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A first–passage time random walk distribution with five transition probabilities: a generalization of the shifted inverse trinomial

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Abstract In this paper a univariate discrete distribution, denoted by GIT, is proposed as a generalization of the shifted inverse trinomial distribution, and is formulated as a first-passage time distribution of a modified random walk on the half-plane with five transition probabilities. In contrast, the inverse trinomial arises as a random walk on the real line with three transition probabilities. The probability mass function (pmf) is expressible in terms of the Gauss hypergeometric function and this offers computational advantage due to its recurrence formula. The descending factorial moment is also obtained. The GIT contains twenty-two possible distributions in total. Special cases include the binomial, negative binomial, shifted negative binomial, shifted inverse binomial or, equivalently, lost-games, and shifted inverse trinomial distributions. A subclass $\mathrm{GIT}_{3,1}$ is a particular member of Kemp's class of convolution of pseudo-binomial variables and its properties such as reproductivity, formulation, pmf, moments, index of dispersion, and approximations are studied in detail. Compound or generalized (stopped sum) distributions provide inflated models. The inflated $\mathrm{GIT}_{3,1}$ extends Minkova's inflated-parameter binomial and negative binomial. A bivariate model which has the GIT as a marginal distribution is also proposed.

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1 Introduction

The inverse trinomial (IT) is a univariate discrete distribution which appears in different contexts such as (a) an exponential model (Jørgensen et al., 1989) derived from the Laplace transform of the basis measure $\mu = \delta_{-1}/2 + a\delta_0 + \delta_1/2$, where δ_i is Dirac mass at i and $a > 0$, (b) the generalized trinomial coefficients (Mohanty and Panny 1990) defined by the coefficient of v^x in the power series $(\alpha v^2 + \beta v + \gamma)^n$, where α, β and γ are constants, (c) trinomial random walks with the use of the difference equation for the generating function (Shimizu and Yanagimoto 1991), and (d) the general Lagrangian expansion (Consul 1994). The shifted IT is the inverse distribution of a three-point distribution in the sense that the cumulant generating function (cgf) of the shifted IT is the inverse function of the cgf of the three-point distribution, while the shifted inverse binomial (Yanagimoto 1989) is the inverse of a binomial (two-point) distribution in this sense. A multivariate extension (Shimizu et al., 1997) of the IT is obtained by the use of the Lagrangian expansion. We now focus on the generation from the random walk to get a univariate generalization of the shifted IT.

The random walk formulation of the IT is as follows. A particle on a straight line starts from the origin and moves with steps $+1, 0, -1$ according to transition probabilities p, q, r $(p, q, r \geq 0; p + q + r = 1)$, respectively until it first reaches the barrier n (positive integer) at the xth step (Fig. 1). Let X be a random variable which represents the number of steps x. For $p > 0$, the distribution of $Y = X - n$ is the proper IT when $p \geq r$ and is denoted by $\mathrm{IT}(n; p, q, r)$ in this paper. The distribution of X is called the shifted IT. The probability generating function (pgf) of the shifted IT is provided by

$$G_n(t) = E(t^X) = \left[\frac{1 - qt - \sqrt{(1 - qt)^2 - 4prt^2}}{2rt} \right]^n$$

and the probability mass function (pmf) by

$$f_n(x) = \sum_{k=0}^{[(x-n)/2]} \frac{n}{x} \binom{x}{n + k, x - n - 2k, k} p^{n+k} q^{x-n-2k} r^k \qquad (1)$$

Fig. 1 Random walk for the shifted IT on the real line

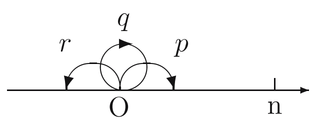
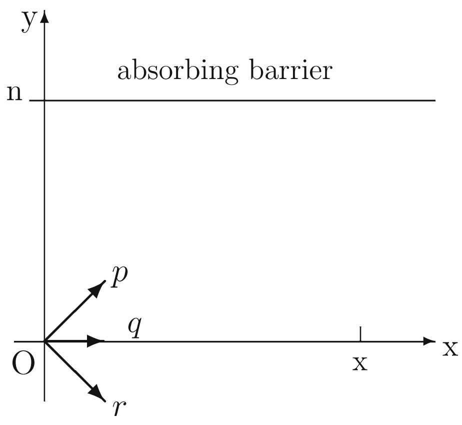


Fig. 2 Random walk for the shifted IT on the half-plane



for $x = n, n+1, n+2, \ldots$, where $[a]$ in (1) denotes the integer part of the number a and $\binom{x}{x_1, x_2, x_3} = x!/(x_1!x_2!x_3!)$, which is the trinomial coefficient. The IT reduces to the inverse binomial or, equivalently, lost-games distribution (Kemp and Kemp 1969) if $q = 0$ and to the negative binomial if $r = 0$.

Figure 2 shows an alternative view of the random walk for the shifted IT on the half-plane ($x \geq 0$). Here a particle starts from the origin and, for non-negative integer x and integer y ($\leq n-1$), the particle moves from (x, y) to $(x+1, y+1), (x+1, y), (x+1, y-1)$ with probabilities p, q, r, respectively until it first reaches the barrier $y = n$. Notice that the particle moves from (x, y) to $(x+1, y+1)$ and $(x+1, y-1)$ directly with probabilities p and r without visiting $(x+1, y)$. When the particle first reaches the barrier, the coordinate x coincides with the number of steps in Fig. 1, and thus the shifted IT is produced from the random walk pictured in Fig. 2. This readily leads to a generalization of the shifted IT if probabilities from (x, y) to $(x, y+1)$ and $(x, y-1)$ are added to the transition probabilities for the shifted IT. The proposed model with five transition probabilities p_1, p_2, p_3, p_4, p_5 ($p_i \geq 0$ for $i = 1, 2, \ldots, 5; \sum_{i=1}^{5} p_i = 1$) and barrier at $y = n$ (positive integer) is shown in Fig. 3. The resulting family of distributions is a generalization of the shifted IT and is denoted by GIT$(n; p_1, p_2, p_3, p_4, p_5)$ or simply GIT.

Section 2 gives the pgf and pmf of the GIT. The pgf is obtained by solving the corresponding difference equation with boundary conditions and the pmf is derived by expanding the pgf. The proof is lengthy and is placed in Appendix A. An alternative expression of the pmf is given in terms of the Gauss hypergeometric function and this offers computational advantage due to its recurrence formula. The inverse distribution of the GIT with respect to the cumulant generating function is studied in Sect. 3 with proof given in Appendix B.

The GIT contains 22 subclasses except one-transition cases. Special cases include the binomial, negative binomial, shifted negative binomial, shifted inverse binomial or, equivalently, lost-games, and shifted inverse trinomial distributions. Section 4 studies GIT$_{3,1}(n; p_1, p_2, p_3)$, a subclass of the GIT family, in detail. (The

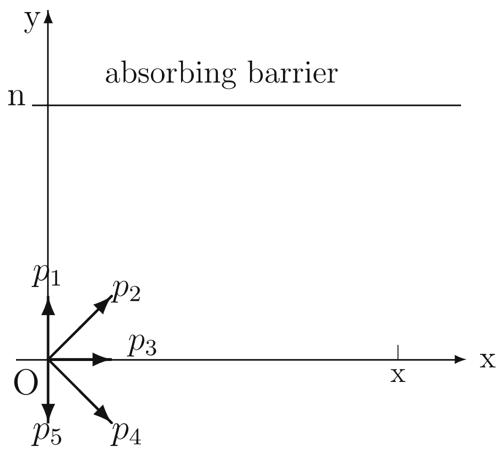


Fig. 3 Random walk for the GIT

first digit in the subscript indicates the number of transition probabilities, while the second digit refers to the combination of transition probabilities p_i given in Table 1.) The $\mathrm{GIT}_{3,1}$ is a particular member of Kemp's class of convolution of pseudo-binomial variables and it has extremely interesting distributional properties of reproductivity, formulation, pmf, moments, index of dispersion, and approximations. Section 5 shows that compound or generalized (stopped sum) distributions (Johnson et al., 2005, p. 381) provide inflated models. The inflated $\mathrm{GIT}_{3,1}$ extends Minkova (2002) inflated-parameter binomial and negative binomial. Finally, a bivariate model which has the GIT as a marginal distribution is proposed in Sect. 6.

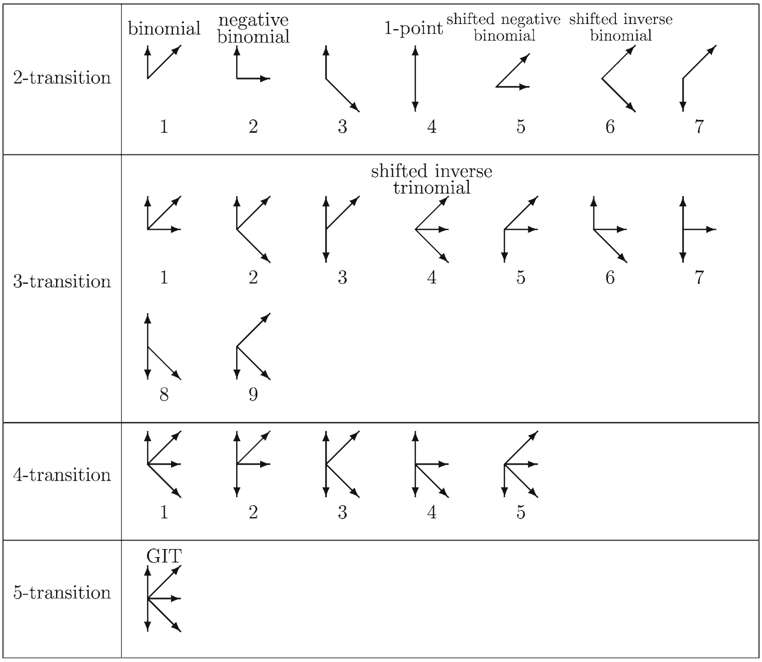
2 Probability generating function and probability mass function

The concept of a modified random walk on the half-plane to generate the GIT is introduced in Sect. 1 (Fig. 3). More precisely, a particle starts from the origin and moves on the lattice of the half-plane as follows. For non-negative integer x and integer y, the particle moves from (x,y) to $(x,y+1)$, $(x+1,y+1)$, $(x+1,y)$, $(x+1,y-1)$, $(x,y-1)$ with five transition probabilities p_1,p_2,p_3,p_4,p_5 ($p_i \geq 0$ for $i=1,2,\ldots,5; \sum_{i=1}^{5} p_i = 1$) respectively. The process ends once the particle reaches the barrier $y=n$ (positive integer). The $\mathrm{GIT}(n;p_1,p_2,p_3,p_4,p_5)$ is the distribution of a random variable X which represents the coordinate of the horizontal axis when the trials end. The pmf $f_n(x)$ of X satisfies the difference equation

$$f_n(x) = p_1 f_{n-1}(x) + p_2 f_{n-1}(x-1) + p_3 f_n(x-1) + p_4 f_{n+1}(x-1) + p_5 f_{n+1}(x)$$

with boundary conditions $f_0(0)=1$, $f_n(-1)=0$ ($n \geq 0$), $f_0(x)=0$ ($x \geq 1$).

Table 1 Subclasses of the GIT



The pgf $G_n(t)$ of $f_n(x)$ is defined by

$$G_n(t) = \sum_{x=0}^{\infty} f_n(x) t^x, \quad n \geq 1,$$

which satisfies the recurrence relation

$$G_n(t) = p_1 G_{n-1}(t) + p_2 t G_{n-1}(t) + p_3 t G_n(t) + p_4 t G_{n+1}(t) + p_5 G_{n+1}(t)$$

or

$$(p_4 t + p_5) G_{n+1}(t) + (-1 + p_3 t) G_n(t) + (p_1 + p_2 t) G_{n-1}(t) = 0, \tag{2}$$

with boundary condition $G_0(t) = 1$. If $p_1 + p_2 > 0$, then the solution of (2) is provided by

$$\begin{aligned} G_n(t) &= \left[\frac{1 - p_3 t - \sqrt{(1 - p_3 t)^2 - 4(p_1 + p_2 t)(p_4 t + p_5)}}{2(p_4 t + p_5)} \right]^n \\ &= \left[\frac{2(p_1 + p_2 t)}{1 - p_3 t + \sqrt{(1 - p_3 t)^2 - 4(p_1 + p_2 t)(p_4 t + p_5)}} \right]^n \end{aligned} \tag{3}$$

for $0 \le t \le 1$. Its proof is given in Appendix A. Note that

$$G_n(1) = \begin{cases} 1, & p_1 + p_2 \ge p_4 + p_5, \\ (p_1 + p_2)/(p_4 + p_5), & p_1 + p_2 < p_4 + p_5, \end{cases}$$

implying that the condition for a proper distribution is $p_1 + p_2 \ge p_4 + p_5$ when (3) is the proper pgf of the GIT. If $p_1 + p_2 < p_4 + p_5$, (3) does not provide a pgf since $\sum_{x=0}^{\infty} f_n(x) = (p_1 + p_2)/(p_4 + p_5) < 1$. In this case the distribution is improper. However, $cf_n(x)$, where $c = (p_4 + p_5)/(p_1 + p_2) \ne 0$, is a proper distribution, or alternatively (3) with $p_1 + p_2 < p_4 + p_5$ gives a distribution if a probability $1 - (p_1 + p_2)/(p_4 + p_5)$ is added at $x = \infty$, which is the probability that the particle never reaches the barrier.

The pmf of the GIT is given by

$$f_n(x) = \sum_{k=0}^{x} \sum_{l=0}^{\infty} \sum_{i=0}^{m} \frac{n}{n + x - i + k + 2l} \binom{n + x - i + k + 2l}{n - i + k + l, i, x - i - k, k, l} \times p_1^{n-i+k+l}\, p_2^i\, p_3^{x-i-k}\, p_4^k\, p_5^l, \tag{4}$$

which is obtained from the expansion of (3) about t, where $m = \min(n + k + l, x - k)$ and $\binom{x}{x_1, x_2, x_3, x_4, x_5} = x!/(x_1!x_2!x_3!x_4!x_5!)$, which is the multinomial coefficient.

An alternative expression of the pmf, which may be more tractable than (4), is obtained as follows. Since it is shown in Appendix A that the pgf of the GIT is written as

$$G_n(t) = \sum_{k=0}^{\infty} \frac{n}{n+k} \binom{n + 2k - 1}{k} \left(\frac{p_1 + p_2 t}{1 - p_3 t}\right)^{n+k} \left(\frac{p_4 t + p_5}{1 - p_3 t}\right)^{k},$$

the pmf is given by

$$f_n(x) = \sum_{k=0}^{\infty} \frac{n}{n+k} \binom{n + 2k - 1}{k} \sum_{i=0}^{x} p_{X_1}(i) p_{X_2}(x - i) \tag{5}$$

with

$$p_{X_1}(i) = \binom{n + k + i - 1}{i} p_1^{n+k} p_3^i\, {}_2F_1\left(-n - k, -i; -n - k - i + 1; -\frac{p_2}{p_1 p_3}\right)$$

and

$$p_{X_2}(i) = \binom{k + i - 1}{i} p_5^k p_3^i\, {}_2F_1\left(-k, -i; -k - i + 1; -\frac{p_4}{p_5 p_3}\right)$$

from the expansion of $H_1(t) = ((p_1 + p_2 t)/(1 - p_3 t))^{n+k}$ and $H_2(t) = ((p_4 t + p_5) / (1 - p_3 t))^k$, where ${}_2F_1$ denotes the Gauss hypergeometric function. Equation (5) offers computational advantage in the following way. Compared to (4), this expression involves only one infinite sum and one finite sum, and only one binomial coefficient. The terms $p_{X_1}(i)$ and $p_{X_2}(i)$ are also easily computed by three-term recurrence formulae (see Sect. 4.2d).

The rth descending factorial moment of the GIT is obtained from

$$\mu'_{[r]} = \left.\frac{\partial^r G_n(t)}{\partial t^r}\right|_{t=1} = \sum_{k=0}^{\infty} \frac{n}{n+k} \binom{n+2k-1}{k} \times \sum_{\ell=0}^{r} \left\{ \left.\frac{\partial^\ell H_1(t)}{\partial t^\ell}\right|_{t=1} \right\} \times \left\{ \left.\frac{\partial^{r-\ell} H_2(t)}{\partial t^{r-\ell}}\right|_{t=1} \right\}.$$

Since

$$\frac{\partial^\ell H_1(t)}{\partial t^\ell} = \sum_{i=0}^{\ell} \binom{\ell}{i} (n+k)(n+k-1)\cdots(n+k-i+1) p_2^i (p_1 + p_2 t)^{n+k-i} \times(-n-k)(-n-k-1)\cdots(-n-k-(\ell-i)+1)(-p_3)^{\ell-i} \times(1-p_3 t)^{-n-k-\ell+i},$$

we have

$$\left.\frac{\partial^\ell H_1(t)}{\partial t^\ell}\right|_{t=1} = (n+k) \sum_{i=0}^{\ell} \binom{\ell}{i} \frac{(n+k+\ell-i-1)!}{(n+k-i)!} p_2^i p_3^{\ell-i} \left(\frac{p_1+p_2}{1-p_3}\right)^{n+k-i} \times(1-p_3)^{-\ell}$$

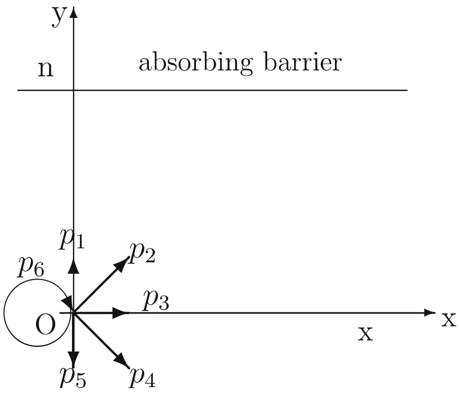
and similarly

$$\left.\frac{\partial^{r-\ell} H_2(t)}{\partial t^{r-\ell}}\right|_{t=1} = k \sum_{j=0}^{r-\ell} \binom{r-\ell}{j} \frac{(k+r-\ell-j-1)!}{(k-j)!} p_4^j p_3^{r-\ell-j} \left(\frac{p_4+p_5}{1-p_3}\right)^{k-j} \times(1-p_3)^{\ell-r}.$$

Thus, the rth descending factorial moment is finally given by

$$\mu'_{[r]} = \sum_{k=0}^{\infty} \frac{n}{n+k} \binom{n+2k-1}{k} \times \sum_{\ell=0}^{r} \mu_1(\ell)\mu_2(\ell) \left(\frac{p_3}{1-p_3}\right)^r,$$

Fig. 4 Random walk for the GIT with stay probability



where

$$\mu_1(\ell) = (n+k)\sum_{i=0}^{\ell}\binom{\ell}{i}\frac{(n+k+\ell-i-1)!}{(n+k-i)!}\left(\frac{p_2}{p_3}\right)^i\left(\frac{p_1+p_2}{1-p_3}\right)^{n+k-i},$$

$$\mu_2(\ell) = k\sum_{j=0}^{r-\ell}\binom{r-\ell}{j}\frac{(k+r-\ell-j-1)!}{(k-j)!}\left(\frac{p_4}{p_3}\right)^j\left(\frac{p_4+p_5}{1-p_3}\right)^{k-j}.$$

Note that the GIT of Fig. 3 is generalized, as Fig. 4, by adding a stay probability p_6 at (x, y), but this model does not produce an extended family of distributions different from the GIT. The reason is as follows. Apparently the new pmf $f_n(x)$ and pgf $G_n(t)$ satisfy the difference equation

$$f_n(x) = p_1 f_{n-1}(x) + p_2 f_{n-1}(x-1) + p_3 f_n(x-1) + p_4 f_{n+1}(x-1) + p_5 f_{n+1}(x) + p_6 f_n(x)$$

and recurrence relation

$$(1-p_6)G_n(t) = p_1 G_{n-1}(t) + p_2 t G_{n-1}(t) + p_3 t G_n(t) + p_4 t G_{n+1}(t) + p_5 G_{n+1}(t)$$

respectively, from which division of both side by $1 - p_6$ leads to

$$G_n(t) = p_1' G_{n-1}(t) + p_2' t G_{n-1}(t) + p_3' t G_n(t) + p_4' t G_{n+1}(t) + p_5' G_{n+1}(t),$$

where $p_i' = p_i/(1-p_6)$ for $i = 1, 2, \ldots, 5$. This recurrence relation is of the form (2). The following problem could arise since the GIT model without a stay probability is indistinguishable from (not identifiable as) the GIT model with a stay probability. Suppose we consider a GIT model with a stay probability p_6. In the context of parameter estimation, it is not possible to estimate p_6 from the estimates for p_i', $i = 1, 2, \ldots, 5$. For identifiability, a stay probability should not be considered in the random walk model for the GIT. It is observed that

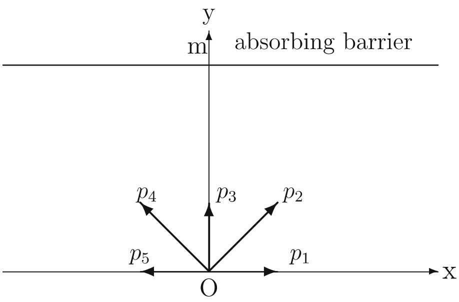


Fig. 5 Random walk for the inverse family of the GIT

the stay probability q in the random walk on the real line for the IT has become the transition probability p_3 along the x-axis in the random walk model on the half-plane (Fig. 2). Thus, the meaning of the stay probabilities is quite different in this sense between IT and GIT.

3 Inverse family of the GIT

When X is distributed as the GIT and its cgf is $C(t) = \log E(\mathrm{e}^{-tX})$, the inverse distribution of the GIT is defined by the distribution whose cgf is given by the inverse function $C^{-1}(t)$ of $C(t)$. Since the cgf of GIT$(n; p_1, p_2, p_3, p_4, p_5)$ is

$$C(t) = -n \log\left[\frac{1-p_3\mathrm{e}^{-t} + \sqrt{(1-p_3\mathrm{e}^{-t})^2 - 4(p_1+p_2\mathrm{e}^{-t})(p_4\mathrm{e}^{-t}+p_5)}}{2(p_1+p_2\mathrm{e}^{-t})}\right], \quad (6)$$

its inverse function is

$$C^{-1}(t) = \log\left[\frac{p_2\mathrm{e}^{-2t/n} + p_3\mathrm{e}^{-t/n} + p_4}{-p_1\mathrm{e}^{-2t/n} + \mathrm{e}^{-t/n} - p_5}\right]. \quad (7)$$

Consider the following modified random walk. A particle starts from the origin and, for integers x and y $(0 \le y \le m-1)$, moves from (x, y) to $(x+1, y), (x+1, y+1), (x, y+1), (x-1, y+1), (x-1, y)$ with five transition probabilities p_1, p_2, p_3, p_4, p_5 $(p_i \ge 0$ for $i = 1, 2, \ldots, 5; \sum_{i=1}^{5} p_i = 1)$, respectively until it first reaches the barrier $y = m$ (Fig. 5). That is, the transition in Fig. 5 is the reflection of that in Fig. 3 with respect to the line $y = x$. (Alternatively, the inverse family of the GIT may be obtained from the random walk model for the GIT in Fig. 3 by setting the absorbing barrier at $x = m$ instead of $y = n$ and then interchanging x and y with reflection about the new x-axis.) Let a random variable Z represent the coordinate x of the horizontal axis when the trials end, then a random variable Z/n in the case $m = 1$ has a distribution whose cgf is $C^{-1}(t)$ in (7). See Appendix B for details. In other words, the GIT is the inverse of the distribution whose cgf is (7).

4 Subclasses of the GIT

4.1 Some examples

There exist twenty-two subclasses of the GIT if one or some of p_1, p_2, p_3, p_4, p_5 are substituted by zero except for one-transition cases where $p_1 = 1,\ p_2 = p_3 = p_4 = p_5 = 0$ and $p_2 = 1,\ p_1 = p_3 = p_4 = p_5 = 0$. Table 1 summarizes the possibility of the distributions. Some known classical distributions which belong to the family are shown below with pgf's and pmf's.

(a) $\mathrm{GIT}_{2,1}(n; p_1, p_2)$, which is the binomial also denoted by $B(n, p_2)$.

$$G_n(t) = (p_1 + p_2 t)^n, \quad f_n(x) = \binom{n}{x} p_2^x\, p_1^{n-x}$$

for $x = 0, 1, \ldots, n$.

(b) $\mathrm{GIT}_{2,2}(n; p_1, p_3)$, which is the negative binomial also denoted by $\mathrm{NB}(n, p_3)$.

$$G_n(t) = \left(\frac{p_1}{1 - p_3 t}\right)^n, \quad f_n(x) = \binom{n + x - 1}{x} p_3^x\, p_1^n$$

for $x = 0, 1, 2, \ldots$.

(c) $\mathrm{GIT}_{2,5}(n; p_2, p_3)$, which is the shifted negative binomial.

$$G_n(t) = \left(\frac{p_2 t}{1 - p_3 t}\right)^n, \quad f_n(x) = \binom{x - 1}{n - 1} p_3^{x-n}\, p_2^n$$

for $x = n, n + 1, n + 2, \ldots$.

(d) $\mathrm{GIT}_{2,6}(n; p_2, p_4)$, which is the shifted inverse binomial or, equivalently, lost-games distribution.

$$G_n(t) = \left[\frac{1 - \sqrt{1 - 4p_2 p_4 t^2}}{2p_4 t}\right]^n, \quad f_n(x) = \frac{n}{x}\binom{x}{(x - n)/2} p_2^{(n+x)/2}\, p_4^{(x-n)/2}$$

for $x = n, n + 2, n + 4, \ldots$.

(e) $\mathrm{GIT}_{3,4}(n; p_2, p_3, p_4)$, which is the shifted inverse trinomial.

$$G_n(t) = \left[\frac{1 - p_3 t - \sqrt{(1 - p_3 t)^2 - 4p_2 p_4 t^2}}{2p_4 t}\right]^n,$$

$$f_n(x) = \sum_{k=0}^{[(x-n)/2]} \frac{n}{x}\binom{x}{n + k, x - n - 2k, k} p_2^{n+k}\, p_3^{x-n-2k}\, p_4^k$$

for $x = n, n + 1, n + 2, \ldots$.

4.2 Properties of $\mathrm{GIT}_{3,1}(n;p_1,p_2,p_3)$

The most interesting subclass may be $\mathrm{GIT}_{3,1}$, which is the distribution such that the transition is invariant under the inversion (with indices exchanged). It is a particular member of Kemp's class of convolution of pseudo-binomial variables (Johnson et al., 1992, p. 147 and references therein; see also Johnson et al., 2005, pp. 140–144) with pgf

$$G(t) = \left(\frac{1-Q_1t}{1-Q_1}\right)^{U_1}\left(\frac{1-Q_2t}{1-Q_2}\right)^{U_2}$$

and generalizes the binomial, negative binomial and shifted negative binomial. Actually the pgf of $\mathrm{GIT}_{3,1}(n;p_1,p_2,p_3)$ is written as

$$G_n(t) = \left(\frac{p_1+p_2t}{1-p_3t}\right)^n = \left(\frac{p_1+p_2t}{p_1+p_2}\right)^n\left(\frac{1-p_3}{1-p_3t}\right)^n, \quad 1-p_3 = p_1+p_2 \tag{8}$$

and reduces to the binomial if $p_3 = 0$, negative binomial if $p_2 = 0$ and shifted negaitve binomial (shifted n steps to the right) if $p_1 = 0$. Some properties of $\mathrm{GIT}_{3,1}(n;p_1,p_2,p_3)$ are summarized below. Compound distributions are studied in Sect. 5. Here X indicates a random variable having $\mathrm{GIT}_{3,1}(n;p_1,p_2,p_3)$.

(a) Reproductive property. If X_1 and X_2 are independent random variables distributed as $\mathrm{GIT}_{3,1}(n;p_1,p_2,p_3)$ and $\mathrm{GIT}_{3,1}(m;p_1,p_2,p_3)$ respectively, then the sum X_1+X_2 is distributed as $\mathrm{GIT}_{3,1}(n+m;p_1,p_2,p_3)$.
(b) The random variable X is expressible as the sum of two independent random variables X_1 and X_2, where X_1 has the binomial $B(n,p_2/(p_1+p_2))$ and X_2 the negative binomial $\mathrm{NB}(n,p_3)$.
(c) The pmf of $\mathrm{GIT}_{3,1}(n;p_1,p_2,p_3)$ is given by

$$f_n(x) = \sum_{i=0}^{\min(n,x)} \frac{n}{n+x-i}\binom{n+x-i}{n-i,i,x-i} p_1^{n-i}\, p_2^i\, p_3^{x-i} \tag{9}$$

for $x = 0,1,2,\ldots$. Equation (9) is also expressed as

$$f_n(x) = \binom{n+x-1}{x} p_1^n p_3^x\, {}_2F_1\left(-n,-x;-n-x+1;-\frac{p_2}{p_1p_3}\right)$$

in terms of the Gauss hypergeometric function. This is the pmf of the negative binomial multiplied by ${}_2F_1$. An alternative expression is

$$
\begin{aligned}
f_n(x) &= \sum_{j=\max(0,x-n)}^{x} \frac{n}{n+j} \binom{n+j}{n-x+j, x-j, j} p_1^{n-x+j} p_2^{x-j} p_3^{j} \\
&= \binom{n}{x} p_1^{n-x} p_2^{x} \, {}_2F_1\left(n, -x; n-x+1; -\frac{p_1 p_3}{p_2}\right) \quad \text{if } n \geq x \\
&= \binom{x-1}{n-1} p_2^{n} p_3^{x-n} \, {}_2F_1\left(x, -n; x-n+1; -\frac{p_1 p_3}{p_2}\right) \quad \text{if } x > n,
\end{aligned}
$$

which is the pmf of the binomial multiplied by ${}_2F_1$.

(d) The pmf $f_n(x)$ of X satisfies the recurrence relation

$$
f_n(x) = \left(a + \frac{b}{x}\right) f_n(x-1) + c\left(1 - \frac{2}{x}\right) f_n(x-2)
$$

for $x \geq 2$ with initial conditions $f_n(0) = p_1^n$, $f_n(1) = np_1^{n-1}(p_1p_3 + p_2)$, which is an example of Sundt (1992) recursion, where $a = (p_1p_3 - p_2)/p_1$, $b = (n(p_1p_3 + p_2) - (p_1p_3 - p_2))/p_1$, $c = p_2p_3/p_1$ with $p_1 > 0$.

(e) The rth descending factorial moment of X is

$$
\mu'_{[r]} = E(X(X-1)\ldots(X-r+1)) = \frac{n}{(p_1+p_2)^r} \sum_{i=0}^{r} \binom{r}{i} \frac{(n+r-i-1)!}{(n-i)!} \times p_2^{i} p_3^{r-i}
$$

for $r \geq 1$. The descending factorial moment has a recursion formula. Beginning with the factorial moment generating function

$$
G(t+1) = \left(\frac{p_1 + p_2 + p_2 t}{1 - p_3 - p_3 t}\right)^n = \sum_{r=0}^{\infty} \frac{\mu'_{[r]}}{r!} t^r,
$$

we have

$$
G'(t+1) = \sum_{r=1}^{\infty} \frac{\mu'_{[r]}}{(r-1)!} t^{r-1} = nG(t+1) \frac{(p_1+p_2)(p_2+p_3)}{(1-p_3-p_3t)(p_1+p_2+p_2t)}.
$$

From the relation

$$
\left\{(p_1+p_2)^2 + (p_1+p_2)(p_2-p_3)t - p_2p_3t^2\right\} G'(t+1) = n(p_1+p_2)(p_2+p_3) \times G(t+1),
$$

we have the recursion formula for the descending factorial moment

$$\frac{(p_1+p_2)^2}{r!}\mu'_{[r+1]}+\frac{(p_1+p_2)(p_2-p_3)}{(r-1)!}\mu'_{[r]}-\frac{p_2p_3}{(r-2)!}\mu'_{[r-1]}$$
$$=\frac{n(p_1+p_2)(p_2+p_3)}{r!}\mu'_{[r]},$$

which leads to

$$(p_1+p_2)^2\mu'_{[r+1]}+\{r(p_1+p_2)(p_2-p_3)-n(p_1+p_2)(p_2+p_3)\}\,\mu'_{[r]}$$
$$-r(r-1)p_2p_3\mu'_{[r-1]}=0$$

with initial conditions

$$\mu'_{[0]}=1,\ \mu'_{[1]}=n\,\frac{p_2+p_3}{p_1+p_2}.$$

(f) The rth moment, $\mu'_r = E(X^r)$, of X about zero satisfies the recurrence relation

$$\mu'_r=\sum_{j=0}^{r-1}\binom{r-1}{j}\{(a+2^{r-j-1}c)\mu'_{j+1}+(a+b)\mu'_j\}$$

for $r \geq 1$ with initial condition $\mu'_0 = 1$ and the understanding that $0! = 1$, where a, b and c are given in (d). This is proved by using the recurrence relation of the pmf in (d) as follows

$$\mu'_r=\sum_{x=1}^{\infty}x^r f(x)$$
$$=f(1)+\sum_{x=0}^{\infty}(x+2)^r f(x+2)$$
$$=\left(a+\frac{b}{1}\right)f(0)+\sum_{x=0}^{\infty}(x+2)^r\left(a+\frac{b}{x+2}\right)f(x+1)$$
$$+\sum_{x=0}^{\infty}(x+2)^r c\left(1-\frac{2}{x+2}\right)f(x)$$
$$=\sum_{x=0}^{\infty}(x+1)^r\left(a+\frac{b}{x+1}\right)f(x)+\sum_{x=0}^{\infty}(x+2)^r c\left(1-\frac{2}{x+2}\right)f(x)$$
$$=\sum_{x=0}^{\infty}(x+1)^r\left(\frac{ax+(a+b)}{x+1}\right)f(x)+\sum_{x=0}^{\infty}(x+2)^r c\left(\frac{x}{x+2}\right)f(x)$$

$$= \sum_{j=0}^{r-1} \sum_{x=0}^{\infty} \binom{r-1}{j} \{ax^{j+1} + (a+b)x^j\} f(x) + \sum_{j=0}^{r-1} \sum_{x=0}^{\infty} \binom{r-1}{j} \times 2^{r-1-j} c x^{j+1} f(x).$$

(g) The mean and variance of X are obtainable from (e) and (f). They are also obtained by using (b). Actually

$$E(X) = E(X_1 + X_2) = n\frac{p_2}{p_1 + p_2} + n\frac{p_3}{1 - p_3} = n\frac{p_2 + p_3}{p_1 + p_2},$$

$$V(X) = V(X_1 + X_2) = n\left(\frac{p_1}{p_1 + p_2}\right)\left(\frac{p_2}{p_1 + p_2}\right) + n\frac{p_3}{(1 - p_3)^2} = n\frac{p_1 p_2 + p_3}{(p_1 + p_2)^2},$$

from which the index of dispersion (ID) is

$$\text{ID} = \frac{V(X)}{E(X)} = \frac{p_1 p_2 + p_3}{(p_1 + p_2)(p_2 + p_3)} \begin{cases} >1, & p_3 > p_2, \\ <1, & p_3 < p_2. \end{cases}$$

If $p_2 = p_3$, then $\text{ID} = 1$. Note that $\text{GIT}_{3,1}(n; 1-2p, p, p)$ with $0 < p < 1/2$ is not a Poisson distribution, but its ID is unity. This gives a quite interesting example because it is known that among the power series distributions, the Poisson distribution is characterized by $\text{ID} = 1$ (Johnson et al., 2005, p. 179).

(h) Normal approximation. The distribution of $(X - E(X))/\sqrt{V(X)}$ goes to a standard normal distribution as n tends to infinity.

(i) Poisson approximation. The distribution $\text{GIT}_{3,1}(n; p_1, p_2, p_3)$ goes to a Poisson distribution with parameter $\lambda_2 + \lambda_3$ as n tends to infinity provided $np_2 = \lambda_2$ and $np_3 = \lambda_3$.

5 Compound distributions

5.1 Inflated model

This section studies the distribution of the random variable $S = X_1 + \cdots + X_N$ with the understanding that $S = 0$ when $N = 0$, where $X_1, X_2, \ldots$ are independent and identically distributed (iid) as $\text{GIT}_{3,1}(1; q_1, q_2, q_3)$, N as GIT $(n; p_1, p_2, p_3, p_4, p_5)$, and N is independent of $X_1, X_2, \ldots$. Let $G_X(t)$ and $G_N(t)$ denote the pgf's of X_i $(i = 1, 2, \ldots)$ and N respectively. Then, from (8) and (3), the pgf $G_S(t)$ of S is provided by

$$\begin{aligned} G_S(t) &= G_N\left(G_X(t)\right) \\ &= \left[\frac{1-\alpha_3 t + \sqrt{(1-\alpha_3 t)^2 - 4(\alpha_1+\alpha_2 t)(\alpha_4 t+\alpha_5)}}{2(\alpha_1+\alpha_2 t)}\right]^{-n}, \end{aligned} \tag{10}$$

which is the pgf of $\text{GIT}(n;\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$ for $0 \le t \le 1$, where $\alpha_1 = (p_1 + p_2q_1)/(1-p_3q_1)$, $\alpha_2 = (-p_1q_3+p_2q_2)/(1-p_3q_1)$, $\alpha_3 = (p_3q_2+q_3)/(1-p_3q_1)$, $\alpha_4 = (p_4q_2 - p_5q_3)/(1-p_3q_1)$, $\alpha_5 = (p_4q_1+p_5)/(1-p_3q_1)$. Note that the range of parameters is extended to $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 = 1$, $-1 \le \alpha_2,\alpha_4 \le 1$, $0 \le \alpha_1,\alpha_3,\alpha_5 \le 1$, $\alpha_1\alpha_3+\alpha_2 \ge 0$, $\alpha_4+\alpha_3\alpha_5 \ge 0$, $\alpha_1+\alpha_2 > 0$ and $\alpha_1+\alpha_2 \ge \alpha_4+\alpha_5$, whereas $p_i \ge 0$ $(i = 1,2,\ldots,5)$, $\sum_{i=1}^5 p_i = 1$, $p_1+p_2 > 0$, $p_1+p_2 \ge p_4+p_5$, and $q_j \ge 0$ $(j = 1,2,3)$, $\sum_{j=1}^3 q_j = 1$. Thus (10) defines an inflated model of the GIT. As a particular case, if $X_1, X_2, \ldots$ are iid as $\text{GIT}_{3,1}(1;q_1,q_2,q_3)$ and N as $\text{GIT}_{3,1}(n;p_1,p_2,p_3)$, and N is independent of $X_1, X_2, \ldots$, then $S = X_1+\cdots+X_N$ has an inflated $\text{GIT}_{3,1}(n;\gamma_1,\gamma_2,\gamma_3)$, where $\gamma_1 = (p_1+p_2q_1)/(1-p_3q_1)$, $\gamma_2 = (-p_1q_3+p_2q_2)/(1-p_3q_1)$, $\gamma_3 = (p_3q_2+q_3)/(1-p_3q_1)$, $\gamma_1+\gamma_2+\gamma_3 = 1$, $-1 \le \gamma_2 \le 1$, $0 \le \gamma_1,\gamma_3 \le 1$.

5.2 A comment

Minkova (2002) studied the family of inflated-parameter generalized power series distributions or generalized power series distributions generalized by the generalizing shifted geometric distribution. If X has a shifted geometric distribution and N a Poisson, binomial, negative binomial, log-series as a member of generalized power series distributions, then the distribution of $S = X_1+\cdots+X_N$ is called the inflated-parameter Poisson, binomial (IBi), negative binomial (INB), log-series respectively. The inflated $\text{GIT}_{3,1}(n;\gamma_1,\gamma_2,\gamma_3)$ in Sect. 5.1 extends the IBi and INB because $\text{GIT}_{3,1}(1;q_1,q_2,q_3)$ includes the shifted geometric distribution $\text{GIT}_{2,5}(1;q_2,q_3)$ and $\text{GIT}_{3,1}(n;p_1,p_2,p_3)$ does the binomial $\text{GIT}_{2,1}(n;p_1,p_2)$ and negative binomial $\text{GIT}_{2,2}(n;p_1,p_3)$. The inflated $\text{GIT}(n;\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$ in Sect. 5.1 extends the inflated $\text{GIT}_{3,1}(n;\gamma_1,\gamma_2,\gamma_3)$.

6 A bivariate model

Consider the random walk model for the GIT (Fig. 3). Let X denote the coordinate of the horizontal axis (i.e. $X \sim$ GIT) and Y the number of transitions. We can consider the bivariate distribution of (X, Y). The joint pmf satisfies the difference equation

$$\begin{aligned} f_n(x,y) &= p_1 f_{n-1}(x,y-1) + p_2 f_{n-1}(x-1,y-1) + p_3 f_n(x-1,y-1) \\ &\quad + p_4 f_{n+1}(x-1,y-1) + p_5 f_{n+1}(x,y-1). \end{aligned}$$

The joint pgf defined by

$$G_n(t,s) = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} t^x s^y f_n(x,y)$$

satisfies the difference equation

$$G_n(t,s) = p_1 s G_{n-1}(t,s) + p_2 ts G_{n-1}(t,s) + p_3 ts G_n(t,s) + p_4 ts G_{n+1}(t,s) + p_5 s G_{n+1}(t,s),$$

which leads to

$$(p_4 ts + p_5 s)G_{n+1}(t,s) + (-1 + p_3 ts)G_n(t,s) + (p_1 s + p_2 ts)G_{n-1}(t,s) = 0.$$

The solution of the form $G_n(t,s) = \{\lambda(t,s)\}^n$ is given by

$$G_n(t,s) = \left[\frac{1 - p_3 ts - \sqrt{(1-p_3 ts)^2 - 4(p_1 s + p_2 ts)(p_4 ts + p_5 s)}}{2(p_4 ts + p_5 s)}\right]^n.$$

The marginal pgf's are: $G_n(t,1)$ is the pgf of the GIT and

$$G_n(1,s) = \left[\frac{1 - p_3 s - \sqrt{(1-p_3 s)^2 - 4(p_1 + p_2)(p_4 + p_5)s^2}}{2(p_4 + p_5)s}\right]^n$$

is the pgf of the shifted IT$(n; p_1 + p_2, p_3, p_4 + p_5)$. The joint pmf of (X, Y) may have an intractable form and is beyond the scope of the present paper.

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Appendix A: The probability mass function of the GIT

The pmf of the GIT is found from the difference equation given by (2)

$$(p_4 t + p_5)G_{n+1}(t) + (-1 + p_3 t)G_n(t) + (p_1 + p_2 t)G_{n-1}(t) = 0$$

with boundary condition $G_0(t) = 1$. We look for particular solutions $G_n(t)$ of the form $G_n(t) = \{\lambda(t)\}^n$. Substitution of this expression into (2) gives the quadratic equation

$$(p_4 t + p_5)\lambda^2(t) + (-1 + p_3 t)\lambda(t) + p_1 + p_2 t = 0, \tag{11}$$

which has the two roots

$$\lambda_1(t) = \frac{1 - p_3 t - \sqrt{(1 - p_3 t)^2 - 4(p_4 t + p_5)(p_1 + p_2 t)}}{2(p_4 t + p_5)},$$

$$\lambda_2(t) = \frac{1 - p_3 t + \sqrt{(1 - p_3 t)^2 - 4(p_4 t + p_5)(p_1 + p_2 t)}}{2(p_4 t + p_5)}.$$

The range of t for which $(1 - p_3 t)^2 - 4(p_4 t + p_5)(p_1 + p_2 t) \geq 0$ is $0 \leq t \leq (b - \sqrt{b^2 - ac})/a$, with $a = p_3^2 - 4p_2 p_4$, $b = p_2 + 2p_2 p_5 + 2p_1 p_4$, $c = 1 - 4p_1 p_5$ and $(b - \sqrt{b^2 - ac})/a > 1$. Since $0 < \lambda_1(t) < 1$ and $\lambda_2(t) > 1$ for $0 < t < 1$, $\{\lambda_2(t)\}^n$ is inappropriate as a solution and $G_n(t) = B(t)\{\lambda_1(t)\}^n$ is a solution to (11). From the boundary condition $G_0(t) = 1$, we obtain $B(t) = 1$. Hence the solution is

$$G_n(t) = \{\lambda_1(t)\}^n. \tag{12}$$

The pmf is provided by expanding (12) about t. From the formula (Abramowitz and Stegun 1972 [15.1.13])

$${}_2F_1\left(a, \frac{1}{2} + a; 1 + 2a; z\right) = 2^{2a}\{1 + (1 - z)^{1/2}\}^{-2a}$$

for $|z| < 1$, where ${}_2F_1$ stands for the Gauss hypergeometric function, (12) is transformed into

$$\begin{aligned}
G_n(t) &= \left[\frac{1 - p_3 t + \sqrt{(1 - p_3 t)^2 - 4(p_4 t + p_5)(p_1 + p_2 t)}}{2(p_1 + p_2 t)}\right]^{-n} \\
&= \left(\frac{1 - p_3 t}{2(p_1 + p_2 t)}\right)^{-n}\left[1 + \sqrt{1 - \frac{4(p_1 + p_2 t)(p_4 t + p_5)}{(1 - p_3 t)^2}}\right]^{-n} \\
&= \left(\frac{1 - p_3 t}{p_1 + p_2 t}\right)^{-n} {}_2F_1\left(\frac{n}{2}, \frac{n+1}{2}; n + 1; \frac{4(p_1 + p_2 t)(p_4 t + p_5)}{(1 - p_3 t)^2}\right) \\
&= \left(\frac{1 - p_3 t}{p_1 + p_2 t}\right)^{-n} \sum_{k=0}^{\infty} \frac{(n/2)_k\,((n+1)/2)_k}{(n+1)_k\,k!}\left(\frac{4(p_4 t + p_5)(p_1 + p_2 t)}{(1 - p_3 t)^2}\right)^k,
\end{aligned}$$

where $(x)_i = x(x+1)\ldots(x+i-1) = \Gamma(x+i)/\Gamma(x)$. From the duplication formula for the gamma function

$$\Gamma(2z) = \frac{1}{(2\pi)^{1/2}} 2^{2z-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right),$$

we obtain

$$G_n(t) = \sum_{k=0}^{\infty} \frac{n}{n+k} \binom{n+2k-1}{k} (p_1 + p_2 t)^{n+k} (p_4 + p_5)^k \left(\frac{1}{1-p_3 t}\right)^{n+2k}$$
$$= \sum_{j}^{\infty} \sum_{k}^{\infty} \sum_{l}^{k} \sum_{i}^{n+k} \frac{n}{n+2k+j} \binom{n+2k+j}{n+k-i, i, j, k-l, l}$$
$$\times p_1^{n+k-i} p_2^i p_3^j p_4^{k-l} p_5^l t^{i+j+k-l},$$

which leads to

$$\sum_{x}^{\infty} \sum_{k'}^{\infty} \sum_{l}^{\infty} \sum_{i}^{n+k'+l} \frac{n}{n+2k'+2l+j} \binom{n+2k'+2l+j}{n+k'+l-i, i, j, k', l}$$
$$\times p_1^{n+k'+l-i} p_2^i p_3^j p_4^{k'} p_5^l t^{i+j+k'}$$

after the replacement $k - l = k'$. The coefficient of t^x, where $x = i + j + k'$, in $G_n(t)$ is (4), the pmf of the GIT.

Appendix B: The inverse distribution of the GIT

We consider the distribution derived from the modified random walk defined in Sect. 3 (Fig. 5). The difference equation of the pmf is

$$h_m(x) = p_1 h_m(x-1) + p_2 h_{m-1}(x-1) + p_3 h_{m-1}(x) + p_4 h_{m-1}(x+1)$$
$$+ p_5 h_m(x+1)$$

with initial condition $h_0(x) = 1$ if $x = 0$, 0 if $x \neq 0$, and the recurrence relation of the corresponding pgf is

$$H_m(t) = p_1 t H_m(t) + p_2 t H_{m-1}(t) + p_3 H_{m-1}(t) + p_4 t^{-1} H_{m-1}(t) + p_5 t^{-1} H_m(t)$$

with initial condition $H_0(t) = 1$. From this we obtain the pgf

$$H_m(t) = \left(\frac{p_2 t^2 + p_3 t + p_4}{-p_1 t^2 + t - p_5}\right)^m.$$

On the other hand to get the inverse of (6), we set $\log E(\mathrm{e}^{-tX}) = s$. Then

$$\frac{1 - p_3 \mathrm{e}^{-t} + \sqrt{(1 - p_3 \mathrm{e}^{-t})^2 - 4(p_1 + p_2 \mathrm{e}^{-t})(p_4 \mathrm{e}^{-t} + p_5)}}{2(p_1 + p_2 \mathrm{e}^{-t})} = e^{-s/n}.$$

If we set $e^{-t} = T$ and $e^{-s/n} = S$, then we have

$$
\begin{aligned}
0 &= S^2(p_1^2 + 2p_1p_2T + p_2^2T^2) + S(p_2p_3T^2 + p_1p_3T - p_2T - p_1) \\
&\quad + (p_2p_4T^2 + p_1p_4T + p_2p_5T + p_1p_5) \\
&= p_2^2\left(S^2 + \frac{p_3}{p_2}S + \frac{p_4}{p_2}\right)T^2 + p_1p_2\left\{2S^2 + \left(\frac{p_3}{p_2} - \frac{1}{p_1}\right)S + \left(\frac{p_4}{p_2} + \frac{p_5}{p_1}\right)\right\}T \\
&\quad + p_1^2\left\{S^2 - \frac{1}{p_1}S + \frac{p_5}{p_1}\right\} \\
&= p_2^2(S^2 + aS + c)T^2 + p_1p_2\left\{2S^2 + (a-b)S + (c+d)\right\}T + p_1^2(S^2 - bS + d) \\
&\equiv AT^2 + BT + C, \qquad (13)
\end{aligned}
$$

where $a = p_3/p_2,\ b = 1/p_1,\ c = p_4/p_2,\ d = p_5/p_1,\ A = p_2^2(S^2 + aS + c),\ B = p_1p_2(2S^2 + (a-b)S + (c+d)),\ C = p_1^2(S^2 - bS + d)$. The solution of (13) is

$$
T = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}
$$

with $B^2 - 4AC = p_1^2p_2^2\{(a+b)S + (c-d)\}^2 \geq 0$. We see that

$$
\begin{aligned}
T &= \frac{-p_1p_2\{2S^2 + (a-b)S + (c+d)\} \pm p_1p_2\{(a+b)S + (c-d)\}}{2(p_2^2S^2 + p_2p_3S + p_2p_4)} \\
&= \frac{-p_1S^2 + S - p_5}{p_2S^2 + p_3S + p_4}, \quad \frac{-p_1p_2S^2 - p_1p_3S - p_1p_4}{p_2^2S^2 + p_2p_3S + p_2p_4}
\end{aligned}
$$

and the second solution is inappropriate since it is negative. Thus we obtain (7).

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