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Estimating a general function of a quadratic function

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Abstract Let $x \in \mathbb{R}^p$ be an observation from a spherically symmetric distribution with unknown location parameter $\theta \in \mathbb{R}^p$. For a general non-negative function c, we consider the problem of estimating $c(\|x-\theta\|^2)$ under the usual quadratic loss. For $p \geq 5$, we give sufficient conditions for improving on the unbiased estimator γ_0 of $c(\|x-\theta\|^2)$ by competing estimators $\gamma_s = \gamma_0 + s$ correcting γ_0 with a suitable function s. The main condition relies on a partial differential inequality of the form $k\,\Delta s + s^2 \leq 0$ for a certain constant $k \neq 0$. Our approach unifies, in particular, the two problems of quadratic loss estimation and confidence statement estimation and allows to derive new results for these two specific cases. Note that we formally establish our domination results (that is, with no recourse to simulation).

Keywords Loss estimation · Confidence statement · Spherically symmetric distribution · Green integral formulas · Sobolev spaces · Differential inequations

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1 Introduction

Let X be a random vector in $\mathbb{R}^p$ from a spherically symmetric distribution around a fixed vector $\theta \in \mathbb{R}^p$. More specifically, we assume that X has a generating function f, that is, X has a density of the form $x \mapsto f(\|x-\theta\|^2)$ where θ is the unknown location parameter. In what follows, as an estimator δ of θ, we only consider the

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least square estimator $\delta(X) = X$. For a given non-negative function c on $\mathbb{R}_+$, we are interested in estimating the quantity $c(\|x - \theta\|^2)$ when x has been observed from X. This problem recovers both the usual case of estimating the quadratic loss $\|x - \theta\|^2$ (c is the identity function as in Johnstone, 1988; Lu and Berger, 1989; and Fourdrinier and Wells, 1995b) and the case of estimating the confidence statement of the usual confidence set $\{\theta \in \mathbb{R}^p \,/\, \|x - \theta\|^2 \leq c_\alpha\}$ with confidence coefficient $1 - \alpha$ (c is the indicator function $\mathbb{1}_{[0,c_\alpha]}$; see Robert and Casella, 1994 for e.g.). This approach is in the framework of the theory of conditional inference formalized by Robinson (1979a, b).

When $E_\theta[c(\|X - \theta\|^2)] < \infty$ (where E_θ denotes the expectation with respect to the density $x \mapsto f(\|x - \theta\|^2)$), a natural estimator is the unbiased estimator $\gamma_0 = E_0[c(\|X\|^2)]$. Since it is a constant estimator, it is natural to search other estimators γ and a simple way of comparison is to use the quadratic risk defined by

$$R(\gamma, \theta) = E_\theta\Big[\big(\gamma - c\big(\|X - \theta\|^2\big)\big)^2\Big]. \tag{1}$$

Then an estimator γ will be better than γ_0 (or will dominate γ_0) if, for any $\theta \in \mathbb{R}^p$,

$$R(\gamma, \theta) \leq R(\gamma_0, \theta)$$

with strict inequality for some θ. Of course, the last inequality makes only sense when $E_0[c^2(\|X\|^2)] < \infty$.

Note that, in lower dimension, γ_0 is still a good estimator with respect to the quadratic risk (1) since it can be shown that γ_0 is admissible for $p \leq 4$. Therefore, in the following, we assume that $p \geq 5$.

Any estimator γ can be written under the form $\gamma = \gamma_s = \gamma_0 + s$ for some function s which can be viewed as a correction of γ_0 (actually $s = \gamma - \gamma_0$). Our goal is then to yield conditions on s such that γ_s dominates γ_0. Our approach consists in developing an upper bound of the risk difference $\delta_\theta = R(\gamma_s, \theta) - R(\gamma_0, \theta)$ between γ_s and γ_0 in terms of the expectation of a differential expression of the form $k\,\Delta s + s^2$ where k is a constant different from 0 and $\Delta s = \sum_{i=1}^p D_{ii} s$ is the Laplacian of s for $D_{ii} = \frac{\partial^2}{\partial x_i^2}$. Although it is not originally present in the risk difference δ_θ, the introduction of the Laplacian of the correction s is the main key of our results. Its intervention relies on a Green formula type which implies the consideration of Sobolev spaces.

Often, in the literature, the domination of γ_s over γ_0 is tackled through Taylor expansions of their risk difference δ_θ. The possible weakness of that technique is that it may be difficult to control the sign of δ_θ, so that formal domination is only obtained for θ around 0 and in a neighborhood of infinity (this is the case in Robert and Casella (1994). The advantage of our approach is that it allows to give a formal proof for all values of θ. A possible drawback is that we work with an upper bound of δ_θ, which may be crude. However, under certain conditions, we are in a position to provide an accurate upper bound.

In Sect. 2, we present the model and give a technical lemma useful to introduce the Laplacian Δs in the risk difference. Then we establish our main result of domination over γ_0. This domination relies on the upper bound $\overline{\delta_\theta} = E_\theta[k\,\Delta s + s^2]$ of the risk difference δ_θ, for a specific value of k, and one expresses the fact that

$\overline{\delta_\theta} \leq 0$, and hence $\delta_\theta \leq 0$, through the differential inequality $k\, \Delta s(x) + s^2(x) \leq 0$ for any $x \in \mathbb{R}^p$. In a second result, we exhibit a smaller upper bound $\check{\delta_\theta}$ for δ_θ corresponding to a greater value of $|k|$. It is obtained at the price of additional conditions on the functions f and c so that the differential inequation mentioned above allows to yield a wider class of corrections s. Section 3 is devoted to several applications: quadratic loss estimation, concave loss estimation and confidence statement estimation. In Sect. 4, we give some conclusions and perspectives. Finally Sect. 5 is an appendix containing technical lemmas with their proofs.

2 Improved estimators of $c(\|x - \theta\|^2)$

The goal is to determine conditions on the function s so that the risk difference $\delta_\theta = R(\gamma_s, \theta) - R(\gamma_0, \theta)$ is non-positive for any $\theta \in \mathbb{R}^p$ and negative for some $\theta \in \mathbb{R}^p$. As previously noticed in Sect. 1, it is necessary to assume that $E_\theta[c^2(\|X - \theta\|^2)] < \infty$. Then it is easy to check through the Schwarz inequality that $R(\gamma_s, \theta) < \infty$ if and only if $E_\theta[s^2] < \infty$. In that case, it is clear that

$$\delta_\theta = E_\theta[2\,(\gamma_0 - c(\|X - \theta\|^2))\, s(X) + s^2(X)]. \tag{2}$$

Our approach consists in introducing the Laplacian of the correction function s, say Δs, under the expectation sign in the right-hand side of (2). We will see that this can be done with the use of a Green formula type

$$\int_{\mathbb{R}^p} u(x)\, \Delta v(x)\, \mathrm{d}x = \int_{\mathbb{R}^p} v(x)\, \Delta u(x)\, \mathrm{d}x \tag{3}$$

for suitable functions u and v. Conditions for Formula (3) to hold are specified in Lemma 1 below. Note that (3) is fundamentally an integration by parts formula which depends on the spaces where the functions u and v live; those are naturally Sobolev spaces. More precisely, we need u to be in the space $W^{2,1}_{loc}(\mathbb{R}^p)$ of the functions twice weakly differentiable from $\mathbb{R}^p$ into $\mathbb{R}$. Recall that a function u from $\mathbb{R}^p$ into $\mathbb{R}$ is said to be weakly differentiable if u is locally integrable and if, for any $i = 1, \ldots, p$, there exists a locally integrable function denoted by $D_i u$ such that, for any function ϕ infinitely differentiable with compact support from $\mathbb{R}^p$ into $\mathbb{R}$,

$$\int_{\mathbb{R}^p} u(x)\, D_i\phi(x)\, \mathrm{d}x = -\int_{\mathbb{R}^p} D_i u(x)\, \phi(x)\, \mathrm{d}x.$$

Although Formula (3) is symmetric in u and v, the assumptions on the function u are not exactly the same as those on the function v. We require v to be in the space $W^{2,\infty}(\mathbb{R}^p)$ of the functions twice weakly differentiable from $\mathbb{R}^p$ into $\mathbb{R}$ and essentially bounded (that is, bounded almost everywhere).

In the following, for any open set Ω in $\mathbb{R}^p$, we denote by $C^2_b(\Omega)$ the space of the functions twice continuously differentiable and bounded on Ω. Furthermore, for any $l \in \mathbb{N}$ and any $r > 0$, the set $\mathcal{S}^{l,r}(\Omega)$ is the space of the functions l times continuously differentiable v on Ω such that

$$\sup_{x \in \Omega\,;\, |\alpha| \leq l\,;\, \beta \leq r} \|x\|^\beta\, |D^\alpha\, v(x)| < \infty,$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)$ denotes a multi-index (i.e. a p-tuple of non-negative integers) such that its length satisfies $|\alpha| = \alpha_1 + \cdots + \alpha_p \leq l$ and D^α is the corresponding partial derivative operator.

Lemma 1 *Let $u \in W^{2,1}_{loc}(\mathbb{R}^p)$ and $v \in W^{2,\infty}(\mathbb{R}^p)$. If there exist $r > 0$ such that $u \in C^2_b(\mathbb{R}^p \setminus B_r)$ and $\epsilon > 0$ such that $v \in \mathcal{S}^{2,p+\epsilon}(\mathbb{R}^p \setminus B_r)$ (where B_r is the ball $\{x \in \mathbb{R}^p \,/\, \|x\| \leq r\}$ of radius r and centered at the origin), then the functions $u\,\Delta v$ and $v\,\Delta u$ are integrable and the corresponding integrals on $\mathbb{R}^p$ are equal, that is,*

$$\int_{\mathbb{R}^p} u(x)\,\Delta v(x)\,\mathrm{d}x = \int_{\mathbb{R}^p} v(x)\,\Delta u(x)\,\mathrm{d}x.$$

Proof See Blouza et al. (2006).

We are now in a position to give a new expression for the risk difference δ_θ in (2).

Theorem 1 *Let s be a function from $\mathbb{R}^p$ into $\mathbb{R}$ such that $E_\theta[s^2] < \infty$. Assume that there exists $r > 0$ such that $s \in W^{2,1}_{loc}(\mathbb{R}^p) \cap C^2_b(\mathbb{R}^p \setminus B_r)$. Assume also that the functions f and c are continuous on $\mathbb{R}^*_+$, except possibly on a finite set T, and there exists $\epsilon > 0$ such that f and $f\,c$ belong to $S^{0,p/2+1+\epsilon}(\mathbb{R}^*_+ \setminus T)$. Then*

$$\delta_\theta = E_\theta\left[\frac{K(\|X-\theta\|^2)}{f(\|X-\theta\|^2)}\,\Delta s(X) + s^2(X)\right], \tag{4}$$

where K is the function depending on f and c defined, for any $t > 0$, by

$$K(t) = \frac{1}{p-2}\int_t^\infty \left[\left(\frac{y}{t}\right)^{p/2-1} - 1\right](\gamma_0 - c(y))\,f(y)\,\mathrm{d}y. \tag{5}$$

Proof According to Formula (2), the proof first relies on the fact that, in Lemma 2 of Appendix, it is shown that, for almost every $x \in \mathbb{R}^p$,

$$\Delta K(\|x-\theta\|^2) = 2(\gamma_0 - c(\|x-\theta\|^2))\,f(\|x-\theta\|^2)$$

and hence

$$E_\theta[2\,(\gamma_0 - c(\|X-\theta\|^2))\,s(X)] = \int_{\mathbb{R}^p} \Delta K(\|x-\theta\|^2)\,s(x)\,\mathrm{d}x.$$

Now, by assumption, $s \in W^{2,1}_{loc}(\mathbb{R}^p) \cap C^2_b(\mathbb{R}^p \backslash B_r)$ for some $r > 0$ and Lemmas 5 and 6 (see Appendix) express that the function $x \longmapsto K(\|x-\theta\|^2)$ is in $W^{2,\infty}(\mathbb{R}^p) \cap \mathcal{S}^{2,\,p+\epsilon}(\mathbb{R}^p \setminus B_r)$ for some $\epsilon > 0$. Therefore Lemma 1 applies and gives

$$\begin{aligned} E_\theta[2(\gamma_0 - c(\|X-\theta\|^2))\,s(X)] &= \int_{\mathbb{R}^p} K(\|x-\theta\|^2)\,\Delta s(x)\,\mathrm{d}x] \\ &= E_\theta\left[\frac{K(\|X-\theta\|^2)}{f(\|X-\theta\|^2)}\,\Delta s(X)\right]. \end{aligned}$$

Finally, as $E_\theta[s^2] < \infty$, the risk difference δ_θ exists and has the desired expression. $\square$

In order to obtain sufficient domination conditions of $\gamma_0 + s(X)$ over γ_0 it is needed to control the behavior of the coefficient $\frac{K(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}$ in (4). Our approach consists in giving conditions on the functions f, c, and s such that

$$E_\theta\left[\frac{K(\|X-\theta\|^2)}{f(\|X-\theta\|^2)}\,\Delta s(X)\right] \le E_\theta[k\,\Delta s(X)]$$

for some constant k different from 0. Before stating these conditions in the following theorem, note that the fact that $f \in S^{0,\,p/2+1+\epsilon}(\mathbb{R}_+^* \setminus T)$ implies that f is bounded from above by a constant $M > 0$.

Theorem 2 *Under the conditions of Theorem 1, assume that the function $\gamma_0 - c$ has only one sign change. In the case where $\gamma_0 - c$ is first negative and then positive (respectively first positive and then negative), assume that the Laplacian of s is subharmonic (respectively superharmonic).*

Then a sufficient condition for γ_s to dominate γ_0 is that s satisfies the partial differential inequality

$$k\,\Delta s + s^2 \le 0, \tag{6}$$

where k is the constant defined by

$$k = \frac{1}{M} E_0[K(\|X\|^2)].$$

Proof Note that, in the case where the function $\gamma_0 - c$ is first negative and then positive, the function K is positive according to Lemma 4 of Appendix and hence $k > 0$. Then Inequality (6) imposes that $\Delta s \le 0$ (that is, the function s is superharmonic). Similarly, when $\gamma_0 - c$ is first positive and then negative, we have $k < 0$ and consequently $\Delta s \ge 0$ (the function s is subharmonic). Therefore, in both cases, for any $x \in \mathbb{R}^p$, the product $K(\|x-\theta\|^2)\,\Delta s(x)$ is non-positive and, as $f \le M$, we have

$$E_\theta\left[\frac{K(\|X-\theta\|^2)}{f(\|X-\theta\|^2)}\,\Delta s(X)\right] \le \frac{1}{M}\,E_\theta[K(\|X-\theta\|^2)\,\Delta s(X)]. \tag{7}$$

Now the last expectation in (7) can be written as

$$\begin{aligned} E_\theta[K(\|X-\theta\|^2)\,\Delta s(X)] &= \int_{\mathbb{R}^p} K(\|x-\theta\|^2)\,\Delta s(x)\, f(\|x-\theta\|^2)\,\mathrm{d}x \\ &= \int_0^\infty \int_{S_{r,\theta}} \Delta s(x)\,\mathrm{d}U_{r,\theta}(x)\, K(r^2)\,\frac{2\pi^{p/2}}{\Gamma(p/2)}\, r^{p-1} f(r^2)\,\mathrm{d}r, \end{aligned} \tag{8}$$

where $U_{r,\theta}$ is the uniform distribution on the sphere $S_{r,\theta} = \{x \in \mathbb{R}^p / \|x - \theta\| = r\}$ of radius r and centered at θ. Note that the function $r \mapsto \frac{2\pi^{p/2}}{\Gamma(p/2)} r^{p-1} f(r^2)$ is the radial density, that is, the density of the radius $R = \|X - \theta\|$.

For simplicity, we only develop the case where $\gamma_0 - c$ is first negative and then positive. By assumption, the superharmonic function s has its Laplacian Δs which is subharmonic (i.e. $\Delta(\Delta s) \geq 0$). So the mean $\int_{S_{\theta,r}} \Delta s(x) \, \mathrm{d}U_{\theta,r}(x)$ is a non-decreasing function of r (see e.g. Doob, 1984). Furthermore, as by Lemma 4 the function K is non-increasing, then, by covariance inequality, it follows from (8) that

$$\begin{aligned} E_\theta[K(\|X - \theta\|^2) \, \Delta s(X)] &\leq \int_0^\infty K(r^2) \frac{2\pi^{p/2}}{\Gamma(p/2)} r^{p-1} f(r^2) \, \mathrm{d}r \\ &\quad \times \int_0^\infty \int_{S_{r,\theta}} \Delta s(x) \, \mathrm{d}U_{r,\theta}(x) \, \frac{2\pi^{p/2}}{\Gamma(p/2)} r^{p-1} f(r^2) \, \mathrm{d}r \\ &= M \, k \, E_\theta[\Delta \, s(X)] \end{aligned}$$

by definition of k.

Now, returning to Inequality (7), we obtain that

$$E_\theta\left[\frac{K(\|X - \theta\|^2)}{f(\|X - \theta\|^2)} \, \Delta s(X) \right] \leq E_\theta[k \, \Delta s(X)]$$

and finally that the risk difference in (4) satisfies

$$\delta_\theta \leq E_\theta[k \, \Delta s(X) + s^2(X)] \leq 0$$

according to (6).

The second case ($\gamma_0 - c$ is first positive and then negative) can be tackled in the same way. Thus γ_s dominates γ_0. □

The proof of Theorem 2 uses, through Inequality (7), the property that the generating function f is bounded by M. This fact leads to a constant k in (6) which may be small and hence may reduce the scope of the possible corrections s generating the improved estimators γ_s. We give, in the next theorem, an additional condition which avoids the use of M; that condition relies on the monotonicity of the ratio $\frac{K}{f}$.

Theorem 3 *Under the conditions of Theorem 2, assume that the functions K and $\frac{K}{f}$ have the same monotonicity (both non-increasing or both non-decreasing).*

Then a sufficient condition for γ_s to dominate γ_0 is that s satisfies the partial differential inequality

$$\kappa \, \Delta s + s^2 \leq 0 \tag{9}$$

with

$$\kappa = E_0\left[\frac{K(\|X\|^2)}{f(\|X\|^2)} \right].$$

Proof We follow the proof of Theorem 2 in the case where $\gamma_0 - c$ is first negative and then positive (hence Δs is subharmonic). The main point is to treat the left hand side of Inequality (7); it equals

$$\int_0^\infty \int_{S_{r,\theta}} \Delta s(x)\, \mathrm{d}U_{r,\theta}(x)\, \frac{K(r^2)}{f(r^2)}\, \frac{2\pi^{p/2}}{\Gamma(p/2)}\, r^{p-1}\, f(r^2)\, \mathrm{d}r$$

$$\leq \int_0^\infty \int_{S_{r,\theta}} \Delta s(x)\, \mathrm{d}U_{r,\theta}(x)\, \frac{2\pi^{p/2}}{\Gamma(p/2)}\, r^{p-1}\, f(r^2)\, \mathrm{d}r$$

$$\times \int_0^\infty \frac{K(r^2)}{f(r^2)}\, \frac{2\pi^{p/2}}{\Gamma(p/2)}\, r^{p-1}\, f(r^2)\, \mathrm{d}r$$

by covariance inequality since $\frac{K}{f}$ is non-increasing (K is non-increasing according to Lemma 4) and $r \mapsto \int_{S_{r,\theta}} \Delta s(x)\, \mathrm{d}U_{r,\theta}(x)$ is non-decreasing by subharmonicity of Δs. Therefore we have obtained

$$E_\theta\left[\frac{K(\|X-\theta\|^2)}{f(\|X-\theta\|^2)}\, \Delta s(X)\right] \leq E_0\left[\frac{K(\|X\|^2)}{f(\|X\|^2)}\right] E_\theta[\Delta s(X)].$$

Finally, the result follows the same way as in the proof of Theorem 2 with

$$\kappa = E_\theta\left[\frac{K(\|X-\theta\|^2)}{f(\|X-\theta\|^2)}\right].$$

□

Remark Theorem 3 gives an improvement on Theorem 2 as far as the constant in front of Δs in (6) and (9) is concerned. Indeed, when $K \geq 0$ (and hence $k > 0$ and $\Delta s \leq 0$), we have

$$\kappa = E_0\left[\frac{K(\|X\|^2)}{f(\|X\|^2)}\right] \geq \frac{1}{M}\, E_0\big[K(\|X\|^2)\big] = k$$

and, when $K \leq 0$ (and hence $k < 0$ and $\Delta s \geq 0$),

$$\kappa = E_0\left[\frac{K(\|X\|^2)}{f(\|X\|^2)}\right] \leq \frac{1}{M}\, E_0\big[K(\|X\|^2)\big] = k.$$

Theorems 1, 2 and 3 specify the spaces in which the correction function s should belong and the question arises naturally as for the existence of such a function. Typically, functions s of the form $s(x) = \frac{a}{b+\|x\|^2}$ where a and b are real constants (with $b \geq 0$) constitute the basis of possible corrections, the particular case where $b = 0$ being of interest. It can be easily shown that, if $s(x) = \frac{a}{\|x\|^2}$, we have $s \in W^{2,1}_{loc}(\mathbb{R}^p)$ for $p \geq 5$ and $s \in C^2_b(\mathbb{R}^p \setminus B_r)$ for any $r > 0$.

Now, for $p \geq 5$, it is easy to see that, for any $x \neq 0$, $\Delta s(x) = \frac{-2\,a\,(p-4)}{\|x\|^4}$ and hence that Inequality (6) is satisfied if and only if $0 \leq a \leq 2\,k\,(p-4)$ when $k > 0$

($2k(p-4) \le a \le 0$ when $k < 0$, respectively). Furthermore, for $p \ge 6$, the bi-Laplacian of s verifies, for any $x \ne 0$, $\Delta(\Delta s(x)) = \frac{8a(p-4)(p-6)}{\|x\|^6}$. Note that the function Δs is subharmonic when $a \ge 0$ and superharmonic when $a \le 0$.

Finally the finiteness risk condition $E_\theta[s^2] < \infty$ reduces to the existence of the second inverse moment for the density $x \mapsto f(\|x-\theta\|^2)$.

3 Applications

3.1 Estimating a loss

Estimating the quadratic loss $\|x-\theta\|^2$ is a natural first application of the previous theory; in that case, the function c is the identity function ($c(t) = t$). Johnstone (1988) treats this problem under the usual normal distribution $\mathcal{N}_p(\theta, I_p)(f(t) = \frac{1}{(2\pi)^{p/2}} \mathrm{e}^{-t/2})$ through a two fold application of Stein's identity. Our approach allows to obtain directly his expression of the risk difference, say

$$\delta_\theta = E_\theta[-2\,\Delta s(X) + s^2(X)]. \tag{10}$$

Indeed, according to (2), the risk difference is

$$\delta_\theta = E_\theta[2\,(p - \|X-\theta\|^2)\,s(X) + s^2(X)]$$

and it is easy to check that, for any $x \in \mathbb{R}^p$,

$$(p - \|x-\theta\|^2)\,\exp\left(-\frac{1}{2}\,\|x-\theta\|^2\right) = -\Delta\exp\left(-\frac{1}{2}\,\|x-\theta\|^2\right)$$

so that a straightforward application of Lemma 1 gives (10).

Fourdrinier and Wells (1995a) address this loss estimation problem in the more general context of spherically symmetric distributions and give a sufficient condition of domination of γ_0 by γ_s of the form (6). Their distributional conditions on f are more technical than ours and it is worth noting that their two examples satisfy the conditions of Theorem 2. However we need here an extra condition on the correction s, that is, Δs is a superharmonic function (nevertheless note that they use the same correction $s(x) = \frac{a}{\|x\|^2}$ as us, and hence this superharmonicity condition is satisfied as above).

Our method typically applies to estimating a loss given through a function of the usual quadratic loss. Brandwein and Strawderman (1980, 1991a, b) and Bock (1985) consider a non-decreasing and concave function c of $\|x-\theta\|^2$ in order to compare various estimators δ of θ. As in the case tackled by Johnstone (1988) and Fourdrinier and Wells (1995b), it is still of interest to assess the loss of $\delta(X) = X$, that is, to estimate $c\left(\|x-\theta\|^2\right)$. When c is non-decreasing, as in Brandwein and Strawderman (1980, 1991a, b) and also in Bock (1985), we are in the case where the function $\gamma_0 - c$ is first positive and then negative; Theorem 2 directly applies and note that concavity of c plays no role. We illustrate that fact with the following examples.

Assume that $c(t) = t^{\beta}$ with $0 < \beta$. Consider the Kotz distribution with generating function

$$f(t) = N_m\, t^m\, \mathrm{e}^{-t/2} \quad \text{with } N_m = \frac{\Gamma(p/2)}{2^m\, \Gamma(p/2+m)}\, \frac{1}{(2\,\pi)^{p/2}} \; m \geq 0. \tag{11}$$

A simple calculation shows that the unbiased estimator $\gamma_0 = E[c(\|X\|^2)]$ equals

$$\gamma_0 = 2^{\beta}\, \frac{\Gamma(p/2+m+\beta)}{\Gamma(p/2+m)}. \tag{12}$$

It is also clear that $E_0[c^2(\|X\|^2)] < \infty$ (actually it is easy to check that this finiteness condition is obtained for $p + 2\,m + 4\,\beta > 0$). Conditions on f and c in Theorem 1 are satisfied since $f \in \mathcal{C}^0(\mathbb{R}_+^*)$ and $c \in \mathcal{C}^0(\mathbb{R}_+^*)$; moreover, due to the form of f, we have $f \in \mathcal{S}^{0,p/2+\epsilon+1}(\mathbb{R}_+^*)$ if and only if $\sup_{t\in\mathbb{R}_+^*} f(t) < \infty$, which is satisfied since $m \geq 0$. Then it is clear that $f\,c \in \mathcal{S}^{0,p/2+\epsilon+1}(\mathbb{R}_+^*)$. Finally, the function $\gamma_0 - c$ is non-increasing and hence has only one sign change.

As for the moment condition of s, that is $E_\theta[s^2] < \infty$, it is satisfied for $s(x) = \frac{a}{b+\|x\|^2}$ since such functions are bounded for $b > 0$. When $b = 0$, that condition reduces to

$$\int_{\mathbb{R}^p} \frac{\|x-\theta\|^{2m}}{\|x\|^4}\, \exp\left(-\frac{1}{2}\,\|x-\theta\|^2\right) \mathrm{d}x < \infty.$$

If $\theta \neq 0$, we have to check that, for any $R > 0$,

$$\int_{B_R} \frac{1}{\|x\|^4}\, \mathrm{d}x < \infty$$

which is satisfied since $p \geq 5$. If $\theta = 0$, the corresponding condition is

$$\int_{B_R} \frac{1}{\|x\|^{4-2m}}\, \mathrm{d}x < \infty$$

which imposes $p + 2m > 4$ and is satisfied since $m \geq 0$ and $p \geq 5$.

We can now calculate the constant k in Theorem 3. First it is easy to check that the constant M equals

$$M = \frac{\Gamma(p/2)}{\Gamma(p/2+m)}\, \frac{1}{(2\,\pi)^{p/2}} \left(\frac{m}{e}\right)^m. \tag{13}$$

Secondly, through the expression of K given by (5), we show in Lemma 7 (see Appendix) that $E_0[K(\|X\|^2)]$ is expressed in terms of hypergeometric functions and finally that

$$
\begin{aligned}
k = {} & \frac{2^{-p/2-2\,m}}{(p-2)\,\Gamma(p/2+m)} \left(\frac{e}{m}\right)^m \\
& \times \left(\frac{\Gamma(p/2+2\,m+1)\,2^{\beta}\,\Gamma(p/2+m+\beta)}{(m+1)\,\Gamma(p/2+m)}\, {}_2F_1(1,\, p/2+2\,m+1;\, m+2,\, 1/2) \right. \\
& - \frac{\Gamma(p/2+2\,m+1)\,2^{\beta}\,\Gamma(p/2+m+\beta)}{\Gamma(p/2+m+1)} \\
& \quad \times {}_2F_1(1,\, p/2+2\,m+1;\, p/2+m+1,\, 1/2) \\
& - \frac{\Gamma(p/2+2\,m+\beta+1)}{m+1}\, {}_2F_1(1,\, p/2+2\,m+\beta+1;\, m+2,\, 1/2) \\
& \left. + \frac{\Gamma(p/2+2\,m+\beta+1)}{p/2+m}\, {}_2F_1(1,\, p/2+2\,m+\beta+1;\, p/2+m+1,\, 1/2) \right).
\end{aligned}
$$

This constant k reduces to a simple form when $\beta = 1$ (that is, we estimate the quadratic loss $\|x - \theta\|^2$) since it can be shown, through Formula 9.137 8. page 1,044 of Gradshteyn and Ryzhik (1980) with $\alpha = 0$, $\beta = p/2 + 2\,m + 1$, $\gamma = m + 1$ and $z = 1/2$, that

$$
\begin{aligned}
& (m+1) + (p/2+m)\, {}_2F_1(1,\, p/2+2\,m+1;\, m+2,\, 1/2) \\
& = (p/2+2\,m+1)\, \frac{1}{2}\, {}_2F_1(1,\, p/2+2\,m+2;\, m+2,\, 1/2).
\end{aligned}
$$

According to the same formula with $\alpha = 0$, $\beta = p/2 + 2\,m + 1$, $\gamma = p/2 + m$ and $z = 1/2$, we have

$$
\begin{aligned}
& (p/2+m) + (m+1)\, {}_2F_1(1,\, p/2+2\,m+1;\, p/2+m+1,\, 1/2) \\
& = (p/2+2\,m+1)\, \frac{1}{2}\, {}_2F_1(1,\, p/2+2\,m+2;\, p/2+m+1,\, 1/2).
\end{aligned}
$$

Then, after simplification, we obtain

$$
\begin{aligned}
k = {} & -2^{-p/2-2\,m} \left(\frac{e}{m}\right)^m \frac{\Gamma(p/2+2\,m+1)}{\Gamma(p/2+m+1)} \\
& \times {}_2F_1(1,\, p/2+2\,m+1;\, p/2+m+1,\, 1/2).
\end{aligned}
$$

In particular, for $m = 1$,

$$
k = -2^{-2-p/2}\, e\,(p+6) \tag{14}
$$

and, when m goes to 0, by a continuity argument we obtain the Gaussian case with

$$
k = -2^{1-p/2}.
$$

Note that this constant k is much smaller in absolute value than the constant 2 exhibited by Johnstone (1988). So it is interesting to seek a better constant turning our attention to Theorem 3 in the case where $\beta = 1$ and $m \geq 0$. It is shown in Lemma 8 (see Appendix) that, for any $t > 0$,

$$
K(t) = -N_m \int_t^{\infty} y^m\, \mathrm{e}^{-y/2}\, \mathrm{d}y = -N_m\, 2^{m+1}\, \Gamma\!\left(m+1,\, \frac{t}{2}\right), \tag{15}
$$

where $\Gamma(a, x)$ denotes the incomplete gamma function

$$\Gamma(a, x) = \int_x^\infty t^{a-1} \, \mathrm{e}^{-t} \, \mathrm{d}t.$$

It follows that

$$\frac{K(t)}{f(t)} = -\frac{\int_t^\infty y^m \, \mathrm{e}^{-y/2} \, \mathrm{d}y}{t^m \, \mathrm{e}^{-t/2}} = -\int_t^\infty \left(\frac{y}{t}\right)^m \mathrm{e}^{-(y-t)/2} \, \mathrm{d}y = -\int_0^\infty \left(1 + \frac{z}{t}\right)^m \mathrm{e}^{-z/2} \, \mathrm{d}y$$

using the change of variable $y = z + t$.

Thus the function $\frac{K}{f}$ is non-decreasing and has the same monotonicity as K (see (15)). Hence Theorem 3 applies with

$$\kappa = -4 \, \frac{p/2 + m}{p} \tag{16}$$

according to Lemma 8. It is worth noting that Theorem 3 leads exactly to the constant given by Johnstone (1988) in the Gaussian case.

We pursue comparing the constants k and κ for any m. Since $K \le 0$, we know that $\kappa \le k < 0$ (see remarks after Theorem 3). More precisely the relative gain using κ instead of k is

$$\begin{aligned}\tau &= \frac{|\kappa| - |k|}{|\kappa|} \\ &= 1 - \frac{2^{-p/2-2m} \left(\frac{e}{m}\right)^m \frac{\Gamma(p/2+2m+1)}{\Gamma(p/2+m+1)} \, {}_2F_1(1, p/2 + 2m + 1; p/2 + m + 1, 1/2)}{4 \, \frac{p/2+m}{p}}.\end{aligned}$$

In particular, for $m = 1$,

$$\tau = 1 - \frac{e \, p \, (p+6)}{2^{p/2+3} \, (p+2)}$$

and, when $m = 0$ (that is the normal case),

$$\tau = 1 - 2^{-p/2}.$$

Note that the gain increases with the dimension p.

Although our results are formally established, we illustrate them through simulations. Figure 1 yields, for the Kotz distribution (11) with $m = 1$ and $p = 8$, what brings Theorem 3 with respect to Theorem 2, and also, what is lost in using the upper bounds $\overline{\delta_\theta} = E_\theta[k \, \Delta s + s^2]$ and $\check{\delta_\theta} = E_\theta[\kappa \, \Delta s + s^2]$ instead of the risk difference $\delta_\theta = R(\gamma_0 + s, \theta) - R(\gamma_0, \theta)$. According to (14) and to (16), $k = -2^{-2-p/2} \, e \, (p+6)$ and $\kappa = -\frac{4}{p} \, (\frac{p}{2} + m)$ respectively, and the correction s is choosen of the form $s(x) = \frac{a}{\|x\|^2}$ with $a = k(p-4)$, since this value of a minimizes

$$k \, \Delta s(x) + s^2(x) = [-2 \, k \, a \, (p-4) + a^2] \, \|x\|^{-4}.$$

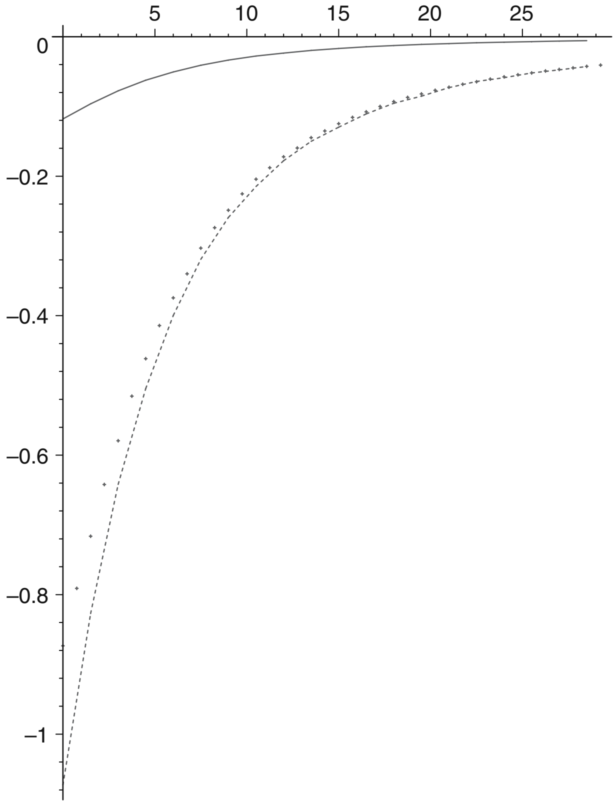

Fig. 1 Estimation of $\|x-\theta\|^2$ when $p=8$ under Kotz distribution with $m=1$: risk difference δ_θ (*dashes*) and its bounds $\overline{\delta_\theta}$ (*solid*) and $\check{\delta_\theta}$ (*crosses*) plotted against $\|\theta\|^2$ (Calculations based on 1,000,000 simulations)

All quantities δ_θ, $\overline{\delta_\theta}$ and $\check{\delta_\theta}$ are plotted against $\|\theta\|^2$. Note that the values at $\theta=0$ can be easily checked since, from (2), it can be shown that

$$\delta_0 = \frac{a}{p}\left(\frac{a}{p-2}+4\right).$$

Now

$$\overline{\delta_0} = [-2\,k\,a\,(p-4)+a^2]\,E_0\left[\|X\|^{-4}\right] = \frac{a}{p}\left(\frac{a}{p-2}-2\,k\,\frac{p-4}{p-2}\right)$$

and also

$$\overline{\delta_0} = [-2\,\kappa\,a\,(p-4)+a^2]\,E_0\left[\|X\|^{-4}\right] = \frac{a}{p}\left(\frac{a}{p-2}-2\,\kappa\,\frac{p-4}{p-2}\right).$$

For the value of a, k and κ mentioned above, with $p=8$, we finally obtain

$$\delta_0 = -1.07, \quad \overline{\delta_0} = -0.118 \quad \text{and } \check{\delta_0} = -0.873.$$

The upper bound $\check{\delta}_\theta$ is significatively below the upper bound $\overline{\delta_\theta}$, so that there is a noticeable improvement in using Theorem 3 instead of Theorem 2. While $\overline{\delta_\theta}$ is far from δ_θ, it is worth noting that $\check{\delta}_\theta$ is very close to δ_θ, indicating that Theorem 3 yields an accurate upper bound for δ_θ.

3.2 Estimating a confidence statement

Another context for estimating a function of the squared norm $c(\|x - \theta\|^2)$ is the confidence statement estimation problem. Consider, for fixed $\alpha \in [0, 1]$, the usual confidence region for the unknown parameter $\theta \in \mathbb{R}^p$ which is given by

$$C_\alpha(X) = \{\theta \in \mathbb{R}^p \,/\, \|X - \theta\|^2 \leq c_\alpha\},$$

where c_α is the constant which guarantees that $C_\alpha(X)$ has confidence coefficient $1 - \alpha$. Robert and Casella (1994) recall the defect of using $1 - \alpha$ as a report confidence statement for $C_\alpha(X)$. They develop, in the normal case, the conditional approach suggested first by Kiefer (1977) and formalized by Robinson (1979a, b). Thus they propose, as a confidence procedure, the couple $(C_\alpha(X), \gamma(X))$ where, if $X = x$ is observed, $\gamma(x)$ is a reported confidence statement for the set $C_\alpha(x)$. In this framework, $\gamma(x)$ is an estimate of the indicator function $\mathbb{1}_{C_\alpha(x)}$ and thus we are reduced to estimate $c\left(\|x - \theta\|^2\right)$ with $c = \mathbb{1}_{[0,\, c_\alpha]}$. Note that the standard estimator γ_0 here is $\gamma_0 = 1 - \alpha$.

We follow first Robert and Casella (1994) in considering the normal case, that is, the case where the generating function f is of the form $f(t) = (2\pi)^{-p/2}\, e^{-t/2}$. Note that these authors give only formal proof of an improvement $\gamma_s = 1 - \alpha + s$ (with $s(x) = \frac{a}{\|x\|^2}$) over $\gamma_0 = 1 - \alpha$ in the case where θ is close to 0 and $\|\theta\|$ is close to infinity. In the other cases, they show improvement of γ_s through simulations. We will see that Theorem 2 applies in this context with a completely specified constant k and gives rise to a formal proof that γ_s dominates $1 - \alpha$ for any value of θ. Actually, Fourdrinier and Lepelletier (2003) yield a theorem, specifically adapted to the confidence statement estimation problem, which guarantees the domination of γ_s over γ_0 through a partial differential inequation $k_1\, \Delta s + s^2 \leq 0$. However, in addition to the specificity of their theorem, their constant k_1 is smaller than the constant k in (6).

First it is clear that the functions f and c satisfy the assumptions of Theorem 1 and that $1 - \alpha - \mathbb{1}_{[0,\, c_\alpha]}$ has only one change sign (being first negative and then positive). Note that the condition $E_0[c^2(\|X\|^2)] < \infty$ is clearly satisfied since $E_0[c^2(\|X\|^2)] = E_0[c(\|X\|^2)] = \gamma_0$.

So, according to Theorem 2, any function $s \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^p) \cap C^2_b(\mathbb{R}^p \setminus B_r)$ (for some $r > 0$) such that $E_\theta[s^2] < \infty$ and such that its Laplacian Δs is subharmonic gives rise to an improved estimator $\gamma_s = 1 - \alpha + s$ as soon as Inequality (6) is satisfied. As recalled in Section 2, a typical correction s is $s(x) = \frac{a}{\|x\|^2}$. Thus, for such a function, straightforward calculations of the left hand side of Inequality (6) show that an improvement is guaranteed if $0 \leq a \leq 2k\,(p - 4)$. According to Lemma 9 and denoting by $\gamma(a, x)$ the incomplete gamma function

$$\gamma(a, x) = \int_0^x t^{a-1} \, \mathrm{e}^{-t} \, \mathrm{d}t,$$

we have

$$k = \frac{\gamma(p/2, c_\alpha) - \gamma(p/2, c_\alpha/2) - 2^{p/2-1} \, \mathrm{e}^{-c_\alpha/2} \, \gamma(p/2, c_\alpha/2)}{(p-2) \, \Gamma(p/2) \, 2^{p/2-2}}$$

and thus the range of values of a is completely specified. Note that, in the neighborhood of 0 for θ, this range can be wider. Indeed Robert and Casella (1994) show that, when $\theta = 0$, γ_s dominates γ_0 if and only if $0 \le a \le 2\,(p-4)\,(\alpha - \nu)$ where ν satisfies $P[\chi^2_{p-2} \le c_\alpha] = 1 - \nu$. Therefore $k \le \alpha - \nu$.

For a Kotz distribution with parameter m (see (11)), improvement of γ_s is still valid with the same type of range for the constant a ($0 \le a \le 2\,k\,(p-4)$). An explicit expression of k is more involved. However, for specific values of m, the corresponding calculation can be made; thus, for $m = 1$, it can be shown that

$$k = \frac{e\left(4\,\gamma(p/2+1, c_\alpha) + 2\,\gamma(p/2+2, c_\alpha) - (p + 6 + 2^{p/2+2}\,\Gamma(2, c_\alpha/2))\,\gamma(p/2+1, c_\alpha/2)\right)}{(p-2)\,p\,\Gamma(p/2)\,2^{p/2}}.$$

As in Robert and Casella (1994), simulations are made for the normal distribution $\mathcal{N}_p(\theta, I_p)$; here $p = 8$ and s is given by $s(x) = \frac{a}{\|x\|^2}$ with $a = k\,(p-4)$. In Fig. 2, the risk difference $\delta_\theta = R(1 - \alpha + s, \theta) - R(1 - \alpha, \theta)$ and its bound $\overline{\delta_\theta} = E_\theta[k\,\Delta s + s^2]$ given by Theorem 2 are plotted against $\|\theta\|^2$.

Values at $\theta = 0$ are, respectively,

$$\delta_0 = \frac{a}{p-2}\left(2\,(1-\alpha) - 2\frac{\gamma(p/2-1, c_\alpha/2)}{\Gamma(p/2-1)} + \frac{a}{p-4}\right)$$

and

$$\overline{\delta_0} = [-2\,k\,a\,(p-4) + a^2]\,E_0\left[\|X\|^{-4}\right] = \frac{a}{p}\left(\frac{a}{p-2} - 2\,k\,\frac{p-4}{p-2}\right),$$

that is, for the value of a and k mentioned above with $p = 8$,

$$\delta_0 = -8.38 \times 10^{-5} \text{ and } \overline{\delta_0} = -0.25 \times 10^{-5}.$$

Clearly the upper bound $\overline{\delta_\theta}$ is crude. Since Theorem 3 does not apply, an alternative would consist in a combination of the two approaches in Theorems 2 and 3, that is, to find a sub-interval on which K and $\frac{K}{f}$ have the same monotonicity and to bound f on the complementary of this interval.

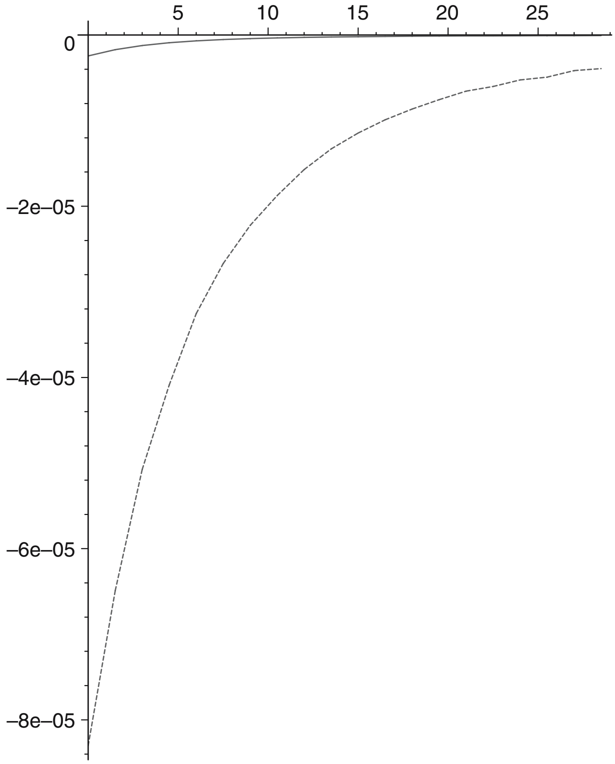

Fig. 2 Estimation of a confidence statement when $p = 8$ under normal distribution: risk difference δ_θ (*dashes*) and its bound $\overline{\delta_\theta}$ (*solid*) plotted against $\|\theta\|^2$ (Calculations based on 1,000,000 simulations)

4 Concluding remarks

We have seen that, in the general estimation problem of a function c of a quadratic function $\|x - \theta\|^2$, improvements of the form $\gamma_s = \gamma_0 + s$ on the unbiased estimator $\gamma_0 = E_0[c(\|X\|^2)]$ can be obtained through a unified approach and via solutions of partial differential inequations of the form $k\,\Delta s + s^2 \leq 0$. This method applies to various setting (in particular to the confidence statement estimation problem, to the loss estimation problem (with $c(t) = t$ and, more generally, $c(t) = t^\beta$ with $\beta > 0$) and to a wide class of sampling distributions (included in the class of the spherically symmetric distributions). This approach is very efficient in the sense that, for a few classical estimation problems, such as the confidence statement estimation problem in the normal case, it brings a formal solution. Recall that, for that problem, Robert and Casella (1994) yield formal proofs in the only cases where $\theta = 0$ and $\|\theta\|$ in a neighbourhood of infinity while, in the other case, they illustrate the improvements of γ_s through simulations.

At first sight, the role of the Laplacian of the correction s is non explicit in the derivation of the risk calculation of γ_s (except in the case where $c(t) = t$ and we estimate $\|x - \theta\|^2$ since Δs appears through repeated uses of Stein's identity as it is shown in Johnstone (1988). However Δs turns out to be crucial in the solution of the problem of finding improvements on γ_0 (even in the case where we estimate a confidence statement with $c(t) = \mathbb{1}_{[0\,;\,c_\alpha]}(t)$).

Our idea was first to introduce the Laplacian in the risk difference δ_θ in (2) in expressing the cross product term as the Laplacian of a function. Then the Laplacian of s can be exhibited through a Green formula type (see Lemma 1). Note that the conditions we need in using such a formula are quite general (and non standard) since the conditions on the function c (such as the indicator function) and on the correction s of the form $s(x) = \frac{a}{\|x\|^2}$ impose a lack of regularity.

Before giving a few perspectives, note that a possible problem with the improved estimators γ_s is that they can take values outside the range of the function c. To avoid such a problem, instead of an estimator γ_s, the use of $\gamma_s^* = \max\{\min\{\sup_{t\in\mathbb{R}_+} c(t), \gamma_s(x)\}, 0\}$ leads to an improved estimator over γ_s as it can be shown through straightforward calculations of their loss difference.

Our examples are centered around the Kotz distributions. However numerous spherically symmetric distributions satisfy the conditions of Theorem 1. Thus it is easy to show that this is the case for the logistic type distribution with generating function $f(t) \propto \frac{\mathrm{e}^{-t}}{(1+\mathrm{e}^{-t})^2}$. More generally generating functions f converging fast enough to infinity are good candidates. It is worth noting that the Student t-distribution with ν degrees of freedom is suitable (as soon as $\nu > 2$ when $c(t) = \mathbb{1}_{[0\,;\,c_\alpha]}(t)$, as soon as $\nu > \max\{4\,\beta\,,\ 2\,\beta + 2\}$ when $c(t) = t^\beta$).

Other extensions are conceivable. Thus, when a residual vector U is available (that is, when the density is of the form $f(\|x-\theta\|^2 + \|u\|^2)$), improved estimation of θ is classical [see Brandwein and Strawderman (1991a) and improved estimators of the quadratic function $\|x-\theta\|^2$ are given in Fourdrinier and Wells (1995a). In this context, estimation of a function of the type $c(\|x-\theta\|^2 + \|u\|^2]$ used in Brandwein and Strawderman (1991b) is a natural perspective.

Finally, as it is clear that our improved estimators are not admissible, a natural question is how to determine Bayesian (formal) estimators $\gamma = \gamma_0 + s$ where the corresponding correcting function s satisfies a differential inequality of the type (6)? We will consider finding prior distributions which lead to such estimators.

Appendix

Most of this appendix is devoted to the properties of the function K defined in (5). It will be convenient to write K under the form

$$K(t) = \frac{-1}{p-2}\left(H(t) + \int_t^\infty G(y)\,\mathrm{d}y\right), \tag{17}$$

where

$$H(t) = \int_0^t \left(\frac{y}{t}\right)^{p/2-1} G(y)\,\mathrm{d}y \tag{18}$$

and

$$G(y) = (\gamma_0 - c(y))\, f(y). \tag{19}$$

Note that $H(t)$ is perfectly defined for any $t > 0$ since, through a change of variable in polar coordinates, it can be easily shown that

$$H(t) = \frac{\Gamma(p/2)}{\pi^{p/2}}\, t^{1-p/2}\, E_\theta\big[(\gamma_0 - c(\|X-\theta\|^2))\, \mathbb{1}_{[0,t]}(\|X-\theta\|^2)\big]$$

the existence of the last expectation being guaranteed since $E_\theta\left[c(\|X-\theta\|^2)\right] < \infty$. Note also that, by definition,

$$\gamma_0 = \frac{\pi^{p/2}}{\Gamma(p/2)} \int_0^\infty y^{p/2-1}\, c(y)\, f(y)\, \mathrm{d}y$$

and, as f is the generating function of a spherically symmetric distribution,

$$1 = \int_0^\infty \frac{\pi^{p/2}}{\Gamma(p/2)}\, y^{p/2-1}\, f(y)\, \mathrm{d}y$$

and hence it follows from (19) that

$$\int_0^\infty y^{p/2-1}\, G(y)\, \mathrm{d}y = 0. \tag{20}$$

Furthermore H can be extended at 0 by $\lim_{t\to 0} H(t) = 0$. Indeed, according to the assumptions of Theorem 1, $|G| = |\gamma_0 - c|\, f$ is bounded on $\mathbb{R}_+^*\backslash T$ by a constant $\nu > 0$ since the functions f and fc belong to $\mathcal{S}^{0,p/2+1+\epsilon}(\mathbb{R}_+^*\backslash T)$. Then, for any $t > 0$,

$$H(t) \le \frac{\nu}{t^{p/2-1}} \int_0^t y^{p/2-1}\, \mathrm{d}y = \frac{2\,\nu}{p}\, t$$

and hence $\lim_{t\to 0} H(t) = 0$.

In the following, setting $T = \{t_1, \ldots, t_m\} \subset \mathbb{R}_+^*$ with $t_1 < \cdots < t_m$, for any $\theta \in \mathbb{R}^p$, we denote by $\mathcal{T}_\theta = \cup_{i=1}^m S_{\sqrt{t_i},\theta}$ where $S_{\sqrt{t_i},\theta}$ is the sphere $\{x \in \mathbb{R}^p /\, \|x-\theta\|^2 = t_i\}$ of radius $\sqrt{t_i}$ and centered at θ.

Lemma 2 *If the functions f and c are continuous (except possibly on T) then the function H is derivable on $\mathbb{R}_+^* \setminus T$ and, for any $t \in \mathbb{R}_+^* \setminus T$, we have*

$$H'(t) = G(t) - \frac{p-2}{2\,t}\, H(t)\,. \tag{21}$$

Furthermore the function K is twice derivable on $\mathbb{R}_+^* \setminus T$ and, for any $t \in \mathbb{R}_+^* \setminus T$,

$$K'(t) = \frac{H(t)}{2\,t} \tag{22}$$

and

$$K''(t) = \frac{G(t)}{2\,t} - \frac{p}{4\,t^2}\,H(t). \tag{23}$$

Finally, for any $\theta \in \mathbb{R}^p$ *and any* $x \in \mathbb{R}^p \setminus \mathcal{T}_\theta$*, we have*

$$\Delta K(\|x-\theta\|^2) = 2\,G(\|x-\theta\|^2). \tag{24}$$

Proof According to (18), setting, for any $y \in \mathbb{R}_+^* \setminus T$, $g(y) = y^{p/2-1}\,G(y)$ we can define, for any $t \in \mathbb{R}_+^*$, $\varphi(t) = \int_0^t g(y)\,\mathrm{d}y$. Then, for fixed $z \in \mathbb{R}_+^*$, the function $g_z = g\ \mathbb{1}_{]0\,,\,z[}$ is in $L^1(\mathbb{R}_+^*)$ since

$$\int_0^\infty |g_z(y)|\,\mathrm{d}y = \int_0^z |g(y)|\,\mathrm{d}y \le \nu \int_0^z y^{p/2-1}\,\mathrm{d}y = \frac{2\,\nu}{p}\,z^{p/2} < \infty$$

using the upper bound ν of $|G|$. Therefore the function φ_z defined, for any $t \in \mathbb{R}_+^*$, by $\varphi_z(t) = \int_0^t g_z(y)\,\mathrm{d}y$ is absolutely continuous and $\varphi_z'(t) = g_z(t) = g(t)\ \mathbb{1}_{]0\,,\,z[}(t)$ a.e. As z has been arbitrarily chosen, we have in fact $\varphi'(t) = g(t)$ a.e. (choose $t < z$).

Now, the function g being continuous on each interval $]0\,,\ t_1[,]t_1\,,\ t_2[, \ldots,]t_{n-1}\,,\ t_n[$, the function φ is derivable on $\mathbb{R}_+^* \setminus T$ and $\varphi'(t) = g(t)$ for any $t \in \mathbb{R}_+^* \setminus T$. Finally, as $H(t) = t^{1-p/2}\,\varphi(t)$, the usual rules of derivation give the stated expression of $H'(t)$.

We turn now our attention to the function K. The integral term in (17) satisfies

$$\begin{aligned}
\left| \int_t^\infty G(y)\,\mathrm{d}y \right| &\le \int_t^\infty |\gamma_0 - c(y)|\,f(y)\,\mathrm{d}y \\
&\le \int_t^\infty \gamma_0\,f(y)\,\mathrm{d}y + \int_t^\infty c(y)\,f(y)\,\mathrm{d}y \\
&\le \int_t^\infty \left(\frac{y}{t}\right)^{p/2-1} \gamma_0\,f(y)\,\mathrm{d}y + \int_t^\infty \left(\frac{y}{t}\right)^{p/2-1} c(y)\,f(y)\,\mathrm{d}y \\
&< \infty
\end{aligned}$$

since $E_\theta[c(\|X-\theta\|^2)] < \infty$. Thus K is well defined on $\mathbb{R}_+^*$ and it is clear from (17) that K is derivable at any $t \in \mathbb{R}_+^* \setminus T$ and

$$K'(t) = -\frac{1}{p-2}\,(H'(t) - G(t)) = \frac{H(t)}{2\,t}$$

according to (21).

Formulas (22) and (21) insure in fact that K is twice derivable and give, for any $t \in \mathbb{R}_+^* \setminus T$,

$$K''(t) = \frac{2\,t\,H'(t) - 2\,H(t)}{4\,t^2} = \frac{2\,t\,(G(t) - \frac{p-2}{2\,t}\,H(t)) - 2H(t)}{4\,t^2} = \frac{G(t)}{2\,t} - \frac{p\,H(t)}{4\,t^2}.$$

Finally, the calculation of the Laplacian of $K(\|x-\theta\|^2)$ can be done as follows. Let $1 \leq i \leq p$ and let $x \in \mathbb{R}^p \setminus \mathcal{T}_\theta$. We have

$$\partial_i K(\|x-\theta\|^2) = 2\,(x_i-\theta_i)\,K'(\|x-\theta\|^2) \tag{25}$$

and

$$\partial_{ii} K(\|x-\theta\|^2) = 2\,K'(\|x-\theta\|^2) + 4\,(x_i-\theta_i)^2\,K''(\|x-\theta\|^2). \tag{26}$$

Using (26), (22) and (23), we obtain

$$\begin{aligned}\Delta K(\|x-\theta\|^2) &= 2\,p\,K'(\|x-\theta\|^2) + 4\,\|x-\theta\|^2\,K''(\|x-\theta\|^2)\\ &= \frac{2\,p\,H(\|x-\theta\|^2)}{2\,\|x-\theta\|^2} + 4\,\|x-\theta\|^2\,\frac{G(\|x-\theta\|^2)}{2\,\|x-\theta\|^2}\\ &\quad -\frac{4\,\|x-\theta\|^2\,p\,H(\|x-\theta\|^2)}{4\,\|x-\theta\|^4}\\ &= 2\,G(\|x-\theta\|^2).\end{aligned}$$

□

We now give conditions for which the function $x \mapsto K(\|x-\theta\|^2)$ belongs to the space $\mathcal{S}^{2,p+\eta}(\mathbb{R}^p \setminus B_{R_0})$, for some $\eta > 0$ and some $R_0 > 0$, and to the space $W^{2,\infty}(\mathbb{R}^p)$. To this end, we recall a few inequalities about the quadratic norm. Let $(x,\theta) \in \mathbb{R}^p \times \mathbb{R}^p$ and $1 \leq i < j \leq p$. We have

$$2\,(x_i-\theta)\,(x_j-\theta_j) \leq \|x-\theta\|^2, \tag{27}$$

$$(x_i-\theta_i)^2 \leq \|x-\theta\|^2, \tag{28}$$

$$|x_i-\theta_i| \leq \max\{\|x-\theta\|^2\,;\,1\}. \tag{29}$$

Furthermore, if $2\,\|\theta\| \leq r$ and $x \notin B_r$, then

$$\|x\| < 2\,\|x-\theta\|. \tag{30}$$

Lemma 3 *If the functions f and $f\,c$ belong to $S^{0,p/2+1+\epsilon}(\mathbb{R}_+^* \setminus T)$ for some $\epsilon > 0$ then*

$$\sup_{t\in\mathbb{R}_+^*}\left|\frac{H(t)}{t}\right| < \infty$$

and $H \in S^{0,p/2+\epsilon}(\mathbb{R}_+^*)$.

Proof By assumption, the functions f and $f\,c$ are bounded from above by a constant M_0. Thus, according to (18) and (19), for any $t > 0$,

$$\begin{aligned}\left|\frac{H(t)}{t}\right| &\leq \frac{1}{t^{p/2}}\int_0^t y^{p/2-1}\,|\gamma_0 - c(y)|\,f(y)\,\mathrm{d}y\\ &\leq \frac{(\gamma_0+1)\,M_0}{t^{p/2}}\int_0^t y^{p/2-1}\,\mathrm{d}y = \frac{2\,(\gamma_0+1)\,M_0}{p}\end{aligned}$$

which gives the first result.

For $0 \leq r \leq \frac{p}{2} + \epsilon$, note that, if $0 \leq t \leq t_m \vee 1$, then

$$t^r \, |H(t)| = t^{r+1} \, \frac{|H(t)|}{t} \leq (t_m \vee 1)^{p/2+\epsilon+1} \times \frac{2\,(\gamma_0 + 1)\, M_0}{p}.$$

Now assume that $t > t_m \vee 1$. Since the functions f and $f\,c$ belong to $S^{0,p/2+1+\epsilon}$ $(\mathbb{R}_+^* \setminus T)$, there exists a constant M_1 such that, for any $y > t_m \vee 1$,

$$y^{p/2+1+\epsilon}\, f(y) < M_1, \quad y^{p/2+1+\epsilon}\, f(y)\, c(y) < M_1$$

and hence

$$y^{p/2+1+\epsilon}\, |G(y)| < (\gamma_0 + 1)\, M_1. \tag{31}$$

Now note that, according to (18) and (20), we have

$$H(t) = \int_t^\infty \left(\frac{y}{t}\right)^{p/2-1} G(y)\, \mathrm{d}y.$$

Hence

$$\begin{aligned}
\left|t^r\, H(t)\right| &\leq t^r \int_t^\infty \left(\frac{y}{t}\right)^{p/2-1} |G(y)|\, \mathrm{d}y \\
&\leq (\gamma_0 + 1)\, M_1\, t^{r+1-p/2} \int_t^\infty y^{-2-\epsilon}\, \mathrm{d}y \\
&= \frac{(\gamma_0 + 1)\, M_1}{1 + \epsilon}\, t^{r-p/2-\epsilon},
\end{aligned}$$

where (31) was used in the second inequality. As $t > 1$ and $0 \leq r \leq \frac{p}{2} + \epsilon$, it follows that

$$|t^r\, H(t)| \leq \frac{(\gamma_0 + 1)\, M_1}{1 + \epsilon}$$

which gives the fact that $H \in S^{0,p/2+\epsilon}(\mathbb{R}_+^*)$. □

Lemma 4 *Assume that the functions f and $f c$ belong to $S^{0,p/2+1+\epsilon}(\mathbb{R}_+^* \setminus T)$ for some $\epsilon > 0$ and that the function $\gamma_0 - c$ has only one sign change. If $\gamma_0 - c$ is first negative and then positive (respectively first positive and then negative) then the function K is non-negative and non-increasing (respectively non-positive and non-decreasing).*

Proof First note that the function H defined in (18) is such that $\lim_{t\to\infty} H(t) = 0$ since, according to Lemma 3, we have $H \in S^{0,p/2+\epsilon}(]t_m \vee 1\,;\,\infty[)$ for some $\epsilon > 0$. Furthermore, as $f \in S^{0,p/2+1+\epsilon}(\mathbb{R}_+^* \setminus T)$ and $f\,c \in S^{0,p/2+1+\epsilon}(\mathbb{R}_+^* \setminus T)$, for any $\beta \leq p/2 + 1 + \epsilon$, the function $y^\beta\,|G(y)|$ is bounded from above. In particular, for $\beta = 1 + \epsilon$, there exists a constant $M_2 > 0$ such that, for any $y > 0$, $y^{1+\epsilon}\,|G(y)| \leq M_2$. Thus we have

$$\left|\int_t^\infty G(y)\,\mathrm{d}y\right| \leq M_2 \int_t^\infty \frac{1}{y^{1+\epsilon}}\,\mathrm{d}y = \frac{M_2}{\epsilon\,t^\epsilon}.$$

Consequently, according to (17), we obtain

$$\lim_{t\to\infty} K(t) = 0.$$

Now assume, for example, that there exists $y_0 > 0$ such that $\gamma_0 - c(y) \leq 0$ for $y \leq y_0$ and $\gamma_0 - c(y) \geq 0$ for $y \geq y_0$. Then it is clear according to (18) that, for $t \leq y_0$, $H(t) \leq 0$. When $t > y_0$, we can write

$$\begin{aligned}
H(t) &= \int_0^{y_o} \left(\frac{y}{t}\right)^{p/2-1} G(y)\,\mathrm{d}y + \int_{y_0}^{t} \left(\frac{y}{t}\right)^{p/2-1} G(y)\,\mathrm{d}y \\
&\leq \int_0^{y_o} \left(\frac{y}{t}\right)^{p/2-1} G(y)\,\mathrm{d}y + \int_{y_0}^{\infty} \left(\frac{y}{t}\right)^{p/2-1} G(y)\,\mathrm{d}y \\
&= 0
\end{aligned}$$

by (20).

Thus the function H is non-positive and hence, according to Lemma 2, we have $K' \leq 0$. Finally the function K is non-increasing and vanishes at infinity; therefore K is non-negative.

The case where the function $\gamma_0 - c$ is first positive and then negative can be treated similarly. □

Lemma 5 *Assume that the functions f and $f c$ belong to $\mathcal{S}^{0,p/2+1+\epsilon}(\mathbb{R}_+^* \setminus T)$ for some $\epsilon > 0$. For any fixed $\theta \in \mathbb{R}^p$ and for $R_0 = \max\{2\,;\,2\,\|\theta\|\,;\,2\sqrt{t_m}\}$, the function $x \mapsto K(\|x-\theta\|^2)$ belongs to $\mathcal{S}^{2,\,p+2\epsilon}(\mathbb{R}^p \setminus B_{R_0})$.*

Proof Let $0 \leq \beta \leq p + 2\epsilon$. The first step consists in showing that

$$\sup_{x \notin B_{R_0}} \left\{\|x\|^\beta \left|K(\|x-\theta\|^2)\right|\right\} < \infty. \tag{32}$$

According to (30), as $R_0 \geq 2\,\|\theta\|$, it suffices to show that

$$\sup_{x \notin B_{R_0}} \left\{\|x-\theta\|^\beta \left|K(\|x-\theta\|^2)\right|\right\} < \infty. \tag{33}$$

Now, for any $x \notin B_{R_0}$, we have

$$\|x-\theta\| > \frac{\|x\|}{2} > \frac{R_0}{2} \geq \sqrt{t_m} \vee 1. \tag{34}$$

Thus

$$\sup_{x \notin B_{R_0}} \left\{ \|x-\theta\|^{\beta} \, |K(\|x-\theta\|^2)| \right\} \leq \sup_{t > \sqrt{t_m} \vee 1} \left\{ t^{\beta} \left| K(t^2) \right| \right\}$$

and to obtain (33) it suffices to show that

$$\sup_{t > t_m \vee 1} \{ t^r \, |K(t)| \} < \infty \tag{35}$$

for $0 \leq r \leq p/2 + \epsilon$.

Fix $t > t_m \vee 1$ and consider the integral term intervening in the expression of $K(t)$ given in (17). Note that, as f and fc belong to $\mathcal{S}^{0,p/2+1+\epsilon}(\mathbb{R}_+^* \setminus T)$, it is clear from (19) that $G \in \mathcal{S}^{0,p/2+1+\epsilon}(\mathbb{R}_+^* \setminus T)$ and hence there exists a constant μ such that, for any $y \in \mathbb{R}_+^* \setminus T$,

$$|G(y)| \, y^{p/2+1+\epsilon} \leq \mu.$$

Then

$$t^r \left| \int_t^{\infty} G(y) \, \mathrm{d}y \right| \leq \mu \, t^r \int_t^{\infty} y^{-p/2-1-\epsilon} \, \mathrm{d}y = \frac{\mu \, t^{r-p/2-\epsilon}}{p/2+\epsilon} \leq \frac{\mu}{p/2+\epsilon}$$

since $r \leq p/2 + \epsilon$ and $t > 1$. Hence, coming back to (17), we have

$$\sup_{t > t_m \vee 1} \left| t^r K(t) \right| \leq \frac{1}{p-2} \left(\sup_{t > 1 \vee t_m} \left| t^r H(t) \right| + \sup_{t > 1 \vee t_m} \left| t^r \int_t^{\infty} G(y) \, \mathrm{d}y \right| \right) < \infty$$

according to Lemma 3. This gives (35) and finally (32) is satisfied.

As a second step, we need to show that, for $1 \leq i \leq p$,

$$\sup_{x \notin B_{R_0}} \left\{ \|x\|^{\beta} \left| \partial_i K(\|x-\theta\|^2) \right| \right\} < \infty. \tag{36}$$

Fix $1 \leq i \leq p$ and $x \notin B_{R_0}$. According successively to (25), (29), (34) and (30), we have

$$\begin{aligned} \|x\|^{\beta} \left| \partial_i K(\|x-\theta\|^2) \right| &= \|x\|^{\beta} \, 2 \left| (x_i - \theta_i) \, K'(\|x-\theta\|^2) \right| \\ &\leq 2 \, \|x\|^{\beta} \, \max \left\{ 1 \, ; \, \|x-\theta\|^2 \right\} \left| K'(\|x-\theta\|^2) \right| \\ &\leq 2 \, \|x\|^{\beta} \, \|x-\theta\|^2 \left| K'(\|x-\theta\|^2) \right| \\ &< 2^{\beta+1} \, \|x-\theta\|^{\beta+2} \left| K'(\|x-\theta\|^2) \right|. \end{aligned} \tag{37}$$

Therefore, using again (34), it suffices to show that

$$\sup_{t > 1 \vee t_m} \left\{ t^{\beta/2+1} \, |K'(t)| \right\} < \infty \tag{38}$$

which is easily checked according to the expression of K' in (22) and the fact that $H \in \mathcal{S}^{0,p/2+\epsilon}(]t_m \vee 1 \, ; \infty[)$ (see Lemma 3).

Finally we turn our attention to the second derivatives of K. Fix $1 \le i, j \le p$ and $x \notin B_{R_0}$. For $i \neq j$, we have

$$\partial_{ij} K(\|x-\theta\|^2) = 4\,(x_i-\theta_i)\,(x_j-\theta_j)\,K''(\|x-\theta\|^2) \tag{39}$$

so that, according to (27) and (30),

$$\|x\|^\beta\,\partial_{ij} K(\|x-\theta\|^2) \le 2^{\beta+1}\,\|x-\theta\|^{\beta+2}\,|K''(\|x-\theta\|)|. \tag{40}$$

Therefore, using (34) and the expression of K'' given in (23) we obtain

$$\begin{aligned}
&\sup_{x\notin B_{R_0}} \left\{\|x\|^\beta \left|\partial_{ij} K(\|x-\theta\|^2)\right|\right\} \\
&\quad \le 2^{\beta+1} \sup_{t>1\vee t_m} \left\{ t^{\beta/2+1} \left| \frac{G(t)}{2\,t} - \frac{p}{4\,t^2}\,H(t) \right| \right\} \\
&\quad \le 2^{\beta} \sup_{t>1\vee t_m} |t^{\beta/2}\,G(t)| + p\,2^{\beta-1} \sup_{t>1\vee t_m} |t^{\beta/2-1}\,H(t)|.
\end{aligned} \tag{41}$$

As previously noticed, $G \in \mathcal{S}^{0,p/2+1+\epsilon}(\mathbb{R}_+^* \setminus T)$ and hence $G \in \mathcal{S}^{0,p/2+\epsilon}(\mathbb{R}_+^* \setminus T)$. Furthermore, according to Lemma 3, $H \in \mathcal{S}^{0,p/2+\epsilon}(\mathbb{R}_+^* \setminus T)$, and hence the right hand side of Inequality (41) is finite.

For $i = j$, we need to show that

$$\sup_{x\notin B_{R_0}} \left\{\|x\|^\beta \left|\partial_{ii} K(\|x-\theta\|^2)\right|\right\} < \infty. \tag{42}$$

According to (26), it suffices that

$$\sup_{x\notin B_{R_0}} \left\{\|x\|^\beta \left|2\,K'(\|x-\theta\|^2)\right|\right\} < \infty \tag{43}$$

and

$$\sup_{x\notin B_{R_0}} \left\{\|x\|^\beta \left|4\,(x_i-\theta_i)^2\,K''(\|x-\theta\|^2)\right|\right\} < \infty. \tag{44}$$

Using (30) and (34), we have

$$\begin{aligned}
\sup_{x\notin B_{R_0}} \left\{\|x\|^\beta \left|2\,K'(\|x-\theta\|^2)\right|\right\} &\le \sup_{x\notin B_{R_0}} \left\{2^\beta\,\|x-\theta\|^\beta\,|2\,K'(\|x-\theta\|^2)|\right\} \\
&\le \sup_{x\notin B_{R_0}} \left\{2^{\beta+1}\,\|x-\theta\|^{\beta+2}\,|K'(\|x-\theta\|^2)|\right\} \\
&\le 2^{\beta+1} \sup_{t>1\vee t_m} \left\{t^{\beta/2+1}\,|K'(t)|\right\}.
\end{aligned} \tag{45}$$

We already showed in (38) that the last term in (45) is finite and hence (43) is satisfied. Now it is clear from Inequalities (28) and (30) that, for obtaining (44), it suffices to show that the upper bound, on the complement of B_{R_0}, of the right hand side of Inequality (40) is finite. This has been already treated above where we proved that the right hand side of (41) is finite.

The finiteness of the left hand side of (40) in addition to (32), (36) and (42) give, finally, that the function $x \mapsto K(\|x-\theta\|^2)$ belongs to $\mathcal{S}^{2,p+2\epsilon}(\mathbb{R}^p \setminus B_{R_0})$, which is the desired result. □

Lemma 6 *Assume the functions f and $f\,c$ belong to $\mathcal{S}^{0,\,p/2+1+\epsilon}(\mathbb{R}_+^* \setminus T)$ for some $\epsilon > 0$. For any fixed $\theta \in \mathbb{R}^p$ the function $x \mapsto K(\|x-\theta\|^2)$ belongs to $W^{2,\infty}(\mathbb{R}^p)$.*

Proof First, Lemma 5 insures that, for some $\epsilon > 0$, the function $x \mapsto K(\|x-\theta\|^2)$ belongs to $\mathcal{S}^{2,\,p+2\epsilon}\big(R^p \setminus B_{R_0}\big)$; hence it belongs to $\mathcal{S}^{2,\,0}\big(R^p \setminus B_{R_0}\big)$ and, finally, to $W^{2,\infty}\big(R^p \setminus B_{R_0}\big)$. Therefore it suffices to show that it belongs to $W^{2,\infty}(B_R)$ for $R > R_0$.

Fix $R > R_0$. The goal is to show that

$$\sup_{x \in B_R\,;\,x\neq\theta} \left|K(\|x-\theta\|^2)\right| < \infty \tag{46}$$

and, for $1 \le i, j \le p$, that

$$\sup_{x \in B_R\setminus\mathcal{T}_\theta\,;\,x\neq\theta} \left|\partial_i K(\|x-\theta\|^2)\right| < \infty \tag{47}$$

and

$$\sup_{x \in B_R\setminus\mathcal{T}_\theta\,;\,x\neq\theta} \left|\partial_{ij} K(\|x-\theta\|^2)\right| < \infty. \tag{48}$$

As, for any $x \in B_R\setminus\{\theta\}$,

$$\left|H(\|x-\theta\|^2)\right| \le (R+\|\theta\|)^2\,\frac{\left|H(\|x-\theta\|^2)\right|}{\|x-\theta\|^2}$$

we have

$$\sup_{x \in B_R\setminus\{\theta\}} \left|H(\|x-\theta\|^2)\right| \le (R+\|\theta\|)^2 \sup_{x \in B_R\setminus\{\theta\}} \frac{\left|H(\|x-\theta\|^2)\right|}{\|x-\theta\|^2} < \infty$$

according to Lemma 3. Now, as $G \in \mathcal{S}^{0,\,p/2+1+\epsilon}(\mathbb{R}_+^*\setminus T)$ (see the proof of Lemma 5), there exists $M_3 > 0$ such that, for any $y \in \mathbb{R}_+^* \setminus T$,

$$|G(y)| \le M_3, \quad \left|y^{1+\epsilon}\,G(y)\right| \le M_3.$$

Then

$$\begin{aligned}\left|\int_{\|x-\theta\|^2}^{\infty} G(y)\,\mathrm{d}y\right| &\le \int_0^1 |G(y)|\,\mathrm{d}y + \int_1^{\infty} |G(y)|\,\mathrm{d}y \\ &\le M_3 + \int_1^{\infty} \frac{M_3}{y^{1+\epsilon}}\,\mathrm{d}y = M_3\left(1+\frac{1}{\epsilon}\right).\end{aligned}$$

Hence it follows from (17) that (46) is satisfied.

As for (47), we can write

$$\begin{aligned}\sup_{x\in B_R\setminus\mathcal{T}_\theta\,;\,x\neq\theta}\left|\partial_i K(\|x-\theta\|^2)\right| &= \sup_{x\in B_R\setminus\mathcal{T}_\theta\,;\,x\neq\theta}\left|2\,(x_i-\theta_i)\,K'(\|x-\theta\|^2)\right|\\ &\le (R+\|\theta\|)\sup_{0<t<(R+\|\theta\|)^2\,;\,t\in T}\left|\frac{H(t)}{t}\right|\\ &< \infty\end{aligned}$$

according to (22) and Lemma 3.

Now consider (48). For $i = j$, we have, according to (22) and (23),

$$\begin{aligned}&\sup_{x\in B_R\setminus\mathcal{T}_\theta\,;\,x\neq\theta}\left|\partial_{ii} K(\|x-\theta\|^2)\right|\\ &\quad= \sup_{x\in B_R\setminus\mathcal{T}_\theta\,;\,x\neq\theta}\left|2\,K'(\|x-\theta\|^2)+4\,(x_i-\theta_i)^2\,K''(\|x-\theta\|^2)\right|\\ &\quad\le \sup_{x\in B_R\setminus\mathcal{T}_\theta\,;\,x\neq\theta}\left|2\,K'(\|x-\theta\|^2)+4\,\|x-\theta\|^2\,K''(\|x-\theta\|^2)\right|\\ &\quad\le \sup_{0<t\le(R+\|\theta\|)^2\,;\,t\notin T}\left|\frac{H(t)}{t}+2\,G(t)-\frac{p\,H(t)}{t}\right|\\ &\quad< \infty\end{aligned}\tag{49}$$

according to Lemma 3 and since $G \in \mathcal{S}^{p/2+1+\epsilon}(\mathbb{R}_+^*\setminus T)$. For $i \neq j$, according to (39) and (27), we can write

$$\begin{aligned}&\sup_{x\in B_R\setminus\mathcal{T}_\theta\,;\,x\neq\theta}\left|\partial_{ij} K(\|x-\theta\|^2)\right|\\ &\quad= \sup_{x\in B_R\setminus\mathcal{T}_\theta\,;\,x\neq\theta}\left|4\,(x_i-\theta_i)\left(x_j-\theta_j\right)K''(\|x-\theta\|^2)\right|\\ &\quad\le \sup_{x\in B_R\setminus\mathcal{T}_\theta\,;\,x\neq\theta}\left|2\,\|x-\theta\|^2\,K''(\|x-\theta\|^2)\right|\\ &\quad\le \sup_{0<t\le(R+\|\theta\|)^2\,;\,t\notin T}\left|G(t)-\frac{p\,H(t)}{2\,t}\right|\\ &\quad< \infty\end{aligned}\tag{50}$$

by similar arguments to these used for (49).

Finally we have shown that the function $x \mapsto K(\|x-\theta\|^2)$ and its partial derivatives are essentially bounded; it remains to prove that it is a twice weakly differentiable function.

First we show that its first partial derivatives exist at θ and on $\mathcal{T}_\theta$ (see Lemma 2). Fix $1 \le i \le p$. For any $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$ and for any $h \in \mathbb{R}$, we denote by $y_{i,h}$ the vector $y_{i,h} = (y_1, \ldots, y_{i-1}, y_i + h, y_{i+1}, \ldots, y_p)$. Then, for $h \neq 0$, we have

$$\begin{aligned}\left|\frac{K(\|\theta_{i,h}-\theta\|^2)-K(0)}{h}\right| &= \left|\frac{K(h^2)-K(0)}{h}\right|\\ &= \frac{1}{p-2}\left|\frac{H(h^2)}{h}+\frac{1}{h}\int_{h^2}^{\infty}G(y)\,\mathrm{d}y-\frac{1}{h}\int_{0}^{\infty}G(y)\,\mathrm{d}y\right|\end{aligned}$$

by definition of K and reminding that $H(0) = 0$. Now, using the upper bound ν of $|G|$, we have

$$\left|\frac{H(h^2)}{h}\right| \leq \frac{1}{|h|} \int_0^{h^2} \left(\frac{y}{h^2}\right)^{p/2-1} |G(y)| \,\mathrm{d}y \leq \frac{\nu}{|h|^{p-1}} \int_0^{h^2} y^{p/2-1} \,\mathrm{d}y = \frac{2\,\nu\,|h|}{p}.$$

Hence

$$\begin{aligned}\left|\frac{K(\|\theta_{i,h} - \theta\|^2) - K(0)}{h}\right| &\leq \frac{1}{p-2}\left(\frac{2\,\nu\,|h|}{p} + \frac{1}{|h|}\int_0^{h^2} |G(y)|\,\mathrm{d}y\right) \\ &\leq \frac{\nu\,|h|}{p-2}\left(\frac{2}{p} + 1\right)\end{aligned}$$

and finally

$$\lim_{h\to 0}\left|\frac{K(\|\theta_{i,h} - \theta\|^2) - K(0)}{h}\right| = 0$$

which proves that the i-th partial derivative of $x \mapsto K(\|x - \theta\|^2)$ at θ exists and equals 0.

As for its continuity, we have, according to (22),

$$\begin{aligned}\lim_{x\to\theta} |2\,(x_i - \theta_i)\,K'(\|x - \theta\|^2)| &= \lim_{x\to\theta}\left|2\,(x_i - \theta_i)\,\frac{H(\|x - \theta\|^2)}{2\,\|x - \theta\|^2}\right| \\ &\leq \sup_{t\in\mathbb{R}_+^*}\left|\frac{H(t)}{t}\right| \lim_{x\to\theta} |x_i - \theta_i| = 0\end{aligned}$$

where Lemma 3 insures the finiteness of the above supremum.

We have already seen, in Lemma 2, that the function $x \mapsto K(\|x - \theta\|^2)$ has a ith partial derivative at $x \in \mathbb{R}^p \backslash (\mathcal{T}_\theta \cup \{\theta\})$ which equals $(x_i - \theta_i)\,\frac{H(\|x-\theta\|^2)}{(\|x-\theta\|^2)}$. We now show its existence on $\mathcal{T}_\theta$ as well with the same expression.

Fix a point $x \in \mathcal{T}_\theta$ such that $\|x - \theta\|^2 = t_k$. According to (17), we have

$$\begin{aligned}&K(\|x_{i,h} - \theta\|^2) - K(\|x - \theta\|^2) \\ &\quad = -\frac{H(\|x_{i,h} - \theta\|^2) - H(\|x - \theta\|^2) + \int_{\|x_{i,h}-\theta\|^2}^{\infty} G(y)\,\mathrm{d}y - \int_{\|x-\theta\|^2}^{\infty} G(y)\,\mathrm{d}y}{p-2}\end{aligned}$$

Now, through (18), we can write

$$
\begin{aligned}
&H(\|x_{i,h}-\theta\|^2)\\
&\quad=\int_0^{\|x_{i,h}-\theta\|^2}\frac{y^{p/2-1}}{\|x_{i,h}-\theta\|^{p-2}}\,G(y)\,\mathrm{d}y\\
&\quad=\int_0^{\|x-\theta\|^2}\frac{y^{p/2-1}}{\|x_{i,h}-\theta\|^{p-2}}\,G(y)\,\mathrm{d}y+\int_{\|x-\theta\|^2}^{\|x_{i,h}-\theta\|^2}\frac{y^{p/2-1}}{\|x_{i,h}-\theta\|^{p-2}}\,G(y)\,\mathrm{d}y\\
&\quad=\frac{\|x-\theta\|^{p-2}}{\|x_{i,h}-\theta\|^{p-2}}\,H(\|x-\theta\|^2)+\int_{\|x-\theta\|^2}^{\|x_{i,h}-\theta\|^2}\frac{y^{p/2-1}}{\|x_{i,h}-\theta\|^{p-2}}\,G(y)\,\mathrm{d}y.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&K(\|x_{i,h}-\theta\|^2)-K(\|x-\theta\|^2)\\
&\quad=-\frac{\left(\frac{\|x-\theta\|^{p-2}}{\|x_{i,h}-\theta\|^{p-2}}-1\right)H(\|x-\theta\|^2)+\int_{\|x-\theta\|^2}^{\|x_{i,h}-\theta\|^2}\left(\frac{y^{p/2-1}}{\|x_{i,h}-\theta\|^{p-2}}-1\right)G(y)\,\mathrm{d}y}{p-2}.
\end{aligned}
$$

Then we can write

$$
\left|\frac{K(\|x_{i,h}-\theta\|^2)-K(\|x-\theta\|^2)}{h}-(x_i-\theta_i)\,\frac{H(\|x-\theta\|^2)}{\|x-\theta\|^2}\right|\le A(h)+B(h),
$$

where

$$
A(h)=\frac{1}{p-2}\left|\frac{1}{h}\int_{\|x-\theta\|^2}^{\|x_{i,h}-\theta\|^2}\left(\frac{y^{p/2-1}}{\|x_{i,h}-\theta\|^{p-2}}-1\right)G(y)\,\mathrm{d}y\right|
$$

and

$$
B(h)=\frac{|H(\|x-\theta\|^2)|}{p-2}\left|\frac{1}{h}\left(\frac{\|x-\theta\|^{p-2}}{\|x_{i,h}-\theta\|^{p-2}}-1+(p-2)\,h\,\frac{x_i-\theta_i}{\|x-\theta\|^2}\right)\right|.
$$

Reminding that the function $|G|$ is bounded by ν, we can bound from above $A(h)$ as follows: in the case where $\|x-\theta\|\le\|x_{i,h}-\theta\|$, it is easy to check that

$$
\begin{aligned}
A(h)&\le\frac{\nu}{p-2}\,\frac{1}{|h|}\int_{\|x-\theta\|^2}^{\|x_{i,h}-\theta\|^2}\left|\frac{\|x-\theta\|^{p-2}}{\|x_{i,h}-\theta\|^{p-2}}-1\right|\mathrm{d}y\\
&\le\frac{\nu}{p-2}\left|\frac{\|x-\theta\|^{p-2}}{\|x_{i,h}-\theta\|^{p-2}}-1\right|\left|\frac{\|x_{i,h}-\theta\|^2-\|x-\theta\|^2}{h}\right|,
\end{aligned}
$$

the other case $\|x-\theta\| \geq \|x_{i,h}-\theta\|$ can be treated in the same way. Note that the limit of the right hand side of this last inequality, when h tends to 0, exists and equals 0 since

$$\lim_{h\to 0}\left|\frac{\|x-\theta\|^{p-2}}{\|x_{i,h}-\theta\|^{p-2}}-1\right|=0$$

and since

$$\lim_{h\to 0}\frac{1}{h}\,\left(\|x_{i,h}-\theta\|^2-\|x-\theta\|^2\right)=2\,(x_i-\theta_i).$$

As for the limit of $B(h)$, it relies on

$$\frac{\|x-\theta\|^{p-2}}{\|x_{i,h}-\theta\|^{p-2}}=1-(p-2)\,h\,\frac{x_i-\theta_i}{\|x-\theta\|^2}+\|x-\theta\|^{p-2}\,o(h^2)$$

and gives

$$\lim_{h\to 0} B(h)=0.$$

Finally we have shown that

$$\lim_{h\to 0}\left|\frac{K(\|x_{i,h}-\theta\|^2)-K(\|x-\theta\|^2)}{h}-(x_i-\theta_i)\,\frac{H(\|x-\theta\|^2)}{\|x-\theta\|^2}\right|=0$$

which implies that the ith partial derivative of $x\mapsto K(\|x-\theta\|^2)$ exists on $\mathcal{T}_\theta$ and equals $(x_i-\theta_i)\,\frac{H(\|x-\theta\|^2)}{\|x-\theta\|^2}$

To prove the continuity of this derivative, it suffices to show that the function H is continuous at t_k. Using the fact that $|G|\leq\nu$ on $\mathbb{R}_+^*\setminus T$ and according to (18), we have

$$\begin{aligned}
&|H(t_k+h)-H(t_k)|\\
&=\left|\int_0^{t_k+h}\left(\frac{y}{t_k+h}\right)^{p/2-1}G(y)\,\mathrm{d}y-\int_0^{t_k}\left(\frac{y}{t_k}\right)^{p/2-1}G(y)\,\mathrm{d}y\right|\\
&=\left|\int_0^{t_k}G(y)\left(\frac{y}{t_k}\right)^{p/2-1}\left(\left(\frac{t_k}{t_k+h}\right)^{p/2-1}-1\right)\mathrm{d}y\right.\\
&\qquad\left.+\int_{t_k}^{t_k+h}\left(\frac{y}{t_k+h}\right)^{p/2-1}G(y)\,\mathrm{d}y\right|\\
&\leq\frac{2\,\nu}{p}\left(\left|\left(\frac{t_k}{t_k+h}\right)^{p/2-1}-1\right|t_k+\left|t_k+h-\frac{t_k^{p/2}}{(t_k+h)^{p/2-1}}\right|\right).
\end{aligned}$$

Then

$$\lim_{h\to 0}|H(t_k+h)-H(t_k)|=0$$

and the partial derivative of the function $x \mapsto K(\|x-\theta\|^2)$ is continuous on $\mathbb{R}^p$, consequently this function is continuously differentiable, and hence weakly differentiable.

To prove that the function $u : x \mapsto \partial_i K(\|x-\theta\|^2)$ is weakly differentiable, it is convenient to use the sufficient condition given by Morrey (1966) page 63, that is, $u \in L^\infty(\mathbb{R}^p)$, u is absolutely continuous in each variable for almost all values of the other variables and its first partial derivatives are in $L^\infty(\mathbb{R}^p)$. The fact that u and its first partial derivatives belong to $L^\infty(\mathbb{R}^p)$ follows from (47) and (48), respectively. As for the absolute continuity part, we have, since $|\partial_j u(x)|$ is symmetric with respect to θ_j,

$$\begin{aligned}\int_{-\infty}^{\infty} \left|\partial_j u(x)\right| \mathrm{d}x_j &= 2\int_{\theta_j}^{\infty} \left|\partial_j u(x)\right| \mathrm{d}x_j\\ &= 2\int_{\theta_j}^{\theta_j+R_0} \left|\partial_j u(x)\right| \mathrm{d}x_j + 2\int_{\theta_j+R_0}^{\infty} \left|\partial_j u(x)\right| \mathrm{d}x_j\end{aligned}$$

with $R_0 = \max\{2\,;\ 2\,\|\theta\|\,;\ 2\sqrt{t_m}\}$ (see Lemma 5). Now

$$\int_{\theta_j}^{\theta_j+R_0} \left|\partial_j u(x)\right| \mathrm{d}x_j < \infty$$

according to (49) and (50) and

$$\int_{\theta_j+R_0}^{\infty} \left|\partial_j u(x)\right| \mathrm{d}x_j < \int_{\theta_j+R_0}^{\infty} \frac{M_4}{\|x-\theta\|^2}\, \mathrm{d}x_j < \infty$$

since, according to Lemma 5, for some constant M_4 and for $x \notin B_{R_0}$, $\|x-\theta\|^2\,|\partial_j u(x)| < M_4$. Therefore we have proved that the function $x_j \mapsto \partial_j u(x)$ is in $L^1(\mathbb{R})$; it follows that

$$x_i \mapsto \int_{-\infty}^{x_i} \partial_j u(x)\,\mathrm{d}x_j = u(x) \quad a.e.$$

is absolutely continuous. Finally $u \in W^{1,\infty}(\mathbb{R}^p)$ and hence the function $x \mapsto K(\|x-\theta\|^2)$ belongs to $W^{2,\infty}(\mathbb{R}^p)$. □

In the case where the sampling distribution is specified as a Kotz distribution, the constant $k = \frac{1}{M}\, E_0[K(\|X\|^2)]$ defined in Theorem 2 can be completely determined.

Lemma 7 *Suppose that X has a Kotz distribution as in* (11) *and that c is a polynomial function of the form $c(t) = t^\beta$ with $\beta > 0$. Then the constant $k = \frac{1}{M} E_0 \left[K(\|X\|^2) \right]$ can be expressed as*

$$
\begin{aligned}
k = {} & \frac{2^{-p/2-2m}}{(p-2)\,\Gamma(p/2+m)} \left(\frac{e}{m}\right)^m \\
& \times \Bigg(\frac{\Gamma(p/2+2m+1)\, 2^\beta\, \Gamma(p/2+m+\beta)}{(m+1)\,\Gamma(p/2+m)} \\
& \times {}_2F_1(1, p/2+2m+1; m+2, 1/2) \\
& - \frac{\Gamma(p/2+2m+1)\, 2^\beta\, \Gamma(p/2+m+\beta)}{\Gamma(p/2+m+1)} \\
& \times {}_2F_1(1, p/2+2m+1; p/2+m+1, 1/2) \\
& - \frac{\Gamma(p/2+2m+\beta+1)}{m+1}\, {}_2F_1(1, p/2+2m+\beta+1; m+2, 1/2) \\
& + \frac{\Gamma(p/2+2m+\beta+1)}{p/2+m} \\
& \quad \times {}_2F_1(1, p/2+2m+\beta+1; p/2+m+1, 1/2) \Bigg).
\end{aligned}
$$

Proof According to (11) and the expression of the function K given by (5), we have

$$
\begin{aligned}
& E_0[K(\|X\|^2)] \\
& = \int_0^\infty \frac{2\,\pi^{p/2}}{\Gamma(p/2)}\, r^{p-1}\, \frac{1}{p-2} \\
& \quad \times \int_{r^2}^\infty \left[\left(\frac{y}{r^2}\right)^{p/2-1} - 1 \right] (\gamma_0 - y^\beta)\, N_m\, y^m\, \mathrm{e}^{-y/2}\, \mathrm{d}y\, N_m\, r^{2m}\, \mathrm{e}^{-r^2/2} \mathrm{d}r \\
& = \frac{2\,\pi^{p/2}}{\Gamma(p/2)}\, \frac{N_m^2}{p-2} \\
& \quad \times \int_0^\infty \int_{r^2}^\infty \Bigg(\frac{\gamma_0}{r^{p-2}}\, y^{p/2+m-1}\, \mathrm{e}^{-y/2} - \gamma_0\, y^m\, \mathrm{e}^{-y/2} \\
& \qquad\qquad - \frac{y^{p/2+m+\beta-1}}{r^{p-2}}\, \mathrm{e}^{-y/2} + y^{m+\beta}\, \mathrm{e}^{-y/2} \Bigg)\, \mathrm{d}y\, r^{p+2m-1}\, \mathrm{e}^{-r^2/2}\, \mathrm{d}r
\end{aligned}
$$

which equals, through the substitution $t = y/2$,

$$\frac{4\pi^{p/2}}{\Gamma(p/2)}\,\frac{N_m^2}{p-2}$$
$$\times\int_0^\infty\Bigg[\frac{\gamma_0}{r^{p-2}}\,2^{p/2+m-1}\,\Gamma\Big(\frac{p}{2}+m,\frac{r^2}{2}\Big)-\gamma_0\,2^m\,\Gamma\Big(m+1,\frac{r^2}{2}\Big)$$
$$-\frac{2^{p/2+m+\beta-1}}{r^{p-2}}\,\Gamma\Big(\frac{p}{2}+m+\beta,\frac{r^2}{2}\Big)+2^{m+\beta}\,\Gamma\Big(m+\beta+1,\frac{r^2}{2}\Big)\Bigg]$$
$$\times r^{p+2m-1}\,\mathrm{e}^{-r^2/2}\,\mathrm{d}r.$$

Then with the change of variable $\frac{r^2}{2}=t$, the expectation $E_0[K(\|X\|^2)]$ equals

$$\frac{2^{p/2+m+1}\pi^{p/2}}{\Gamma(p/2)}\,\frac{N_m^2}{p-2}$$
$$\times\int_0^\infty\gamma_0\,2^m\,t^m\,\Gamma\Big(\frac{p}{2}+m,t\Big)\,\mathrm{e}^{-t}-\gamma_0\,2^m\,t^{p/2+m-1}\,\Gamma(m+1,t)\,\mathrm{e}^{-t}$$
$$-2^{m+\beta}t^m\,\Gamma\Big(\frac{p}{2}+m+\beta,t\Big)\,\mathrm{e}^{-t}+2^{m+\beta}\,t^{p/2+m-1}\,\Gamma\,(m+\beta+1,t)\,\mathrm{e}^{-t}\,\mathrm{d}t.$$

Finally, according to (11), (12), (13) and Formula 6.4551. page 663 of Gradshteyn and Ryzhik (1980), we have the desired result. □

As for the constant $\kappa=E_0[\frac{K(\|X\|^2)}{f(\|X\|^2)}]$ defined in Theorem 3, we have the following lemma.

Lemma 8 *Suppose that X has a Kotz distribution as in* (11) *and that $c(t)=t$. Then the constant $\kappa=E_0\left[\frac{K(\|X\|^2)}{f(\|X\|^2)}\right]$ can be expressed as*

$$\kappa=-4\,\frac{p/2+m}{p}.$$

Proof As β equals 1 and according to (12), we have $\gamma_0=p+2\,m$. Expanding the expression of K given in (5), we obtain

$$K(t)=\frac{N_m}{p-2}\left(\frac{p+2m}{t^{p/2-1}}\int_t^\infty y^{p/2+m-1}\,\mathrm{e}^{-y/2}\,\mathrm{d}y-\frac{1}{t^{p/2-1}}\int_t^\infty y^{p/2+m}\,\mathrm{e}^{-y/2}\,\mathrm{d}y\right.$$
$$\left.-(p+2\,m)\int_t^\infty y^m\,\mathrm{e}^{-y/2}\,\mathrm{d}y+\int_t^\infty y^{m+1}\,\mathrm{e}^{-y/2}\,\mathrm{d}y\right).$$

Through an integration by parts ($u = e^{-y/2}$) in the second and in the fourth integral, the function K equals

$$K(t) = \frac{N_m}{p-2}\left(-(p+2m)\int_t^\infty y^m \,\mathrm{e}^{-y/2}\,\mathrm{d}y + (2m+2)\int_t^\infty y^m\,\mathrm{e}^{-y/2}\,\mathrm{d}y\right)$$
$$= -N_m \int_t^\infty y^m\,\mathrm{e}^{-y/2}\,\mathrm{d}y$$
$$= -N_m\, 2^{m+1}\,\Gamma\left(m+1, \frac{t}{2}\right).$$

According to the definition of κ and to that expression of K, we have

$$\kappa = \frac{2\pi^{p/2}}{\Gamma(p/2)}\int_0^\infty K(r^2)\,r^{p-1}\,\mathrm{d}r = -\frac{2\pi^{p/2}}{\Gamma(p/2)}\,N_m\,2^{m+1}\int_0^\infty \Gamma\left(m+1, \frac{r^2}{2}\right) r^{p-1}\,\mathrm{d}r.$$

Finally, through the substitution $r^2 = z$ and according to (11) and to Formula 6.4551. page 663 of Gradshteyn and Ryzhik (1980) with $\mu = \frac{p}{2}$, $\beta = 0$, $\nu = m+1$ and $\alpha = \frac{1}{2}$, we obtain

$$\kappa = -\frac{\pi^{p/2}}{\Gamma(p/2)} N_m 2^{m+1}\,\frac{1}{2^{m+1}}\,\frac{\Gamma(p/2+m+1)}{\frac{p}{2}\left(\frac{1}{2}\right)^{p/2+m+1}}\;{}_2F_1(1, p/2+m+1; p/2+1; 0)$$
$$= -4\,\frac{p/2+m}{p}.$$

□

The next lemma is devoted to the confidence statement problem in the normal case.

Lemma 9 *Suppose that X has a Gaussian distribution $\mathcal{N}_p(\theta, I_p)$ and that $c(t) = \mathbb{1}_{[0\,;\,c_\alpha]}(t)$. Define, for any $t > 0$,*

$$K_1(t) = \frac{1-\alpha}{(p-2)\,(2\pi)^{p/2}}\left(\frac{2^{p/2}}{t^{p/2-1}}\,\Gamma\left(\frac{p}{2}, \frac{t}{2}\right) - 2\,\mathrm{e}^{-t/2}\right)$$

and

$$K_2(t) = \frac{1}{(p-2)(2\pi)^{p/2}} \times\left(\frac{2^{p/2}}{t^{p/2-1}}\,\Gamma\left(\frac{p}{2}, \frac{t}{2}\right) - \frac{2^{p/2}}{t^{p/2-1}}\,\Gamma\left(\frac{p}{2}, \frac{c_\alpha}{2}\right) + 2\,\mathrm{e}^{-c_\alpha/2} - 2\,\mathrm{e}^{-t/2}\right).$$

Then the function K defined in (5) *verifies*

$$K(t) = K_1(t) - K_2(t) \qquad \textit{if } t \le c_\alpha.$$

and

$$K(t) = K_1(t) \quad \text{if } t > c_\alpha.$$

Furthermore the constant k can be expressed as

$$k = \frac{\gamma(p/2, c_\alpha) - \gamma(p/2, c_\alpha/2) - 2^{p/2-1}\,\mathrm{e}^{-c_\alpha/2}\,\gamma(p/2, c_\alpha/2)}{(p-2)\,\Gamma(p/2)\,2^{p/2-2}}.$$

Proof If c is the indicator function $\mathbb{1}_{[0\,;\,c_\alpha]}$, then γ_0 equals $1-\alpha$ and, according to the expression of K given in (5), we have

$$K(t) = \frac{1}{p-2}\int_t^\infty \left(\left(\frac{y}{t}\right)^{p/2-1} - 1\right)\left(1 - \alpha - \mathbb{1}_{[0\,;\,c_\alpha]}(y)\right)\frac{\mathrm{e}^{-y/2}}{(2\,\pi)^{p/2}}\,\mathrm{d}y.$$

The indicator function leads to separate the two cases $t > c_\alpha$ and $t \le c_\alpha$. When $t > c_\alpha$ we have

$$K(t) = \frac{1-\alpha}{(p-2)\,(2\,\pi)^{p/2}}\left(\int_t^\infty \left(\frac{y}{t}\right)^{p/2-1}\mathrm{e}^{-y/2}\,\mathrm{d}y - \int_t^\infty \mathrm{e}^{-y/2}\,\mathrm{d}y\right) = K_1(t)$$

through the change of variable $z = \frac{y}{2}$. Now it is clear that, when $t \le c_\alpha$, we have

$$\begin{aligned} K(t) &= K_1(t) - \frac{1}{p-2}\left(\int_t^{c_\alpha}\left(\frac{y}{t}\right)^{p/2-1}\frac{\mathrm{e}^{-y/2}}{(2\,\pi)^{p/2}}\,\mathrm{d}y - \int_t^{c_\alpha}\frac{\mathrm{e}^{-y/2}}{(2\,\pi)^{p/2}}\,\mathrm{d}y\right)\\ &= K_1(t) - K_2(t) \end{aligned}$$

through the same change of variable.

This expression of K allows us to evaluate the constant $k = \frac{1}{M}\,E_0[K(\|X\|^2)]$. Note that, in the normal case, $M = \frac{1}{(2\,\pi)^{p/2}}$. The expectation of the function K equals

$$\begin{aligned} &E_0[K(\|X\|^2)]\\ &\quad = \int_0^\infty \frac{2\,\pi^{p/2}}{\Gamma(p/2)}\,r^{p-1}\,K(r^2)\,\frac{\mathrm{e}^{-r^2/2}}{(2\,\pi)^{p/2}}\,\mathrm{d}r\\ &\quad = \int_0^{\sqrt{c_\alpha}} \frac{2\,\pi^{p/2}}{\Gamma(p/2)}\,r^{p-1}\,(K_1(r^2) - K_2(r^2))\,\frac{\mathrm{e}^{-r^2/2}}{(2\,\pi)^{p/2}}\,\mathrm{d}r\\ &\qquad + \int_{\sqrt{c_\alpha}}^\infty \frac{2\,\pi^{p/2}}{\Gamma(p/2)}\,r^{p-1}\,K_1(r^2)\,\frac{\mathrm{e}^{-r^2/2}}{(2\,\pi)^{p/2}}\,\mathrm{d}r\\ &\quad = \frac{(1-\alpha)\,I_1 - (1-\alpha)\,I_2 - I_3 + I_4 - I_5 + I_6}{(p-2)\,\pi^{p/2}\,\Gamma(p/2)\,2^{p-1}}, \end{aligned} \tag{51}$$

where

$$I_1 = \int_0^\infty 2^{p/2}\, r\, \Gamma\left(\frac{p}{2}, \frac{r^2}{2}\right)\, \mathrm{e}^{-r^2/2}\, \mathrm{d}r,$$

$$I_2 = 2\int_0^\infty r^{p-1}\, \mathrm{e}^{-r^2}\, \mathrm{d}r = \Gamma\left(\frac{p}{2}\right),$$

$$I_3 = 2^{p/2} \int_0^{\sqrt{c_\alpha}} r\, \Gamma\left(\frac{p}{2}, \frac{r^2}{2}\right)\, \mathrm{e}^{-r^2/2}\, \mathrm{d}r,$$

$$I_4 = 2^{p/2} \int_0^{\sqrt{c_\alpha}} r\, \Gamma\left(\frac{p}{2}, \frac{c_\alpha}{2}\right)\, \mathrm{e}^{-r^2/2}\, \mathrm{d}r = 2^{p/2}\, \Gamma\left(\frac{p}{2}, \frac{c_\alpha}{2}\right)\, \left(1 - \mathrm{e}^{-c_\alpha/2}\right),$$

$$I_5 = 2\, \mathrm{e}^{-c_\alpha/2} \int_0^{\sqrt{c_\alpha}} r^{p-1}\, \mathrm{e}^{-r^2/2}\, \mathrm{d}r = 2^{p/2}\, \mathrm{e}^{-c_\alpha/2}\, \gamma\left(\frac{p}{2}, \frac{c_\alpha}{2}\right)$$

and

$$I_6 = 2\int_0^{\sqrt{c_\alpha}} r^{p-1}\, \mathrm{e}^{-r^2}\, \mathrm{d}r = \gamma\left(\frac{p}{2}, c_\alpha\right).$$

Now I_1 and I_3 can be reexpressed. First, using the change of variable $z = \frac{r^2}{2}$ and according to 6.451 2. p 662 of Gradshteyn and Ryzhik (1980), we have

$$I_1 = \int_0^\infty 2^{p/2}\, r\, \Gamma\left(\frac{p}{2}, \frac{r^2}{2}\right)\, \mathrm{e}^{-r^2/2}\, \mathrm{d}r = \Gamma\left(\frac{p}{2}\right)\, (2^{p/2} - 1).$$

As for I_3, using the change of variable $z = \frac{r^2}{2}$, by definition of the incomplete gamma function and by Fubini theorem, we have

$$\begin{aligned} I_3 &= 2^{p/2} \int_0^{\sqrt{c_\alpha}} r\, \Gamma\left(\frac{p}{2}, \frac{r^2}{2}\right)\, \mathrm{e}^{-r^2/2}\, \mathrm{d}r \\ &= 2^{p/2} \int_0^{c_\alpha/2} \int_z^\infty t^{p/2-1}\, \mathrm{e}^{-t}\, \mathrm{d}t\, \mathrm{e}^{-z}\, \mathrm{d}z \\ &= 2^{p/2} \int_0^\infty t^{p/2-1}\, \mathrm{e}^{-t} \int_0^{t \wedge c_\alpha/2} \mathrm{e}^{-z}\, \mathrm{d}z\, \mathrm{d}t \\ &= 2^{p/2}\, \Gamma\left(\frac{p}{2}\right) - \gamma\left(\frac{p}{2}, c_\alpha\right) - 2^{p/2}\, \mathrm{e}^{-c_\alpha/2}\, \Gamma\left(\frac{p}{2}, \frac{c_\alpha}{2}\right). \end{aligned}$$

In order to complete the last expression in (51), note that

$$1-\alpha=\gamma_0=E_0[c(\|X\|^2)]=\int_0^\infty \frac{2\,\pi^{p/2}}{\Gamma(p/2)}\,r^{p-1}\;\mathbb{1}_{[0;c_\alpha]}(r^2)\,\frac{\mathrm{e}^{-r^2/2}}{(2\,\pi)^{p/2}}\,\mathrm{d}r.$$

Then, using the change of variable $z=\frac{r^2}{2}$, we obtain

$$1-\alpha=\frac{\gamma(p/2,c_\alpha/2)}{\Gamma(p/2)}. \tag{52}$$

Finally, according to (51) and (52) and after simplifications, we obtain the desired expression of constant k. □

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