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Multivariate measures of concordance

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Abstract In 1984 Scarsini introduced a set of axioms for measures of concordance of ordered pairs of continuous random variables. We exhibit an extension of these axioms to ordered n-tuples of continuous random variables, $n \geq 2$. We derive simple properties of such measures, give examples, and discuss the relation of the extended axioms to multivariate measures of concordance previously discussed in the literature.

Keywords Copula · Concordance · Measures of association · Multivariate measures of association

1 Introduction

If we are given an n-tuple of random variables, we think of their "concordance" as being their tendency to all be large or to all be small simultaneously. The idea of comparing the concordances of different n-tuples of random variables is considered in, for example, Kimeldorf and Sampson (1987, 1989) and Joe (1990, 1997). A second way to investigate concordance is to attach to an n-tuple $(X_1, \ldots, X_n)$ of random variables a real number $\kappa_{X_1,\ldots,X_n}$, such as Kendall's tau or Spearman's rho, that "measures" their concordance.

A set of axioms was given in Scarsini (1984) for measures of concordance κ_{X_1,X_2} for ordered pairs of random variables. Scarsini's measures of concordance are invariant under a.s. increasing transformations of the random variables, hence they can be treated as operating on two-copulas. [One is also referred to Nelsen (1999) for a nice exposition of these ideas].

How might one generalize Scarsini's axioms to more than two random variables? This is the question we consider.

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Here are some points connected to this question: (1) One way to try to generalize a measure of concordance to an ordered n-tuple $(X_1, \ldots, X_n)$ of random variables is to compute bivariate measures of concordance for all ordered pairs (X_i, X_j), with $i < j$, and then average the results; see Hays (1960) and Joe (1997). (2) In another approach, Joe (1990) and Nelsen (2002) constructed examples of measures of concordance of a different character which in some sense consider all the random variables of $(X_1, \ldots, X_n)$ simultaneously. (3) In our investigations of Scarsini's axioms in Edwards et al. (2004, 2005), a fundamental role is played by symmetries of the unit square. Thus it seemed reasonable to expect symmetries of the unit n-dimensional cube, I^n, where $I = [0, 1]$, to play an important role in a higher-dimensional theory.

The generalized axioms given here were most strongly influenced by the examples of Nelsen (2002) and by our emphasis on symmetries of I^n. The greatest difference from Scarsini's formulation lies in the last two axioms. One of these, the Reflection Symmetry Property, says, in effect, that if one runs through all possible variations of orientation of the components of a random vector, then the average measure of concordance is zero. The other one, the Transposition Property, provides a link between measures of concordance for n random variables and those for $n - 1$.

We state our axioms now in terms of random variables and later in terms of copulas.

By a *measure of concordance* κ we mean a function that attaches to every n-tuple of continuous random variables $(X_1, \ldots, X_n)$ defined on a common probability space, where $n \geq 2$, a real number $\kappa(X_1, \ldots, X_n)$ satisfying the following:

A1. (Normalization) $\kappa(X_1, \ldots, X_n) = 1$ if each X_i is a.s. an increasing function of every other X_j, and $\kappa(X_1, \ldots, X_n) = 0$ if $X_1, \ldots, X_n$ are independent.
A2. (Monotonicity) If $X_1, \ldots, X_n$ is less concordant than $Y_1, \ldots, Y_n$ in the sense of $\prec_C$ in Joe (1990), then $\kappa(X_1, \ldots, X_n) \leq \kappa(Y_1, \ldots, Y_n)$.
A3. (Continuity) If F_k is the joint distribution function of the random vector $(X_{k1}, \ldots, X_{kn})$ and F is the distribution function for $(X_1, \ldots, X_n)$ and $F_k \to F$, then $\kappa(X_{k1}, \ldots, X_{kn}) \to \kappa(X_1, \ldots, X_n)$.
A4. (Permutation Invariance) If $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$, then $\kappa(X_{i_1}, \ldots, X_{i_n}) = \kappa(X_1, \ldots, X_n)$.
A5. (Duality) $\kappa(-X_1, \ldots, -X_n) = \kappa(X_1, \ldots, X_n)$.
A6. (Reflection Symmetry Property; RSP) $\sum_{\epsilon_1,\ldots,\epsilon_n=\pm 1} \kappa(\epsilon_1 X_1, \ldots, \epsilon_n X_n) = 0$ where it is to be understood that the sum is over the 2^n vectors of the form $(\epsilon_1 X_1, \ldots, \epsilon_n X_n)$ for which each $\epsilon_i = 1$ or -1.
A7. (Transition Property; TP) There exists a sequence of numbers $\{r_n\}$, where $n \geq 2$, such that for every n-tuple of continuous random variables $(X_1, \ldots, X_n)$, we have

$$r_{n-1}\kappa(X_2, \ldots, X_n) = \kappa(X_1, X_2, \ldots, X_n) + \kappa(-X_1, X_2, \ldots, X_n).$$

It can be shown as in the proof of Theorem 1 of Scarsini (1984) that the measure of concordance described by these axioms is invariant under a.s. increasing transformations of the random variables. Thus the axioms can be stated in terms of copulas and will later be reformulated that way.

A limitation of the present paper is that we concern ourselves only with population versions of measures of concordance. We do not consider sample versions of measures of concordance, but we shall make a further comment on this in the last section.

In what follows, we develop the language of copulas and symmetries and motivate our axioms. We give examples of measures of concordance and derive some of their simple properties. Two of our arguments were greatly improved by suggestions of a referee. Finally we consider examples of higher dimensional measures of concordance in the literature and the extent to which they do or do not satisfy our axioms. We look at examples from Nelsen (2002), Joe (1990), Úbeda Flores (2005), and Dolati and Úbeda Flores (2004).

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2 Basic tools

2.1 Copulas

Take X to be the random vector $(X_1, \ldots, X_n)$ where each X_i is a continuous random variable. To every such X we associate a unique n-copula $C : I^n \to I$, where $I = [0, 1]$, defined by the equation $F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$ where F is the joint distribution function of $(X_1, \ldots, X_n)$, each F_i is the one-dimensional distribution function of X_i, and $x_i \in \mathbb{R}$.

We can, if we wish, always take the X_i's to be uniformly distributed over I.

To each n-copula C we may associate a probability measure μ_C satisfying $\mu_C(A \times I^{n-1}) = \mu_C(I \times A \times I^{n-2}) = \cdots = \mu_C(I^{n-1} \times A) = \lambda(A)$ where A is a Borel set of I and λ is one-dimensional Lebesgue measure. The connection is given by $C(x_1, \ldots, x_n) = \mu_C([0, x_1] \times \cdots \times [0, x_n])$ where the last quantity is also interpreted as the probability $P(X < x) = P(X_1 < x_1, \ldots, X_n < x_n)$, each X_i being considered as uniformly distributed over I and $x = (x_1, \ldots, x_n) \in I^n$.

We define the *survival function* $\overline{C}$ of C by

$$\overline{C}(x_1, \ldots, x_n) = P(X > x) = P(X_1 > x_1, \ldots, X_n > x_n).$$

$\overline{C}$ is not in general a copula.

Two important standard copulas are

$$M^n(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\} \quad \text{and} \quad \Pi^n(x_1, \ldots, x_n) = x_1 x_2 \ldots x_n.$$

M^n is known as the Fréchet–Höffding upper bound copula because we have $C \le M^n$ for every $C \in \mathrm{Cop}(n)$. The mass (probability) of M^n is distributed uniformly along the line segment $x_1 = \cdots = x_n$ that runs from $(0, \ldots, 0)$ to $(1, \ldots, 1)$ in I^n. M^n is the n-copula corresponding to each X_i of X being an a.s. increasing function of every other X_j; this can be thought of as a state of "maximal concordance." Π^n is the independence copula because the associated $X_1, \ldots, X_n$ are independent, a state of total lack of concordance. Its mass is uniformly distributed over I^n. If the choice of n seems clear, we feel free to write M for M^n and Π for Π^n.

Another important function on I^n is $W^n(x_1, \ldots, x_n) = \max\{x_1 + \cdots + x_n - n + 1, 0\}$. This has the property that $W^n \leq C$ for every n-copula C, W^2 is a two-copula, and W^n fails to be a copula for $n \geq 3$.

A useful "trick" that we shall appeal to is that if $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ are continuous, independent random vectors with n-copulas A and B respectively, then $P(X_1 < Y_1, \ldots, X_n < Y_n) = \int_{I^n} A \, \mathrm{d}B$.

We denote the set of n-copulas as $\mathrm{Cop}(n)$. More detailed discussions of copulas and their properties can be found in Schweizer and Sklar (1983) and Nelsen (1999).

2.2 Symmetries of the unit n-cube

By a *symmetry of* I^n we understand a one-to-one, onto map $\phi : I^n \to I^n$ of the form $\phi(x_1, \ldots, x_n) = (u_1, \ldots, u_n)$ where for each i we have $u_i = x_{k_i}$ or $1 - x_{k_i}$ and where $(k_1, \ldots, k_n)$ is a permutation of $(1, \ldots, n)$. By $\mathcal{S}(I^n)$ we mean the group of such symmetries under the operation of composition.

We say that ϕ is a *permutation* if for each i we have $u_i = x_{k_i}$ and is a *reflection* if for each i we have $u_i = x_i$ or $1 - x_i$. The sets of permutations and reflections constitute subgroups of $\mathcal{S}(I^n)$ that we label $\mathcal{P}_n$ and $\mathcal{R}_n$, respectively. If $\theta : I^n \to I^n$ is the permutation $\theta(x_1, \ldots, x_n) = (x_{k_1}, \ldots, x_{k_n})$, then it is uniquely associated with the permutation $(1, \ldots, n) \mapsto (k_1, \ldots, k_n)$ of $\{1, 2, \ldots, n\}$, and we use the symbol θ for this second permutation as well. Thus $\theta(x_1, \ldots, x_n) = (x_{\theta(1)}, \ldots, x_{\theta(n)})$. We define the *elementary reflections* $\sigma_1, \sigma_2, \ldots, \sigma_n$ of I^n by

$$\sigma_i(x_1, \ldots, x_n) = (u_1, \ldots, u_n) \quad \text{where } u_j = \begin{cases} 1 - x_j & if\, j = i \\ x_j & \text{otherwise.} \end{cases}$$

By σ^n we mean the reflection $\sigma_1\sigma_2 \cdots \sigma_n$; that is, $\sigma^n(x_1, \ldots, x_n) = (1 - x_1, \ldots, 1 - x_n)$. If the choice of n is clear, we shall write σ for σ^n.

$\mathcal{R}_n$ is an abelian group and every element of $\mathcal{R}_n$ is its own inverse. This is not true of either $\mathcal{P}_n$ or $\mathcal{S}(I^n)$. Using the fact that for every permutation θ we have $\sigma_i\theta = \theta\sigma_{\theta(i)}$, it is easily shown that every symmetry ζ of I^n has a unique representation of the form $\zeta = \sigma_{i_1} \cdots \sigma_{i_k}\theta$ (or, equivalently, $\zeta = \theta'\sigma_{j_1} \cdots \sigma_{j_k}$) where θ (or θ') is a permutation and $i_1 < \cdots < i_k$ (or $j_1 < \cdots < j_k$). Because of this, we may define the *length of a symmetry* ζ by $|\zeta| = k$. We may think of the length of ζ as being the minimal number of elementary reflections needed to write it.

To every reflection ψ we can associate a unique *dual* reflection $\sigma\psi$, the composition of the two reflections. If ψ has the form $\sigma_{i_1} \cdots \sigma_{i_k}$, this means that the dual reflection will have the form $\sigma\psi = \sigma_{j_1} \cdots \sigma_{j_{n-k}}$ where $j_1 < \cdots < j_{n-k}$ and $\{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\}$. Of course this has the consequence that $|\psi| + |\sigma\psi| = n$.

We can think of symmetries of I^n as operating on copulas to produce new copulas: for each n-copula C and each element ξ of $\mathcal{S}(I^n)$ we define an n-copula C^ξ whose probability measure is given by

$$\mu_{C^\xi}(S) = \mu_C(\xi(S)) \tag{1}$$

where S is a Borel set of I^n.

Suppose θ is the permutation $(1, \ldots, n) \mapsto (i_1, \ldots, i_n)$. We see that $C^\theta(u_1, \ldots, u_n) = \mu_C(\theta([0, u_1] \times \cdots \times [0, u_n])) = P(X_{\theta^{-1}(1)} < x_1, \ldots, X_{\theta^{-1}(n)} < x_n)$, where $u_j = F_j(x_j)$. Thus C^θ must be the n-copula of $(X_{\theta^{-1}(1)}, \ldots, X_{\theta^{-1}(n)})$.

Next suppose ψ is a reflection. Consider the particularly simple case where $\psi = \sigma_1$. We can show that $C^{\sigma_1}(u_1, u_2, \ldots, u_n) = P(-X_1 < x_1, X_2 < x_2, \ldots, X_n < x_n)$, where $u_1 = P(-X_1 < x_1)$ and $u_k = P(X_k < x_k)$ for $2 \le k \le n$, so that C^{σ_1} is seen to be the copula of $(-X_1, X_2, \ldots, X_n)$. More generally, if $\psi = \sigma_{i_1} \cdots \sigma_{i_k}$ and $i_1, \ldots, i_k$ are distinct, then C^ψ is the copula of $(Z_1, \ldots, Z_n)$ where

$$Z_j = \begin{cases} -X_j \text{ if } j = \quad \text{some } i_r, \\ X_j \quad \text{otherwise.} \end{cases}$$

If $X_1, \ldots, X_n$ are uniformly distributed on I, we can show that $C^{\sigma_1}(x_1, x_2, \ldots, x_n) = P(1-X_1 < x_1, X_2 < x_2, \ldots, X_n < x_n)$, so that C^{σ_1} is seen to be the copula of $(1 - X_1, X_2, \ldots, X_n)$. In particular, C^σ is the copula of $(1 - X_1, \ldots, 1 - X_n)$ and

$$\begin{aligned} \overline{C}(x_1, \ldots, x_n) &= P(1 - X_1 < 1 - x_1, \ldots, 1 - X_n < 1 - x_n) \\ &= C^\sigma(1 - x_1, \ldots, 1 - x_n). \end{aligned} \tag{2}$$

The standard definition of concordance ordering for n-copulas (see Joe, 1997) amounts to $A \prec_C B$ if and only if $A \le B$ and $\overline{A} \le \overline{B}$. We see from (2) that an alternate way to say this is

$$A \prec_C B \quad \text{if and only if} \quad A \le B \text{ and } A^\sigma \le B^\sigma. \tag{3}$$

3 Extending Scarsini's axioms

We give a set of axioms for bivariate measures of concordance equivalent to those of Scarsini (1984) but stated in terms of copulas and symmetries of I^n: By a *measure of concordance* (in the sense of Scarsini) we mean a function $\kappa : \mathrm{Cop}(2) \to \mathbb{R}$ satisfying the following:

S1. $\kappa(M^2) = 1$ and $\kappa(\Pi^2) = 0$.
S2. If $C_m \to C$ uniformly, then $\kappa(C_m) \to \kappa(C)$ as $m \to \infty$.
S3. $\kappa(C^\theta) = \kappa(C)$ whenever θ is a permutation.
S4. If $A \le B$, then $\kappa(A) \le \kappa(B)$.
S5. $\kappa(C^{\sigma_1}) = \kappa(C^{\sigma_2}) = -\kappa(C)$.

When Scarsini gave these axioms, he had in mind standard, already familiar measures of concordance such as Spearman's rho and Kendall's tau which were readily seen to satisfy the axioms.

How can we extend these axioms to n variables?

The first three axioms generalize trivially.

To generalize **S4**, we replace $A \le B$ by $A \prec_C B$. We want to have both $A \le B$ and $\overline{A} \le \overline{B}$ (or, equivalently, $A^\sigma \le B^\sigma$) because we want to talk about the tendency of X and Y to both be "large simultaneously" and both be "small simultaneously."

In dimension $n = 2$ it is sufficient to talk about $A \leq B$ because in that dimension, and that one only, $A \leq B$ and $A \prec_{\mathcal{C}} B$ are equivalent.

To justify our generalization of **S5**, we present an argument suggested by one of the referees that is much shorter than our original one.

Let $X = (X_1, \ldots, X_n)$ be a continuous random vector and $\delta_1, \ldots, \delta_n$ be independent random variables that are also independent of X such that each δ_i takes on the values of 1 and -1 with probability 1/2. If we switch back to talking about concordance of random vectors, then it seems reasonable to assume that $\kappa(\delta_1 X_1, \ldots, \delta_n X_n) = 0$. Now let D be the set of all sequences $(d_1, \ldots, d_n)$ where each $d_i = 1$ or -1. For any $(d_1, \ldots, d_n)$, the probability that $(\delta_1, \ldots, \delta_n) = (d_1, \ldots, d_n)$ is $1/2^n$. For a fixed $(X_1, \ldots, X_n)$, we may now regard $\kappa(\delta_1 X_1, \ldots, \delta_n X_n)$ as a random variable on D with induced probability measure Q. Then

$$0 = \int_D \kappa(\delta_1 X_1, \ldots, \delta_n X_n)\, dQ = \frac{1}{2^n} \sum_{d_1,\ldots,d_n} \kappa(d_1 X_1, \ldots, d_n X_n).$$

If C is the n-copula for $(X_1, \ldots, X_n)$, then for each $(d_1, \ldots, d_n)$ there is a unique reflection ξ of I^n such that C^ξ is the n-copula of $(d_1 X_1, \ldots, d_n X_n)$. Thus we should expect that

$$\sum_{\xi \in \mathcal{R}_n} \kappa(C^\xi) = 0. \tag{4}$$

This does not quite generalize **S5**; we need an extra axiom, namely that $(X_1, \ldots, X_n)$ and $(-X_1, \ldots, -X_n)$ have the same measure of concordance, or, equivalently,

$$\kappa_n(C^\sigma) = \kappa_n(C). \tag{5}$$

(We did not need it when using Scarsini's axioms because by **S5** we had $\kappa_2(C) = -\kappa_2(C^{\sigma_1}) = \kappa_2(C^{\sigma_1\sigma_2}) = \kappa_2(C^\sigma)$.) This is a weak assumption since any κ that satisfied our previously considered axioms could be modified to a κ' that also satisfies (5) by setting $\kappa'(C) = \frac{1}{2}(\kappa(C) + \kappa(C^\sigma))$.

We check that we have now generalized **S5**: For $n = 2$, (4) becomes $\kappa_2(C) + \kappa_2(C^{\sigma_1}) + \kappa_2(C^{\sigma_2}) + \kappa_2(C^\sigma) = 0$. By our new axiom that $\kappa_2(C^\sigma) = \kappa_2(C)$ for all two-copulas C, we have $\kappa_2(C^{\sigma_1}) = \kappa_2(C^{\sigma_1\sigma}) = \kappa_2(C^{\sigma_2})$, so (4) becomes $2(\kappa_2(C) + \kappa_2(C^{\sigma_1})) = 0$, which amounts to Scarsini's axiom **S5**.

We now add an axiom different from any of Scarsini's. It arises from the following consideration: how should measures of concordance for $n-1$ variables be related to those for n variables? To justify this axiom, we again present an argument which was suggested by one of the referees and which is shorter than our original argument.

One way to extend a bivariate measure of concordance to a multivariate version is to construct the average over all distinct bivariate margins of a random vector. For example, if we start with the bivariate Kendall's tau, we extend it to a multivariate measure of concordance κ by setting

$$\kappa(X_1, \ldots, X_n) = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \tau(X_i, X_j).$$

(We prove later that this extension procedure always results in a multivariate measure of concordance satisfying our axioms.) Since τ satisfies **S5**, notice that we have

$$\kappa(X_1, X_2, \ldots, X_n) + \kappa(-X_1, X_2, \ldots, X_n) = \frac{2}{\binom{n}{2}} \sum_{2 \le i < j \le n} \tau(X_i, X_j)$$
$$= r_{n-1}\kappa(X_2, \ldots, X_n),$$

where $r_{n-1} = 2(n-2)/n$.

This example leads us to propose as an axiom relating measures of concordance for $n-1$ and n random variables, where $n \geq 3$, that there is a constant r_{n-1} such that

$$r_{n-1}\kappa(B) = \kappa(C) + \kappa(C^{\sigma_1})$$

for all copulas B and C related by $B(x_2, x_3, \ldots, x_n) = C(1, x_2, x_3, \ldots, x_n)$.

4 Axioms, examples, and properties

4.1 Axioms for a multivariate measure of concordance

We first collect and explicitly state our axioms for a multivariate measure of concordance.

By a *measure of concordance* κ we mean a sequence of maps $\kappa_n : \mathrm{Cop}(n) \to \mathbb{R}$ and a sequence of numbers $\{r_n\}$, such that if A, B, C, and C_m are n-copulas and $n \geq 2$, then the following hold:

A1. (Normalization) $\kappa_n(M^n) = 1$ and $\kappa_n(\Pi^n) = 0$.
A2. (Monotonicity) If $A \prec_{\mathcal{C}} B$, then $\kappa_n(A) \leq \kappa_n(B)$.
A3. (Continuity) If $C_m \to C$ uniformly, then $\kappa_n(C_m) \to \kappa_n(C)$ as $m \to \infty$.
A4. (Permutation Invariance) $\kappa_n(C^\theta) = \kappa_n(C)$ whenever θ is a permutation.
A5. (Duality) $\kappa_n(C^{\sigma^n}) = \kappa_n(C)$.
A6. (Reflection Symmetry Property; RSP) $\sum_{\psi \in \mathcal{R}_n} \kappa_n(C^\psi) = 0$.
A7. (Transition Property; TP) $r_n\kappa_n(C) = \kappa_{n+1}(E) + \kappa_{n+1}(E^{\sigma_1})$ whenever E is an $(n+1)$-copula such that $C(x_1, \ldots, x_n) = E(1, x_1, \ldots, x_n)$.

When we wish to indicate a measure of concordance κ along with its attendant κ_n's and r_n's, we shall sometimes use the notation $\kappa = (\{\kappa_n\}, \{r_n\})$.

There is one way in which we have not perfectly extended Scarsini's axioms. Scarsini requires that $-1 \leq \kappa(C) \leq 1$. We have not listed that requirement in our version of Scarsini's axioms, **S1–S5**, however it does follow from them since if C is a two-copula, then $W^2 \leq C \leq M^2$ so that $-1 = \kappa((M^2)^{\sigma_1}) = \kappa(W^2) \leq \kappa(C) \leq \kappa(M^2) = 1$. For $n \geq 3$ the most we can manage is $\kappa_n(C) \leq 1$. We know this much is true because we always have $C \leq M^n$ and $C^\sigma \leq M^n = (M^n)^\sigma$. That is, we always have $C \prec_{\mathcal{C}} M^n$ and $\kappa_n(M^n) = 1$. However there does not seem to be an obvious proof that $-1 \leq \kappa_n(C)$ for $n \geq 3$.

4.2 Some examples

We first show how to extend a bivariate measure of concordance to a multivariate one. This result was brought to our attention by one of the referees. (We also received some useful remarks about this topic from A. Dolati).

We require a notation for the marginal of a copula. Let C be an n-copula and S be a nonempty subset of $\{1, 2, \ldots, n\}$. If S has cardinality $n-k$, then we define the k-marginal of C determined by S to be the map $C_S : I^k \to I$ defined by $C_S(x_{i_1}, \ldots, x_{i_k}) = C(x_1, \ldots, x_n)$ where $i_1 < \cdots < i_k$, each $i_r \notin S$, and $x_j = 1$ if $j \in S$. Of course this C_S is an k-copula provided $k \geq 2$. If, for example, C is a five-copula and $S = \{1, 3, 4\}$, then C_S is a two-copula and $C_S(x_2, x_5) = C(1, x_2, 1, 1, x_5)$.

Theorem 1 *Suppose that κ_2: $Cop(2) \to \mathbb{R}$ is a bivariate measure of concordance in the sense of Scarsini. For $p = 1, 2, \ldots$, let us define $\kappa_{p+2} : Cop(p+2) \to \mathbb{R}$ and r_{1+p} by*

$$\kappa_{p+2}(C) = \frac{1}{\binom{p+2}{2}} \sum_{card(S)=p} \kappa_2(C_S) \quad and \quad r_{1+p} = \frac{2p}{2+p}$$

where it is understood that the summation is over all subsets S of $\{1, 2, \ldots, p+2\}$ of size p, hence over all two-marginals C_S of C. Then $\kappa = (\{\kappa_n\}, \{r_n\})$ is a multivariate measure of concordance in our sense.

Proof Because the two-marginals of M^{p+2} are M^2 and those of Π^{p+2} are Π^2, the Normalization property is trivial.

The Monotonicity property follows from the fact that if A and B are $(p+2)$-copulas such that $A \prec_C B$, then $A_S \leq B_S$ for all two-marginals.

The Continuity property and Permutation Invariance are easily seen.

It is straightforward to show that if S is a subset of $\{1, 2, \ldots, p+2\}$ of cardinality p, then $(C^{\sigma^{p+2}})_S = (C_S)^{\sigma^2}$. Since $\kappa_2((C_S)^{\sigma^2}) = \kappa_2(C_S)$, the Duality property holds.

To establish RSP, first notice that for a $(p+2)$-copula C we can write $\sum_{\xi \in \mathcal{R}_{p+2}} \kappa_{p+2}(C^\xi) = \left(1/\binom{p+2}{2}\right) \sum_{\text{card}(S)=p} \sum_\xi \kappa_2((C^\xi)_S)$. Fix S, a subset of $\{1, 2, \ldots, p+2\}$ with cardinality p, and suppose that $\{1, 2, \ldots, p+2\} - S = \{i, j\}$ where $i < j$. Notice that the number of reflections $\xi : I^{p+2} \to I^{p+2}$ such that ξ either replaces both x_i and x_j by their reflected values $1 - x_i$ and $1 - x_j$ or else ξ affects neither x_i nor x_j, is 2^{p+1}. For each such ξ we have $\kappa_2((C^\xi)_S) = \kappa_2(C_S)$. On the other hand, the number of reflections $\xi : I^{p+2} \to I^{p+2}$ which replaces precisely one of the components x_i, x_j by its reflected value $1 - x_k$ and leaves the other one alone, is 2^{p+1}. For each such ξ we have $\kappa_2((C^\xi)_S) = -\kappa_2(C_S)$. Thus for fixed S we see that $\sum_\xi \kappa_2((C^\xi)_S) = 0$. Hence RSP holds.

Finally we consider TP. Notice that for a $(p+2)$-copula C we have $\kappa_2((C^{\sigma_1})_S)$ equal to $\kappa_2(C_S)$ if $1 \in S$ and equal to $-\kappa_2(C_S)$ if $1 \notin S$. We then calculate

$$\begin{aligned}
\kappa_{p+2}(C) + \kappa_{p+2}(C^{\sigma_1}) &= \frac{1}{\binom{p+2}{2}} \left[\sum_{\text{card}(S)=p} \kappa_2(C_S) + \sum_{\text{card}(S)=p} \kappa_2((C^{\sigma_1})_S) \right] \\
&= \frac{2}{\binom{p+2}{2}} \sum_{\substack{\text{card}(S)=p \\ 1 \in S}} \kappa_2(C_S)
\end{aligned}$$

$$= \frac{2}{\binom{p+2}{2}} \binom{p+1}{2} \frac{1}{\binom{p+1}{2}} \sum_{\substack{\mathrm{card}(T)=p-1 \\ T \subseteq \{2,3,\ldots,p+2\}}} \kappa_2(C_{\{1\} \cup T})$$

$$= \frac{2p}{p+2} \;\kappa_{p+1}(C_{\{1\}}).$$

Since $(C_{\{1\}})(x_2, \ldots, x_{p+2}) = C(1, x_2, \ldots, x_{p+2})$, we have established TP and hence the theorem.

We next prove a theorem which generalizes the form Nelsen used in his version of Spearman's rho in Nelsen (2002), but first we need some terminology.

We say that a measure μ on the Borel sets of I^n is I^n-invariant if $\mu(\xi(B)) = \mu(B)$ for every symmetry ξ of I^n and every Borel set B of I^n. Next define a projection map $\pi_i^n : I^n \to I^{n-1}$, where $n \geq 3$, by $\pi_i^n(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Let μ_n be a measure on the Borel sets of I^n for $n = 2, 3, \ldots$. We say that the sequence $\{\mu_n\}$ is *projective* if $\mu_n((\pi_i^n)^{-1}(B)) = \mu_{n-1}(B)$ for $1 \leq i \leq n$, for all Borel sets B of I^{n-1}, and for $n \geq 3$.

Notice that if we take each μ_n to be Lebesgue measure on I^n, then $\{\mu_n\}$ is projective, each μ_n is I^n-invariant, and $\mu_n((0,1)^n) > 0$ for all n. We give other examples of $\{\mu_n\}$ having all three of these properties in Examples 2, 3, 4, and 5 and we will be concerned with examples of this sort in the next theorem. However it is not clear how to construct such examples in general. One cannot simply start with μ_2 and work upwards. If, for instance, one begins with μ_2 whose mass is distributed uniformly over the inscribed circle in I^2, then it is easily seen that there is no μ_3 on I^3 that projects correctly onto μ_2. This is because, in this case, we are making three different requirements for the relationship between μ_3 and μ_2, namely, $\mu_3(B_1 \times B_2 \times I) = \mu_3(B_1 \times I \times B_2) = \mu_3(I \times B_1 \times B_2) = \mu_2(B_1 \times B_2)$ for all Borel sets B_1 and B_2 of I.

Theorem 2 *Let $\{\mu_n\}$ be a sequence of probability measures on $(I^n, \mathcal{B}(I^n))$, where $n \geq 2$, such that each μ_n is I^n-invariant, $\mu_n((0,1)^n) > 0$ for all n, and the sequence $\{\mu_n\}$ is projective. Then $2\int_{I^n} M \,\mathrm{d}\mu_n - \frac{1}{2^{n-1}} > 0$ for $n \geq 2$; and if we define*

$$\alpha_n = \frac{1}{2\int_{I^n} M \,\mathrm{d}\mu_n - \frac{1}{2^{n-1}}}, \tag{6}$$

$$\kappa_n(C) = \alpha_n \left(\int_{I^n} (C + C^\sigma)\, d\mu_n - \frac{1}{2^{n-1}} \right), \tag{7}$$

$$r_n = \frac{\alpha_{n+1}}{\alpha_n}, \tag{8}$$

where $n \geq 2$ and C is an n-copula, then the sequences of maps $\{\kappa_n\}$ and numbers $\{r_n\}$ define a measure of concordance κ.

Note that (6) and (7) are for satisfying A1 and A6, and (8) is for satisfying A7.

Proof In order to show that $\alpha_n > 0$, and hence that we can define κ_n by (7), and in order to establish the Normalization axiom, we first show that that for every reflection $\psi \in \mathcal{R}_n$ we have

$$\int_{I^n} \Pi^\psi \,\mathrm{d}\mu_n = \frac{1}{2^n}. \tag{9}$$

Suppose $X_1, \ldots, X_n$ are random variables that are independent and uniformly distributed over I, that $Y_1, \ldots, Y_n$ are random variables distributed in I such that they generate the probability measure μ_n, and that $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ are independent. We now perform a simple calculation:

$$\begin{aligned}1 &= P(X_1 < Y_1) + P(1 - X_1 < 1 - Y_1)\\ &= P(X_1 < Y_1,\ X_2 < Y_2) + P(X_1 < Y_1,\ 1 - X_2 < 1 - Y_2)\\ &\quad + P(1 - X_1 < 1 - Y_1,\ X_2 < Y_2) + P(1 - X_1 < 1 - Y_1,\ 1 - X_2 < 1 - Y_2)\\ &= \ldots\\ &= P(X_1 < Y_1, \ldots, X_n < Y_n) + \cdots + P(1 - X_1 < 1 - Y_1, \ldots, 1 - X_n\\ &< 1 - Y_n).\end{aligned} \tag{10}$$

From our discussion of the way random variables and copulas are linked, we see that for any $\psi \in \mathcal{R}_n$ we can choose $Z_i = X_i$ or $1 - X_i$ in such a way that $\int_{I^n} \Pi^{\psi}\,\mathrm{d}\mu_n = P(Z_1 < Y_1, \ldots, Z_n < Y_n)$. Thus we have shown that $\sum_{\psi \in \mathcal{R}_n} \left(\int_{I^n} \Pi^{\psi}\,\mathrm{d}\mu_n - \frac{1}{2^n}\right) = 0$. But now notice that because $Z_1, \ldots, Z_n$ must be independent and have Π as their copula, we also have $P(Z_1 < Y_1, \ldots, Z_n < Y_n) = \int_{I^n} \Pi\,\mathrm{d}\mu_n$. Therefore (9) must hold.

Since $M^n - \Pi^n$ is positive on $(0, 1)^n$ and $\mu_n((0, 1)^n) > 0$, it follows that $\int_{I^n} (M^n - \Pi^n)\,\mathrm{d}\mu_n > 0$. Because (9) holds, we see that we can define α_n by $\alpha_n = 1/(2 \int_{I^n} M^n\,\mathrm{d}\mu_n - \frac{1}{2^{n-1}})$ and that $\alpha_n > 0$. Thus we can define $\kappa_n : \mathrm{Cop}(n) \to \mathbb{R}$ by (7).

We now begin to verify the axioms for a measure of concordance.

Since $M^{\sigma} = M$, we have $\int_{I^n} (M + M^{\sigma})\mathrm{d}\mu_n = 2 \int_{I^n} M\,\mathrm{d}\mu_n$, so that $\kappa_n(M^n) = 1$ trivially. By (9), we also have $\kappa_n(\Pi^n) = \int_{I^n} (\Pi + \Pi^{\sigma} - \frac{1}{2^{n-1}})\mathrm{d}\mu_n = 0$. Thus the Normalization property holds.

The Monotonicity property follows from (3), while the Continuity property is a consequence of the dominated convergence theorem.

To verify Permutation Invariance, we first choose a permutation θ and note that

$$\kappa_n(C^{\theta}) = \alpha_n \left(\int_{I^n} (C^{\theta} + C^{\theta\sigma})\mathrm{d}\mu_n - \frac{1}{2^{n-1}} \right). \tag{11}$$

Now $C^{\theta}(x_1, \ldots, x_n) = P(X_1 < x_{\theta(1)}, \ldots, X_n < x_{\theta(n)}) = (C \circ \theta)(x_1, \ldots, x_n)$. So by the symmetry of μ_n we have $\int_{I^n} C^{\theta}\mathrm{d}\mu_n = \int_{I^n} C\,\mathrm{d}\mu_n$. Next, since $\theta\sigma = \sigma\theta$, we see that $\int_{I^n} C^{\theta\sigma}\mathrm{d}\mu_n = \int_{I^n} C^{\sigma\theta}\mathrm{d}\mu_n = \int_{I^n} C^{\sigma}\mathrm{d}\mu_n$ by our previous calculation. It now follows from (11) that $\kappa_n(C^{\theta}) = \kappa_n(C)$.

The Duality property is trivially true by the form of κ_n.

We now turn our attention to RSP and TP. Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ be independent random vectors in I^n such that C is the copula of X and Y induces the probability measure μ_n.

To establish RSP, we simply repeat the calculations of (10) to obtain this time $\sum_{\psi \in \mathcal{R}_n} \left(\int_{I^n} C^{\psi}\mathrm{d}\mu_n - \frac{1}{2^n}\right) = 0$. This implies that $\sum_{\psi \in \mathcal{R}_n} \left(\int_{I^n} (C^{\psi} + C^{\psi\sigma})\mathrm{d}\mu_n - \frac{1}{2^{n-1}}\right) = 0$, which amounts to RSP.

To establish TP, assume $n \geq 3$ and that B is the $(n-1)$-copula defined by $B(x_2, \ldots, x_n) = C(1, x_2, \ldots, x_n)$. Notice that it follows from the fact that $(Y_1, \ldots, Y_n)$ generates μ_n on I^n and the fact that the sequence of measures $\mu_2, \mu_3, \ldots$ is projective that $(Y_2, \ldots, Y_n)$ must induce the measure μ_{n-1} on I^{n-1}. Now

$$\begin{aligned}\int_{I^n} C\,\mathrm{d}\mu_n + \int_{I^n} C^{\sigma_1}\mathrm{d}\mu_n &= P(X_1 < Y_1, X_2 < Y_2, \ldots, X_n < Y_n)\\ &\quad + P(1 - X_1 < Y_1, X_2 < Y_2, \ldots, X_n < Y_n)\\ &= P(X_2 < Y_2, \ldots, X_n < Y_n) = \int_{I^{n-1}} B\,\mathrm{d}\mu_{n-1}.\end{aligned}$$

Similarly, $\int_{I^n} C^{\sigma}\mathrm{d}\mu_n + \int_{I^n} C^{\sigma\sigma_1}\mathrm{d}\mu_n = \int_{I^{n-1}} B^{\sigma}\mathrm{d}\mu_{n-1}$. It then follows from the definition of κ_n that we have $\kappa_n(C) + \kappa_n(C^{\sigma_1}) = r_{n-1}\kappa_{n-1}(B)$ where $r_{n-1} = \alpha_n/\alpha_{n-1}$.

Example 1 The generalization of Spearman's rho from Nelsen (2002) is

$$\rho_n(C) = \alpha_n \left(\int_{I^n} (C + C^{\sigma})\,\mathrm{d}\Pi - \frac{1}{2^{n-1}} \right).$$

By Theorem 2 with each μ_n equal to Lebesgue measure on I^n, we see that this is a measure of concordance in our sense. It is known that $\alpha_n = ((n+1)2^{n-1})/(2^n - (n+1))$, hence

$$r_n = \frac{\alpha_{n+1}}{\alpha_n} = 2\left(\frac{n+2}{n+1}\right)\left(\frac{2^n - (n+1)}{2^{n+1} - (n+2)}\right).$$

Example 2 (Product measure examples) Let ν be a probability measure on the Borel sets of I that is symmetric about $\frac{1}{2}$. We take μ_n, our measure on I^n, to be $\nu \times \cdots \times \nu$. These measures are easily seen to satisfy the hypotheses of Theorem 2 so that κ as defined in that theorem is a measure of concordance.

One simple instance of this follows from letting ν assign positive masses $m_1, \ldots, m_k$ symmetrically about 1/2 to a finite number of points in I where $m_1 + \cdots + m_k = 1$. Then there exists a finite lattice of points L_n in I^n such that for each C, an n-copula, $\int_{I^n} C\,\mathrm{d}\mu_n = \sum_{p \in L_n} C(p)\,\mu_n(p)$ where the mass of each p, namely $\mu_n(p)$, has the form $m_{i_1}m_{i_2}\cdots m_{i_n}$.

Another instance is obtained by choosing a nonnegative, Lebesgue integrable function f on I that is symmetric about 1/2 and satisfies $\int_0^1 f(t)\,\mathrm{d}t = 1$. We then take μ_n to be the measure on I^n having density $f(x_1)\cdots f(x_n)$ at $(x_1, \ldots, x_n)$. In this case, $\int_{I^n} C\,\mathrm{d}\mu_n = \int_{I^n} C(x_1, \ldots, x_n)\,f(x_1)\cdots f(x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n$.

Example 3 (Blomqvist's coefficient) A particularly simple instance of the last example, one that can be considered as a generalization of Blomqvist's coefficient (see Nelsen 1999), is obtained by taking ν to be a unit mass at $\frac{1}{2}$ and hence μ_n to be a unit mass at the point $(\frac{1}{2}, \ldots, \frac{1}{2})$ in I^n. In this case we obtain

$$\alpha_n = \frac{2^{n-1}}{2^{n-1} - 1} \quad \text{and} \quad r_n = \frac{2(2^{n-1} - 1)}{2^n - 1}. \tag{12}$$

(This version of Blomqvist's coefficient was constructed earlier and independently in Úbeda Flores 2005).

Example 4 (Gini's coefficient) We next construct a generalization of Gini's coefficient. We know from Nelsen (1999) and Edwards et al. (2004) that Gini's coefficent for two random variables with copula C is given by

$$\gamma(C) = 8 \int_{I^2} C \, \mathrm{d}\left(\frac{M+W}{2}\right) - 2.$$

The mass that defines the measure of concordance is equally and uniformly distributed along the two diagonals $x_1 = x_2$ and $x_1 + x_2 = 1$ of I^2. We therefore construct our probability measure μ_n on I^n by distributing a unit mass over the 2^{n-1} diagonals of I^n, each diagonal receiving the same amount of mass as every other diagonal, and the distribution along each diagonal being uniform. Thus for any copula C, we have $\int_{I^n} C \, \mathrm{d}\mu_n = \int_{I^n} C \, \mathrm{d}\big(\frac{1}{2^n} \sum_{\xi \in \mathcal{R}_n} M^{\xi}\big)$. Invariance under symmetries of I^n holds by construction, and it is straightforward to check that $\{\mu_n\}$ is projective.

It can be shown that

$$\alpha_n = \frac{2^n}{2^{n-1} - 1} \quad \text{and} \quad r_n = 2\left(\frac{2^{n-1} - 1}{2^n - 1}\right). \tag{13}$$

Example 5 (Some examples generated by Archimedean copulas) Let $\phi : [0, 1] \to [0, \infty]$ be such that $\phi^{(-1)}$, the quasi-inverse of ϕ, is completely monotone or the Laplace transform of a distribution function on $[0, \infty)$. Then $A_n(x_1, \ldots, x_n) = \phi^{(-1)}(\phi(x_1) + \cdots + \phi(x_n))$ is the associated Archimedean n-copulas with generator ϕ. (See Nelsen 1999 for a discussion of Archimedean copulas.) If ν_n is the probability measure associated with A_n, then it is trivially invariant under permutations, and it follows from $A_n(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = A_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ for $n \geq 3$ that the sequence $\{\nu_n\}$ is projective. However the ν_n's are not necessarily I^n-invariant. Therefore we introduce the n-copulas $B_n = (1/2^n) \sum_{\xi \in \mathcal{R}_n} A_n^{\xi}$ which are trivially invariant under reflections.

Let μ_n be the probability measure associated with B_n. We would like to know that the sequence $\{\mu_n\}$ is projective; we indicate how this works for the case $n = 3$ by replacing x_3 by 1 in B_3. For a three-copula C, let C^* be the two-copula $(x_1, x_2) \mapsto C(x_1, x_2, 1)$. We calculate

$$A_3^* = A_3^{\sigma_3 *} = A_2, \quad A_3^{\sigma_1 *} = A_3^{\sigma_1 \sigma_3 *} = A_2^{\sigma_1}, \quad A_3^{\sigma_2 *} = A_3^{\sigma_2 \sigma_3 *} = A_2^{\sigma_2},$$
$$\text{and} \quad A_3^{\sigma_1 \sigma_2 *} = A_3^{\sigma_1 \sigma_2 \sigma_3 *} = A_2^{\sigma_1 \sigma_2}.$$

It follows from the definition of B_n that $B_3(x_1, x_2, 1) = B_2(x_1, x_2)$. The general proof is similar.

It then follows from Theorem 2 that $\kappa_n(C) = \alpha_n\big(\int_{I^n}(C + C^{\sigma}) \, \mathrm{d}B_n - 1/2^{n-1}\big)$, where $n \geq 2$ and α_n is chosen to satisfy $\kappa_n(M^n) = 1$, defines a measure of concordance. Examples of generators ϕ that produce n-copulas for all $n \geq 2$ can be found in Sect. 4.6 of Nelsen (1999); alternatively, suitable examples of ϕ^{-1} can be obtained as Laplace transforms of distribution functions on $[0, \infty)$ by consulting Tricomi (1954) or Zwillenger (2002). It appears that B_n does not, in general have a computationally simple form.

4.3 Some simple properties

Theorem 3 *For every measure of concordance* $\kappa = (\{\kappa_n\}, \{r_n\})$*, the following is true:*

(a) *If C and E are $(n-1)$- and n-copulas respectively (where $n \geq 3$) such that*

$$E(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = C(x_1, \ldots, , x_{i-1}, x_{i+1}, \ldots, x_n),$$

then $r_{n-1}\kappa_{n-1}(C) = \kappa_n(E) + \kappa_n(E^{\sigma_i})$.
(b) $r_{n-1} = 1 + \kappa_n((M^n)^{\psi})$ *whenever* $|\psi| = 1$ *or* $n-1$ *and* $n \geq 3$.
(c) $r_2 = \frac{2}{3}$ *and* $\kappa_3((M^3)^{\psi}) = -\frac{1}{3}$ *whenever* $|\psi| = 1$ *or* 2.

Proof We shall prove only (b) and (c).

(b) By the Transition Property with $C = M^n$ and part (a) of this proposition we have $r_{n-1} = 1 + \kappa_n((M^n)^{\sigma_i})$. Any symmetry ψ of I^n having $|\psi| = 1$ can be written in the form $\sigma_i\theta$ where θ is a permutation, and $\kappa_n((M^n)^{\sigma_i\theta}) = \kappa_n((M^n)^{\sigma_i})$, so that takes care of the case $|\psi| = 1$. The case where $|\psi| = n-1$ is handled by Duality and a similar argument since such a ψ can be written in the form $\sigma_i\sigma\,\theta$, where θ is a permutation.

(c) Taking $M = M^3$, by the Reflection Symmetry Property, we must have

$$\kappa_3(M) + \kappa_3(M^{\sigma_1}) + \kappa_3(M^{\sigma_2}) + \kappa_3(M^{\sigma_3}) + \kappa_3(M^{\sigma_1\sigma_2}) + \kappa_3(M^{\sigma_1\sigma_3})$$
$$+\kappa_3(M^{\sigma_2\sigma_3}) + \kappa_3(M^{\sigma}) = 0.$$

By Duality and Normalization we have $\kappa_3(M) = \kappa_3(M^{\sigma}) = 1$ and $\kappa_3(M^{\sigma_i}) = \kappa_3(M^{\sigma_j\sigma_k})$. This leads to $2 + 6\,\kappa_3(M^{\sigma_i}) = 2 + 6\,\kappa_3(M^{\sigma_k\sigma_j}) = 0$. It follows that $\kappa_3(M^{\psi}) = -1/3$ whenever $|\psi| = 1$ or 2. Then by part (b) of this proposition, we have $r_2 = 1 + \kappa_3(M^{\sigma_i}) = 2/3$.

Theorem 4 *Let C be an n-copula which is permutation symmetric (that is, $C^{\xi} = C$ for all permutations ξ of I^n). Then for all measures of concordance κ and for all symmetries ψ and ξ of I^n we have $\kappa_n(C^{\psi}) = \kappa_n(C^{\xi})$ whenever $|\psi| = |\xi|$ or $|\psi| + |\xi| = n$.*

Proof We first consider the case $|\psi| = |\xi|$.

Recall our notation for marginals of copulas. Let us take $S = \{i\}$ and $T = \{j\}$. By the Transition Property we have $r_{n-1}\kappa_{n-1}(C_S) = \kappa_n(C) + \kappa_n(C^{\sigma_i})$ and $r_{n-1}\kappa_{n-1}(C_T) = \kappa_n(C) + \kappa_n(C^{\sigma_j})$. Since C is permutation symmetric, it follows that all its $(n-1)$-marginals are identical so that $C_S = C_T$ and thus $\kappa_n(C^{\sigma_i}) = \kappa_n(C^{\sigma_j})$. It is then a short step, using Permutation Invariance, to conclude that $\kappa_n(C^{\psi}) = \kappa_n(C^{\xi})$ whenever $|\psi| = |\xi| = 1$.

The rest of the $|\psi| = |\xi|$ case is established by induction. The general argument and its validity is easily seen by considering what happens when $|\psi| = |\xi| = 2$. Let $S = \{i, j\}$ and $T = \{k, l\}$ where $i \neq j$ and $k \neq l$. Next set $S' = \{i\}$ and $T' = \{k\}$. By the Transition Property we have $r_{n-2}\kappa_{n-2}(C_S) = \kappa_{n-1}(C_{S'}) + \kappa_{n-1}((C^{\sigma_j})_{S'})$ and $r_{n-2}\kappa_{n-2}(C_T) = \kappa_{n-1}(C_{T'}) + \kappa_{n-1}((C^{\sigma_l})_{T'})$. Since C is permutation symmetric, we must have $C_S = C_T$ and $C_{S'} = C_{T'}$, and thus $\kappa_{n-1}((C^{\sigma_j})_{S'}) = \kappa_{n-1}((C^{\sigma_l})_{T'})$. We apply the Transition Property once more to obtain $r_{n-1}\kappa_{n-1}((C^{\sigma_j})_{S'}) = \kappa_n(C^{\sigma_j}) + \kappa_n(C^{\sigma_j\sigma_i})$ and $r_{n-1}\kappa_{n-1}((C^{\sigma_l})_{T'}) = \kappa_n(C^{\sigma_l}) + \kappa_n(C^{\sigma_l\sigma_k})$. It follows from this and the $|\psi| = |\xi| = 1$ case that $\kappa_n(C^{\sigma_j\sigma_i}) = \kappa_n(C^{\sigma_l\sigma_k})$. By appealing to Permutation Invariance, we now see that $\kappa_n(C^\psi) = \kappa_n(C^\xi)$ whenever $|\psi| = |\xi| = 2$.

Finally suppose $|\psi| + |\xi| = n$. Then $|\xi\,\sigma| = |\psi|$ so that by Duality we have $\kappa_n(C^\psi) = \kappa_n(C^{\xi\,\sigma}) = \kappa_n(C^\xi)$.

Corollary 1 *For all $n \geq 2$ and all ψ and ξ such that $|\psi| = |\xi|$ or $|\psi| + |\xi| = n$, we have $\kappa_n(M^\psi) = \kappa_n(M^\xi)$.*

5 Examples and counterexamples from the literature

5.1 Another example from Nelsen

Example 6 (Blomqvist's coefficient) In Nelsen (2002), Blomqvist's coefficient is generalized to

$$\beta_n(C) = \alpha_n \left(C\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) - \frac{1}{2^n} \right).$$

It is straightforward to see that β satisfies all of our axioms except for Duality. The arguments justifying RSP and TP are essentially those used in Theorem 2. The fact that Duality fails can easily be seen by constructing a three-copula C with different amounts of mass in $[0, 1/2]^3$ and $[1/2, 1]^3$ so that $C(1/2, 1/2, 1/2) \neq C^\sigma(1/2, 1/2, 1/2)$.

For this particular example,

$$\alpha_n = \frac{2^n}{2^{n-1} - 1} \quad \text{and} \quad r_n = 2\left(\frac{2^{n-1} - 1}{2^n - 1}\right).$$

5.2 Joe's examples

We shall describe Joe's examples from Joe (1990) in terms of copulas and in ways that do not always match his notation; nevertheless, they will be the same measures of concordance.

Example 7 (Blomqvist's coefficient) Let C be the n-copula of the random vector $X = (X_1, \ldots, X_n)$ where each X_i is uniformly distributed over I. We define

$$\beta_n(C) = \sum_{k=0}^{n} w_{n,k} \sum_{|\xi|=k} C^\xi(1/2) = \sum_{k=0}^{n} w_{n,k} \sum_{|\xi|=k} P(\xi(X) < \frac{1}{2}),$$

where the symbol $\sum_{|\xi|=k}$ indicates a summation over all reflections ξ (elements of $\mathcal{R}_n$ only, not general symmetries of I^n) such that $|\xi| = k$, where $1/2$ really stands for the constant vector $(1/2, \ldots, 1/2)$, and where $\{w_{n,k}\}$ is a collection of coefficients, $n \geq 2$ and $0 \leq k \leq n$, such that

(β1) $w_{n,0} = w_{n,n} = 1$.
(β2) $w_{n,k} = w_{n,n-k}$.

(β3) $\sum_{k=0}^{n} \binom{n}{k} w_{n,k} = 0$.

(β4) $w_{n,0} \geq w_{n,1} \geq \cdots \geq w_{n,m}$ where $m = \lfloor \frac{n+1}{2} \rfloor$, the greatest integer less than or equal to $(n+1)/2$.

It is straightforward to verify that such β_n satisfy the axioms of Normalization, Continuity, Permutation Invariance, Duality, and RSP.

Only Monotonicity and the Transition Property need to be checked. Joe gives a condition in Joe (1990) on $\{w_{n,k}\}$ that insures Monotonicity, but necessary and sufficient conditions are not known. One sufficient choice that Joe gives is

$$w_{n,k} = \begin{cases} 1 & \text{if } k = 0, n, \\ -\left(\dfrac{1}{2^{n-1} - 1}\right) & \text{otherwise.} \end{cases}$$

It should also be borne in mind that Joe (1990) assumes n is fixed and does not focus on the question of how measures of concordance are related for different n. It turns out that the TP can be seen to hold if we put one more condition on $\{w_{n,k}\}$:

(β5) $r_{n-1} w_{n-1,j} = w_{n,j+1} + w_{n,j}$.

For the $w_{n,k}$'s shown above, they satisfy (β5) if $r_n = (2^n - 2)/(2^n - 1)$.

Thus we have a situation in which it is possible to give an example of $\{w_{n,k}\}$ such that the generalization of Blomqvist's coefficient given here is a measure of concordance in our sense.

(The reader is also invited to compare this with other versions of Blomqvist's coefficient in this paper.)

Example 8 (Kendall's tau) Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ be two independent random vectors each having C as its copula and such that X_i and Y_i are uniformly distributed over I. We define

$$\tau_n(C) = \sum_{k=0}^{n} w_{n,k} \sum_{|\xi|=k} P(\xi(X) < \xi(Y)) = \sum_{k=0}^{n} w_{n,k} \sum_{|\xi|=k} \int_{I^n} C^{\xi} \mathrm{d}C^{\xi}$$

where, as before, the ξ's range over reflections having $|\xi| = k$ and the $w_{n,k}$'s satisfy (β1)–(β4). As in the example of Blomqvist's coefficient, this will satisfy all our axioms except possibly Monotonicity and TP. It is shown in Joe (1990) that it will satisfy Monotonicity for certain choices of $\{w_{n,k}\}$. In order to satisy TP, we must again introduce the extra condition of (β5).

For the same choice of $\{r_n\}$ and $\{w_{n,k}\}$ as in our last example, all of our axioms of a measure of concordance will hold. Thus, as in the last example, Joe's generalization of Kendall's tau is compatible with our axioms for at least one choice of $\{w_{n,k}\}$.

Example 9 (Spearman's rho) Joe develops two versions of Spearman's rho. The first version is defined by

$$\rho_n(C) = \alpha_n \left(\int_{I^n} \Pi \, \mathrm{d}C - \frac{1}{2^n} \right) \quad \text{where} \quad \alpha_n = \frac{(n+1)2^n}{2^n - (n+1)}.$$

It is straightforward to verify all of the axioms of a measure of concordance except for Duality. (It turns out that as in previous examples we have $r_n = \alpha_{n+1}/\alpha_n$.) Duality amounts to $\rho_n(C^\sigma) = \rho_n(C)$ which reduces first to $\int \Pi \, \mathrm{d}(C^\sigma) = \int \Pi \, \mathrm{d}C$ and then to $P(Y > X) = P(Y < X)$ where X and Y are independent, continuous random vectors having C and Π respectively as their copulas. This last equation holds for $n = 2$ but not in general for $n \geq 3$.

The second version of Spearman's rho is $\overline{\rho}_n(C) = \alpha_n \left(\int_{I^n} \overline{\Pi} \, dC - \frac{1}{2^n} \right)$. The same remarks apply as for the first version.

Thus this last example misses fitting our definition only with respect to Duality. However, if we take $\frac{1}{2}(\rho_n + \overline{\rho}_n)$, this turns out to be Nelsen's generalization of Spearman's rho which satisfies all the axioms.

5.3 Work by Dolati and Úbeda Flores

The work of Dolati and Úbeda Flores on measures of concordance, Úbeda Flores (2005) and Dolati and Úbeda Flores (2004, 2006), was kindly brought to our attention by M. Úbeda Flores. In Dolati and Úbeda Flores (2006) they give an alternative set of axioms for multivariate measures of concordance. It coincides with our own set except that it does not include RSP or TP and it explicitly requires measures of concordance to be bounded below by -1. Their work forms a contrasting and independent line of development to our own and features several interesting examples of measures of concordance.

A generalization of Blomqvist's coefficient is given in Úbeda Flores (2005) that is identical with the one we have given in Example 3 but is earlier than ours. A generalization of Spearman's footrule coefficient is given in the same paper. This turns out to be a particular instance of the class of AOD measures of concordance (see Dolati and Úbeda Flores 2006) which we discuss below.

In Dolati and Úbeda Flores (2004) a multivariate version of Gini's rank association coefficient is defined and investigated. The definition involves the formal construction of an analog to the survival function for W in higher dimensions with no obvious probabilistic interpretation. It is not clear how to apply the methods of analysis used in this paper, and we have not determined if this version of Gini's coefficient is a measure of concordance.

Example 10 In Dolati and Úbeda Flores (2006), the authors define an *average orthant dependent (AOD) measure of concordance* to be one of the form

$$\omega_n(C) = \alpha_n \int_{I^n} (\overline{C} - \overline{\Pi}^n + C - \Pi^n)\mathrm{d}A,$$

where A is a fixed n-copula such that

$$\int_{I^n} (\overline{M}^n - \overline{\Pi}^n + M - \Pi^n)\mathrm{d}A > 0, \tag{14}$$

$$A^\sigma = A, \tag{15}$$

$$A^\theta = A \text{ for all permutations } \theta \text{ of } I^n, \tag{16}$$

and α_n is chosen to satisfy $\omega_n(M) = 1$. AOD measures of concordance satisfy the axioms that Dolati and Úbeda Flores give for measures of concordance in Dolati and Úbeda Flores (2006).

The requirement $\int_{I^n}(\overline{M}^n - \overline{\Pi}^n + M^n - \Pi^n)\,\mathrm{d}A > 0$ is trivially satisfied for all n-copulas A since $M^n - \Pi^n$ is positive on $(0, 1)^2$.

AOD measures of concordance do not in general satisfy the Reflection Symmetry Property (RSP). We can check on this by considering (14) with $A = M = M^n$. We can show that $\alpha_n = (n+1)/(n-1)$ and then that, for example, $\sum_{\xi \in R_3} \omega_3(M^\xi) = 2$.

There is no clear or convincing way, in this setting, of passing from κ_n to κ_{n-1}. Therefore for AOD measures of concordance, RSP fails and TP looks unlikely.

6 Questions

Many questions suggest themselves for further study. We present a few.

(i) Can it be shown to follow from our axioms that $-1 \le \kappa_n(C)$? Or better yet, that $-1 < \kappa_n(C)$ if $n \ge 3$? (One of the referees thinks it likely that the minimum value of κ_n will approach 0 from below as $n \to \infty$. We are, at the present, of the same mind.)

(ii) Can it be shown that for every measure of concordance, $r_n > 0$ for $n > 2$?

(iii) Can one find interesting procedures for constructing sequences $\{\mu_n\}$ of measures on $I^n, n \ge 2$, such that each μ_n is I^n-invariant, $\mu_n((0, 1)^n) > 0$ for all n, and $\{\mu_n\}$ is projective? (The construction of Example 5 using Archimedean copulas deserves more consideration).

(iv) In Theorem 2 we require the measures μ_n to be invariant under all symmetries of I^n. This is most likely a stronger symmetry than needed to construct measures of concordance of the type considered in the theorem. What is the weakest symmetry required on μ_n for a theorem of this type to hold?

(v) An important question, though it is not considered here, is this: What can be said about sample versions of multivariate measures of concordance? In connection with this, it is worth quoting a comment of one of the referees: "The sample version could be considered as $\kappa(F)$ with F replaced by an empirical distribution F_N. So for practical use, there needs to be one more 'axiom': $\kappa(F_N)$ should be easy to compute."

(vi) The examples of measures of concordance examined here have the property that if A and B are n-copulas and $0 \le t \le 1$, then $\kappa_n((1-t)A + tB)$ is either a first or second degree polynomial in t. One can clearly extend this idea to talk about measures of concordance of degree m. Is it possible to characterize measures of concordance of a fixed degree, to exhibit some sort of canonical form for them? (Our colleague H. Edwards has done this, Edwards (2004), for degree one bivariate measures of concordance).

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