connection between the observation intervals Π and the variance–covariance structure of the given processes, $(v^1(\cdot), v^2(\cdot), v(\cdot))$. The constant c in (4) serves as the asymptotic variance of the (rescaled) proposed estimator. In fact, when there is no drift in the underlying processes, one can show that

$$E^{\Pi}\left[U_n^2\right] = \sum_{i,j} v^1\left(I^i\right) v^2\left(J^j\right) 1_{\{I^i \cap J^j \neq \emptyset\}} + \sum_i v\left(I^i\right)^2 + \sum_j v\left(J^j\right)^2 - \sum_{i,j} v\left(I^i \cap J^j\right)^2 + \theta^2 \tag{5}$$

and $E^{\Pi}[U_n] = \theta$ (see Appendix); hence, the l.h.s. of (4) is exactly equal to $b_n^{-1} Var^{\Pi}[U_n]$ so far as $\mu^l \equiv 0$, $l = 1, 2$. Here, $E^{\Pi}[\cdot]$ and $Var^{\Pi}[\cdot]$ denote respectively the conditional expectation and variance, given the partition Π. In case the drift is non-zero, the condition (C4) below guarantees that, for sufficiently large n, the l.h.s. of (4) well approximates the (rescaled) conditional variance. The rescaling factor b_n^{-1} may be interpreted as the 'average number' of the observation times (i.e., the size of the partitions Π^1 and Π^2), or equivalently, b_n may be the 'average length' of the observation intervals.

The reader may wonder if the condition (C2) is mild enough to cover practical, meaningful cases. A non-trivial, example that satisfies (C2) will be studied later, i.e., the Poisson observation times case in Sect. 3.2, where b_n is (set to) n^{-1} and c is found concretely.

Remark (C2) looks rather complicated to check and apply in practice. An alternative, slightly more stringent but amiable condition is explored in a subsequent paper by the authors, Hayashi and Yoshida (2005a), which accordingly draws a stronger result than Theorem 2.

Next, we allow the random mesh size r_n of Π to tend to zero slowly relative to the (deterministic) b_n, but not too slowly.

Condition (C3): *There exists some* $\alpha \in (0, 1/4)$ *such that*

$$r_n = o_P\left(b_n^{\frac{3}{4}+\alpha}\right).$$

Now, for a continuous stochastic process X, we define, for each $\omega \in \Omega$ and $\Delta > 0$, the *modulus of continuity* on $[0, T]$, by

$$\delta(X(\omega); \Delta) := \sup\{|X_t(\omega) - X_s(\omega)|\,;\, |t-s| \le \Delta, 0 \le s, t \le T\}.$$

The following is a condition stating that the (random) drifts of the underlying processes are sufficiently smooth so that their contribution to U_n in (3) would be asymptotically negligible and that asymptotic normality for the zero drift case would be generalized to the non-zero drift case.

Condition (C4): *For $l = 1, 2$, μ^l is continuous and adapted, such that*

$$\delta(\mu^l; h) = O_P\left(h^k\right) \quad \text{as } h \downarrow 0$$

for some $k \in (\frac{1}{6}, \frac{1}{2})$.

Remark Clearly, (C4) holds for processes with the same Hölder continuity as Brownian sample paths.

An alternative condition, whose appearance is slightly artificial, is the following.

Condition (C4′): *For $l = 1, 2$, μ^l is continuous and adapted, such that*

$$\delta(\mu^l; r_n) = O_P\left(r_n^{\frac{1}{2}} b_n^{-\left(\frac{1}{4}+\alpha'\right)}\right) \tag{6}$$

for some $\alpha' \in (0, \alpha)$, where α is given in (C3).

Remark Suppose (C4) holds with some $k \in (\frac{1}{4}, \frac{1}{2})$, which is slightly stronger than stated. Let $N_n := \#(\Pi_1) \vee \#(\Pi_2)$, where $\#(A)$ counts the number of elements in a given set A. Assume that $E[N_n] \uparrow \infty$ and that $b_n = \kappa_n (E[N_n])^{-1}$ for some positive, bounded sequence (κ_n), i.e., loosely speaking, b_n^{-1} represents (a multiple of) the average interval size (cf. the Poisson sampling example of Sect. 4.2). Then, (C4′) is implied. See Appendix A.2 for details.

Remark The condition

$$\delta(\mu^l; r_n) = O_P\left(r_n^{\beta} b_n^{-\left(\frac{1}{4}+\alpha\right)}\right),$$

for some $\beta \in \left(1/2, \alpha\,(3/4+\alpha)^{-1} + 1/2\right)$, together with (C3), implies (6) of (C4′).

Here is the main result of the paper.

Theorem 2 *Under Conditions* (C1) *through* (C3), *together with either* (C4) *or* (C4′), *as $n \to \infty$,*

$$b_n^{-1/2}(U_n - \theta) \xrightarrow{\mathcal{L}} N(0, c).$$

Remark The case of random volatility: When $\sigma^l_\cdot$, $l = 1, 2$, are independent of W^1 and W^2, then, by conditioning, $\sigma^l_\cdot$ become deterministic, hence the argument in this paper essentially carries over. In particular, asymptotic mixed normality will be obtained. On the other hand, when $\sigma^l_\cdot$, $l = 1, 2$, are depending on W^1 and W^2, e.g., when they have feedback from P^1 and P^2 (cf. Hayashi and Kusuoka 2004), establishing asymptotic (mixed) normality will be a rather challenging task. Nevertheless, it can still be shown; we will write on it in a separate paper.

3 Proof for the main theorem

Preceding the proof, we need to prepare some technical Lemmas. For this, we introduce auxiliary symbols as follows.

Put $K_{ij} := 1_{\{I^i \cap J^j \neq \emptyset\}}$, $i, j \geq 1$, for ease of writing. We define a sequence of positive numbers (a_n) by

$$a_n := b_n^{\frac{1}{2}+2\alpha}$$

for α specified in (C3). That is, (a_n) goes to zero faster than $(b_n^{1/2})$ but slower than (b_n). Note then that

$$\frac{r_n^2}{a_n b_n} = \frac{r_n^2}{b_n^{\frac{3}{2}+2\alpha}} = o_P(1) \quad \text{and} \quad \frac{r_n}{a_n} = \frac{r_n}{b_n^{\frac{1}{2}+2\alpha}} = \frac{r_n}{b_n^{\frac{3}{4}+\alpha}} b_n^{\frac{1}{4}-\alpha} = o_P(1).$$

So, the deterministic (a_n) is chosen so that it is likely to go to zero slower than the random (r_n). To sum up, (C3) implies the following.

Condition (C3′): *As $n \to \infty$, (i) $r_n^2 = o_P(a_n b_n)$, and (ii) $r_n = o_P(a_n)$.*

Let us define, for each $I \in \mathcal{B}_{[0,T]}$,

$$\Delta P^l(I) := \int_0^T 1_I(t)\sigma^l_t \mathrm{d}W^l_t, \quad l = 1, 2.$$

Define a (random) set function $u : \mathcal{B}_{[0,T]} \times \mathcal{B}_{[0,T]} \times \Omega \to \mathbb{R}$ by

$$u(A, B)(\omega) := \Delta P^1(A)(\omega)\Delta P^2(B)(\omega) - v(A \cap B), \quad A, B \in \mathcal{B}_{[0,T]}, \ \omega \in \Omega. \tag{7}$$

(Notice that u is 'bilinear.') Then, the quantity of interest is expressible as

$$\Psi_n := b_n^{-1/2}(U_n - \theta) = b_n^{-1/2} \sum_{i,j} \left(\Delta P^1(I^i)\Delta P^2(J^j) - v(I^i \cap J^j) \right)$$
$$K_{ij} = b_n^{-1/2} \sum_{i,j} u\left(I^i, J^j\right) K_{ij},$$

owing to the fact that $(I^i \cap J^j)$ partitions $(0, T]$ and $\theta = \sum_{i,j} v(I^i \cap J^j)K_{ij}$. Accordingly, because $Var^{\Pi}[\Psi_n] \equiv b_n^{-1} Var^{\Pi}[U_n]$, in the case of zero drift (C2) may be restated as:

Condition (C2′): *There exist a sequence of positive numbers* (b_n) *and some constant* $c \in (0, \infty)$ *such that, as* $n \to \infty$, $b_n \to 0$ *and*

$$Var^{\Pi}[\Psi_n] \xrightarrow{P} c,$$

provided that $\mu^l \equiv 0$, $l = 1, 2$.

In what follows, the first four Lemmas 1 through 4 are prepared to show asymptotic normality in the zero drift case. The last two, 5 and 6, are to deal with the non-zero drift case.

Throughout the paper, for sequences (x_n) and (y_n), $x_n \lesssim y_n$ means that there exists a constant $C \in [0, \infty)$ such that $x_n \leq C y_n$ for large n; $x_n \sim y_n$ means that $x_n \lesssim y_n$ and $y_n \lesssim x_n$ at the same time.

Lemma 1 *Suppose* $\mu^l \equiv 0$, $l = 1, 2$. *For* $A, B \in \mathcal{B}_{[0,T]}$,

$$E\left[u(A, B)^2\right] = v^1(A)v^2(B) + v\,(A \cap B)^2 .$$

Proof Because $\sigma_\cdot^l$ and $\rho_\cdot$ are deterministic, the random vector

$$\left(\int_0^T f^1(t)\sigma_t^1 \mathrm{d}W_t^1, \int_0^T f^2(t)\sigma_t^2 \mathrm{d}W_t^2\right)$$

is jointly normal for any deterministic and bounded, measurable functions f^1 and f^2. Therefore, putting $f^1(t) = 1_A(t)$ and $f^2(t) = 1_B(t)$, one has

$$E\,[u(A, B)] = E\left[\Delta P^1(A)\Delta P^2(B)\right] - v\,(A \cap B) = 0$$

and

$$E\left[u(A, B)^2\right] = E\left[\Delta P^1(A)^2 \Delta P^2(B)^2\right] - v\,(A \cap B)^2 = v^1\,(A)\,v^2\,(B) + v\,(A \cap B)^2 .$$

We have used the fact that, for any jointly normal random variables X_1 and X_2 with the respective mean and variance, 0 and v_k, $k = 1, 2$, and with covariance $v_{1,2}$, $E\left[X_1^2 X_2^2\right] = 2{v_{1,2}}^2 + v_1 v_2$. □

Fix $n \in \mathbb{N}$. Define the intervals of size a_n by $A_n^k := ((k-1)a_n, k a_n]$, $k \in \mathbb{N}$. Put $K(n) := \min\{k \in \mathbb{N}; k a_n \geq T\}$, the smallest number of intervals (A_n^k) needed to cover the whole observation period $(0, T]$. For convenience, we re-interpret the setup on the two-dimensional plane. We furnish the (x, y)-coordinate system in an obvious manner. Once Π is fixed, any pair $\left(I^i, J^j\right)$ of intervals is representable as a rectangle $I^i \times J^j$, the overall aggregation of which amounts to the square $\mathbb{S} :=$

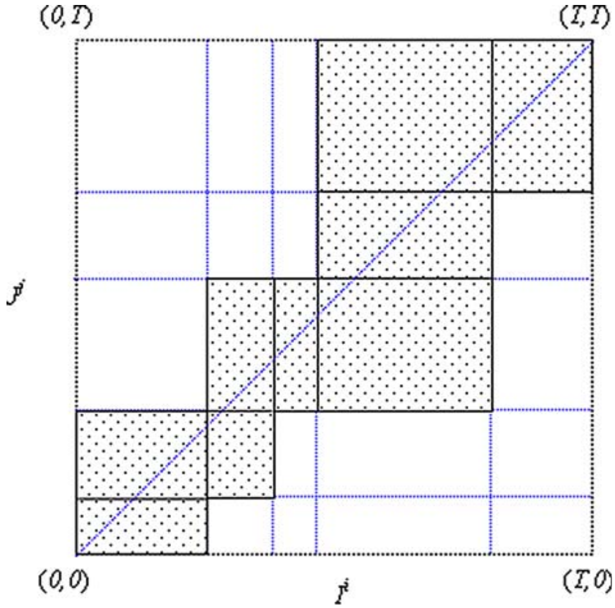

Fig. 1 *Active rectangles* (associated with a realization Π) on the total square $\mathbb{S} := (0, T] \times (0, T]$

$(0, T] \times (0, T]$. A pair $\left(I^i, J^j\right)$ that intersects each other ($K_{ij} = 1$) and hence contributes to the sum U_n in (3) forms a rectangle $I^i \times J^j$ that intersects the 45° line connecting $(0, 0)$ and (T, T). We call *active rectangles* the rectangles with $K_{ij} = 1$ (Fig. 1). Besides, we refer to points $\{(ka_n, ka_n)\,, k = 1, \ldots, K(n) - 1\}$ as *markers*, which are vertices of any of the squares $A_n^k \times A_n^k$ situated in a row on the 45° line. We will refer to such squares $\mathbb{A} := \cup_{k=1}^{K(n)} \left\{A_n^k \times A_n^k\right\} \cap \mathbb{S}$ *diagonal squares,* which will serve collectively as a 'filter,' being utilized on constructing an approximation to Ψ_n as explained in the following paragraph (Fig. 2). Notice that equipping $\mathbb{A}$ on the two-dimensional plane corresponds to setting the regular intervals with size a_n on the one-dimensional time axis. The remaining area, $\mathbb{S} \setminus \mathbb{A}$, will be referred to as *the residual region*.

For the proof of the theorem, we propose to approximate Ψ_n by

$$\Phi_n := \sum_{k=1}^{K(n)} \eta_n^k,$$

where

$$\eta_n^k := b_n^{-1/2} \sum_{i,j} u\left(I^i \cap A_n^k, J^j \cap A_n^k\right) K_{ij}, \quad k = 1, \ldots, K(n). \tag{8}$$

The usefulness of this approximation for proving the asymptotic normality stems from the fact that $\left\{\eta_n^k, k = 1, \ldots, K(n)\right\}$ are independent conditionally on Π. Notice that Φ_n collects contributions only from $I^i \cap J^j$s in the diagonal squares.

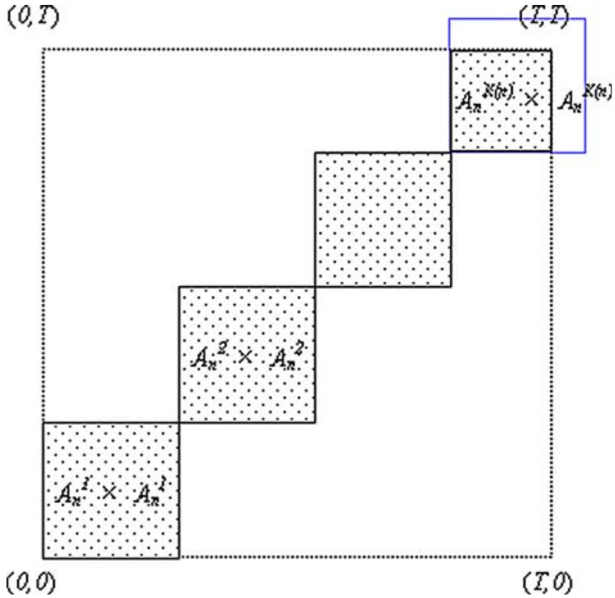

Fig. 2 *The diagonal squares* $\mathbb{A} := \cup_{k=1}^{K(n)} \left\{ A_n^k \times A_n^k \right\} \cap \mathbb{S}$

Let us define the approximation error of Φ_n relative to Ψ_n by

$$\mathcal{R}_n := \Psi_n - \Phi_n.$$

According to (C3′)(ii), r_n, the maximum length of any edge of rectangles, is asymptotically negligible compared to a_n, the length of each edge of the diagonal squares, hence, the (conditional) variance of the approximation error $\mathcal{R}_n$, namely, the 'energy' on the residual region, should vanish eventually. This conjecture indeed leads to the following claim:

Lemma 2 *Suppose* $\mu^l \equiv 0$, $l = 1, 2$. *As* $n \to \infty$,

$$E^\Pi \left[(\mathcal{R}_n)^2 \right] \xrightarrow{P} 0.$$

Proof Beforehand, observe that, if $r_n < \frac{a_n}{2}$, then one can see easily that *no* active rectangle except for the ones covering any of the markers crosses *more than one* edge of the diagonal squares, because the length of each edge is a_n while the maximum edge length among all the rectangle $I^i \times J^j$ is r_n. In this case, we divide all the elements of $\mathcal{R}_n$ into four groups in light of the direction (relative to the corresponding markers) in which each element is positioned. Inevitably any active rectangle that contributes to $\mathcal{R}_n$ must intersect with at least one edge of the diagonal squares; however, when $r_n < \frac{a_n}{2}$, those rectangles (except for the ones covering the markers) intersect *exactly* once. (Recall that $\mathcal{R}_n$ is the aggregate contribution from the residual region, $\mathbb{S} \setminus \mathbb{A}$.) Let $G_n := \left\{ r_n < \frac{a_n}{2} \right\}$.

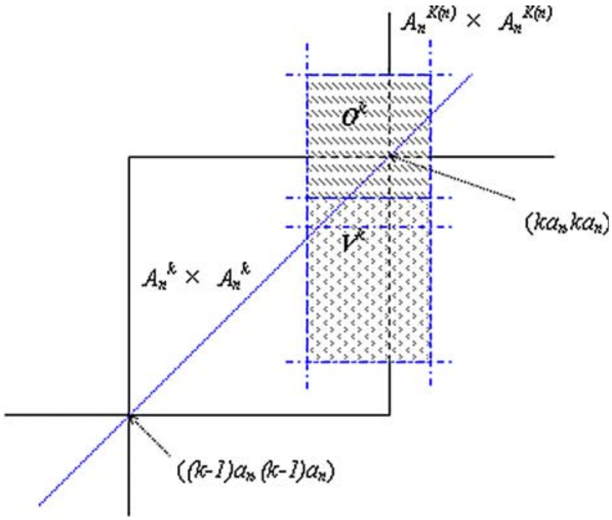

Fig. 3 Active rectangles intersecting with the kth diagonal square $A_n^k \times A_n^k$ in the "North-South" direction of the kth marker (ka_n, ka_n)

Define index sets, for $k = 1, \ldots K(n) - 1$,

$$\mathbf{O}^k := \left\{(i, j) \in \mathbb{N}^2; K_{ij} = 1, ka_n \in I^i, ka_n \in J^j\right\},$$

$$\mathbf{V}^k := \left\{(i, j) \in \mathbb{N}^2; K_{ij} = 1, ka_n \in I^i, ka_n \notin J^j\right\},$$

$$\mathbf{H}^k := \left\{(i, j) \in \mathbb{N}^2; K_{ij} = 1, ka_n \notin I^i, ka_n \in J^j\right\}.$$

$\mathbf{O}^k$ corresponds to the *unique* active rectangle that contains the kth marker, $\mathbf{V}^k$ to those (except the one in $\mathbf{O}^k$) crossing the line $x = ka_n$ (stacked in the 'North-South' direction), while $\mathbf{H}^k$ corresponds to those (except the one in $\mathbf{O}^k$) crossing $y = ka_n$ ('East-West' direction); see Fig. 3.

By construction, $\left\{\mathbf{O}^k, \mathbf{V}^k, \mathbf{H}^k, k = 1, \ldots K(n) - 1\right\}$ are mutually disjoint. Moreover, for $r_n < \frac{a_n}{2}$, $\left\{\mathbf{V}^k, \mathbf{H}^k, k = 1, \ldots K(n) - 1\right\}$ represent all the active rectangles in $\mathbb{S}$ that intersect with any edge of the diagonal squares *exactly once*. Thus, for $r_n < \frac{a_n}{2}$, one can decompose $\mathcal{R}_n \equiv \mathcal{R}_{n,S} + \mathcal{R}_{n,N} + \mathcal{R}_{n,E} + \mathcal{R}_{n,W}$, where

$$\mathcal{R}_{n,S} := \sum_{k=1}^{K(n)-1} b_n^{-1/2} \sum_{(i,j)\in \mathbf{V}^k \cup \mathbf{O}^k} u\left(I^i \cap A_n^{k+1}, J^j \cap A_n^k\right) (\text{'South' region rel. to } \mathbf{O}^k),$$

$$\mathcal{R}_{n,N} := \sum_{k=1}^{K(n)-1} b_n^{-1/2} \sum_{(i,j)\in \mathbf{V}^k \cup \mathbf{O}^k} u\left(I^i \cap A_n^k, J^j \cap A_n^{k+1}\right) (\text{'North' region rel. to } \mathbf{O}^k),$$

$$\mathcal{R}_{n,E} := \sum_{k=1}^{K(n)-1} b_n^{-1/2} \sum_{(i,j)\in\mathbf{H}^k} u\left(I^i \cap A_n^{k+1}, J^j \cap A_n^k\right) \text{ ('East' region rel. to } \mathbf{O}^k\text{)},$$

$$\mathcal{R}_{n,W} := \sum_{k=1}^{K(n)-1} b_n^{-1/2} \sum_{(i,j)\in\mathbf{H}^k} u\left(I^i \cap A_n^{k}, J^j \cap A_n^{k+1}\right) \text{ ('West' region rel. to } \mathbf{O}^k\text{)}.$$

Consider now $\mathcal{R}_{n,S} \equiv \sum_{k=1}^{K(n)-1} \mathcal{R}_{n,S}^k$, where

$$\mathcal{R}_{n,S}^k := b_n^{-1/2} \sum_{(i,j)\in\mathbf{V}^k\cup\mathbf{O}^k} u\left(I^i \cap A_n^{k+1}, J^j \cap A_n^k\right), \quad k = 1, \ldots, K(n) - 1.$$

Fix $k \in \{k = 1, \ldots, K(n) - 1\}$. Let $i^+(k)$ be the index $i \in \mathbb{N}$ such that $ka_n \in I^i$ and $\mathbf{J}^+(k) := \left\{j : (i^+(k), j) \in \mathbf{V}^k \cup \mathbf{O}^k\right\}$. Then,

$$\begin{aligned} b_n^{1/2}\mathcal{R}_{n,S}^k &= \sum_{j\in\mathbf{J}^+(k)} u\left(I^{i^+(k)} \cap A_n^{k+1}, J^j \cap A_n^k\right) \\ &= u\left(I^{i^+(k)} \cap A_n^{k+1}, \left(\cup_{j\in\mathbf{J}^+(k)} J^j\right) \cap A_n^k\right), \end{aligned}$$

because u is 'bilinear' and $\{J^j, j \in \mathbf{J}^+(k)\}$ are disjoint.

Because by assumption $\mu^l \equiv 0$, $l = 1, 2$, by use of Lemma 1, on G_n, one has

$$\begin{aligned} b_n E^\Pi\left[\left(\mathcal{R}_{n,S}^k\right)^2\right] &= v^1\left(I^{i^+(k)} \cap A_n^{k+1}\right) v^2\left(\left(\cup_{j\in\mathbf{J}^+(k)} J^j\right) \cap A_n^k\right) \\ &\quad + v\left(I^{i^+(k)} \cap A_n^{k+1} \cap \left(\cup_{j\in\mathbf{J}^+(k)} J^j\right) \cap A_n^k\right) \\ &\leq r_n v_{\max}^1 \cdot 3r_n v_{\max}^2, \end{aligned}$$

noting that $A_n^{k+1} \cap A_n^k = \emptyset$. Here we have put $v_{\max}^l := \max_{0\leq t\leq T}\left(\sigma_t^l\right)^2$, $l = 1, 2$.

Now, observe that, conditionally on Π, $\left\{\mathcal{R}_{n,S}^k, k = 1, \ldots, K(n) - 1\right\}$ are independent. Also, 1_{G_n} is deterministic on Π. So, for some constant $C < \infty$,

$$E^\Pi\left[(\mathcal{R}_{n,S})^2 1_{G_n}\right] = \sum_{k=1}^{K(n)-1} E^\Pi\left[\left(\mathcal{R}_{n,S}^k\right)^2\right] 1_{G_n} \leq C(K(n) - 1)\frac{r_n^2}{b_n} 1_{G_n} \sim \frac{1}{a_n}\frac{r_n^2}{b_n} 1_{G_n},$$

whenever n is sufficiently large. However, the r.h.s. goes to zero in probability as $n \to \infty$ by (C3′)(i).

The same argument can apply to $\mathcal{R}_{n,N}$, $\mathcal{R}_{n,E}$, and $\mathcal{R}_{n,W}$. Therefore,

$$E^\Pi\left[(\mathcal{R}_n)^2 1_{G_n}\right] \xrightarrow{P} 0 \quad \text{as } n \to \infty$$

so that

$$E^{\Pi}\left[(\mathcal{R}_n)^2\right] \xrightarrow{P} 0 \quad \text{as } n \to \infty$$

because $1_{G_n} \xrightarrow{P} 1$ as $n \to \infty$ according to (C3′)(ii). □

Next, observe that, conditionally on Π, $\left\{\eta_n^k, k = 1, \ldots, K(n)\right\}$ are independent, and that $E^{\Pi}\left[\eta_n^k\right] = 0$, $k = 1, \ldots, K(n)$, hence

$$\sum_{k=1}^{K(n)} \mathrm{Var}^{\Pi}\left[\eta_n^k\right] = \mathrm{Var}^{\Pi}\left[\Phi_n\right].$$

Moreover, the previous Lemma 2 (the vanishing property of the residual energy) will imply the asymptotic equivalence of the conditional variances of Φ_n and Ψ_n. Specifically,

Lemma 3 *Suppose* $\mu^l \equiv 0$, $l = 1, 2$. *As* $n \to \infty$,

$$\mathrm{Var}^{\Pi}\left[\Phi_n\right] \xrightarrow{P} c. \tag{9}$$

Proof By definition,

$$\mathrm{Var}^{\Pi}\left[\Phi_n\right] = \mathrm{Var}^{\Pi}\left[\Psi_n - \mathcal{R}_n\right] = \mathrm{Var}^{\Pi}\left[\Psi_n\right] + \mathrm{Var}^{\Pi}\left[\mathcal{R}_n\right] - 2\mathrm{Cov}^{\Pi}\left[\Psi_n, \mathcal{R}_n\right],$$

where $\mathrm{Cov}^{\Pi}[\cdot,\cdot]$ is the conditional covariance, given Π. Note that, for every fixed Π,

$$\left|\mathrm{Cov}^{\Pi}\left[\Psi_n, \mathcal{R}_n\right]\right| \le \left\{\mathrm{Var}^{\Pi}\left[\Psi_n\right]\right\}^{\frac{1}{2}} \left\{\mathrm{Var}^{\Pi}\left[\mathcal{R}_n\right]\right\}^{\frac{1}{2}}.$$

Therefore, Lemma 2 and (C2′) implies the assertion. □

The following Lindeberg-type condition will be used later when invoking the central limit theorem.

Lemma 4 *Suppose* $\mu^l \equiv 0$, $l = 1, 2$. *As* $n \to \infty$,

$$\sum_{k=1}^{K(n)} E^{\Pi}\left[\left|\eta_n^k\right|^2 1_{\{|\eta_n^k| > \epsilon\}}\right] \xrightarrow{P} 0 \tag{10}$$

for any $\epsilon > 0$.

Proof Put $\mathbf{J}(i) := \{j \ge 1, K_{ij} = 1\}$ for each $i \ge 1$. First note that, for any $p > 2$,

$$E^{\Pi}\left[\left|\eta_n^k\right|^2 1_{\{|\eta_n^k| > \epsilon\}}\right] \le \frac{1}{\epsilon^{p-2}} E^{\Pi}\left[\left|\eta_n^k\right|^p\right],$$

hence it suffices to show that

$$\sum_{k=1}^{K(n)} E^{\Pi}\left[\left|\eta_n^k\right|^p\right] \xrightarrow{P} 0$$

for some $p > 2$.

To this end, observe that, from (8) and 'bilinearity' of u, for $k = 1, \ldots, K(n)$,

$$\begin{aligned} b_n^{1/2}\eta_n^k &= \sum_{i,j} u\left(I^i \cap A_n^k, J^j \cap A_n^k\right) K_{ij} \\ &= \sum_i u\left(I^i \cap A_n^k, \left(\cup_{j\in\mathbf{J}(i)} J^j\right) \cap A_n^k\right) = \sum_i u\left(B^i, C^i\right), \end{aligned}$$

where $B^i := I^i \cap A_n^k$ and $C^i := \left(\cup_{j\in\mathbf{J}(i)} J^j\right) \cap A_n^k$. It should be noted that $\cup_{i=1}^{\infty} B^i = A_n^k$ and $\cup_{i=1}^{\infty} C^i = A_n^k$.

By definition (7) of u,

$$\left|\sum_i u\left(B^i, C^i\right)\right| \leq \left|\sum_i \Delta P^1(B^i)\Delta P^2(C^i)\right| + \left|\sum_i v\left(B^i \cap C^i\right)\right|. \tag{11}$$

Because $B^i \cap C^i = I^i \cap A_n^k = B^i$, the second term on the r.h.s. of (11) is evaluated as, by putting $v_{\max}^{12} := \max_{0\leq t\leq T}\left|\sigma_t^1\sigma_t^2\rho_t\right|$,

$$\left|\sum_i v\left(B^i \cap C^i\right)\right| = \left|v(A_n^k)\right| \leq v_{\max}^{12} \cdot a_n.$$

Now, regarding the first term on the r.h.s. of (11), because

$$\pm 2\sum_i \Delta P^1(B^i)\Delta P^2(C^i) \leq \sum_i \left(\Delta P^1(B^i)\right)^2 + \sum_i \left(\Delta P^2(C^i)\right)^2,$$

one has, for $p > 2$,

$$\begin{aligned} 2E^{\Pi}\left[\left|\sum_i \Delta P^1(B^i)\Delta P^2(C^i)\right|^p\right] &\leq E^{\Pi}\left[\left\{\sum_i \left(\Delta P^1(B^i)\right)^2\right\}^p\right] \\ &\quad + E^{\Pi}\left[\left\{\sum_i \left(\Delta P^2(C^i)\right)^2\right\}^p\right]. \end{aligned}$$

Recalling that $P^l, l = 1, 2$, are martingales, the Burkholder and Doob's maximal inequalities imply that

$$E^{\Pi}\left[\left\{\sum_i \left(\Delta P^l(B^i)\right)^2\right\}^p\right] \le K_{2p} E^{\Pi}\left[\sup_{t\in A_n^k}\left|P_t^l - P_{(k-1)a_n}^l\right|^{2p}\right]$$
$$\le K_{2p}\left(\frac{2p}{2p-1}\right)^{2p} E^{\Pi}\left[\left|P_{ka_n}^l - P_{(k-1)a_n}^l\right|^{2p}\right],\ l=1,2,$$

where K_{2p} is the Burkholder constant. Because $P_{ka_n}^l - P_{(k-1)a_n}^l \sim N(0, v^l(a_n))$, $l = 1, 2$, one has

$$E^{\Pi}\left[\left\{\sum_i \left(\Delta P^l(B^i)\right)^2\right\}^p\right] \le K_{2p}\left(\frac{2p}{2p-1}\right)^{2p} C_p\left(v^l(a_n)\right)^p,$$

where $C_p = 1\cdot 3\cdot\cdots\cdot(2p-1)$ (Note: in case of $l = 2$, 'B^i' should be read as 'C^i' in the left-most hand side of the preceding two inequalities.) Noting that $v^l(a_n) \le v_{\max}^l \cdot a_n$, one has

$$E^{\Pi}\left[\left|\sum_i \Delta P^1(B^i)\Delta P^2(C^i)\right|^p\right] \le C a_n^p$$

for some $C > 0$.

Because $K(n) \sim a_n^{-1}$, one can conclude that

$$\sum_{k=1}^{K(n)} E^{\Pi}\left[\left|\eta_n^k\right|^p\right] = \sum_{k=1}^{K(n)} \frac{1}{b_n^{p/2}} E^{\Pi}\left[\left|\sum_i u\left(B^i, C^i\right)\right|^p\right] \le C\frac{a_n^{p-1}}{b_n^{p/2}} \sim b_n^{2\alpha p - 2\alpha - 1/2} \to 0$$

provided that $p > 1 + \frac{1}{4\alpha}$. In fact, because $\alpha > 0$ under Condition (C3), one can always choose such $p > 2$ at his / her disposal. □

The lemmas established so far are valid only in the zero drift case. Now, we need to derive additional results that will deal with the non-zero drift case. For an interval I^i, let $J(I^i) := \cup_{j\in\mathbf{J}(i)} J^j$, the minimal, combined interval of (J^j) that covers I^i. Let $A_\cdot^l := \int_0^\cdot \mu_t^l dt$, $M_\cdot^l := \int_0^\cdot \sigma_t^l \mathrm{d}W_t^l$, $l = 1, 2$, and

$$B_0 := \sum_{i,j} \Delta M^1(I^i)\Delta M^2(J^j)K_{ij},\ B_1 := \sum_{i,j} \Delta M^1(I^i)\Delta A^2(J^j)K_{ij},$$
$$B_2 := \sum_{i,j} \Delta A^1(I^i)\Delta M^2(J^j)K_{ij},\ \ B_3 := \sum_{i,j} \Delta A^1(I^i)\Delta A^2(J^j)K_{ij}, \tag{12}$$

where $\Delta A^l(\cdot)$ and $\Delta M^l(\cdot)$ are defined similarly to $\Delta P^l(\cdot)$.

For every realized Π, we construct the *reduced design with respect to* $\Pi^1 \equiv \left(I^i\right)_{i\ge 1}$ in the following manner. For each $j = 1, 2, \ldots$, collect all I^is such that $I^i \subset J^j$ and combine them into a new interval; if such I^i does not exist, do nothing. Collecting all such intervals and re-labeling them from left to right yield the most 'economical' partition of (0, T], denoted $\left(\overline{I}^{\overline{i}}\right)_{\overline{i}\geq 1}$. Let $K_{\overline{i}j}$ and $\mathbf{J}(\overline{i})$ be defined as before with the obvious amendment. Analogously, we denote by $J\left(\overline{I}^{\overline{i}}\right)$ the minimal covering of a given $\overline{I}^{\overline{i}}$ with $\left(J^j\right)_{j\geq 1}$.

The following observations are useful: due to 'bilinearity' U_n is *invariant* under the 'Π^1-reduction'; i.e.,

$$U_n \equiv \sum_{i,j} \Delta P^1(I^i)\Delta P^2(J^j)K_{ij} = \sum_{\overline{i},j} \Delta P^1(\overline{I}^{\overline{i}})\Delta P^2(J^j)K_{\overline{i}j},$$

moreover, the resulting B_k, $k = 0, \dots, 4$, are invariant as well. By construction, r_n also remains intact, that is, $r_n = \max_{1\leq \overline{i}<\infty} \left|\overline{I}^{\overline{i}}\right| \vee \max_{1\leq j<\infty} \left|J^j\right|$. Besides,

Lemma 5 *For every fixed* Π*, the corresponding* Π^1*-reduced design* $\left(\left(\overline{I}^{\overline{i}}\right)_{\overline{i}\geq 1}, \left(J^j\right)_{j\geq 1}\right)$ *satisfies*

$$\sum_{\overline{i}=1}^{\infty} \left|J\left(\overline{I}^{\overline{i}}\right)\right| \leq 3T.$$

Proof It should be noted that under the Π^1-reduced design $\left(\left(\overline{I}^{\overline{i}}\right)_{\overline{i}\geq 1}, \left(J^j\right)_{j\geq 1}\right)$, for each fixed j, J^j intersects with *at most three* $\overline{I}^{\overline{i}}$s. Therefore, $\sum_{\overline{i}=1}^{\infty} K_{\overline{i}j} \leq 3$ for each fixed j.

By definition of $J\left(\overline{I}^{\overline{i}}\right)$,

$$\sum_{\overline{i}=1}^{\infty} \left|J\left(\overline{I}^{\overline{i}}\right)\right| = \sum_{\overline{i}=1}^{\infty} \left|\cup_{j\in\mathbf{J}(\overline{i})} J^j\right| = \sum_{\overline{i}=1}^{\infty} \left(\sum_{j=1}^{\infty} \left|J^j\right| K_{\overline{i}j}\right) = \sum_{j=1}^{\infty} \left|J^j\right| \left(\sum_{\overline{i}=1}^{\infty} K_{\overline{i}j}\right) \leq 3T.$$

□

Then, we claim:

Lemma 6 *As* $n \to \infty$,

$$b_n^{-1/2} B_k = o_P(1), \quad k = 1, 2, 3.$$

Proof Consider B_1 first. We exploit the invariance property of B_1 under the Π^1-reduction. In particular, without loss of generality, we may assume that Π^1-reduction will be made immediately after Π is fixed (note: this assumption will be used when $B_{1,2}$ defined below will be evaluated). However, for notational simplicity we write as $\left(I^i\right)_i$ for $\left(\overline{I}^{\bar{i}}\right)_{\bar{i}}$ throughout the proof.

Once Π is fixed, then, because, for every fixed i,

$$\sum_{j=1}^{\infty} \Delta A^2(J^j)K_{ij} = \Delta A^2\left(\cup_{j\in\mathbf{J}(i)} J^j\right) = \int_{J(I^i)} \mu_t^2 \mathrm{d}t = \mu^2_{T^{1,i-1}}\left|J(I^i)\right| + R_i,$$

with

$$R_i := \int_{J(I^i)} \left(\mu_t^2 - \mu^2_{T^{1,i-1}}\right)\mathrm{d}t,$$

one may decompose

$$B_1 = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} \Delta A^2(J^j)K_{ij}\right)\Delta M^1(I^i) =: B_{1,1} + B_{1,2}, \tag{13}$$

where

$$B_{1,1} := \sum_i \mu^2_{T^{1,i-1}}\left|J(I^i)\right|\Delta M^1(I^i); \quad B_{1,2} := \sum_i R_i \Delta M^1(I^i).$$

We are going to show that

$$b_n^{-1/2} B_{1,1} = o_P(1). \tag{14}$$

To this end, suppose for now that $\sup_{0\le t\le T}\left|\mu_t^2\right| \in L^2$. Observing that, for fixed Π, $B_{1,1}$ is a sum of (L^2-)martingale differences, one has

$$\begin{aligned} E^{\Pi}\left[\left(B_{1,1}\right)^2\right] &= E^{\Pi}\left[\sum_i \left(\mu^2_{T^{1,i-1}}\left|J(I^i)\right|\right)^2 v^1(I^i)\right] \\ &\le 9r_n^2 E^{\Pi}\left[\sum_i \left(\mu^2_{T^{1,i-1}}\right)^2 v^1(I^i)\right] \qquad \left(\left|J(I^i)\right| \le 3r_n\right). \end{aligned} \tag{15}$$

Now, because

$$\sum_i \left(\mu^2_{T^{1,i-1}}\right)^2 v^1(I^i) = \sum_i \left(\mu^2_{T^{1,i-1}}\right)^2 \int_{I^i}\left(\sigma_t^1\right)^2 \mathrm{d}t \le \sup_{0\le t\le T}\left|\mu_t^2\right|^2 \cdot v^1\left((0,T]\right),$$

the r.h.s. of which is independent of Π, the r.h.s. of (15) is dominated by

$$9r_n^2 E\left[\sup_{0\le t\le T}\left|\mu_t^2\right|^2\right] v^1\left((0,T]\right),$$

so that

$$b_n^{-1}E^{\Pi}\left[\left(B_{1,1}\right)^2\right] \le Cb_n^{-1}r_n^2 \xrightarrow{P} 0$$

for some C. Therefore,

$$P^{\Pi}\left[b_n^{-1}\left(B_{1,1}\right)^2 \ge \epsilon\right] \le \frac{1}{\epsilon}E^{\Pi}\left[b_n^{-1}\left(B_{1,1}\right)^2\right] \xrightarrow{P} 0,$$

which implies that

$$P\left[b_n^{-1}\left(B_{1,1}\right)^2 \ge \epsilon\right] = EE^{\Pi}\left[1_{\left\{b_n^{-1}\left(B_{1,1}\right)^2\ge\epsilon\right\}}\right] \to 0.$$

Relax the L^2 assumption for μ^2; because μ^2 is continuous, the usual stopping-time argument can apply. Specifically, for $\epsilon > 0$ and $K > 0$,

$$\begin{aligned}P\left[b_n^{-1}\left(B_{1,1}\right)^2 \ge \epsilon\right] &\le P\left[T_K \le T\right] + P\left[b_n^{-1}\left(B_{1,1}\right)^2 \ge \epsilon, T_K > T\right]\\ &\le P\left[T_K \le T\right] + P\left[b_n^{-1}\left(B_{1,1}^{(K)}\right)^2 \ge \epsilon\right],\end{aligned}$$

where T_K is the first time of $\left|\mu^2\right|$ to hit the given level K and $B_{1,1}^{(K)}$ is the corresponding value of $B_{1,1}$ based on the stopped process $\mu^2_{\cdot\wedge T_K}$. Letting $n \to \infty$ then $K \to \infty$, (14) is obtained as desired.

Next we are going to show that

$$b_n^{-1/2}B_{1,2} = o_P(1). \tag{16}$$

Recall that Π^1-reduction has been made prior to obtaining $B_{1,2}$. One has

$$\left|B_{1,2}\right| = \left|\sum_i R_i \Delta M^1(I^i)\right| \le \sum_i |R_i| \left|\Delta M^1(I^i)\right| \le \max_i \left|\Delta M^1(I^i)\right| \sum_i |R_i|.$$

Obviously, $\max_i\left|\Delta M^1(I^i)\right| \le \delta\left(M^1; \max_i\left|I^i\right|\right) \le \delta\left(M^1; r_n\right)$. On the other hand, because $\left|J\left(I^i\right)\right| \le 3r_n$,

$$\sum_i |R_i| \leq \sum_i \int_{J(I^i)} \left|\mu_t^2 - \mu_{T^{1,i-1}}^2\right| \mathrm{d}t \leq \sum_i \delta\left(\mu^2; \left|J\left(I^i\right)\right|\right)\left|J\left(I^i\right)\right|$$
$$\leq \delta\left(\mu^2; 3r_n\right) \sum_i \left|J\left(I^i\right)\right| \leq \delta\left(\mu^2; 3r_n\right) \cdot 3T$$

by Lemma 5, therefore,

$$\left|B_{1,2}\right| \leq 3T\delta\left(M^1; r_n\right)\delta\left(\mu^2; 3r_n\right).$$

Notice the fact that $\delta\left(M^1; r_n\right) = O_P(r_n^{1/2-\xi})$ for any $\xi \in (0, 1/2)$ (this is a direct consequence of *local Hölder continuity* of Brownian paths with exponent parameter γ, for every $\gamma \in (0, 1/2)$, cf. pp.53–54 of Karatzas and Shreve (1991), along with the facts that M^1 is a time-changed Brownian motion and that σ^1 is bounded).

In case (C4) is assumed, if one puts $\lambda := \frac{1}{2} - k \in \left(0, \frac{1}{3}\right)$,

$$b_n^{-1/2}\left|B_{1,2}\right| = O_P\left(b_n^{-\frac{1}{2}} \cdot r_n^{\frac{1}{2}-\xi} \cdot r_n^{\frac{1}{2}-\lambda}\right) = o_P\left(b_n^{-\frac{1}{2}} \cdot r_n^{1-(\xi+\lambda)}\right)$$
$$= o_P\left(b_n^{-\frac{1}{2}+(1-(\xi+\lambda))\left(\frac{3}{4}+\alpha\right)}\right),$$

together with (C3). Notice that the exponent can be made as

$$\frac{1}{4} + \alpha\left(1 - (\xi + \lambda)\right) - \frac{3}{4}(\xi + \lambda) > 0,$$

because $\frac{3}{4}(\xi + \lambda) < 1/4$ so far as one chooses ξ sufficiently small.

If instead (C4′) is assumed, then, by the same token, one has

$$b_n^{-1/2}\left|B_{1,2}\right| = O_P\left(b_n^{-\frac{1}{2}} \cdot r_n^{\frac{1}{2}-\xi} \cdot r_n^{\frac{1}{2}} b_n^{-\left(\frac{1}{4}+\alpha'\right)}\right) = o_P\left(b_n^{-\xi\left(\frac{3}{4}+\alpha\right)+\eta}\right),$$

where $\eta := \alpha - \alpha'(> 0)$ is a fixed constant, determined from (C3) and (C4′). It follows that, by taking ξ sufficiently small, one can always make the exponent of b_n strictly positive. In either case (16) is shown, as desired.

Secondly, $b_n^{-1/2}B_2 = o_P(1)$ can be shown by symmetry. Finally, regarding B_3, by a similar argument, it can be shown that

$$B_3 = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} \Delta A^2(J^j)K_{ij}\right)\Delta A^1(I^i) = \sum_{i=1}^{\infty} \Delta A^2(J\left(I^i\right))\Delta A^1(I^i) = O_p\left(r_n\right),$$

hence, that $b_n^{-1/2}B_3 = O_p(b_n^{-1/2}r_n) = o_P(1)$ due to (C3′). This completes the proof. □

Now we are ready to prove the main theorem.

Proof of Theorem 2 Suppose $\mu^l \equiv 0, l = 1, 2$ for the time being. Recall that $b_n^{-1/2}(U_n - \theta) \equiv \Psi_n \equiv \Phi_n + \mathcal{R}_n$. Because Lemma 2 implies that $\mathcal{R}_n \xrightarrow{P} 0$ as $n \to \infty$, to get the desired result one only needs to prove that $\Phi_n \xrightarrow{\mathcal{L}} N(0, c)$ as $n \to \infty$ in light of the Slutsky theorem. In particular, if one can show that, for every fixed $u \in \mathbb{R}$, as $n \to \infty$

$$\varphi_n(u) := E^\Pi\left[e^{iu\Phi_n}\right] \xrightarrow{P} e^{-\frac{c}{2}u^2}, \tag{17}$$

then the bounded convergence theorem will imply

$$E\left[e^{iu\Phi_n}\right] \to e^{-\frac{c}{2}u^2},$$

thus asymptotic normality of Φ_n is obtained.

To this end, recall that, conditionally on Π, $\left\{\eta_n^k, k = 1, \ldots, K(n)\right\}$ are independent, and that $E^\Pi\left[\eta_n^k\right] = 0$ for $k = 1, \ldots, K(n)$. According to the standard subsequence argument, the convergence results (9) (in Lemma 3) and (10) (in Lemma 4),

$$(\textit{asymptotic variance condition}) \quad \alpha_n := \mathrm{Var}^\Pi[\Phi_n] \xrightarrow{P} c,$$

$$(\textit{Lindeberg condition}) \quad \beta_n := \sum_{k=1}^{K(n)} E^\Pi\left[\left|\eta_n^k\right|^2 1_{\{|\eta_n^k|>\epsilon\}}\right] \xrightarrow{P} 0,$$

imply that an arbitrary subsequence $\{\alpha_{n'}, \beta_{n'}, n' \in \mathbb{N}'\}$, $\mathbb{N}' \subset \mathbb{N}$, contains a further subsequence $\{\alpha_{n''}, \beta_{n''}, n'' \in \mathbb{N}''\}$, $\mathbb{N}'' \subset \mathbb{N}'$, that converges to the same limit *almost surely.*

Then, the Lindeberg-Feller central limit theorem implies that

$$\varphi_{n''}(u) \to e^{-\frac{c}{2}u^2}, \quad n'' \in \mathbb{N}'', \tag{18}$$

almost surely. Because, for every fixed $u \in \mathbb{R}$, every subsequence $\{\varphi_{n'}(u), n' \in \mathbb{N}'\}$ has a further subsequence $\{\varphi_{n''}(u), n'' \in \mathbb{N}''\}$ with the almost sure convergence to the *unique* constant in (18), (by taking the reverse direction of the subsequence argument) the original sequence $\{\varphi_n(u), n \in \mathbb{N}\}$ must tend to the same limit in probability, i.e., (17) is obtained.

Finally, we relax the restriction of the zero drift, more specifically, assume that $\mu^l, l = 1, 2$, satisfy (C4). Then, because $\Psi_n \equiv b_n^{-1/2}(U_n - \theta) = b_n^{-1/2}(B_0 - \theta) + b_n^{-1/2}(B_1 + B_2 + B_3)$, where B_k, $k = 0, 1, 2, 3$, are defined by (12), the Slutsky theorem and Lemma 6 imply the conclusion. □

4 Case study

Two important examples are considered in this section. Another example—a deterministic, nonsynchronous case is treated in Hayashi and Yoshida (2005a). To invoke

Theorem 2, the set of Conditions (C1) through (C3) and either (C4) or (C4′) needs to be checked. The main difficulty is to check (C2), which is stated in a general way. In this section, we will impose a supplementary condition as follows, which will be utilized in identifying the constant c appearing in (4) of (C2). Recall that c serves as the asymptotic variance of the (rescaled) covariance estimator.
Condition (C5): σ^l, $l = 1, 2$, *and* ρ *are continuous in* t.

4.1 Synchronous sampling

Suppose *synchronous* and *equidistant* sampling, $I^i \equiv J^i$, $|I^i| \equiv \frac{T}{n}$. (C1) is trivially satisfied. (C3) holds automatically. Then, without difficulty we can show that, if either (C4) or (C4′), and (C5) hold, then, as $n \to \infty$,

$$n^{1/2}\,(U_n - \theta) \xrightarrow{\mathcal{L}} N\left(0, \int_0^T \left(\sigma_t^1 \sigma_t^2\right)^2 (1 + \rho_t^2)\mathrm{d}t\right).$$

(Recall that $\theta \equiv \int_0^T \sigma_t^1 \sigma_t^2 \rho_t \mathrm{d}t$.)

In this synchronous case, the estimator U_n reduces to the realized covariance. The asymptotic distribution has been known in the literature.

4.2 Poisson sampling

Consider the Poisson sampling case (Poisson random sampling scheme), together with the additional assumption that the Poisson processes N^1 and N^2 are mutually independent. For simplicity of notation, throughout the section we denote as $T^{l,i}$ in place of $\widetilde{T}^{l,i}$, where $\widetilde{T}^{l,i}$ is the ith arrival time of the lth Poisson process, $l = 1, 2$.

We have the following result.

Theorem 3 *In the Poisson sampling case with* N^1 *and* N^2 *mutually independent, if* (C1)*, either* (C4) *or* (C4′)*, and* (C5) *hold at the same time, then, as* $n \to \infty$*,*

$$n^{1/2}\,(U_n - \theta) \xrightarrow{\mathcal{L}} N(0, c),$$

where

$$c := \left(\frac{2}{p^1} + \frac{2}{p^2}\right) \int_0^T \left(\sigma_t^1 \sigma_t^2\right)^2 \mathrm{d}t + \left(\frac{2}{p^1} + \frac{2}{p^2} - \frac{2}{p^1 + p^2}\right) \int_0^T \left(\sigma_t^1 \sigma_t^2 \rho_t\right)^2 \mathrm{d}t. \tag{19}$$

Special case: If P^ls are standard Brownian motions with constant correlation, i.e., $P^l := W^l$, $l = 1, 2$, with $d\left\langle W^1, W^2\right\rangle = \rho \mathrm{d}t$, we have the asymptotic variance of the form

$$c := \left(\frac{2T}{p^1} + \frac{2T}{p^2}\right) + \rho^2 \left(\frac{2T}{p^1} + \frac{2T}{p^2} - \frac{2T}{p^1 + p^2}\right).$$

Remark Notice that even in this simple case—independent Poisson sampling—the result, as well as the proof provided below, is not straightforward. The reader may be convinced of the *difficult nature of the nonsynchronous sampling problems*. At the best of our knowledge a result of this kind is new in the literature. Continual efforts should be made to upgrade the conditions and results obtained in this paper; see Hayashi and Yoshida (2005a, 2006).

4.3 Proof for the Poisson sampling case

Preceding the proof of Theorem 3, we are going to demonstrate the following proposition.

Proposition 1 *Under the assumption of Theorem 3, as $n \to \infty$,*

$$(a)\ n\sum_{i=1}^{\infty} v\left(I^i\right)^2 \xrightarrow{P} \frac{2}{p^1}\int_0^T \left(\sigma_t^1\sigma_t^2\rho_t\right)^2 \mathrm{d}t,\ (b)\ n\sum_{i=1}^{\infty} v\left(J^j\right)^2 \xrightarrow{P} \frac{2}{p^2}\int_0^T \left(\sigma_t^1\sigma_t^2\rho_t\right)^2 \mathrm{d}t,$$

$$(c)\ n\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} v\left(I^i\cap J^j\right)^2 \xrightarrow{P} \frac{2}{p^1+p^2}\int_0^T \left(\sigma_t^1\sigma_t^2\rho_t\right)^2 \mathrm{d}t,\ and$$

$$(d)\ n\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} v^1\left(I^i\right) v^2\left(J^j\right) K_{ij} \xrightarrow{P} \left(\frac{2}{p^1}+\frac{2}{p^2}\right)\int_0^T \left(\sigma_t^1\sigma_t^2\right)^2 \mathrm{d}t.$$

Let $\sigma_t^l := \sigma_{t\wedge T}^l$ for $0 \le t < \infty$, $l = 1, 2$. Accordingly, $v(\cdot)$ and $v^l(\cdot)$, $l = 1, 2$, are to be defined on the Borel σ-field on $[0, \infty)$. Let $\widetilde{\Pi}^1 := \left\{T^{1,i}; i \ge 0\right\}$. Preceding the proof we are going to introduce the auxiliary notation as follows. The rationale behind this is to deal with the fact that $\left|I^i\right|$ and $\left|J^j\right|$ are *not* i.i.d. (exponential) due to the truncation at time T. (Notice that some were defined previously but are re-defined here to be the same just for the readers' convenience.):

$$I^i := (T^{1,i-1}, T^{1,i}] \cap (0, T], \quad J^i := (T^{2,i-1}, T^{2,i}] \cap (0, T];$$
$$K_{ij} := 1_{\{I^i\cap J^j \ne \emptyset\}}, \quad i, j \ge 1.$$
$$\widetilde{I}^i := (T^{1,i-1}, T^{1,i}],\ \widetilde{J}^j := (T^{2,j-1}, T^{2,j}]; \quad \widetilde{K}_{ij} := 1_{\{\widetilde{I}^i\cap\widetilde{J}^j \ne \emptyset\}},\ i, j \ge 1.$$
$$J(I^i) := \cup_{j\in\mathbf{J}(i)} J^j, \quad \widetilde{J}(\widetilde{I}^i) := \cup_{j\in\widetilde{\mathbf{J}}(i)} \widetilde{J}^j; \quad \mathbf{J}(i) := \{j \ge 1; K_{ij} = 1\},$$
$$\widetilde{\mathbf{J}}(i) := \{j \ge 1; \widetilde{K}_{ij} = 1\},\ i \ge 1.$$
$$i^* := \min\left\{i \ge 1; T^{1,i} \ge T\right\}, \quad j^* := \min\left\{j \ge 1; T^{2,j} \ge T\right\}.$$
$$\widetilde{j}_+(i) := \max \widetilde{\mathbf{J}}(i), \quad \widetilde{j}_-(i) := \min \widetilde{\mathbf{J}}(i), \quad i \ge 1.$$
$$r_n := \max_{1\le i<\infty}\left|I^i\right| \vee \max_{1\le j<\infty}\left|J^j\right|, \quad \widetilde{r}_n := \max_{1\le i\le(\lambda(n)\vee i^*)}\left|\widetilde{I}^i\right| \vee \max_{1\le j\le\widetilde{j}_+(\lambda(n)\vee i^*)}\left|\widetilde{J}^j\right|.$$
$$v_{\max}^l := \sup_{0\le t<\infty}\left(\sigma_t^l\right)^2, l = 1, 2; \quad v_{\max}^{12} := \sup_{0\le t<\infty}\left|\sigma_t^1\sigma_t^2\rho_t\right|.$$

We have put $\lambda(n) := \lceil np^1 T \rceil$, where $\lceil x \rceil$ denotes the largest integer that does not exceed x. Recall that, for sequences (x_n) and (y_n), $x_n \lesssim y_n$ means that there exists a constant $C \in [0, \infty)$ such that $x_n \leq C y_n$ for large n; $x_n \sim y_n$ means that $x_n \lesssim y_n$ and $y_n \lesssim x_n$ at the same time. Besides, for random sequences (X_n) and (Y_n), $X_n \simeq Y_n$ means that $X_n - Y_n \xrightarrow{P} 0$ as $n \to \infty$.

The following simple facts will be used later:

1. (a) $\widetilde{j}_+(i)$, the index associated with the first jump time of N^2 after and including the ith observation time $T^{1,i}$ of N^1, is a discrete-time stopping time with respect to the filtration $\{\mathcal{G}_k; k \geq 0\}$ defined by $\mathcal{G}_k := \sigma\left\langle S^{2,j}; 1 \leq j \leq k\right\rangle \vee \sigma\left\langle \widetilde{\Pi}^1\right\rangle$, where $S^{2,k} := T^{2,k} - T^{2,k-1}$, i.i.d. exponential random variables. In fact, for every $k \geq 0$,

$$\left\{\widetilde{j}_+(i) = k\right\} = \left\{T^{2,k-1} < T^{1,i}, T^{2,k} \geq T^{1,i}\right\} \in \mathcal{G}_k.$$

   Similarly, $\widetilde{j}_-(i)$, the index associated with the first jump time of N^2 after the $(i-1)$th observation time $T^{1,i-1}$ of N^1, is a (discrete-time) $\{\mathcal{G}_k\}$-stopping time.
2. (a) $(i^* - 1)$ is a Poisson random variable with intensity $\lambda^1 T \equiv np^1 T$. (b) Since, for every $k \geq 1$,

$$\left\{\widetilde{j}_+(i) = k\right\} = \left\{N^2 \text{ jumps exactly } (k-1) \text{ times prior to } T^{1,i}\right\},$$

   $\left(\widetilde{j}_+(i) - 1\right)$ is Poisson with intensity $\lambda^2 T^{1,i} \equiv np^2 T^{1,i}$, conditionally on $\widetilde{\Pi}^1$. Similarly, $\left(\widetilde{j}_-(i) - 1\right)$ is Poisson with $\lambda^2 T^{1,i-1} \equiv np^2 T^{1,i-1}$, conditionally on $\widetilde{\Pi}^1$.
3. (a) $\left(T^{1,i^*} - T\right)$ is an exponential random variable with intensity $\lambda^1 \equiv np^1$. (b) Conditionally on $\widetilde{\Pi}^1$, both $\left(T^{2,\widetilde{j}_+(i)} - T^{1,i}\right)$ and $\left(T^{2,\widetilde{j}_-(i)} - T^{1,i-1}\right)$ are exponential with intensity $\lambda^2 \equiv np^2$. Similarly, conditionally on $\widetilde{\Pi}^1$, $\left(T^{2,\widetilde{j}_+(i^*)} - T^{1,i^*}\right)$ is exponential with intensity $\lambda^2 \equiv np^2$.
4. $\left(T^{1,i+l} - T^{1,i}\right) \equiv \sum_{k=i+1}^{i+l} |\widetilde{I}^k|$ is a gamma random variable with shape parameter l and scale parameter $\left(\lambda^1\right)^{-1} \equiv \left(np^1\right)^{-1}$, which may be denoted as $\Gamma\left(l, \left(\lambda^1\right)^{-1}\right)$ by convention.

We are going to derive some technical lemmas as follows, which will be used in the proof for Proposition 1 provided later. Let $P^{\widetilde{\Pi}^1}$ denote the conditional probability measure given $\widetilde{\Pi}^1$.

Lemma 7 *For every $i \geq 1$,*

$$E^{\widetilde{\Pi}^1}\left[\left|\widetilde{J}(\widetilde{I}^i)\right|\right] = \left|\widetilde{I}^i\right| + \frac{2}{np^2} - \frac{1}{np^2}\exp\left\{-np^2 T^{1,i-1}\right\}.$$

Proof Because $\left|\widetilde{J}(\widetilde{I}^i)\right| = \sum_{j=\widetilde{j}_-(i)}^{\widetilde{j}_+(i)} \left|\widetilde{J}^j\right| = \sum_{j=1}^{\widetilde{j}_+(i)} \left|\widetilde{J}^j\right| - \sum_{j=1}^{\widetilde{j}_-(i)} \left|\widetilde{J}^j\right| + \left|\widetilde{J}^{\widetilde{j}_-(i)}\right|$, Facts 1 and 2 (i.e., $\widetilde{j}_+(i)$ and $\widetilde{j}_-(i)$ are $\{\mathcal{G}_k\}$-stopping times with finite expectations under $P^{\widetilde{\Pi}^1}$) and the fact that $\left|\widetilde{J}^j\right|(\equiv S^{2,j})$ are i.i.d. under $P^{\widetilde{\Pi}^1}$ imply

$$E^{\widetilde{\Pi}^1}\left[\left|\widetilde{J}(\widetilde{I}^i)\right|\right] = E^{\widetilde{\Pi}^1}\left[\widetilde{j}_+(i) - \widetilde{j}_-(i)\right] E^{\widetilde{\Pi}^1}\left[\left|\widetilde{J}^j\right|\right] + E^{\widetilde{\Pi}^1}\left[\left|\widetilde{J}^{\widetilde{j}_-(i)}\right|\right],$$

by means of the Wald identity.

From Fact 2 one can evaluate as

$$E^{\widetilde{\Pi}^1}\left[\widetilde{j}_+(i)\right] = np^2 T^{1,i} + 1, \quad E^{\widetilde{\Pi}^1}\left[\widetilde{j}_-(i)\right] = np^2 T^{1,i-1} + 1,$$

while $E^{\widetilde{\Pi}^1}\left[\left|\widetilde{J}^j\right|\right] = \left(np^2\right)^{-1}$. We now claim that

$$E^{\widetilde{\Pi}^1}\left[\left|\widetilde{J}^{\widetilde{j}_-(i)}\right|\right] = \frac{2}{np^2} - \frac{1}{np^2}\exp\left\{-np^2 T^{1,i-1}\right\},$$

from which the assertion will be obtained. To this end, notice that

$$\left|\widetilde{J}^{\widetilde{j}_-(i)}\right| = \left(T^{2,\widetilde{j}_-(i)} - T^{1,i-1}\right) + \left(T^{1,i-1} - T^{2,\widetilde{j}_-(i)-1}\right).$$

The conditional expectation of the first term (elapsed time from $T^{1,i-1}$ to the corresponding first observation time of N^2) is found easily from Fact 3 above, while that of the second term (elapsed time to $T^{1,i-1}$ from the corresponding last observation time of N^2) can be evaluated with, for instance, the Eq. (2.2) of Hayashi and Yoshida (2005b) (p. 365). □

Lemma 8 *For any* $q \in [1, \infty)$,

$$E\left[\widetilde{r}_n^q\right] = o(n^{-\alpha}) \quad \textit{for any } \alpha \in (0, q).$$

Proof Let $q \in (0, \infty)$ be fixed.

(i) Let $k(n) \geq 1$ be an integer-valued, positive function of n, with at most polynomial order of q. For any $\delta \in (0, 1)$,

$$\begin{aligned} E\left[\max_{1\leq i\leq k(n)}\left|\widetilde{I}^i\right|^q\right] &\leq E\left[1_{\{\max_{1\leq i\leq k(n)}|\widetilde{I}^i|\geq n^{-(1-\delta)}\}}\max_{1\leq i\leq k(n)}\left|\widetilde{I}^i\right|^q\right] + n^{-q(1-\delta)} \\ &\leq \sum_{i=1}^{k(n)} E\left[1_{\{|\widetilde{I}^i|\geq n^{-(1-\delta)}\}}\left|\widetilde{I}^i\right|^q\right] + n^{-q(1-\delta)} \\ &\leq \sum_{i=1}^{k(n)} E\left[\left|\widetilde{I}^i\right|^{p+q}\right] n^{p(1-\delta)} + n^{-q(1-\delta)} \\ &= Ck(n)n^{-p\delta-q} + n^{-q(1-\delta)} \end{aligned} \tag{20}$$

for any $p > 0$, with C a constant. The last equality holds because $\left|\widetilde{I}^i\right|$ are i.i.d. exponential with intensity np^1, hence $E\left[\left|\widetilde{I}^i\right|^{p+q}\right] = \Gamma(p+q+1)\left(np^1\right)^{-(p+q)}$.

Note that, for any given $q \in (0, \infty)$, for any $\alpha \in (0, q)$, one can always choose small enough $\delta_\alpha \in (0, 1)$ and large enough $p_\alpha \in (0, \infty)$. In particular, in case $K(n) \sim n$ and $q = 1$, choose $\delta < 1 - \alpha$ and $p > \alpha/2$.

(ii) Now, for the purpose of evaluating $E\left[\max_{1\le i\le i^*}\left|\widetilde{I}^i\right|^q\right]$, a simple application of (20) together with the law of iterated expectation (with respect to i^*) will not work (note that $\{i^* = k\} = \left\{T^{1,k} \ge T, T^{1,k-1} < T\right\}$, which indicates that $\left|\widetilde{I}^i\right| \equiv T_1^i, -T_1^{i-1}$, are no more i.i.d. once i^* is conditioned on). To deal with the situation, one may alternatively evaluate as, for $q \ge 1$,

$$E\left[\max_{1\le i\le i^*}\left|\widetilde{I}^i\right|^q\right] \le E\left[\max_{1\le i\le \lceil 3np^1T\rceil}\left|\widetilde{I}^i\right|^q\right] + E\left[\max_{\lceil 3np^1T\rceil+1\le i\le i^*}\left|\widetilde{I}^i\right|^q; i^* > \left\lceil 3np^1T\right\rceil\right].$$

The first term on the r.h.s. is bounded (up to constant) by $n^{1-p\delta-q} + n^{-q(1-\delta)}$ from (20) (put $k(n) = \left\lceil 3np^1T\right\rceil$ $(\sim n)$). For the second term, via the Hölder inequality

$$\begin{aligned}
&E\left[\max_{\lceil 3np^1T\rceil+1\le i\le i^*}\left|\widetilde{I}^i\right|^q; i^* > \left\lceil 3np^1T\right\rceil\right] \\
&\quad\le \left\{E\left[\max_{1\le i\le i^*}\left|\widetilde{I}^i\right|^{2q}\right]\right\}^{\frac{1}{2}}\left\{P\left[i^* > \left\lceil 3np^1T\right\rceil\right]\right\}^{\frac{1}{2}}.
\end{aligned}$$

Clearly the first factor on the r.h.s. is bounded (in fact, $\max_{1\le i\le i^*}\left|\widetilde{I}^i\right| \le \left(T^{1,i^*} - T\right) + T$ with $\left(T^{1,i^*} - T\right)$ being exponentially distributed with intensity $\lambda^1 \equiv np^1$). The second term may be evaluated as, according to the argument in the Appendix of Hayashi and Yoshida (2005b), for constant C',

$$\begin{aligned}
P\left[i^* > \left\lceil 3np^1T\right\rceil\right] &\le P\left[N_T^1 \ge \left\lceil 3np^1T\right\rceil\right] = \sum_{k=\lceil 3np^1T\rceil}^{\infty} \mathrm{e}^{-np^1T}\frac{\left(np^1T\right)^k}{k!} \\
&\le \sum_{k=\lceil 3np^1T\rceil}^{\infty} \mathrm{e}^{-np^1T}\frac{\left(np^1T\right)^k}{k^k\mathrm{e}^{-k}} \\
&\le \mathrm{e}^{-np^1T}\sum_{k=\lceil 3np^1T\rceil}^{\infty}\left(\frac{np^1T}{\left\lceil 3np^1T\right\rceil}e\right)^k \le C'\mathrm{e}^{-np^1T}, \qquad (21)
\end{aligned}$$

which goes to zero exponentially fast as $n \to \infty$. Here, we have used the fact that $k! > k^k\mathrm{e}^{-k}$ for all $k \ge 1$. Therefore,

$$E\left[\max_{1\le i\le i^*}\left|\widetilde{I}^i\right|^q\right] \lesssim n^{1-p\delta-q} + n^{-q(1-\delta)}.$$

(iii) We will now evaluate $E\left[\max_{1\le j\le \widetilde{j}_+(i^*)}\left|\widetilde{J}^j\right|^q\right]$. The same argument as (ii) can apply with slight modification. That is, first note the inequality

$$E\left[\max_{1\le j\le \widetilde{j}_+(i^*)}\left|\widetilde{J}^j\right|^q\right]\le E\left[\max_{1\le j\le\lceil 3np^2T^{1,i^*}\rceil}\left|\widetilde{J}^j\right|^q\right] \\ +E\left[\max_{\lceil 3np^2T^{1,i^*}\rceil+1\le j\le \widetilde{j}_+(i^*)}\left|\widetilde{J}^j\right|^q;\ \widetilde{j}_+\left(i^*\right)>\left\lceil 3np^2T^{1,i^*}\right\rceil\right]. \tag{22}$$

Recall that $\widetilde{\Pi}^1\equiv\left\{T^{1,i};\,i\ge 1\right\}$. Noting the independence of $\widetilde{\Pi}^1$ and $\widetilde{J}^j$s, one can apply the same argument as (20) to the first term to obtain

$$\begin{aligned} E\left[\max_{1\le j\le\lceil 3np^2T^{1,i^*}\rceil}\left|\widetilde{J}^j\right|^q\right] &= EE^{\widetilde{\Pi}^1}\left[\max_{1\le j\le\lceil 3np^2T^{1,i^*}\rceil}\left|\widetilde{J}^j\right|^q\right]\\ &\le CE\left[\left\lceil 3np^2T^{1,i^*}\right\rceil\right]n^{-p\delta-q}+n^{-q(1-\delta)}\\ &\lesssim n^{1-p\delta-q}+n^{-q(1-\delta)}. \end{aligned}$$

To evaluate the second on the r.h.s. of (22), apply the Hölder inequality

$$\begin{aligned} &E\left[\max_{\lceil 3np^2T^{1,i^*}\rceil+1\le j\le \widetilde{j}_+(i^*)}\left|\widetilde{J}^j\right|^q;\ \widetilde{j}_+\left(i^*\right)>\left\lceil 3np^2T^{1,i^*}\right\rceil\right]\\ &\le\left\{E\left[\max_{\lceil 3np^2T^{1,i^*}\rceil+1\le j\le \widetilde{j}_+(i^*)}\left|\widetilde{J}^j\right|^{2q}\right]\right\}^{\frac12}\left\{P\left[\widetilde{j}_+\left(i^*\right)>\left\lceil 3np^2T^{1,i^*}\right\rceil\right]\right\}^{\frac12}. \end{aligned}$$

Following (21),

$$\begin{aligned} P^{\widetilde{\Pi}^1}\left[\widetilde{j}_+\left(i^*\right)>\left\lceil 3np^2T^{1,i^*}\right\rceil\right] &\le P^{\widetilde{\Pi}^1}\left[N^2_{T^{1,i^*}}\ge\left\lceil 3np^2T^{1,i^*}\right\rceil\right]\\ &\le \mathrm{e}^{-np^2T^{1,i^*}}\sum_{k=\lceil 3np^2T^{1,i^*}\rceil}^{\infty}\left(\frac{np^2T^{1,i^*}}{\left\lceil 3np^2T^{1,i^*}\right\rceil}e\right)^k\\ &\le C'\mathrm{e}^{-np^2T^{1,i^*}}\le C'\mathrm{e}^{-np^2T}, \end{aligned}$$

for every fixed $\widetilde{\Pi}^1$, where C' is a constant. Hence, $P\left[\widetilde{j}_+\left(i^*\right)>\left\lceil 3np^2T^{1,i^*}\right\rceil\right]\lesssim e^{-np^2T}$.

Besides,

$$
\begin{aligned}
&E\left[\max_{\lceil 3np^2T^{1,i^*}\rceil+1\le j\le \widetilde{j}_+(i^*)}\left|\widetilde{J}^j\right|^{2q}\right]\\
&\le E\left[\left(T^{\widetilde{j}_+(i^*)}\right)^{2q}\right]\\
&\lesssim E\left[\left(T^{2,\widetilde{j}_+(i^*)}-T^{1,i^*}\right)^{2q}+\left(T^{1,i^*}-T\right)^{2q}+T^{2q}\right]\sim T^{2q}
\end{aligned}
$$

because $\left(T^{2,\widetilde{j}_+(i^*)}-T^{1,i^*}\right)$ is exponential with intensity $\lambda^2\equiv np^2$ conditionally on $\widetilde{\Pi}^1$ (Fact 3) so that its $2q$th moment equals to $\Gamma(2q+1)\left(np^2\right)^{-2q}$.

It follows that the second term on the r.h.s. of (22) tends to zero exponentially fast, thus the whole r.h.s. is bounded as

$$
E\left[\max_{1\le j\le \widetilde{j}_+(i^*)}\left|\widetilde{J}^j\right|^{q}\right]\lesssim n^{1-p\delta-q}+n^{-q(1-\delta)}.
$$

(iv) The inequality

$$
E\left[\max_{1\le j\le \widetilde{j}_+(\lambda(n))}\left|\widetilde{J}^j\right|^{q};\, i^*\le\lambda(n)\right]\lesssim n^{1-p\delta-q}+n^{-q(1-\delta)}.
$$

can be derived analogously to (ii) and (iii). It is left as an exercise to the reader. (Hint: replace i^* with $\lambda(n)$ in the argument in (iii).)

Putting all (i)–(iv) together, one can confirm that, for any $q\in[1,\infty)$ and $\alpha\in(0,q)$,

$$
E\left[n^{\alpha}\widetilde{r}_n^{q}\right]\lesssim n^{\alpha+1-p\delta-q}+n^{\alpha-q(1-\delta)}.
$$

In particular, one can choose δ and p arbitrarily within the region $-(q-\alpha)+q\delta<0$ and $-(q-\alpha)-p\delta+1<0$, that is, choose any $\delta\in(0,1-\frac{\alpha}{q})$ and $p\in\left(\frac{1-(q-\alpha)}{\delta}\vee 0,\infty\right)$. The lemma is proved. □

For a given $\epsilon\in(1/2,1)$, let $\Omega_0^n:=\{|\lambda(n)-i^*|\le n^{\epsilon}\}$. We claim that it will eventually fill up the whole space. Formally,

Lemma 9 *As $n\to\infty$,*

$$
P\left[\Omega_0^n\right]=1-o\left(1\right).
$$

Proof Recall that $i^*\equiv\min\left\{i\ge 1;\,T^{1,i}\ge T\right\}$ and $N_t^1=\sum_{i=1}^{\infty}1_{\{T^{1,i}\le t\}}$, $t\ge 0$. So, i^* satisfies the equation

$$
i^*(\omega)=N_T^1(\omega)+1_{\{\omega;\,T^{1,i^*(\omega)}\gneq T\}}(\omega),\quad \omega\in\Omega.
$$

Note also that $np^1T - 1 < \lambda(n) \equiv \lceil np^1T \rceil \le np^1T$. Hence, for every $\epsilon > \frac{1}{2}$,

$$\begin{aligned} P\left[i^* - \left\lceil np^1T \right\rceil > n^\epsilon\right] &\le P\left[N_T^1 + 1_{\{T^{1,i^*} \gneq T\}} - np^1T + 1 > n^\epsilon\right] \\ &\le P\left[N_T^1 + 2 - np^1T > n^\epsilon\right] \\ &= P\left[\frac{N_T^1 - np^1T}{\sqrt{np^1T}} > \frac{n^{\epsilon-\frac{1}{2}}}{\sqrt{p^1T}} - 2\frac{n^{-\frac{1}{2}}}{\sqrt{p^1T}}\right] = o(1), \end{aligned}$$

as $n \to \infty$, by use of normal approximation to the Poisson distribution. Filling in details is an easy exercise. Similarly,

$$P\left[i^* - \left\lceil np^1T \right\rceil < -n^\epsilon\right] \le P\left[\frac{N_T^1 - np^1T}{\sqrt{np^1T}} < -\frac{n^{\epsilon-\frac{1}{2}}}{\sqrt{p^1T}}\right] = o(1),$$

as $n \to \infty$. The assertion follows. □

It should be noted that we have additionally imposed $\epsilon < 1$, which will be used in the proofs for the next Lemma 10 and for Proposition 1.

Lemma 10 *It holds that* $n\sum_{i=1}^\infty \left|I^i\right|^2 = O_P(1)$, $n\sum_{j=1}^\infty \left|J^j\right|^2 = O_P(1)$, $n\sum_{i=1}^\infty\sum_{j=1}^\infty \left|I^i \cap J^j\right|^2 = O_P(1)$, *and that* $n\sum_{i=1}^\infty\sum_{j=1}^\infty \left|I^i\right|\left|J^j\right| K_{ij} = O_P(1)$, *as* $n \to \infty$.

Proof First we claim that

$$n\sum_{i=1}^{\infty}\left|I^i\right|^2 = n\sum_{i=1}^{\lambda(n)}\left|\widetilde{I}^i\right|^2 + o_P(1),$$

as $n \to \infty$. To this end, we compare

$$n\sum_{i=1}^{\infty}\left|I^i\right|^2 = n\sum_{i=1}^{i^*}\left|I^i\right|^2 \overset{(i)}{\simeq} n\sum_{i=1}^{i^*}\left|\widetilde{I}^i\right|^2 \overset{(ii)}{\simeq} n\sum_{i=1}^{\lambda(n)}\left|\widetilde{I}^i\right|^2.$$

(i) To evaluate the absolute difference of the second and third terms, recalling that $\widetilde{I}^i \equiv I^i$ for $1 \le i \le i^* - 1$,

$$n\left(\left|\widetilde{I}^{i^*}\right|^2 - \left|I^{i^*}\right|^2\right) \le 2n\widetilde{r}_n^2,$$

the r.h.s. of which goes to zero in probability as $n \to \infty$, according to Lemma 8.

(ii) The absolute difference of the last two terms is evaluated as

$$\Delta^{(ii)} := n\sum_{i=(\lambda(n)\wedge i^*)+1}^{\lambda(n)\vee i^*}\left|\widetilde{I}^i\right|^2 \le n^{1+\epsilon}\widetilde{r}_n^2$$

on Ω_0^n, $\epsilon \in (1/2, 1)$. Thus, Lemma 8 implies that as $n \to \infty$, $\Delta^{(ii)} 1_{\Omega_0^n} \xrightarrow{P} 0$, therefore $\Delta^{(ii)} \xrightarrow{P} 0$ by Lemma 9, as desired.

Now, because $\left|\widetilde{I}^i\right|$ are i.i.d. exponential with intensity $\lambda^1 = np^1$, in light of the weak law of large numbers,

$$n \sum_{i=1}^{\lambda(n)} \left|\widetilde{I}^i\right|^2 = \frac{\lambda(n)}{n} \left\{ \frac{1}{\lambda(n)} \sum_{i=1}^{\lambda(n)} \left(n \left|\widetilde{I}^i\right| \right)^2 \right\} \xrightarrow{P} \frac{2T}{p^1}$$

as $n \to \infty$. Therefore, the first assertion $n \sum_{i=1}^{\infty} \left|I^i\right|^2 = O_P(1)$ is obtained. The second assertion can be shown by the same way.

To show the third, one only needs to recall that $(\widetilde{I}^i)$ and $(\widetilde{J}^j)$ are, respectively, the inter-arrival times of the independent Poisson processes N^1 and N^2, and hence that $\left(\widetilde{I}^i \cap \widetilde{J}^j\right)$ form the inter-arrival times of the aggregate Poisson process $\left(N^1 + N^2\right)$. Thereafter, a similar argument to the first can apply.

Finally, note that

$$\begin{aligned}
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left|I^i\right| \left|J^j\right| K_{ij} &= \sum_{i=1}^{\infty} \left|I^i\right| \sum_{j=1}^{\infty} \left|J^j\right| K_{ij} 1_{\{|I^i| \geq |J^j|\}} + \sum_{j=1}^{\infty} \left|J^j\right| \sum_{i=1}^{\infty} \left|I^i\right| K_{ij} 1_{\{|I^i| < |J^j|\}} \\
&\leq 3 \sum_{i=1}^{\infty} \left|I^i\right|^2 + 3 \sum_{j=1}^{\infty} \left|J^j\right|^2 .
\end{aligned}$$

Hence, the first two assertions imply the last. □

Proof of Proposition 1 Consider (d) only. Proving of the other cases is simpler; it is left to the reader as an exercise. Put $\theta^{(c)} := \left(\frac{2}{p^1} + \frac{2}{p^2}\right) \int_0^T \left(\sigma_t^1 \sigma_t^2\right)^2 dt$. Recall that $v_{\max}^l \equiv \max_{0 \leq t < \infty} \left(\sigma_t^l\right)^2, l = 1, 2$. The basic strategy for proof is to approximate as

$$\begin{aligned}
V_n &:= n \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} v^1\left(I^i\right) v^2\left(J^j\right) K_{ij} \\
&\overset{(A)}{\simeq} n \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sigma_{T^{1,i-1}}^1 \sigma_{T^{1,i-1}}^2\right)^2 \left|I^i\right| \left|J^j\right| K_{ij} (=: V_n^{(A)}) \\
&\overset{(B)}{\simeq} n \sum_{i=1}^{i^*} \sum_{j=1}^{j^*} \left(\sigma_{T^{1,i-1}}^1 \sigma_{T^{1,i-1}}^2\right)^2 \left|\widetilde{I}^i\right| \left|\widetilde{J}^j\right| \widetilde{K}_{ij} \\
&\equiv n \sum_{i=1}^{i^*} \left(\sigma_{T^{1,i-1}}^1 \sigma_{T^{1,i-1}}^2\right)^2 \left|\widetilde{I}^i\right| \left|\widetilde{J}(\widetilde{I}^i)\right| \ (=: V_n^{(B)}) \\
&\overset{(C)}{\simeq} n \sum_{i=1}^{\lambda(n)} \left(\sigma_{T^{1,i-1}}^1 \sigma_{T^{1,i-1}}^2\right)^2 \left|\widetilde{I}^i\right| \left|\widetilde{J}(\widetilde{I}^i)\right| \ (=: V_n^{(C)}),
\end{aligned}$$

where (A), (B) and (C) mean that the difference goes to zero in probability, as defined earlier. Thereafter, we will show that

$$(D) : V_n^{(C)} \xrightarrow{P} \theta^{(c)}$$

to obtain the conclusion.

(A): We are going to show

$$\Delta^{(A)} := \left| V_n - V_n^{(A)} \right| \xrightarrow{P} 0.$$

To this end, note first that, for each i,

$$v^1\left(I^i\right) \equiv \int_{I^i} \left(\sigma_t^1\right)^2 \mathrm{d}t = \left(\sigma^1_{T^{1,i-1}}\right)^2 \left|I^i\right| + \int_{I^i} \left(\left(\sigma_t^1\right)^2 - \left(\sigma^1_{T^{1,i-1}}\right)^2\right) \mathrm{d}t,$$

where the second term of the r.h.s. is bounded by

$$\delta\left(\left(\sigma^1\right)^2; \left|I^i\right|\right)\left|I^i\right| \le \delta\left(\left(\sigma^1\right)^2; r_n\right)\left|I^i\right|.$$

In the meantime, for $j \in \mathbf{J}(i) \equiv \{j \ge 1;\, K_{ij} = 1\}$ with i given,

$$v^2\left(J^j\right) \equiv \int_{J^j} \left(\sigma_t^2\right)^2 \mathrm{d}t = \left(\sigma^2_{T^{1,i-1}}\right)^2 \left|J^j\right| + \int_{J^j} \left(\left(\sigma_t^2\right)^2 - \left(\sigma^2_{T^{1,i-1}}\right)^2\right) \mathrm{d}t.$$

Although it may be possible that $T^{1,i-1} \notin J^j$, it is always true that $T^{1,i-1} \in J(I^i)$ and $J^j \subset J(I^i)$ for $j \in \mathbf{J}(i)$; therefore, the second term of the r.h.s. is bounded by

$$\delta\left(\left(\sigma^2\right)^2; \left|J(I^i)\right|\right)\left|J^j\right| \le \delta\left(\left(\sigma^2\right)^2; 3r_n\right)\left|J^j\right|.$$

Thus,

$$\begin{aligned}\Delta^{(A)} \le\, & n v^1_{\max} \delta\left(\left(\sigma^2\right)^2; 3r_n\right) \sum_{i,j} \left|I^i\right| \left|J^j\right| K_{ij} + n v^2_{\max} \delta\left(\left(\sigma^1\right)^2; r_n\right) \sum_{i,j} \left|I^i\right| \left|J^j\right| K_{ij} \\ & + n\delta\left(\left(\sigma^1\right)^2; r_n\right) \delta\left(\left(\sigma^2\right)^2; 3r_n\right) \sum_{i,j} \left|I^i\right| \left|J^j\right| K_{ij},\end{aligned}$$

which goes to zero in probability by the continuity of σ^l, $l = 1, 2$ ((C5)), and from Lemma 10.

(B): We are going to show

$$\Delta^{(B)} := \left| V_n^{(A)} - V_n^{(B)} \right| \xrightarrow{P} 0.$$

For economy of space we put $s(i) := \sigma^1_{T^{1,i-1}}\sigma^2_{T^{1,i-1}}$ throughout the rest of the proof. Note that

$$V_n^{(A)} \equiv n\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} s(i)^2 \left|I^i\right|\left|J^j\right| K_{ij} = n\sum_{i=1}^{i^*}\sum_{j=1}^{j^*} s(i)^2 \left|I^i\right|\left|J^j\right| K_{ij}.$$

Also, notice the fact that $K_{ij} \equiv \widetilde{K}_{ij}$ for $1 \le i \le i^*$ and $1 \le j \le j^*$; $\widetilde{I}^i \equiv I^i$, $1 \le i \le i^*-1$, $\widetilde{J}^j \equiv J^j$, $1 \le j \le j^*-1$; $\widetilde{I}^{i^*} \supseteq I^{i^*}$, $\widetilde{J}^{j^*} \supseteq J^{j^*}$. Hence, always $V_n^{(A)} \le V_n^{(B)}$, and the difference is made up of the terms involving either i^* or j^*, or both.

Now we put $\widetilde{J}(I^i) := \cup_{j\in \mathbf{J}(i)}\widetilde{J}^j$ and $\widetilde{I}(J^j) := \cup_{i\in \mathbf{I}(j)}\widetilde{I}^i$. Note that $\widetilde{J}(I^{i^*})$ is the sub-collection of $\{\widetilde{J}^j, 1 \le j \le j^*\}$ that 'covers' I^{i^*} and that its length is bounded by $\left|\widetilde{J}(I^{i^*})\right| \le 2\max_{1\le j\le j^*}\left|\widetilde{J}^j\right| + \left|I^{i^*}\right| \le 3\widetilde{r}_n$. Similarly, $\left|\widetilde{I}(J^{j^*})\right| \le 3\widetilde{r}_n$. Because $s(i)^2 \le v^1_{\max}v^2_{\max}$ for all i, one has

$$\Delta^{(B)} = V_n^{(B)} - V_n^{(A)} \le n v^1_{\max}v^2_{\max}\left|\widetilde{J}(I^{i^*})\right|\left|\widetilde{I}(J^{j^*})\right| \le 9 v^1_{\max}v^2_{\max} n\widetilde{r}_n^2.$$

Therefore, $\Delta^{(B)} \xrightarrow{P} 0$ by Lemma 8.

(C): We are going to show

$$\Delta^{(C)} := \left|V_n^{(B)} - V_n^{(C)}\right| \xrightarrow{P} 0.$$

To this end, note that

$$\begin{aligned}\Delta^{(C)} &= n\left|\sum_{i=1}^{i^*} s(i)^2\left|\widetilde{I}^i\right|\left|\widetilde{J}(\widetilde{I}^i)\right| - \sum_{i=1}^{\lambda(n)} s(i)^2\left|\widetilde{I}^i\right|\left|\widetilde{J}(\widetilde{I}^i)\right|\right| \le n\sum_{i=(\lambda(n)\wedge i^*)+1}^{\lambda(n)\vee i^*} s(i)^2\left|\widetilde{I}^i\right|\left|\widetilde{J}(\widetilde{I}^i)\right| \\ &\le n v^1_{\max}v^2_{\max}\sum_{i=(\lambda(n)\wedge i^*)+1}^{\lambda(n)\vee i^*}\left|\widetilde{I}^i\right|\left|\widetilde{J}(\widetilde{I}^i)\right|.\end{aligned}$$

Because $\left|\widetilde{I}^i\right| \le \widetilde{r}_n$ and $\left|\widetilde{J}\left(\widetilde{I}^i\right)\right| \le 3\widetilde{r}_n$ for $i \le \lambda(n)\vee i^*$ and $(\lambda(n)\vee i^* - \lambda(n)\wedge i^*) \le n^{\epsilon}$ on Ω_0^n, one has

$$0 \le \Delta^{(C)}1_{\Omega_0^n} \le 3v^1_{\max}v^2_{\max}n^{1+\epsilon}\left(\widetilde{r}_n\right)^2 \xrightarrow{P} 0$$

in light of Lemma 8. Together with Lemma 9, one has $\Delta^{(C)} \xrightarrow{P} 0$ as intended.

(D): We are going to prove that

$$V_n^{(C)} \xrightarrow{P} \theta^{(c)} \equiv \left(\frac{2}{p^1} + \frac{2}{p^2}\right)\int_0^{T^1}\left(\sigma^1_t\sigma^2_t\right)^2 \mathrm{d}t$$

as $n \to \infty$.

To this end, first recall that, according to Lemma 7, for $1 \le i \le \lambda(n)$,

$$E^{\widetilde{\Pi}^1}\left[\left|\widetilde{J}(\widetilde{I}^i)\right|\right] = \left|\widetilde{I}^i\right| + \frac{2}{np^2} - \frac{1}{np^2}\exp\left\{-np^2T^{1,i-1}\right\} = \left|\widetilde{I}^i\right| + \frac{2}{np^2} - Q_i,$$

where $Q_i := \frac{1}{np^2}\exp\left\{-np^2T^{1,i-1}\right\}$ for ease of notation.

(D1) Let

$$\begin{aligned}
\overline{V}_n^{(C)} &:= n\sum_{i=1}^{\lambda(n)} s(i)^2\left|\widetilde{I}^i\right|\left(E^{\widetilde{\Pi}^1}\left[\left|\widetilde{J}(\widetilde{I}^i)\right|\right] + Q_i\right)\\
&= n\sum_{i=1}^{\lambda(n)} s(i)^2\left|\widetilde{I}^i\right|^2 + \sum_{i=1}^{\lambda(n)} s(i)^2\left|\widetilde{I}^i\right|\frac{2}{p^2}, \text{ and}\\
M_n' &:= V_n^{(C)} - \overline{V}_n^{(C)}\\
&= n\sum_{i=1}^{\lambda(n)} s(i)^2\left|\widetilde{I}^i\right|\left(\left|\widetilde{J}(\widetilde{I}^i)\right| - E^{\widetilde{\Pi}^1}\left[\left|\widetilde{J}(\widetilde{I}^i)\right|\right] - Q_i\right) = n\sum_{i=1}^{\lambda(n)} h(i),
\end{aligned}$$

where

$$\begin{aligned}
h(i) &:= s(i)^2\left|\widetilde{I}^i\right|\left(\left|\widetilde{J}(\widetilde{I}^i)\right| - E^{\widetilde{\Pi}^1}\left[\left|\widetilde{J}(\widetilde{I}^i)\right|\right] - Q_i\right)\\
&= s(i)^2\left|\widetilde{I}^i\right|\left(\left|\widetilde{J}(\widetilde{I}^i)\right| - \left|\widetilde{I}^i\right| - \frac{2}{np^2}\right).
\end{aligned}$$

We claim that $M_n' \xrightarrow{P} 0$ so that $V_n^{(C)} \simeq \overline{V}_n^{(C)}$ as $n \to \infty$. For this, for an arbitrary $\varepsilon \in (0,1)$ (which can be different from ϵ defined for Lemma 9), decompose

$$\begin{aligned}
E\left[\left(M_n'\right)^2\right] &= n^2\sum_{i=1}^{\lambda(n)}\sum_{i'=1:|i-i'|\le n^\varepsilon}^{\lambda(n)} E\left[h(i)h(i')\right]\\
&\quad + n^2\sum_{i=1}^{\lambda(n)}\sum_{i'=1:|i-i'|> n^\varepsilon}^{\lambda(n)} E\left[h(i)h(i')\right] =: E_1 + E_2.
\end{aligned}$$

Regarding E_1,

$$n^{-2}\left|E_1\right| \le \sum_{i=1}^{\lambda(n)}\sum_{i'=i-\lceil n^\varepsilon\rceil\vee 1}^{(i+\lceil n^\varepsilon\rceil)\wedge\lambda(n)} E\left[\left|h(i)h(i')\right|\right] \lesssim \sum_{i=1}^{\lambda(n)}\sum_{i'=i-\lceil n^\varepsilon\rceil\vee 1}^{(i+\lceil n^\varepsilon\rceil)\wedge\lambda(n)} E\left[\widetilde{r}_n^4\right].$$

Recalling that $\lambda(n) = \lceil np_1T\rceil$, one has

$$\left|E_1\right| \lesssim n^2\cdot n(2n^\varepsilon + 1)\cdot E\left[\widetilde{r}_n^4\right] \to 0$$

as $n \to \infty$, by Lemma 8.

Next consider E_2. Let $H_{ii'} := \left\{\tau^{2,i} < T^{1,i'-1}\right\}$ (recall that $\tau^{2,i} \equiv T^{2,\widetilde{j}_+(i)}$). Decompose

$$\frac{1}{2}E_2 \equiv n^2 \sum_{i=1}^{\lambda(n)} \sum_{i'=1:i'-i>n^\varepsilon}^{\lambda(n)} E\left[h(i)h(i')1_{H_{ii'}}\right]$$
$$+n^2 \sum_{i=1}^{\lambda(n)} \sum_{i'=1:i'-i>n^\varepsilon}^{\lambda(n)} E\left[h(i)h(i')1_{H_{ii'}^c}\right] =: E_{2-1} + E_{2-2}.$$

We claim that E_{2-2} is negligible. By Hölder's inequality,

$$\left|E\left[h(i)h(i')1_{H_{ii'}^c}\right]\right| \le \left\{E\left[\left(h(i)h(i')\right)^2\right]\right\}^{\frac{1}{2}} \left\{P\left[H_{ii'}^c\right]\right\}^{\frac{1}{2}} \lesssim \left\{E\left[\widetilde{r}_n^8\right]\right\}^{\frac{1}{2}} \cdot \left\{P\left[H_{ii'}^c\right]\right\}^{\frac{1}{2}}.$$

For i and i' with $i' - i > n^\varepsilon$, it should be noted that $\tau^{2,i} \ge T^{1,i'-1} \ge T^{1,\lceil n^\varepsilon\rceil+i}$ on $H_{ii'}^c \equiv \left\{\tau^{2,i} \ge T^{1,i'-1}\right\}$ since $i' \ge \lceil n^\varepsilon\rceil + i + 1$, thus, using the Markov property at $T^{1,i}$,

$$P\left[H_{ii'}^c\right] = P\left[\tau^{2,i} \ge T^{1,i'-1}\right] \le P\left[\tau^{2,i} - T^{1,i} \ge T^{1,\lceil n^\varepsilon\rceil+i} - T^{1,i}\right]$$
$$= E\left[\exp\left\{-np^2\left(T^{1,\lceil n^\varepsilon\rceil+i} - T^{1,i}\right)\right\}\right].$$

Since $\left(T^{1,\lceil n^\varepsilon\rceil+i} - T^{1,i}\right) \sim \Gamma\left(\lceil n^\varepsilon\rceil, \left(np^1\right)^{-1}\right)$ (Fact 4),

$$E\left[\exp\left\{-np^2\left(T^{1,\lceil n^\varepsilon\rceil+i} - T^{1,i}\right)\right\}\right] = \gamma^{\lceil n^\varepsilon\rceil},$$

where $\gamma := \frac{p^1}{p^1+p^2}$. It follows that

$$|E_{2-2}| \lesssim n^2 \cdot \lambda(n)^2 \cdot E\left[\widetilde{r}_n^8\right]^{\frac{1}{2}} \cdot \gamma^{\frac{\lceil n^\varepsilon\rceil}{2}} \to 0$$

because $\gamma \in (0, 1)$ and by Lemma 8.

We are going to show that E_{2-1} is also negligible. Recall Fact 1, i.e., $\widetilde{j}_+(i)$ is a discrete-time stopping time relative to $\{\mathcal{G}_k; k \ge 0\}$ where $\mathcal{G}_k \equiv \sigma\left\langle S^{2,j}; 1 \le j \le k\right\rangle \vee \sigma\left\langle\widetilde{\Pi}^1\right\rangle$ and $S^{2,k} \equiv T^{2,k} - T^{2,k-1}$. Hence, the σ-field $\mathcal{G}_{\widetilde{j}_+(i)}$ is well-defined in the usual manner. Notice that $H_{ii'}$ is $\mathcal{G}_{\widetilde{j}_+(i)}$-measurable, so is $h(i)$.

Note at the same time that, whenever $i < i'$, $\tau^{2,i} \equiv T^{2,\widetilde{j}_+(i)} \le T^{2,\widetilde{j}_-(i')-1}$ on $H_{ii'} \equiv \left\{\tau^{2,i} < T^{1,i'-1}\right\}$, i.e., the interval $\widetilde{J}(\widetilde{I}^{i'})$ has not 'started' yet by time $t = \tau^{2,i}$.

Accordingly, for $i < i'$,

$$\begin{aligned}
&E\left[h(i)h(i')1_{H_{ii'}}\right] \\
&\quad = E\left[h(i)1_{H_{ii'}}E\left[h(i')\middle|\mathcal{G}_{\widetilde{j}_+(i)}\right]\right] \\
&\quad = E\left[h(i)1_{H_{ii'}}s(i')^2\left|\widetilde{I}^{i'}\right|E\left[\left|\widetilde{J}(\widetilde{I}^{i'})\right| - E^{\widetilde{\Pi}^1}\left[\left|\widetilde{J}(\widetilde{I}^{i'})\right|\right] - Q_{i'}\middle|\mathcal{G}_{\widetilde{j}_+(i)}\right]\right] \\
&\quad = -E\left[h(i)1_{H_{ii'}}s(i')^2\left|\widetilde{I}^{i'}\right|\frac{1}{np^2}\exp\left\{-np^2\left(T^{1,i'-1}-\tau^{2,i}\right)\right\}\right].
\end{aligned}$$

The last equality is due to the fact

$$E\left[\left|\widetilde{J}(\widetilde{I}^{i'})\right|\middle|\mathcal{G}_{\widetilde{j}_+(i)}\right]1_{H_{ii'}} = \left(\left|\widetilde{I}^{i'}\right| + \frac{2}{np^2} - \frac{1}{np^2}\exp\left\{-np^2\left(T^{1,i'-1}-\tau^{2,i}\right)\right\}\right)1_{H_{ii'}},$$

which can be shown analogously to Lemma 7. Moreover,

$$\begin{aligned}
\left|E\left[h(i)h(i')1_{H_{ii'}}; \tilde{r}_n > n^{-1+\frac{\varepsilon}{2}}\right]\right| &\lesssim n^{-1}E\left[\widetilde{r}_n^6\right]^{\frac{1}{2}} \\
&\quad\cdot E\left[\exp\left\{-2np^2\left(T^{1,i'-1}-\tau^{2,i}\right)\right\}1_{H_{ii'}}; \tilde{r}_n > n^{-1+\frac{\varepsilon}{2}}\right]^{\frac{1}{2}},
\end{aligned}$$

as well as

$$\begin{aligned}
\left|E\left[h(i)h(i')1_{H_{ii'}}; \tilde{r}_n \le n^{-1+\frac{\varepsilon}{2}}\right]\right| &\lesssim n^{-4+\frac{3}{2}\varepsilon} \\
&\quad\cdot E\left[\exp\left\{-np^2\left(T^{1,i'-1}-\tau^{2,i}\right)\right\}1_{H_{ii'}}; \tilde{r}_n \le n^{-1+\frac{\varepsilon}{2}}\right].
\end{aligned}$$

Therefore,

$$|E_{2-1}| \lesssim E_{2-1-1} + E_{2-1-2},$$

where

$$E_{2-1-1} := nE\left[\widetilde{r}_n^6\right]^{\frac{1}{2}}\sum_{i=1}^{\lambda(n)}\sum_{i'=1:i'-i>n^\varepsilon}^{\lambda(n)}E\left[\exp\left\{-2np^2\left(T^{1,i'-1}-\tau^{2,i}\right)\right\}1_{H_{ii'}}; \widetilde{r}_n > n^{-1+\frac{\varepsilon}{2}}\right]^{\frac{1}{2}},$$

$$E_{2-1-2} := n^{-2+\frac{3}{2}\varepsilon}\sum_{i=1}^{\lambda(n)}\sum_{i'=1:i'-i>n^\varepsilon}^{\lambda(n)}E\left[\exp\left\{-np^2\left(T^{1,i'-1}-\tau^{2,i}\right)\right\}1_{H_{ii'}}; \widetilde{r}_n \le n^{-1+\frac{\varepsilon}{2}}\right].$$

Since $-np^2\left(T^{1,i'-1}-\tau^{2,i}\right) < 0$ on $H_{ii'}$, one has

$$E\left[\exp\left\{-2np^2\left(T^{1,i'-1}-\tau^{2,i}\right)\right\}1_{H_{ii'}}; \widetilde{r}_n > n^{-1+\frac{\varepsilon}{2}}\right] \le P\left[\widetilde{r}_n > n^{-1+\frac{\varepsilon}{2}}\right] \le E\left[\widetilde{r}_n\right]n^{1-\frac{\varepsilon}{2}}$$

by Markov's inequality. Pick arbitrary $\alpha' \in \left(0, \frac{\varepsilon}{2}\right)$. Then, the last inequality implies that

$$E_{2-1-1} \lesssim nE\left[\widetilde{r}_n^6\right]^{\frac{1}{2}} \cdot \lambda(n)^2 \cdot \left\{E\left[\widetilde{r}_n\right]n^{1-\frac{\varepsilon}{2}}\right\}^{\frac{1}{2}}$$
$$\sim \left(n^{6-\alpha'}E\left[\widetilde{r}_n^6\right]^{\frac{1}{2}}\right) \cdot \left(E\left[\widetilde{r}_n\right]n^{1-\frac{\varepsilon}{2}+\alpha'}\right)^{\frac{1}{2}} \to 0$$

as $n \to \infty$ by Lemma 8.

In the meantime, since $\tau^{2,i} \le T^{1,i} + \widetilde{r}_n$, as well as $i'-1 \ge i + \lceil n^\varepsilon \rceil$ for $i'-i > n^\varepsilon$,

$$\begin{aligned}
&E\left[\exp\left\{-np^2\left(T^{1,i'-1} - \tau^{2,i}\right)\right\} 1_{H_{ii'}}; \widetilde{r}_n \le n^{-1+\frac{\varepsilon}{2}}\right] \\
&\le E\left[\exp\left\{-np^2\left(T^{1,i'-1} - T^{1,i} - \widetilde{r}_n\right)\right\} 1_{H_{ii'}}; \widetilde{r}_n \le n^{-1+\frac{\varepsilon}{2}}\right] \\
&\le \exp\left(p^2 n^{\frac{\varepsilon}{2}}\right) E\left[\exp\left\{-np^2\left(T^{1,i'-1} - T^{1,i}\right)\right\}\right] \\
&\le \exp\left(p^2 n^{\frac{\varepsilon}{2}}\right) E\left[\exp\left\{-np^2\left(T^{1,\lceil n^\varepsilon \rceil + i} - T^{1,i}\right)\right\}\right] = \exp\left(p^2 n^{\frac{\varepsilon}{2}}\right) \cdot \gamma^{\lceil n^\varepsilon \rceil},
\end{aligned}$$

where γ ($\in (0, 1)$) has been defined as above. Therefore,

$$E_{2-1-2} \lesssim n^{-2+\frac{3}{2}\varepsilon} \cdot \lambda(n)^2 \cdot \exp\left(p^2 n^{\frac{\varepsilon}{2}}\right) \cdot \gamma^{\lceil n^\varepsilon \rceil}$$
$$\lesssim n^{\frac{3}{2}\varepsilon} \cdot \exp\left(p^2 n^{\frac{\varepsilon}{2}}\right) \cdot \gamma^{\lceil n^\varepsilon \rceil} \to 0,$$

as $n \to \infty$. Notice that by taking the logarithm of the second factor,

$$\frac{3}{2}\varepsilon \log n + p^2 n^{\frac{\varepsilon}{2}} + \lceil n^\varepsilon \rceil \log \gamma \to -\infty$$

as $n \to \infty$ because $\log \gamma < 0$.

It follows that $E_{2-1} \to 0$, as $n \to \infty$, which in turn implies $E_2 \to 0$. Therefore, $M'_n \xrightarrow{P} 0$ so that $V_n^{(C)} \simeq \overline{V}_n^{(C)}$, as claimed.

(D2) Let

$$\overline{W}_n^{(C)} := \sum_{i=1}^{\lambda(n)} s(i)^2 \left|\widetilde{I}^i\right| \frac{2}{p^1}, \quad W_n^{(C)} := n\sum_{i=1}^{\lambda(n)} s(i)^2 \left|\widetilde{I}^i\right|^2, \text{ and}$$
$$M''_n := W_n^{(C)} - \overline{W}_n^{(C)} = n\sum_{i=1}^{\lambda(n)} s(i)^2 \left|\widetilde{I}^i\right| \left(\left|\widetilde{I}^i\right| - \frac{2}{np^1}\right).$$

Then, since $\left|\widetilde{I}^i\right|$ are i.i.d. exponential with intensity np^1,

$$E\left[s(i)^2 \left|\widetilde{I}^i\right| \left(\left|\widetilde{I}^i\right| - \frac{2}{np^1}\right) \middle| \mathcal{G}^1_{i-1}\right] = 0$$

where $\mathcal{G}^1_k := \sigma\left\{\left|\widetilde{I}^i\right|, 1 \le i \le k\right\}$, hence, $\left(M''_n\right)$ is a $\left(\mathcal{G}^1_k\right)$-martingale with mean 0.

Accordingly,

$$E\left[\left(M_n''\right)^2\right] = n^2 \sum_{i=1}^{\lambda(n)} E\left[s(i)^4 \left\{\left|\widetilde{I}^i\right|\left(\left|\widetilde{I}^i\right| - \frac{2}{np^1}\right)\right\}^2\right] \lesssim n^2 \cdot \lambda(n) \cdot E\left[\widetilde{r}_n^4\right] \to 0,$$

which implies that $W_n^{(C)} \simeq \overline{W}_n^{(C)}$ as $n \to \infty$.

It follows that

$$\begin{aligned}
\overline{V}_n^{(C)} &\equiv W_n^{(C)} + \sum_{i=1}^{\lambda(n)} s(i)^2 \left|\widetilde{I}^i\right| \frac{2}{p^2} \simeq \overline{W}_n^{(C)} + \sum_{i=1}^{\lambda(n)} s(i)^2 \left|\widetilde{I}^i\right| \frac{2}{p^2} \\
&= \sum_{i=1}^{\lambda(n)} s(i)^2 \left|\widetilde{I}^i\right| \frac{2}{p^1} + \sum_{i=1}^{\lambda(n)} s(i)^2 \left|\widetilde{I}^i\right| \frac{2}{p^2} \\
&\simeq \frac{2}{p^1} \int_0^{T^{1,\lambda(n)}} \left(\sigma_t^1 \sigma_t^2\right)^2 \mathrm{d}t + \frac{2}{p^2} \int_0^{T^{1,\lambda(n)}} \left(\sigma_t^1 \sigma_t^2\right)^2 \mathrm{d}t \\
&\to \theta^{(c)} \equiv \left(\frac{2}{p^1} + \frac{2}{p^2}\right) \int_0^{T^1} \left(\sigma_t^1 \sigma_t^2\right)^2 \mathrm{d}t,
\end{aligned}$$

as desired.

The whole proof of Lemma 1 is completed. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3 To invoke Theorem 2, Conditions (C2) and (C3) are to be checked, whereas (C1) and either (C4) or (C4′) are hypothesized. (C3) holds since $b_n = n^{-1}$ and $r_n = O_p(\log n/n)$.

Showing (C2) demands one to identify the limit c that will serve as the asymptotic variance of the quantity of interest, $\Psi_n \equiv \sqrt{n}(U_n - \theta)$. This can be done simply by recalling the equality (5) and applying Proposition 1 with the additional condition (C5). □

5 Concluding remarks

This paper is a sequel to Hayashi and Yoshida (2005b), which proposed the covariance estimator (3) for two diffusion-type processes when the processes are observed at discrete times in a nonsynchronous manner. Specifically, we have demonstrated asymptotic normality of the estimator as the observation interval shrinks to zero, when the variance/covariance structure is deterministic and the observation times are independent of the processes. As a non-trivial example, we have studied the case when observation times are independent Poisson arrival times; we have found the explicit limiting distribution of the estimator.

From practical viewpoints it is important to have *joint* asymptotic normality of the proposed estimator with *realized volatility* estimators, which has been treated in a successive paper, Hayashi and Yoshida (2005a). In the same paper, asymptotic normality of *correlation estimators* is discussed. Besides, in order to improve applicability of asymptotic normality, the paper introduces a more convenient (but more stringent) condition than (C2), which also leads to a stronger result.

It is also of practical importance to find a way to estimate the asymptotic variance c of the (rescaled) covariance estimator, which depends on the variance–covariance structure of the underlying processes and hence is unknown. It can be done, for instance, with bootstrapping.

Hayashi and Kusuoka (2004) extended Hayashi and Yoshida (2005b) to the general case where underlying processes are *continuous semimartingales* and observation times are *stopping times,* which is also of practical importance. They showed that consistency is preserved. Limiting distribution under this general case has been studied by the current authors (Hayashi and Yoshida 2006).

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A Appendix

A.1 Derivation of (5)

We are going to provide a sketch for the derivation of (5), following the argument in the proof of Theorem 3.1 in Hayashi and Yoshida (2005b). See the reference for details.

Suppose $\mu^l \equiv 0, l = 1, 2$. To verify (5), one may decompose first as

$$\begin{aligned} E^{\Pi}\left[U_n^2\right] &= \sum_{i,j,i',j'} E^{\Pi}\left\{\Delta P^1(I^i)\Delta P^2(J^j)\Delta P^1(I^{i'})\Delta P^2(J^{j'})\right\} K_{ij}K_{i'j'} \\ &= \sum_{\substack{i,j,i',j':\\ i'=i,j'=j}} + \sum_{\substack{i,j,i',j':\\ i'=i,j'\neq j}} + \sum_{\substack{i,j,i',j':\\ i'\neq i,j'=j}} + \sum_{\substack{i,j,i',j':\\ i'\neq i,j'\neq j}} =: D_1 + D_2 + D_3 + D_4. \end{aligned}$$

Using the orthogonality of the increments repeatedly, one finds that

$$\begin{aligned} D_1 &= \sum_{i,j} v^1\left(I^i\right) v^2\left(J^j\right) K_{ij} + 2\sum_{i,j} v\left(I^i \cap J^j\right)^2 K_{ij}, \\ D_2 &= 2\sum_{i} v\left(I^i\right)^2 - 2\sum_{i}\sum_{j} v\left(I^i \cap J^j\right)^2 K_{ij}, \end{aligned}$$

$$D_3 = 2\sum_i v\left(J^j\right)^2 - 2\sum_i \sum_j v\left(I^i \cap J^j\right)^2 K_{ij},$$

$$D_4 = \theta^2 + \sum_{i,j} v\left(I^i \cap J^j\right)^2 K_{ij} - \sum_i v\left(I^i\right)^2 - \sum_j v\left(J^j\right)^2.$$

Putting all together, one obtains

$$E^\Pi\left[U_n^2\right] = \sum_{i,j} v^1\left(I^i\right) v^2\left(J^j\right) K_{ij} + \sum_i v\left(I^i\right)^2 + \sum_i v\left(J^j\right)^2 - \sum_{i,j} v\left(I^i \cap J^j\right)^2 K_{ij} + \theta^2$$

as desired.

A.2 Supplement to page 10

We are going to show that (C4) implies (C4′) if (C4) holds with some $k \in (\frac{1}{4}, \frac{1}{2})$, in the case where $b_n = \kappa_n (E[N_n])^{-1}$ for some positive, bounded sequence (κ_n) and $E[N_n] \uparrow \infty$. Recall that $N_n \equiv \#(\Pi_1) \vee \#(\Pi_2)$, where $\#(A)$ counts the number of elements in a given set A.

First note that, for some $\zeta \in (0, \frac{1}{4})$, for almost all $\omega \in \Omega$,

$$\widetilde{K}(\omega) := \sup_n \frac{\delta(\mu^l(\omega); r_n(\omega))}{r_n(\omega)^{\frac{1}{2}-\zeta}} < \infty.$$

Then, since $r_n \geq \frac{T}{N_n}$, one has, for almost all $\omega \in \Omega$, for all n,

$$\delta(\mu^l(\omega); r_n(\omega)) \leq r_n(\omega)^{\frac{1}{2}-\zeta} \widetilde{K}(\omega) \leq r_n(\omega)^{\frac{1}{2}} N_n^\zeta(\omega) \widetilde{K}'(\omega),$$

where $\widetilde{K}'(\omega) := T^{-\zeta} \widetilde{K}(\omega)$. Now, if we put $G_n := \left\{ N_n^\zeta \leq b_n^{-\left(\frac{1}{4}+\alpha'\right)} \right\}$ for arbitrary $\alpha' \in (0, \alpha)$, then Markov's inequality implies that

$$\begin{aligned} P[G_n] &= P\left[N_n^\zeta \leq \kappa_n^{-\left(\frac{1}{4}+\alpha'\right)} (E[N_n])^{\left(\frac{1}{4}+\alpha'\right)} \right] \\ &= P\left[\frac{N_n}{E[N_n]} \leq \kappa_n^{-\frac{\frac{1}{4}+\alpha'}{\zeta}} (E[N_n])^{\frac{\frac{1}{4}+\alpha'}{\zeta}-1} \right] \\ &\geq 1 - \kappa_n^{\frac{\frac{1}{4}+\alpha'}{\zeta}} (E[N_n])^{-\frac{\frac{1}{4}+\alpha'}{\zeta}+1} \quad \uparrow 1, \end{aligned}$$

as $n \to \infty$, since $\frac{1}{4} + \alpha' > \zeta$ and $E[N_n] \uparrow \infty$.

Moreover,

$$
\begin{aligned}
P\left[G_n\right] &= P\left[\left\{\delta(\mu^l; r_n) \leq r_n^{\frac{1}{2}} N_n^{\zeta} \widetilde{K}'\right\} \cap G_n\right] \\
&\leq P\left[\left\{\delta(\mu^l; r_n) \leq r_n^{\frac{1}{2}} b_n^{-\left(\frac{1}{4}+\alpha'\right)} \widetilde{K}'\right\} \cap G_n\right] \\
&\leq P\left[\delta(\mu^l; r_n) \leq r_n^{\frac{1}{2}} b_n^{-\left(\frac{1}{4}+\alpha'\right)} \widetilde{K}'\right],
\end{aligned}
$$

which implies that

$$
1 \leq \liminf_{n\to\infty} P\left[\delta(\mu^l; r_n) \leq r_n^{\frac{1}{2}} b_n^{-\left(\frac{1}{4}+\alpha'\right)} \widetilde{K}'\right].
$$

Finally, for arbitrary constant $M < \infty$,

$$
\begin{aligned}
&\liminf_{n\to\infty} P\left[\delta(\mu^l; r_n) \leq r_n^{\frac{1}{2}} b_n^{-\left(\frac{1}{4}+\alpha'\right)} M\right] \\
&\quad\geq \liminf_{n\to\infty} P\left[\delta(\mu^l; r_n) \leq r_n^{\frac{1}{2}} b_n^{-\left(\frac{1}{4}+\alpha'\right)} \widetilde{K}', \widetilde{K}' \leq M\right] \\
&\quad\geq \liminf_{n\to\infty} P\left[\delta(\mu^l; r_n) \leq r_n^{\frac{1}{2}} b_n^{-\left(\frac{1}{4}+\alpha'\right)} \widetilde{K}'\right] - P\left[\widetilde{K}' > M\right] = 1 - P\left[\widetilde{K}' > M\right].
\end{aligned}
$$

Letting $M \uparrow \infty$ leads to (C4′), as desired.

References

Andersen, T. G., Bollerslev, T., Diebold, F. X., Labys, P. (2001). The distribution of realized exchange rate volatility, *Journal of the American Statistical Association*, *96(453)*, 42–55.

Dacorogna, M. M., Gençay, R., Müller, U. A., Olsen, R. B., Pictet, O. V. (2001). *An Introduction to High-Frequency Finance*. San Diego: Academic Press.

Dacunha-Castelle, D., Florens-Zmirou, D. (1986). Estimation of the coefficients of diffusion from discrete observations. *Stochastics, 19(4)*, 263–284.

Hayashi, T., Kusuoka, S. (2004). Nonsynchronous covariation measurement for continuous semimartingales. Preprint 2004-21, Graduate School of Mathematical Sciences, University of Tokyo.

Hayashi, T., Yoshida, N. (2004). Asymptotic normality of nonsynchronous covariance estimators for diffusion processes. Preprint.

Hayashi, T., Yoshida, N. (2005a). Estimating correlations with missing observations in continuous diffusion models. Preprint.

Hayashi, T., Yoshida, N. (2005b). On covariance estimation of non-synchronously observed diffusion processes. *Bernoulli, 11(2)*, 359–379.

Hayashi, T., Yoshida, N. (2006). Nonsynchronous covariance estimator and limit theorem. Research Memorandum No. 1020, Institute of Statistical Mathematics.

Karatzas, I., Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd edn. New York: Springer.

Zhang, L., Mykland, P. A., Aït-Sahalia, Y. (2005). A tale of two time scales: determining integrated volatility with noisy high-frequency data, *Journal of the American Statistical Association, 100(472)*, 1394–1411.