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Empirical process approach to some two-sample problems based on ranked set samples

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Abstract We study the asymptotic properties of both the horizontal and vertical shift functions based on independent ranked set samples drawn from continuous distributions. Several tests derived from these shift processes are developed. We show that by using balanced ranked set samples with bigger set sizes, one can decrease the width of the confidence band and hence increase the power of these tests. These theoretical findings are validated through small-scale simulation studies. An application of the proposed techniques to a cancer mortality data set is also provided.

Keywords Shift function · Q–Q plot · P–P plot · Bootstrap · ROC curve · Wilcoxon–Mann–Whitney test

1 Introduction

Let X and Y be two random variables with cumulative distribution functions F and G respectively. Let $S(F) = \{x : 0 < F(x) < 1\}$ be the support of F. The horizontal shift function from F to G at x is defined as

$$\Delta(x) = G^{-1} \circ F(x) - x, \quad x \in S(F)$$

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and the vertical shift function at p as

$$\Lambda(p) = G \circ F^{-1}(p) - p, \quad 0 \le p \le 1,$$

where, for any nondecreasing function Ψ, the generalized inverse is given by

$$\Psi^{-1}(t) = \inf\{u : \Psi(u) \ge t\}.$$

These functions measure the distances between the 45° line and the quantities plotted in the Q–Q and P–P plots respectively, which are useful graphical tools in ascertaining how two distributions differ. For example, a straight line plot of the horizontal shift function indicates a location-scale shift. The vertical shift function is also related to the ROC curve, which is a tool used to assess the performance of a diagnostic test with a continuous marker. The relation is given by

$$\mathrm{ROC}(p) = p - \Lambda(1-p), \quad 0 \le p \le 1.$$

Examples of the use of ROC analysis can be found in speech recognition, disease detection, image analysis and a variety of other fields.

Doksum (1974) first investigated the asymptotic behavior of the horizontal shift process and constructed distribution-free confidence bands based on simple random samples. The results obtained were extended to the nonparametric Bayesian framework by Wells and Tiwari (1989). Lu et al. (1994) further extended these results to the case of right censored data and used the bootstrap to construct simultaneous confidence bands. Li et al. (1996, 1999) derived the asymptotic distribution of the vertical shift process in the presence of right censoring. Hsieh and Turnbull (1996) and Li et al. (1999) have discussed nonparametric and semiparametric estimation of the ROC curve based on simple random samples.

Obtaining an RSS consists of sampling in multiple stages as follows: an SRS of k units is drawn from the underlying population and the units are then ranked according to the characteristic of interest. From this set, the smallest unit is identified and then measured. Another SRS of k units is drawn (independent of the first sample), the units ranked, and the second smallest unit is measured. The process is continued, until at the kth stage, a random sample of k units is taken, the units are ordered and the largest unit is measured. This completes one cycle and the k measurements so obtained constitute a ranked set sample of size k from the population of interest. Note that although k^2 units were screened, the RSS consists of only k observations. The entire cycle can be repeated m times to get m replicates of each order statistic. The sample so obtained is called a balanced ranked set sample (BRSS) of set size k with m replications. Thus, a BRSS consists of equal numbers of independent copies of all the order statistics, arising from independent samples, each of the same size.

The above procedure can be generalized to the situation where one gets multiple independent copies of various order statistics which are not necessarily based on samples of the same size or where different order statistics may get unequally represented. This is called an unbalanced ranked set sample or a Generalized Ranked Set Sample (GRSS). GRSS occurs naturally in many situations such as nomination sampling where one always observes independent copies of the same order statistic (see, for example, Willemain, 1980; Boyles and Samaniego, 1986; Wells and Tiwari, 1991). One such example is the failure times of independent r-out-of-k

systems. For more on ranked set samples, see Kaur et al. (1995), Patil et al. (1999) and the recent book by Chen et al. (2004). Also see Özturk and Wolfe (2000) and Chen et al. (2004) for examples on applications.

In this article, we first study the asymptotic properties of the two shift functions based on independent GRSS. We derive the limiting distributions of the two shift processes and show that when one uses BRSS, the pointwise variances get smaller as one increases the set size k. Hence, one always improves by using a ranked set sample instead of a simple random sample, provided sampling costs are negligible compared to measurement costs. Various two-sample tests are developed based on these shift functions and their properties are studied. Examples include the control percentile test, the Wilcoxon–Mann–Whitney test and a Kolmogorov–Smirnov type test based on the bootstrap.

In what follows, we will use the notation $\xrightarrow{d}$ to denote "convergence in distribution" or "weak convergence", $\stackrel{\mathrm{d}}{=}$ to denote "equality in distribution", $\xrightarrow{P}$ to denote "convergence in probability" and $\xrightarrow{\text{a.s.}}$ to denote "almost sure convergence". Following the notation of Billingsley (1968), we will say that a sequence of random elements $\{F_n\}$ converges in distribution to a random element F if the corresponding probability measures converge weakly (i.e., $F_n \xrightarrow{d} F$ if and only if $P_n \Longrightarrow P$). We will use the notation $(a \wedge b)$ to denote the minimum of a and b, $[x]$ to denote the biggest integer less than or equal to x, $D[a,\ b]$ to denote the space of all right continuous functions on $[a,\ b]$ with left limit and $C[a,\ b]$ to denote the space of all continuous functions on $[a,\ b]$.

This paper is organized as follows. The asymptotic properties of the proposed estimators of the horizontal and vertical shift functions are presented in Sect. 2. In Sect. 3, we study various test statistics constructed from results in Sect. 2. In Sect. 4, we investigate the construction of confidence bands using the bootstrap and in Sect. 5, we present the results of a small simulation study and analyze a data set. In particular, we compare the distributions of prostate cancer mortality rates for the years 1991–1992 and and 1999–2000 to examine if the two distributions are equal or not as a result of the introduction of the prostate specific antigen (PSA) screening test. Finally, Sect. 6 is devoted to the conclusion and discussion. The proofs of the results in Sects. 2 and 4 are deferred to the Appendix.

2 Asymptotic properties

Suppose we have a generalized ranked set sample (GRSS) $\boldsymbol{X}$ from F given by

$$\boldsymbol{X} = \left\{ \begin{array}{cccc} X_{(r_{11}:k_{11})1} & X_{(r_{11}:k_{11})2} & \cdots & X_{(r_{11}:k_{11})m_{11}} \\ X_{(r_{12}:k_{12})1} & X_{(r_{12}:k_{12})2} & \cdots & X_{(r_{12}:k_{12})m_{12}} \\ \vdots & \vdots & \vdots & \vdots \\ X_{(r_{1n_1}:k_{1n_1})1} & X_{(r_{1n_1}:k_{1n_1})2} & \cdots & X_{(r_{1n_1}:k_{1n_1})m_{1n_1}} \end{array} \right\}. \tag{1}$$

Here $X_{(r:k)j}$ denotes the jth replicate of the rth order statistic based on a sample of size k from the underlying distribution F. For a BRSS with set size k_1 and m_1 replications, we have $r_{1i} = i,\ k_{1i} = k_1,\ m_{1i} = m_1$ and $n_1 = k_1$. We will denote this BRSS by $\boldsymbol{X}_{k_1 \times m_1}$. Independently, let us have a GRSS $\boldsymbol{Y}$ from G given by

$$\boldsymbol{Y} = \left\{ \begin{array}{cccc} Y_{(r_{21}:k_{21})1} & Y_{(r_{21}:k_{21})2} & \cdots & Y_{(r_{21}:k_{21})m_{21}} \\ Y_{(r_{22}:k_{22})1} & Y_{(r_{22}:k_{22})2} & \cdots & Y_{(r_{22}:k_{22})m_{22}} \\ \vdots & \vdots & \vdots & \vdots \\ Y_{(r_{2n_2}:k_{2n_2})1} & Y_{(r_{2n_2}:k_{2n_2})2} & \cdots & Y_{(r_{2n_2}:k_{2n_2})m_{2n_2}} \end{array} \right\}. \tag{2}$$

We use the notations

$$\begin{aligned} M_1 &= m_{11} + m_{12} + \cdots + m_{1n_1}, \\ M_2 &= m_{21} + m_{22} + \cdots + m_{2n_2}, \\ m_1 &= (m_{11} \wedge m_{12} \wedge \cdots \wedge m_{1n_1}), \\ m_2 &= (m_{21} \wedge m_{22} \wedge \cdots \wedge m_{2n_2}), \\ M &= M_1 + M_2, \\ m &= (m_1 \wedge m_2). \end{aligned}$$

For $i = 1, \ldots, n_1$, let $\frac{m_{1i}}{M_1} \to q_{1i}$ as $m_1 \to \infty$ and similarly, for $i = 1, \ldots, n_2$, let $\frac{m_{2i}}{M_2} \to q_{2i}$ as $m_2 \to \infty$.

Following Chen (2001, 2003), we define

$$F_{\boldsymbol{q}_1}(x) = \sum_{i=1}^{n_1} q_{1i} F_{(r_{1i}:k_{1i})}(x), \tag{3}$$

where $F_{(r:k)}(x)$ is the cdf of $X_{(r:k)}$ based on F. Denoting the Beta$(r,\ k-r+1)$ cdf by

$$B_{r,\,k}(x) = \frac{\Gamma(k+1)}{\Gamma(r)\Gamma(k-r+1)} \int_0^x u^{r-1}(1-u)^{k-r} \mathrm{d}u,$$

we can write $F_{(r:k)}(x) = B_{r,\,k} \circ F(x)$ and rewrite (3) as

$$F_{\boldsymbol{q}_1}(x) = h_1 \circ F(x),$$

where $h_1 : [0,\ 1] \mapsto [0,\ 1]$ is given by

$$h_1(u) = \sum_{i=1}^{n_1} q_{1i} B_{r_{1i},\,k_{1i}}(u).$$

Note that for a BRSS $\boldsymbol{X}_{k \times m}$, we have $h_1(u) = u$. Also, for a maxima-nomination sample (i.e., $r_{1i} = k,\ k_{1i} = k,\ m_{1i} = m,\ n_1 = 1$), we have $h_1(u) = u^k$ whereas for a minima-nomination sample (i.e., $r_{1i}{=}1,\ k_{1i}{=}k,\ m_{1i}{=}m,\ n_1{=}1$), we have $h_1(u) = 1 - (1-u)^k$.

Since $h_1'(u) > 0$ for all $u \in (0,\ 1)$, $h_1(\cdot)$ is continuous and strictly increasing, and has a unique inverse, $h_1^{-1}(\cdot)$. We thus write

$$F(x) = h_1^{-1} \circ F_{\boldsymbol{q}_1}(x).$$

Let

$$\hat{F}_{(r_{1i}:k_{1i})}(x) = \frac{1}{m_{1i}} \sum_{j=1}^{m_{1i}} I_{[X_{(r_{1i}:k_{1i})j},\ \infty)}(x), \quad i = 1, \ \ldots, \ n_1. \tag{4}$$

Define

$$\hat{F}_{\boldsymbol{q}_1}(x) = \sum_{i=1}^{n_1} q_{1i} \hat{F}_{(r_{1i}:k_{1i})}(x) \tag{5}$$

and consequently,

$$\hat{F}(x) = h_1^{-1} \circ \hat{F}_{\boldsymbol{q}_1}(x). \tag{6}$$

Similarly, define $G_{\boldsymbol{q}_2}(x) = \sum_{i=1}^{n_2} q_{2i} G_{(r_{2i}:k_{2i})}(x)$ and write

$$G_{\boldsymbol{q}_2}(x) = h_2 \circ G(x),$$

where

$$h_2(u) = \sum_{i=1}^{n_2} q_{2i} B_{r_{2i},\ k_{2i}}(u).$$

As before, $h_2(\cdot)$ is invertible and thus

$$G(x) = h_2^{-1} \circ G_{\boldsymbol{q}_2}(x).$$

Also let,

$$\hat{G}_{(r_{2i}:k_{2i})}(x) = \frac{1}{m_{2i}} \sum_{j=1}^{m_{2i}} I_{[Y_{(r_{2i}:k_{2i})j},\ \infty)}(x), \quad i = 1, \ \ldots, \ n_2, \tag{7}$$

$$\hat{G}_{\boldsymbol{q}_2}(x) = \sum_{i=1}^{n_2} q_{2i} \hat{G}_{(r_{2i}:k_{2i})}(x) \tag{8}$$

and

$$\hat{G}(x) = h_2^{-1} \circ \hat{G}_{\boldsymbol{q}_2}(x). \tag{9}$$

Note that we have

$$\hat{F}_{\boldsymbol{q}_1}(x) = EDF_{\boldsymbol{X}}(x) + o_p(1),$$

where $EDF_{\boldsymbol{X}}$ is the empirical distribution function of $\boldsymbol{X}$. A similar result holds for $\boldsymbol{Y}$.

Lemma 1 *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be independent GRSS given by* (1) *and* (2) *respectively. Then, as* $m \to \infty$,

$$\sqrt{M_1}(\hat{F} - F) \xrightarrow{d} \frac{\mathbb{W}_F}{h_1' \circ F} \tag{10}$$

and

$$\sqrt{M_2}(\hat{G} - G) \xrightarrow{d} \frac{\mathbb{W}_G}{h_2' \circ G}, \tag{11}$$

where $\mathbb{W}_F$ *and* $\mathbb{W}_G$ *are independent zero-mean Gaussian processes with covariance kernels*

$$K_F(x,\ y) = F_{\boldsymbol{q}_1}(x \wedge y) - \sum_{i=1}^{n_1} q_{1i} F_{(r_{1i}:k_{1i})}(x) F_{(r_{1i}:k_{1i})}(y) \tag{12}$$

and

$$K_G(x,\ y) = G_{\boldsymbol{q}_2}(x \wedge y) - \sum_{i=1}^{n_2} q_{2i} G_{(r_{2i}:k_{2i})}(x) G_{(r_{2i}:k_{2i})}(y), \tag{13}$$

respectively. Consequently,

$$\sqrt{M}[(\hat{F}, \hat{G}) - (F,\ G)] \xrightarrow{d} \left(\frac{\mathbb{W}_F}{\sqrt{\lambda} h_1' \circ F},\ \frac{\mathbb{W}_G}{\sqrt{1-\lambda} h_2' \circ G} \right) \tag{14}$$

as $m \to \infty$, *where* $\lambda = \lim_{m \to \infty} \frac{M_1}{M}$.

Define the estimators of the horizontal and vertical shift functions to be

$$\hat{\Delta}(x) = \hat{G}^{-1} \circ \hat{F}(x) - x, \quad x \in S(F)$$

and

$$\hat{\Lambda}(p) = \hat{G} \circ \hat{F}^{-1}(p) - p, \quad 0 \le p \le 1,$$

respectively.

Theorem 1 *For independent* GRSS $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *from* F *and* G, *respectively, as* $m \to \infty$,

$$\sqrt{M}(\hat{\Delta} - \Delta) \xrightarrow{d} \frac{\mathbb{Z}_{\Delta,\mathrm{GRSS}}}{g \circ G^{-1} \circ F},$$

where $\mathbb{Z}_{\Delta,\mathrm{GRSS}}$ *is a zero-mean Gaussian process with covariance kernel*

$$K_{\Delta,\mathrm{GRSS}}(x,\ y) = \left[\frac{K_F(x,\ y)}{\lambda h_1' \circ F(x) h_1' \circ F(y)} \right. \\ \left. + \frac{K_G(G^{-1} \circ F(x),\ G^{-1} \circ F(y))}{(1-\lambda) h_2' \circ F(x) h_2' \circ F(y)} \right]$$

and $\frac{M_1}{M} \to \lambda$.

Corollary 1 *For independent* BRSS $\boldsymbol{X}_{k_1 \times m_1}$ *and* $\boldsymbol{Y}_{k_2 \times m_2}$ *from* F *and* G *respectively, as* $(m_1 \wedge m_2) \to \infty$,

$$\sqrt{k_1 m_1 + k_2 m_2}(\hat{\Delta} - \Delta) \xrightarrow{d} \frac{\mathbb{Z}_{\Delta,\mathrm{BRSS}}}{g \circ G^{-1} \circ F},$$

where $\mathbb{Z}_{\Delta,\mathrm{BRSS}}$ *is a zero-mean Gaussian process with covariance kernel*

$$K_{\Delta,\mathrm{BRSS}}(x,\ y) = \frac{1}{\lambda}\left\{F(x \wedge y) - \frac{1}{k_1}\sum_{i=1}^{k_1} F_{(i:k_1)}(x) F_{(i:k_1)}(y)\right\} + \frac{1}{1-\lambda}\left\{F(x \wedge y) - \frac{1}{k_2}\sum_{i=1}^{k_2} F_{(i:k_2)}(x) F_{(i:k_2)}(y)\right\}$$

and $\frac{k_1 m_1}{k_1 m_1 + k_2 m_2} \to \lambda$.

Theorem 4.1 of Doksum (1974) becomes a special case of Corollary 1 by taking $k_1 = k_2 = 1$, $m_1 = m$ and $m_2 = n$.

Remark 1 Since

$$\frac{1}{k}\sum_{i=1}^{k} F^2_{(i:k)}(x) \geq \left\{\frac{1}{k}\sum_{i=1}^{k} F_{(i:k)}(x)\right\}^2 = F^2(x),$$

we have for all x,

$$K_{\Delta,\mathrm{BRSS}}(x,\ x) \leq K_{\Delta,\mathrm{SRS}}(x,\ x).$$

Hence, a pointwise confidence band for Δ based on BRSS would be narrower than that based on SRS.

Theorem 2 *Let* $\frac{M_1}{M} \to \lambda$ *as* $m \to \infty$. *Fix* $[a,\ b] \subset (0,\ 1)$. *Then, as* $m \to \infty$,

$$\sqrt{M}(\hat{\Lambda} - \Lambda) \xrightarrow{d} \mathbb{Z}_{\Lambda,\mathrm{GRSS}},$$

on $[a,\ b]$, *where*

$$\mathbb{Z}_{\Lambda,\mathrm{GRSS}} \stackrel{d}{=} \frac{g \circ F^{-1}}{f \circ F^{-1}} \times \frac{\mathbb{W}_F \circ F^{-1}}{\sqrt{\lambda} h_1'} + \frac{\mathbb{W}_G \circ F^{-1}}{\sqrt{1-\lambda} h_2' \circ G \circ F^{-1}}.$$

Note that $\mathbb{Z}_{\Lambda,\mathrm{GRSS}}$ *is a zero-mean Gaussian process with covariance kernel*

$$K_{\Lambda,\mathrm{GRSS}}(x,\ y) = \frac{g \circ F^{-1}(x) \times g \circ F^{-1}(y)}{f \circ F^{-1}(x) \times f \circ F^{-1}(y)} \times \frac{1}{\lambda h_1'(x) \times h_1'(y)} \times \left\{h_1(x \wedge y) - \sum_{i=1}^{n_1} q_{1i} F_{(r_{1i}:k_{1i})} \circ F^{-1}(x) F_{(r_{1i}:k_{1i})} \circ F^{-1}(y)\right\}$$

$$+\frac{1}{(1-\lambda)h_2' \circ G \circ F^{-1}(x) \times h_2' \circ G \circ F^{-1}(y)}$$
$$\times \left\{ h_1 \circ G \circ F^{-1}(x \wedge y) - \sum_{i=1}^{n_2} q_{2i} G_{(r_{2i}:k_{2i})} \circ F^{-1}(x) \times G_{(r_{2i}:k_{2i})} \circ F^{-1}(y) \right\},$$

for $0 < a \leq x,\ y \leq b < 1$.

Corollary 2 *For independent* BRSS $\boldsymbol{X}_{k_1 \times m_1}$ *and* $\boldsymbol{Y}_{k_2 \times m_2}$ *from* F *and* G *respectively, as* $(m_1 \wedge m_2) \to \infty$, *we have*

$$\sqrt{k_1 m_1 + k_2 m_2}(\hat{\Lambda} - \Lambda) \xrightarrow{d} \mathbb{Z}_{\Lambda,\mathrm{BRSS}}$$

on $[a,\ b] \subset (0,\ 1)$ *where* $\mathbb{Z}_{\Lambda,\mathrm{BRSS}}$ *is a zero-mean Gaussian process with covariance kernel*

$$K_{\Lambda,\mathrm{BRSS}}(x,\ y) = \frac{g \circ F^{-1}(x) \times g \circ F^{-1}(y)}{f \circ F^{-1}(x) \times f \circ F^{-1}(y)} \times \frac{1}{\lambda}$$
$$\times \left\{ x \wedge y - \frac{1}{k_1} \sum_{i=1}^{k_1} F_{(i:k_1)} \circ F^{-1}(x) \times F_{(i:k_1)} \circ F^{-1}(y) \right\}$$
$$+\frac{1}{(1-\lambda)} \left\{ G \circ F^{-1}(x \wedge y) - \frac{1}{k_2} \sum_{i=1}^{k_2} G_{(i:k_2)} \circ F^{-1}(x) \times G_{(i:k_2)} \circ F^{-1}(y) \right\}$$

for $0 < a \leq x,\ y \leq b < 1$.

Remark 2 As in Remark 1, a pointwise confidence band for Λ is narrower when using BRSS, instead of SRS.

3 Some tests

In this section, we use the results presented earlier to develop tests for various hypotheses of interest. First, we present a test based on the horizontal shift function at a fixed point and its generalization to multiple points. Next, we discuss comparison of one or several quantiles of the two distribution functions. Finally, we present the Wilcoxon–Mann–Whitney Statistic that aggregates quantile comparisons of the two distributions.

As seen in Theorem 2, under the null hypothesis $H_0 : F = G$, the limiting process $\mathbb{Z}_\Lambda$ is distribution free, unlike the horizontal shift process, where the limiting distribution depends on F. Hence, it is more convenient to use $\hat{\Lambda}$ (or its functionals) to perform tests of equality of the two distributions.

3.1 Test for location-scale shift

The following is a direct consequence of Theorem 1.

Corollary 3 *For any positive integer p and points $x_1, \ldots, x_p \in S(F)$,*

$$\sqrt{M}\left(\begin{pmatrix}\hat{\Delta}(x_1)\\ \hat{\Delta}(x_2)\\ \vdots\\ \hat{\Delta}(x_p)\end{pmatrix} - \begin{pmatrix}\Delta(x_1)\\ \Delta(x_2)\\ \vdots\\ \Delta(x_p)\end{pmatrix}\right) \xrightarrow{d} N_p(\mathbf{0},\ \Sigma),$$

where $\Sigma = ((\sigma_{uv}))$ is given by

$$\sigma_{uv} = [g \circ G^{-1} \circ F(x_u) g \circ G^{-1} \circ F(x_v)]^{-1}$$
$$\times \left[\frac{h_1 \circ F(x_u \wedge x_v) - \sum_{i=1}^{n_1} q_{1i} B_{r_{1i},k_{1i}} \circ F(x_u) B_{r_{1i},k_{1i}} \circ F(x_v)}{\lambda h_1' \circ F(x_u) h_1' \circ F(x_v)}\right.$$
$$\left. + \frac{h_2 \circ F(x_u \wedge x_v) - \sum_{i=1}^{n_2} q_{2i} B_{r_{2i},k_{2i}} \circ F(x_u) B_{r_{2i},k_{2i}} \circ F(x_v)}{\lambda h_2' \circ F(x_u) h_2' \circ F(x_v)}\right].$$

That is,

$$\sqrt{M}(\hat{\mathbf{\Delta}} - \mathbf{\Delta}) \xrightarrow{d} N_p(\mathbf{0},\ \Sigma).$$

Assume $p \geq 3$. Let

$$A = \begin{pmatrix} -1 & 1 & 0 \cdots & \cdots & 0 \\ 0 & -1 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \, \vdots \\ 0 & 0 \cdots & & 0 & -1 \; 1 \end{pmatrix}_{(p-1)\times p},$$

$$B = \mathrm{diag}\left(\frac{1}{x_2 - x_1}, \ \ldots, \ \frac{1}{x_p - x_{p-1}}\right)_{(p-1)\times(p-1)}$$

and

$$C = \begin{pmatrix} -1 & 1 & 0 \cdots & \cdots & 0 \\ 0 & -1 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \, \vdots \\ 0 & 0 \cdots & & 0 & -1 \; 1 \end{pmatrix}_{(p-2)\times(p-1)}.$$

Under the null-hypothesis of location-scale shift $H_0 : F(x) = G((x - \mu)/\sigma)$, we have $\mathrm{CBA}\mathbf{\Delta} = \mathbf{0}$. Hence,

$$\sqrt{M}\mathrm{CBA}\hat{\mathbf{\Delta}} \xrightarrow{d} N_{p-2}(\mathbf{0},\ (\mathrm{CBA})\Sigma(\mathrm{CBA})').$$

One can construct a χ^2-test for location-scale shift based on the above. For the special case $p = 3$, this is equivalent to a Z-test. Note that in practice, the elements of $\Sigma = (\sigma_{uv})$ will need to be replaced by their corresponding consistent estimators. Estimation of the variance is discussed in Sect. 3.5.

3.2 Confidence interval for $\Delta(x)$

Putting $p = 1$ in Corollary 3, we see that for any $x \in S(F)$,

$$\frac{\sqrt{M}(\hat{\Delta}(x) - \Delta(x))}{\sigma_\Delta(x)} \xrightarrow{d} N(0,\ 1),$$

where

$$\sigma_\Delta^2(x) = \left[\frac{\sum_{i=1}^{n_1} q_{1i} \left\{ B_{r_{1i},k_{1i}} \circ F(x) - B^2_{r_{1i},k_{1i}} \circ F(x) \right\}}{\lambda \left\{ \sum_{i=1}^{n_1} q_{1i} b_{r_{1i},k_{1i}} \circ F(x) \right\}^2} + \frac{\sum_{i=1}^{n_2} q_{2i} \left\{ B_{r_{2i},k_{2i}} \circ F(x) - B^2_{r_{2i},k_{2i}} \circ F(x) \right\}}{(1-\lambda) \left\{ \sum_{i=1}^{n_2} q_{2i} b_{r_{2i},k_{2i}} \circ F(x) \right\}^2} \right] \Big/ \left[g \circ G^{-1} \circ F(x) \right]^2 .$$

Suppose $\hat{\sigma}_\Delta^2(x)$ is a consistent estimator of the above variance. Then, by Slutsky's Theorem, an asymptotic $100(1-\alpha)\%$ pointwise confidence interval for $\Delta(x)$ is $\hat{\Delta}(x) \pm z_{\alpha/2} \frac{\hat{\sigma}_\Delta(x)}{\sqrt{M}}$. A simultaneous χ_p^2-test for H_0:$F(x_i) = G(x_i)$ at some pre-specified points $x_1, \ldots, x_k$ can also be developed.

3.3 Control percentile test

Suppose F is the control population and G is the treatment population. One may be interested in testing whether at a specified percentile value, the control and treatment populations differ. The following Corollary to Theorem 2 is useful.

Corollary 4 *For any $p \in [a,\ b]$, we have*

$$\frac{\sqrt{M}(\hat{\Lambda}(p) - \Lambda(p))}{\sigma_\Lambda(p)} \xrightarrow{d} N(0,\ 1),$$

where

$$\sigma_\Lambda^2(p) = \left\{ \frac{g \circ F^{-1}(p)}{f \circ F^{-1}(p)} \right\}^2 \frac{\sum_{i=1}^{n_1} q_{1i} \left\{ B_{r_{1i},k_{1i}}(p) - B^2_{r_{1i},k_{1i}}(p) \right\}}{\lambda \left\{ h_1'(p) \right\}^2} + \frac{\sum_{i=1}^{n_2} q_{2i} \left\{ B_{r_{2i},k_{2i}} \circ G \circ F^{-1}(p) - B^2_{r_{2i},k_{2i}} \circ G \circ F^{-1}(p) \right\}}{(1-\lambda) \left\{ h_2' \circ G \circ F^{-1}(p) \right\}^2}.$$

As before, if $\hat{\sigma}_\Lambda^2(p)$ is a consistent estimator of the variance, by Slutsky's Theorem, $\hat{\Lambda}(p) \pm \frac{\hat{\sigma}_\Lambda(p)}{\sqrt{M}} z_{\alpha/2}$ is an approximate $100(1-\alpha)\%$ confidence interval for $\Lambda(p)$. Since the covariance kernel of the limiting process in Theorem 2 is distribution free under the null hypothesis H_0: $F = G$, a Z-test or χ^2-test for equality of a fixed number of percentiles would not require estimation of the associated variance. Estimation of $\sigma_\Lambda^2(p)$ under the alternative is discussed in Sect. 3.5.

3.4 The Wilcoxon–Mann–Whitney statistic

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be independent GRSS from F and G, respectively. Let

$$T = \frac{1}{M}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} I(Y_{(r_{2i}:k_{2i})} < X_{(r_{1j}:k_{1j})}) = \frac{1}{M}\#(Y < X) \tag{15}$$

denote the WMW statistic. We now present the asymptotic distribution of this statistic.

Theorem 3 *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be independent* GRSS *from F and G, respectively, and T be defined as in* (15)*. Then,*

$$\sqrt{M}\,(T-\theta) \xrightarrow{d} N(0,\ \sigma^2),$$

where

$$\theta \equiv \theta(F,\ G) = \int_0^1 h_2 \circ G \circ F^{-1} \circ h_1^{-1}(p)\mathrm{d}p,$$

$$\sigma^2 = \frac{\sigma_1^2}{\lambda} + \frac{\sigma_2^2}{1-\lambda},$$

with

$$\sigma_1^2 = \int_0^1\int_0^1 U_1(x,\ y) \times U_2(x,\ y) \times U_3(x,\ y)\mathrm{d}x\mathrm{d}y,$$

$$U_1(x,\ y) = x \wedge y - \sum_{i=1}^{n_1} q_{1i} B_{r_{1i},\ k_{1i}}(h_1^{-1}(x)) B_{r_{1i},\ k_{1i}}(h_1^{-1}(y)),$$

$$U_2(x,\ y) = \frac{h_2' \circ G \circ F^{-1} \circ h_1^{-1}(x) \times h_2' \circ G \circ F^{-1} \circ h_1^{-1}(y)}{h_1' \circ h_1^{-1}(x) \times h_1' \circ h_1^{-1}(y)},$$

$$U_3(x,\ y) = \frac{g \circ F^{-1} \circ h_1^{-1}(x) \times g \circ F^{-1} \circ h_1^{-1}(y)}{f \circ F^{-1} \circ h_1^{-1}(x) \times f \circ F^{-1} \circ h_1^{-1}(y)}$$

and

$$\sigma_2^2 = \int_0^1\int_0^1 \Bigg[h_2 \circ G \circ F^{-1} \circ h_1^{-1}(x \wedge y) - \sum_{j=1}^{n_2} q_{2j} B_{r_{2j},\ k_{2j}}(G \circ F^{-1} \circ h_1^{-1}(x)) B_{r_{2j},\ k_{2j}}(G \circ F^{-1} \circ h_1^{-1}(y))\mathrm{d}x\mathrm{d}y \Bigg].$$

In particular, under $H_0 : F = G$, we have $\theta = \int_0^1 h_2 \circ h_1^{-1}(p)\mathrm{d}p$,

$$\sigma_1^2 = \int_0^1 \int_0^1 \frac{x \wedge y - \sum_{i=1}^{n_1} q_{1i} B_{r_{1i},\, k_{1i}}(h_1^{-1}(x)) B_{r_{1i},\, k_{1i}}(h_1^{-1}(y))}{h_1' \circ h_1^{-1}(x) \times h_1' \circ h_1^{-1}(y)} \times h_2' \circ h_1^{-1}(x) \times h_2' \circ h_1^{-1}(y) \mathrm{d}x\mathrm{d}y$$

and

$$\sigma_2^2 = \int_0^1 \int_0^1 \left[h_2 \circ h_1^{-1}(x \wedge y) - \sum_{j=1}^{n_2} q_{2j} B_{r_{2j},\, k_{2j}}(h_1^{-1}(x)) B_{r_{2j},\, k_{2j}}(h_1^{-1}(y)) \mathrm{d}x\mathrm{d}y \right].$$

For independent balanced ranked set samples, we have the following result.

Corollary 5 *Let $\mathbf{X}_{k_1 \times m_1}$ and $\mathbf{Y}_{k_2 \times m_2}$ be independent* BRSS *from F and G, respectively. Then,*

$$\sqrt{k_1 m_1 + k_2 m_2}(T - P(Y < X)) \xrightarrow{d} N(0,\ \sigma^2),$$

where

$$\sigma^2 = \frac{\sigma_1^2}{\lambda} + \frac{\sigma_2^2}{1-\lambda},$$

$$\sigma_1^2 = \int_0^1 \int_0^1 \left\{ x \wedge y - \frac{1}{k_1} \sum_{i=1}^{k_1} B_{i,\, k_1}(x) B_{i,\, k_1}(y) \right\} \times \frac{g \circ F^{-1}(x) g \circ F^{-1}(y)}{f \circ F^{-1}(x) f \circ F^{-1}(y)} \mathrm{d}x\mathrm{d}y$$

and

$$\sigma_2^2 = \int_0^1 \int_0^1 \left\{ x \wedge y - \frac{1}{k_2} \sum_{i=1}^{k_2} B_{i,\, k_2}(G \circ F^{-1}(x)) B_{i,\, k_2}(G \circ F^{-1}(y)) \right\} \mathrm{d}x\mathrm{d}y.$$

In particular, under $H_0 : F = G$, we have

$$\sqrt{k_1 m_1 + k_2 m_2}\left(T - \frac{1}{2}\right) \xrightarrow{d} N\left(0,\ \frac{1}{6}\left\{\frac{1}{\lambda(k_1+1)} + \frac{1}{(1-\lambda)(k_2+1)}\right\}\right). \tag{16}$$

Furthermore, if $k_1 = k_2 = k$, (16) *is equivalent to*

$$\sqrt{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1}}\left(T - \frac{1}{2}\right) \xrightarrow{d} N\left(0,\ \frac{1}{6k(k+1)}\right). \tag{17}$$

It is easily verified that the results in Corollary 5 are asymptotically equivalent to those obtained in Theorem 3.2 and Corollary 3.3 of Bohn and Wolfe (1992). Note, however that λ in our calculations is related to the λ in their calculations (denoted by λ^* here) through $\lambda = \frac{k_1\lambda^*}{k_1\lambda^*+k_2(1-\lambda^*)}$.

Also note from Corollary 5 that,

$$\mathrm{ARE}(T_{\mathrm{BRSS}}(k),\ T_{\mathrm{SRS}}) = \frac{k(k+1)}{2}.$$

Thus, the WMW Statistic based on BRSS with $k_1 = k_2 = 2$ is 150% efficient compared to SRS. It should be kept in mind that T_{SRS} is based on effective sample size $m_1 + m_2$ whereas T_{BRSS} is based on a sample of size $k(m_1 + m_2)$. Estimation of σ^2 is discussed below.

3.5 Estimation of variances

3.5.1 Horizontal shift

Let $Q(p) = G^{-1}(p)$ and $q(p)$ be its derivative. To obtain a consistent estimator of the asymptotic variance $\sigma^2_\Delta(x)$ or of quantities σ_{uv}, the first step is to obtain a uniformly consistent estimator of $q(p)$. Let $k(\cdot)$ be a non-negative kernel function that vanishes outside $[-1,\ 1]$ and satisfies

$$\int k(u)\mathrm{d}u = 1, \quad \int |u|k(u)\mathrm{d}u < \infty \quad \text{and} \quad \int |k'(u)|\mathrm{d}u < \infty,$$

where $k'(\cdot)$ is the first derivative of $k(\cdot)$. Define

$$\hat{q}(p) = \frac{-1}{b_m^2}\int_0^1 \hat{G}^{-1}(s)k'\left(\frac{s-p}{b_m}\right)\mathrm{d}s, \quad 0 < p < 1.$$

This is the derivative of the smoothed estimator of $q(p)$ given by

$$\tilde{Q}(p) = \frac{1}{b_m}\int_0^1 \hat{G}^{-1}(s)k\left(\frac{s-p}{b_m}\right)\mathrm{d}s.$$

The bandwidth b_m is chosen to converge to 0.

Under certain regularity conditions and bandwidth choices, it can be shown using arguments similar to Lu et al. (1994) that $\hat{q}$ weakly uniformly converges to q. Also, $\hat{F}$ is weakly uniformly consistent for F. Hence,

$$\hat{\sigma}^2_\Delta(x) = \left[\frac{\sum_{i=1}^{n_1} q_{1i}\left\{B_{r_{1i},k_{1i}} \circ \hat{F}(x) - B^2_{r_{1i},k_{1i}} \circ \hat{F}(x)\right\}}{\lambda\left\{\sum_{i=1}^{n_1} q_{1i} b_{r_{1i},k_{1i}} \circ \hat{F}(x)\right\}^2} + \frac{\sum_{i=1}^{n_2} q_{2i}\left\{B_{r_{2i},k_{2i}} \circ \hat{F}(x) - B^2_{r_{2i},k_{2i}} \circ \hat{F}(x)\right\}}{(1-\lambda)\left\{\sum_{i=1}^{n_2} q_{2i} b_{r_{2i},k_{2i}} \circ \hat{F}(x)\right\}^2}\right] \Bigg/ \left\{\hat{q}(\hat{F}(x))\right\}^2$$

is a consistent estimator of $\sigma^2_\Delta(x)$. Estimation of σ_{uv} follows similarly.

3.5.2 Vertical shift

Let $\tilde{F}$ and $\tilde{G}$ be the kernel-smoothed versions of $\hat{F}$ and $\hat{G}$ and $\tilde{f},\ \tilde{g}$ be the corresponding densities. Hence, for example,

$$\tilde{F}(x) = h_1^{-1}\left(\frac{1}{M_1}\sum_{i=1}^{M_1} K((x - X_i)/b_1)\right)$$

and

$$\tilde{f}(x) = \frac{1}{h_1'(\tilde{F}(x))}\frac{1}{b_1 M_1}\sum_{i=1}^{M_1} k\left((x - X_i)/b_1\right)$$

where $K(\cdot)$ is the cdf corresponding to the kernel $k(\cdot)$. Using arguments similar to Hall et al. (2004), we estimate the variance to be

$$\hat{\sigma}_\Lambda^2(p) = \left\{\frac{\tilde{g}(\tilde{F}^{-1}(p))}{\tilde{f}(\tilde{F}^{-1}(p))}\right\}^2 \frac{\sum_{i=1}^{n_1} q_{1i}\left\{B_{r_{1i},k_{1i}}(p) - B^2_{r_{1i},k_{1i}}(p)\right\}}{\lambda\left\{h_1'(p)\right\}^2}$$
$$+\frac{\sum_{i=1}^{n_2} q_{2i}\left\{B_{r_{2i},k_{2i}}(\tilde{G}(\tilde{F}^{-1}(p))) - B^2_{r_{2i},k_{2i}}(\tilde{G}(\tilde{F}^{-1}(p)))\right\}}{(1-\lambda)\left\{h_2'(\tilde{G}(\tilde{F}^{-1}(p)))\right\}^2}.$$

Then, the interval $\hat{\Lambda}(p) \pm \frac{\hat{\sigma}_\Lambda(p)}{\sqrt{M}} z_{\alpha/2}$ optimizes the coverage probability. Implementation of this procedure requires choosing 10 bandwidths which are done according to the suggestions of Hall et al. (2004). They also suggest the use of the smoothed versions of $\hat{F}$ and $\hat{G}$ instead of the unsmoothed versions in the estimated vertical shift curve, to give it a more regular appearance, especially in the case of small sample sizes.

3.5.3 WMW test

Let $R(p) = G \circ F^{-1}(p)$ and $r(p)$ be its derivative. Let the smoothed estimate of R be

$$\tilde{R}(p) = \frac{1}{b_m}\int_0^1 \hat{G} \circ \hat{F}^{-1}(s) k\left(\frac{s-p}{b_m}\right).$$

Define the estimated derivative of R as

$$\hat{r}(p) = \frac{-1}{b_m^2}\int_0^1 \hat{G} \circ \hat{F}^{-1}(s) k'\left(\frac{s-p}{b_m}\right) \mathrm{d}s.$$

Under regularity conditions similar to Theorem 2 of Li et al. (1996), both $\tilde{R}$ and $\hat{r}$ are weakly uniformly consistent for the respective quantities. Since $h_1,\ h_2$ are continuously differentiable, a consistent estimator of σ^2 is given by

$$\hat{\sigma}^2 = \frac{\hat{\sigma}_1^2}{\lambda} + \frac{\hat{\sigma}_2^2}{1-\lambda}$$

with

$$\hat{\sigma}_1^2 = \int_0^1 \int_0^1 \frac{x \wedge y - \sum_{i=1}^{n_1} q_{1i} B_{r_{1i},\, k_{1i}}(h_1^{-1}(x)) B_{r_{1i},\, k_{1i}}(h_1^{-1}(y))}{h_1' \circ h_1^{-1}(x) \times h_1' \circ h_1^{-1}(y)}$$
$$\times \hat{r} \circ h_1^{-1}(x) \times \hat{r} \circ h_1^{-1}(y) \times h_2' \circ \tilde{R} \circ h_1^{-1}(x) \times h_2' \circ \tilde{R} \circ h_1^{-1}(y) \mathrm{d}x \mathrm{d}y,$$

and

$$\hat{\sigma}_2^2 = \int_0^1 \int_0^1 \left[h_2 \circ \tilde{R} \circ h_1^{-1}(x \wedge y) \right.$$
$$\left. - \sum_{j=1}^{n_2} q_{2j} B_{r_{2j},\, k_{2j}}(\tilde{R} \circ h_1^{-1}(x)) B_{r_{2j},\, k_{2j}}(\tilde{R} \circ h_1^{-1}(y)) \mathrm{d}x \mathrm{d}y \right].$$

This estimator of σ may be used to estimate the power of the WMW test for $H_0 : F = G$.

4 Bootstrapped shift functions

Since the distributions of the limiting processes in Theorems 1 and 2 depend on the unknown distributions F and G, the previous section used consistent estimators of the variance function. For practical applications, one can use resampling techniques such as the bootstrap to approximate these limiting distributions. In this section, we introduce the bootstrap for GRSS and present some relevant results. We will use ideas similar to Bickel and Freedman (1981). See also Chen (2001) and Chen et al. (2004) for a short discussion on bootstrap for ranked set samples.

For a fixed i, we generate a bootstrap sample $X^*_{(r_{1i}:k_{1i})1},\ X^*_{(r_{1i}:k_{1i})2}, \ldots,$ $X^*_{(r_{1i}:k_{1i})m_{1i}}$ from $\hat{F}_{(r_{1i}:k_{1i})}$. Repeating this over for $i = 1, \ldots,\ n_1$, we get a bootstrapped GRSS $\boldsymbol{X}^*$ from $\boldsymbol{X}$. Similarly, we generate $\boldsymbol{Y}^*$ from $\boldsymbol{Y}$. Based on $\boldsymbol{X}^*$, define the bootstrapped versions of (5) and (6) as

$$\hat{F}^*_{\boldsymbol{q}_1}(x) = \sum_{i=1}^{n_1} q_{1i} \hat{F}^*_{(r_{1i}:k_{1i})}(x)$$

and

$$\hat{F}^*(x) = h_1^{-1} \circ \hat{F}^*_{\boldsymbol{q}_1}(x),$$

where $\hat{F}^*_{(r_{1i}:k_{1i})}(\cdot)$ is the edf of the bootstrap sample $\{X^*_{(r_{1i}:k_{1i})j}\}_{j=1}^{m_{1i}}$. Also, based on $\boldsymbol{Y}^*$, define the bootstrapped versions of (8) and (9) as

$$\hat{G}^*_{\boldsymbol{q}_2}(x) = \sum_{i=1}^{n_2} q_{2i}\hat{G}^*_{(r_{2i}:k_{2i})}(x)$$

and

$$\hat{G}^*(x) = h_2^{-1} \circ \hat{G}^*_{\boldsymbol{q}_2}(x),$$

where $\hat{G}^*_{(r_{2i}:k_{2i})}(\cdot)$ is the edf of the bootstrap sample $\{Y^*_{(r_{2i}:k_{2i})j}\}_{j=1}^{m_{2i}}$.

Define the bootstrapped horizontal and vertical shift functions as

$$\hat{\Delta}^*(x) = \hat{G}^{*-1}\hat{F}^*(x) - x, \quad x \in S(F)$$

and

$$\hat{\Lambda}^*(p) = \hat{G}^* \circ \hat{F}^{*-1}(p) - p, \quad p \in [0,\ 1],$$

respectively.

Theorem 4 *Suppose $c_\alpha(\Delta)$ is chosen such that for $0 < \alpha < 1$,*

$$P\left(\sqrt{M}\sup_{x\in S(F)}|\hat{\Delta}^*(x) - \hat{\Delta}(x)| \le c_\alpha(\Delta)\middle|\boldsymbol{X},\ \boldsymbol{Y}\right) = 1-\alpha.$$

If $\frac{M_1}{M} \to \lambda$ as $m \to \infty$, then

$$P\left(\hat{\Delta}(x) - \frac{c_\alpha(\Delta)}{\sqrt{M}} \le \Delta(x) \le \hat{\Delta}(x) + \frac{c_\alpha(\Delta)}{\sqrt{M}}\forall x \in S(F)\right) \to 1-\alpha.$$

Theorem 5 *Suppose $c_\alpha(\Lambda)$ is chosen such that for $0 < \alpha < 1$,*

$$P\left(\sqrt{M}\sup_{0\le p\le 1}|\hat{\Lambda}^*(p) - \hat{\Lambda}(p)| \le c_\alpha(\Lambda)\middle|\boldsymbol{X},\ \boldsymbol{Y}\right) = 1-\alpha.$$

If $\frac{M_1}{M} \to \lambda$ as $m \to \infty$, then

$$P\left(\hat{\Lambda}(p) - \frac{c_\alpha(\Lambda)}{\sqrt{M}} \le \Lambda(p) \le \hat{\Lambda}(p) + \frac{c_\alpha(\Lambda)}{\sqrt{M}}\forall p \in [0,\ 1]\right) \to 1-\alpha.$$

Now, to construct a simultaneous $100(1-\alpha)\%$ confidence band for Δ, one would proceed as follows: first calculate $\hat{\Delta}$ based on $\boldsymbol{X}$ and $\boldsymbol{Y}$. Draw bootstrap samples $\boldsymbol{X}^*$ and $\boldsymbol{Y}^*$ and use them to calculate $\sup_{x\in S(F)}\sqrt{M}|\hat{\Delta}^*(x) - \hat{\Delta}(x)|$. Repeat this procedure for a large number (B, say) of times to get B such numbers. $c_\alpha(\Delta)$ is calculated to be the $100(1-\alpha)$th percentile of these numbers. The required bootstrap confidence band is then $\hat{\Delta} \pm \frac{c_\alpha(\Delta)}{\sqrt{M}}$.

Construction of a simultaneous $100(1-\alpha)\%$ confidence band for Λ proceeds similarly.

5 Numerical studies

5.1 A simulation study

To further investigate the properties of the proposed test procedures, we resorted to simulation studies. First, we used the χ^2 test described in Sect. 3.2 to test for equality of the two distributions. We chose $F \sim$ Weibull$(\theta,\ 1)$ and $G \sim$ Exponential(1). Testing was done based on balanced ranked set samples from the respective populations. We chose $p = 3$ points to be the quartiles of the X-sample. Various combinations of set size k and the Weibull shape parameter θ were used. The results of an empirical power study based on 1,000 Monte Carlo simulations are presented in Table 1. It is evident from the table that even by using BRSS with $k_1 = k_2 = 2$, the power of the proposed test shows a drastic improvement over SRS both at the null and alternative hypotheses. The improvement is more evident with higher set size.

For this simulation study, we used the biweight kernel

$$k(x) = \frac{15}{16}(1 - x^2)^2, \quad -1 < x < 1$$

with the bandwidth b_m for estimating $g \circ G^{-1}$ chosen as

$$b_m = \min(\text{IQR}/1.349, \text{sd}) \times (4/(3M_1))^{1/3}$$

where IQR and sd are the interquartile-range and standard deviation, respectively of the X-sample.

Next, we generated independent BRSS from $F \sim$ Weibull$(1.5,\ 1)$ and $G \sim$ Exponential(1). Two cases were investigated: $k_1 = k_2 = 1$ and $k_1 = k_2 = 10$. In each case, the BRSS used $m_1 = m_2 = 30$ replications. The resulting empirical vertical shift functions (smoothed version), the 95% pointwise confidence limits and 95% bootstrap confidence bands were also obtained. The bootstrap confidence bands were based on 10, 000 replications. We also plotted the theoretical shift functions in each case. The results are shown in Fig. 1. When $k = 1$, the 95% confidence bands for the two plots include the $y = 0$ line and hence the data does not provide enough evidence to conclude that the two distributions are different. However, when $k = 10$, the bands become narrower and cross the $y = 0$ line, implying that the distributions are different at 5% level of significance. The pointwise confidence intervals were constructed using bandwidths chosen according to

Table 1 Simulated power of the χ_3^2 test for $H_0 : F = G$. Results are based on 1,000 simulations with F ~Weibull$(\theta,\ 1)$ and G ~Exponential(1). All calculations are based on balanced RSS with $k_1 = k_2 = k$ and $m_1 = m_2 = 30$. Test used $\alpha = .05$

k	θ			
	0.5	1.0	1.5	2.0
1	0.574	0.114	0.181	0.333
2	0.778	0.074	0.265	0.638
5	0.998	0.048	0.719	0.991
10	1.000	0.046	0.999	1.000

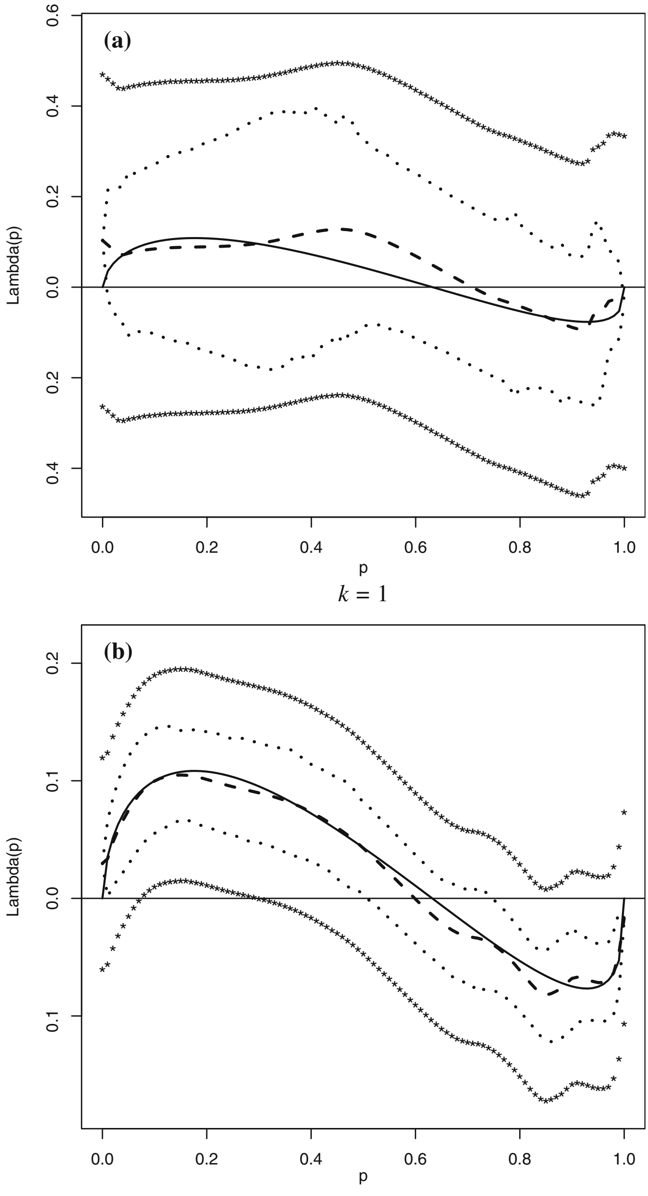

Fig. 1 Estimating the vertical shift function Λ based on BRSS from $F \sim$ Weibull(1.5, 1) and $G \sim$ Exponential(1). *continuous line*: $\Lambda(x)$, *dashed line*: smoothed $\hat{\Lambda}(x)$, *dotted line*: 95% pointwise confidence limits and ****: 95% bootstrap confidence bands

the recommendations in Hall and Hyndman (2003). The biweight kernel mentioned earlier was used in all smoothing operations. As expected, the pointwise confidence intervals are narrower than the confidence bands, but still may fail to reject H_0 when $k_1 = k_2 = 1$.

5.2 Prostate cancer example

One of the measures of cancer burden is the number of deaths due to the disease. With the introduction of the PSA screening test, the number of deaths due to prostate cancer has dramatically gone down, thanks to the earlier detection (and hence treatment). PSA was introduced in the early to mid 1990s, so its effectiveness can be measured by comparing the mortality rates due to prostate cancer before and after introduction of the test. We obtained the rates of prostate cancer deaths in the USA (by county) for the two year-groups: 1990–1992 and 1999–2001 using Seer*Stat software available from http://www.seer.cancer.gov/seerstat/. The mortality data are provided by the National Center for Health Statistics (NCHS) (http://www.cdc.gov/nchs) which collects it from the death certificates filed for each death. For each of the 2 year-groups, we found that there were a small percentage (e.g. $\sim 1{-}5\%$) of counties with zero death rates and few counties with missing death rates. For this example, we ignored the counties with zero or missing prostate cancer death rates, since the theoretical development of the shift functions assumes that F and G are continuous. As a result, we obtained 3,041 counties for 1990–1992 and 3,017 counties from 1999 to 2001 with non-missing positive death rates.

Let F denote the 1990–1992 prostate cancer mortality distribution and G denote that for 1999–2001. First, we selected independent BRSS from the two populations with $k_1 = k_2 = 1$ and $m_1 = m_2 = 30$. The WMW test for the equality of the two distributions gave a z-value (corresponding p-value in parenthesis) of 1.881(.0599) and the Z-test for location-scale shift gave a z-value of $-0.3457(0.7296)$. We repeated the same process with independent BRSS with $k_1 = k_2 = 10$ and $m_1 = m_2 = 30$. The corresponding z-values came out to be 23.1175(0) and 0.075(0.9402), respectively. Thus, based on independent SRS, we are unable to conclude that the distributions are different. However, the independent BRSS with $k_1 = k_2 = 10$ provide us with strong evidence that the two distributions are different but are location-scale shifts of each other. The findings are supported by the side-by-side boxplots of the two populations given in Fig. 2.

6 Discussion and conclusion

In this article, we have discussed the theoretical aspects of shift-functions in nonparametric two-sample problems based on independent ranked set samples. Our results generalize those already known for SRS and show that one can improve upon the inferential procedures by increasing the ranked set size “k” in BRSS, whether one is working with the shift function as a whole, the function evaluated at specific points, or, with its integral. The limiting distributions of the two shift processes depend on the unknown F and G; hence one has to use resampling techniques to construct tests or confidence bands. However, under the null hypothesis $H_0 : F = G$, the vertical shift function is asymptotically distribution free and exact cut-offs can be obtained without resorting to resampling.

If one is interested in comparing he distributions F and G at the p th quantile instead of over their common entire support, $\Lambda(p) = G \circ F^{-1}(p) - p$ is the quantity of interest. Based on independent BRSS $\boldsymbol{X}_{k\times m}$ and $\boldsymbol{Y}_{k\times m}$, we have from

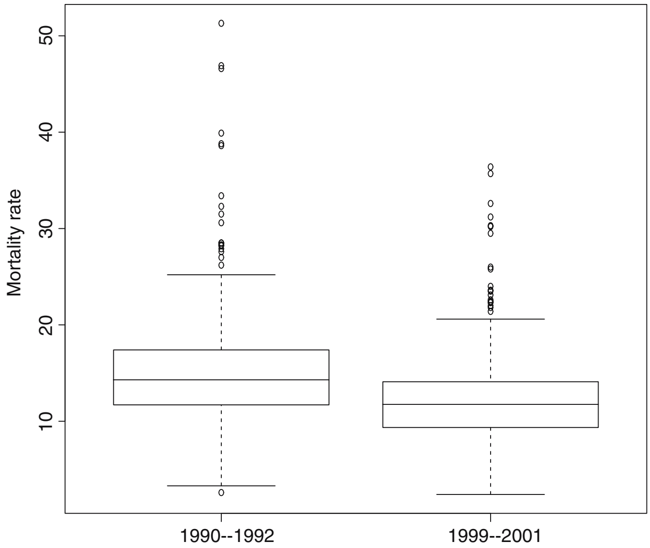

Fig. 2 Boxplots comparing prostate cancer mortality rates in the USA during 1990–1992 and 1999–2001

Corollary 2

$$\sqrt{km}(\hat{\Lambda}_{\text{BRSS}}(p) - \Lambda(p)) \xrightarrow{d} N(0,\ 2\sigma^2_{\text{BRSS}}(p)),$$

where

$$\sigma^2_{\text{BRSS}}(p) = \left\{ p - \frac{1}{k} \sum_{i=1}^{k} B^2_{i,\ k}(p) \right\}.$$

On the other hand, if one decides to use independent nomination samples consisting of the sample pth quantiles (based on samples of size k) as the nominee, we have from Theorem 2

$$\sqrt{m}(\hat{\Lambda}_{\text{NOM}}(p) - \Lambda(p)) \xrightarrow{d} N(0,\ 2\sigma^2_{\text{NOM}}(p)),$$

where

$$\sigma^2_{\text{NOM}}(p) = \left\{ \frac{B_{[pk]+1,\ k}(p) - B^2_{[pk]+1,\ k}(p)}{b^2_{[pk]+1,\ k}(p)} \right\}$$

and $b_{m,\ n}(x)$ denotes the Beta$(m,\ n-m+1)$ density at x. The asymptotic relative efficiency (ARE) of $\hat{\Lambda}_{\text{NOM}}(p)$ with respect to $\hat{\Lambda}_{\text{BRSS}}(p)$ is given by

$$\text{ARE}(\hat{\Lambda}_{\text{NOM}}(p),\ \hat{\Lambda}_{\text{BRSS}}(p)) = \frac{\sigma^2_{\text{BRSS}}(p)}{k\sigma^2_{\text{NOM}}(p)} = h(p) \text{ say.}$$

A plot of $h(p)$ against p for different values of k appears in Fig. 3a, suggesting that BRSS is asymptotically more efficient than NOM. Note, however, that BRSS is based on an "effective sample size" of $2km$ while NOM is based on an "effective sample size" of $2m$. Hence, to make the ARE comparison fair, we should have NOM based on km replicates (instead of m). The revised plot incorporating this correction appears in Fig. 3b. It is apparent that in this adjusted setup, NOM is more efficient than BRSS. Thus, if one is interested in comparing a particular percentile point of the two distributions, it is more advantageous to use nomination sampling based on the same sample size as BRSS.

Note that in each of the graphs, the ARE is maximized at $p = 1/2$ and increases with k. Table 2 gives the ARE at $p = 1/2$ for selected values of k. It is interesting to note that the jump in efficiency is twofold by moving from SRS to the case with $k = 2$. The case with $k = 2$ is the most practical one to use, since it requires sorting only two observations at a time.

The asymptotic results obtained in this paper can be used to derive the limiting distributions of various test statistics that are based on the notion of "divergence" between F and G such as the functional $\int\{G \circ F^{-1}(p)\}^2 \mathrm{d}p$ or the Kolmogorov–Smirnov distance $\sup_x |F(x) - G(x)|$. One can also use the results to obtain the asymptotic distribution of the crossing point of two distributions as discussed in Hawkins and Kochar (1991). The results may also be extended to the case of multiple comparisons, where instead of two samples, one may have independent ranked set samples from several distributions, possibly along the lines of Nair (1982). It would be interesting to investigate the behavior of these processes under imperfect ranking or to extend the results when F is a nonparametric distribution and G is a parametric distribution or to the case when the two GRSS are randomly right censored. The latter extensions should be straightforward and can be carried out along the lines of Lu et al. (1994) and Li et al. (1996, 1999).

Note that the horizontal shift functions have easily distinguishable features for location-scale models and are natural candidates for detecting them. However, the limiting distribution of the horizontal shift function, even under the null hypothesis $F = G$ depends on the unknown F and G, which need to be estimated from the data. In contrast, the vertical shift process, under the null hypothesis, is asymptotically distribution free, making it statistically more reliable.

In the numerical studies in Sect. 5, we compared results based on SRS and BRSS with $k = 10$. Our calculations showed that inferences based on BRSS were more powerful than those based on SRS of the same size. It should be noted that this is based on the assumption that cost of sampling (and ranking) a unit is negligible compared to measuring it based on the attribute of interest. Hence, even though one needs to sample k^2 units to obtain a BRSS of size k, the cost is no more greater than obtaining an SRS of size k. In situations where that is not the case, one would

Table 2 Asymptotic relative efficiency of NOM with respect to BRSS in estimating vertical shift at the median. Calculations are based on equal sample sizes

k	ARE(NOM, BRSS)
1	100.00
2	200.00
5	288.96
10	440.19

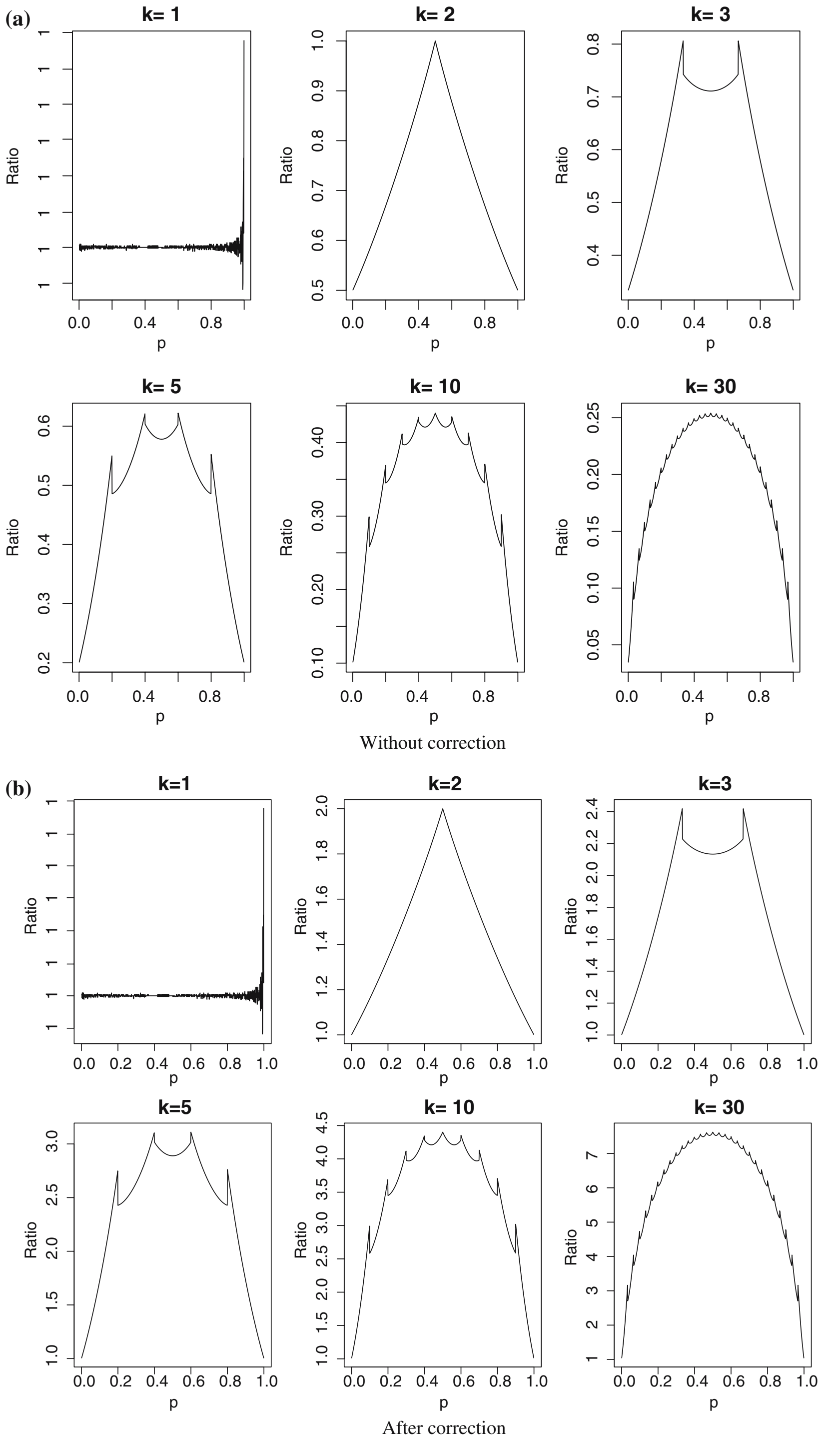

Fig. 3 Efficiency comparison for various k, with and without correcting for the effective sample sizes

need to take into account the cost considerations to make an appropriate decision. Further research is needed to incorporate the cost structure into the decision-making process and is beyond the scope of the current paper.

Finally, it would be more interesting to think of other applications such as comparing the cancer incidences of two groups where complete data is not available due to lack of a national registry and to apply the proposed method.

Appendix: Proofs

Proof of Lemma 1 By standard result on empirical processes (see, for example Billingsley, 1968, Theorem 16.4), as $m_1 \to \infty$, we have independently for each i,

$$\sqrt{m_{1i}}(\hat{F}_{(r_{1i}:k_{1i})} - F_{(r_{1i}:k_{1i})}) \xrightarrow{d} \mathbb{W}_i,$$

where $\mathbb{W}_i$ is a zero-mean Gaussian process with covariance kernel

$$K_i(x,\ y) = F_{(r_{1i}:k_{1i})}(x \wedge y) - F_{(r_{1i}:k_{1i})}(x)F_{(r_{1i}:k_{1i})}(y).$$

Since the map $(x_1,\ \ldots,\ x_{n_1}) \mapsto x_1 + \cdots + x_{n_1}$ is continuous, we get (see Billingsley, 1968, Theorem 5.1)

$$\begin{aligned}\sqrt{M_1}(\hat{F}_{\boldsymbol{q}_1} - F_{\boldsymbol{q}_1}) &= \sqrt{M_1}\sum_{i=1}^{n_1} q_{1i}(\hat{F}_{(r_{1i}:k_{1i})} - F_{(r_{1i}:k_{1i})}) \\ &= \sum_{i=1}^{n_1} \sqrt{m_{1i}}(\hat{F}_{(r_{1i}:k_{1i})} - F_{(r_{1i}:k_{1i})})\frac{q_{1i}}{\sqrt{m_{1i}/M_1}} \\ &\xrightarrow{d} \sum_{i=1}^{n_1} \sqrt{q_{1i}}\ \mathbb{W}_i \ \stackrel{defn}{=} \mathbb{W}_F, \text{ say.} \end{aligned} \tag{18}$$

It is easily verified that $\mathbb{W}_F$ is a zero-mean Gaussian process with covariance kernel given by (12).

Since h_1 is differentiable on $(0,\ 1)$ with a non-vanishing derivative and is continuous at both the end points, by the mean value theorem, for all x

$$\begin{aligned}\hat{F}(x) - F(x) &= h_1^{-1} \circ \hat{F}_{\boldsymbol{q}_1}(x) - h_1^{-1} \circ F_{\boldsymbol{q}_1}(x) \\ &= (\hat{F}_{\boldsymbol{q}_1}(x) - F_{\boldsymbol{q}_1}(x))\frac{1}{h_1' \circ h_1^{-1} \circ \tilde{F}_{\boldsymbol{q}_1}(x)},\end{aligned}$$

where $\tilde{F}_{\boldsymbol{q}_1}(x) \in l(\hat{F}_{\boldsymbol{q}_1}(x),\ F_{\boldsymbol{q}_1}(x))$, the line segment joining $\hat{F}_{\boldsymbol{q}_1}(x)$ and $F_{\boldsymbol{q}_1}(x)$. By Glivenko–Cantelli theorem, as $m_1 \to \infty$, we have for each i,

$$\sup_x \left|\hat{F}_{(r_{1i}:k_{1i})}(x) - F_{(r_{1i}:k_{1i})}(x)\right| \xrightarrow{\text{a.s.}} 0.$$

Hence,

$$\begin{aligned}\sup_x \left|\hat{F}_{\boldsymbol{q}_1}(x) - F_{\boldsymbol{q}_1}(x)\right| &= \sup_x \left|\sum_{i=1}^{n_1} q_{1i}\,(\hat{F}_{(r_{1i}:k_{1i})}(x) - F_{(r_{1i}:k_{1i})}(x))\right| \\ &\leq \sum_{i=1}^{n_1} q_{1i} \sup_x |\hat{F}_{(r_{1i}:k_{1i})}(x) - F_{(r_{1i}:k_{1i})}(x)| \\ &\overset{\text{a.s.}}{\to} 0. \end{aligned} \tag{19}$$

Hence,

$$\sup_x \left|\tilde{F}_{\boldsymbol{q}_1}(x) - F_{\boldsymbol{q}_1}(x)\right| \leq \sup_x \left|\hat{F}_{\boldsymbol{q}_1}(x) - F_{\boldsymbol{q}_1}(x)\right| \overset{\text{a.s.}}{\to} 0. \tag{20}$$

By continuity of $h_1' \circ h_1^{-1}$ on $[0,\ 1]$, we have

$$\begin{aligned}&\sup_x \left|\frac{1}{h_1' \circ h_1^{-1} \circ \tilde{F}_{\boldsymbol{q}_1}(x)} - \frac{1}{h_1' \circ F(x)}\right| \\ &= \sup_x \left|\frac{1}{h_1' \circ h_1^{-1} \circ \tilde{F}_{\boldsymbol{q}_1}(x)} - \frac{1}{h_1' \circ h_1^{-1} \circ F_{\boldsymbol{q}_1}(x)}\right| \\ &\overset{\text{a.s.}}{\to} 0. \end{aligned} \tag{21}$$

Combining (18) and (21) yields

$$\sqrt{M_1}(\hat{F} - F) \overset{d}{\to} \frac{\mathbb{W}_F}{h_1' \circ F}.$$

The proof of (11) is identical. The proof of (14) follows directly from the independence of the two limiting processes. □

Lemma 2 *Let $\phi : D(\overline{\mathbb{R}}) \times D(\overline{\mathbb{R}}) \to D(\overline{\mathbb{R}})$ be defined by*

$$\phi(F,\ G) = G^{-1} \circ F.$$

Then the Hadamard derivative of ϕ at $(F,\ G)$ tangentially to $(h,\ k)$ is given by

$$d\phi(F,\ G)\,.\,(h,\ k) = \frac{h - k \circ G^{-1} \circ F}{g \circ G^{-1} \circ F}.$$

Proof of Lemma 2 Let $B_1 = D(\overline{\mathbb{R}})$, $B_2 = D(\overline{\mathbb{R}})$. Take any sequence $(h_n,\ k_n) \in B_1 \times B_2$ and $a_n \in \mathbb{R}$ such that $(h_n,\ k_n) \overset{\|\cdot\|_\infty}{\to} (h,\ k) \in C([F^{-1}(a),\ F^{-1}(b)]) \times$

$C([F^{-1}(a),\ F^{-1}(b)])$ and $a_n \to 0$ as $n \to \infty$. Now,

$$
\begin{aligned}
&\left\| \frac{\phi((F,\ G)+a_n(h_n,\ k_n))-\phi(F,\ G)}{a_n} - \frac{h-k\circ G^{-1}\circ F}{g\circ G^{-1}\circ F} \right\| \\
&\quad \le \sup_t \left| \frac{(G+a_nk_n)^{-1}\circ(F+a_nh_n)(t)-G^{-1}\circ(F+a_nh_n)(t)}{a_n} \right. \\
&\qquad\qquad \left. + \frac{k\circ G^{-1}\circ F(t)}{g\circ G^{-1}\circ F(t)} \right| \\
&\qquad\qquad + \sup_t \left| \frac{G^{-1}\circ(F+a_nh_n)(t)-G^{-1}\circ F(t)}{a_n} - \frac{h(t)}{g\circ G^{-1}\circ F(t)} \right| \\
&\quad = \sup_t \left| \left(\frac{(G+a_nk_n)^{-1}-G^{-1}}{a_n} + \frac{k\circ G^{-1}}{g\circ G^{-1}} \right)\circ(F+a_nh_n)(t) \right| \\
&\qquad\qquad + \sup_t \left| \left(\frac{k\circ G^{-1}}{g\circ G^{-1}} \right)\circ(F+a_nh_n)(t) - \left(\frac{k\circ G^{-1}}{g\circ G^{-1}} \right)\circ F(t) \right| \\
&\qquad\qquad + \sup_t \left| \frac{G^{-1}\circ(F+a_nh_n)(t)-G^{-1}\circ F(t)}{a_n} - \frac{h(t)}{g\circ G^{-1}\circ F(t)} \right| \\
&\quad \le \left\| \frac{(G+a_nk_n)^{-1}-G^{-1}}{a_n} + \frac{k\circ G^{-1}}{g\circ G^{-1}} \right\|_\infty \\
&\qquad\qquad + \sup_t \left| \left(\frac{k\circ G^{-1}}{g\circ G^{-1}} \right)\circ(F+a_nh_n)(t) - \left(\frac{k\circ G^{-1}}{g\circ G^{-1}} \right)\circ F(t) \right| \\
&\qquad\qquad + \sup_t \left| \frac{h_n(t)}{g\circ G^{-1}\circ \tilde{F}(t)} - \frac{h(t)}{g\circ G^{-1}\circ F(t)} \right|,
\end{aligned} \tag{22}
$$

where $\tilde{F}(t) \in l(F(t),\ F(t)+a_nh_n(t))$. Since $h_n \overset{\|\cdot\|_\infty}{\to} h$, $a_n \to 0$ and g and k are continuous, we have all the three terms in (22) converging to zero. $\square$

Lemma 3 *Let $B_1 = D(\overline{\mathbb{R}})$, $B_2 = D[F^{-1}(a),\ F^{-1}(b)]$ and $B_3 = D[a,\ b]$, where $\overline{\mathbb{R}} = [-\infty,\ \infty]$ and $0 < a < b < 1$. Let $C(\overline{\mathbb{R}})$ denote the subspace of continuous functions in B_1. Define $\phi : B_1 \times B_2 \to B_3$ by*

$$\phi(F,\ G) = G \circ F^{-1}.$$

Suppose F is continuously differentiable on $\mathbb{R}$ with positive derivative, and G is continuously differentiable on $\mathbb{R}$. Then, ϕ is compactly differentiable at $(F,\ G)$ tangentially to $C([F^{-1}(a),\ F^{-1}(b)]) \times C([F^{-1}(a),\ F^{-1}(b)])$ and the derivative is given by

$$d\phi(F,\ G)\,.\,(h,\ k) = \frac{-h\circ F^{-1}}{f\circ F^{-1}} g\circ F^{-1} + k\circ F^{-1}$$

for $(h,\ k) \in C([F^{-1}(a),\ F^{-1}(b)]) \times C([F^{-1}(a),\ F^{-1}(b)])$, where f is the ordinary derivative of F and g is the ordinary derivative of G.

Proof of Lemma 3 Take any sequence $(h_n,\ k_n) \in B_1 \times B_2$ and $a_n \in \mathbb{R}$ such that $(h_n,\ k_n) \stackrel{\|\cdot\|_\infty}{\to} (h,\ k) \in C([F^{-1}(a),\ F^{-1}(b)]) \times C([F^{-1}(a),\ F^{-1}(b)])$ and $a_n \to 0$ as $n \to \infty$. Now,

$$\begin{aligned}
&\frac{\phi((F,\ G) + a_n(h_n,\ k_n)) - \phi(F,\ G)}{a_n} \\
&\quad = \frac{(G + a_n k_n) \circ (F + a_n h_n)^{-1} - G \circ F^{-1}}{a_n} \\
&\quad = \frac{G \circ (F + a_n h_n)^{-1} + a_n k_n \circ (F + a_n h_n)^{-1} - G \circ F^{-1}}{a_n} \\
&\quad = \frac{G \circ (F + a_n h_n)^{-1} - G \circ F^{-1}}{a_n} + k_n \circ (F + a_n h_n)^{-1} \\
&\quad = \frac{(F + a_n h_n)^{-1} - F^{-1}}{a_n} g \circ F_n^{-1} + k_n \circ (F + a_n h_n)^{-1},
\end{aligned}$$

where F_n^{-1} is a function on $[a,\ b]$ such that $F_n^{-1}(p)$ is between $F^{-1}(p)$ and $(F + a_n h_n)^{-1}(p)$.

Let $\tilde{F}(t) = F(t) + a_n h_n(t)$ and $J = [F^{-1}(a),\ F^{-1}(b)]$. Since $h_n \stackrel{\|\cdot\|_\infty}{\to} h$, we have

$$\sup_{t \in J} \left| \frac{\tilde{F}(t) - F(t)}{a_n} - h(t) \right| \to 0.$$

Hence,

$$\sup_{p \in [a,\ b]} \left| \frac{\tilde{F} \circ F^{-1}(p) - F \circ F^{-1}(p)}{a_n} - h \circ F^{-1}(p) \right| \to 0.$$

That is,

$$\sup_{p \in [a,\ b]} \left| \frac{\tilde{F} \circ F^{-1}(p) - p}{a_n} - h \circ F^{-1}(p) \right| \to 0.$$

Since h is continuous, using arguments similar to Vervaat (1972), we have for every $\epsilon > 0$,

$$\sup_{p \in [a,\ b-\epsilon]} \left| \frac{F \circ \tilde{F}^{-1}(p) - p}{a_n} + h \circ F^{-1}(p) \right| \to 0. \tag{23}$$

Note that, by the mean value theorem,

$$\begin{aligned}
F \circ \tilde{F}^{-1}(p) - p &= F \circ \tilde{F}^{-1}(p) - F \circ F^{-1}(p) \\
&= (\tilde{F}^{-1}(p) - F^{-1}(p)) f(\eta_p),
\end{aligned} \tag{24}$$

where $\eta_p \in l(\tilde{F}^{-1}(p),\ F^{-1}(p))$. From (23), we also have

$$\sup_{p \in [a,\ b-\epsilon]} |F \circ \tilde{F}^{-1}(p) - p| \to 0,$$

from which it follows that

$$\sup_{p\in[a,\ b-\epsilon]} |\eta_p - F^{-1}(p)| \to 0. \tag{25}$$

From (23), (24) and (25), we get

$$\sup_{p\in[a,\ b-\epsilon]} \left| \frac{\tilde{F}^{-1}(p) - F^{-1}(p)}{a_n} f(\eta_p) + h \circ F^{-1}(p) \right| \to 0.$$

Since f is continuous and positive,

$$\sup_{p\in[a,\ b-\epsilon]} \left| \frac{\tilde{F}^{-1}(p) - F^{-1}(p)}{a_n} + \frac{h \circ F^{-1}(p)}{f \circ F^{-1}(p)} \right| \to 0. \tag{26}$$

Also, since the above is true for every $\epsilon > 0$, we have

$$\frac{(F + a_n h_n)^{-1} - F^{-1}}{a_n} \stackrel{\|\cdot\|_\infty}{\to} -\frac{h \circ F^{-1}}{f \circ F^{-1}}.$$

Since $\tilde{F}^{-1} \stackrel{\|\cdot\|_\infty}{\to} F^{-1}$ and g is continuous, we have from (26)

$$g \circ \tilde{F}^{-1} \stackrel{\|\cdot\|_\infty}{\to} g \circ F^{-1}.$$

Finally,

$$\begin{aligned}
&\sup_{p\in[a,\ b-\epsilon]} |k_n \circ (F + a_n h_n)^{-1}(p) - k \circ F^{-1}(p)| \\
&\quad \le \sup_{p\in[a,\ b-\epsilon]} |k_n \circ (F + a_n h_n)^{-1}(p) - k \circ (F + a_n h_n)^{-1}(p)| \\
&\qquad + \sup_{p\in[a,\ b-\epsilon]} |k \circ (F + a_n h_n)^{-1}(p) - k \circ F^{-1}(p)| \\
&\quad \le \|k_n - k\|_\infty + \sup_{p\in[a,\ b-\epsilon]} |k \circ (F + a_n h_n)^{-1}(p) - k \circ F^{-1}(p)| \to 0
\end{aligned}$$

since $k_n \stackrel{\|\cdot\|_\infty}{\to} k$, k is continuous and $(F + a_n h_n)^{-1} \stackrel{\|\cdot\|_\infty}{\to} F^{-1}$. Combining all of the above, the result follows. □

Proof of Theorem 1 Use Lemmas 1, 3 and the functional δ method of Gill (1989) and Andersen et al. (1992). □

Proof of Theorem 2 Use Lemmas 1, 2 and the functional δ method of Gill (1989) and Andersen et al. (1992). □

Proof of Theorem 3 Note that T can be alternately written as

$$T = \int_0^1 EDF_{\boldsymbol{Y}} \circ EDF_{\boldsymbol{X}}^{-1}(p)\mathrm{d}p.$$

Also recall that

$$EDF_{\boldsymbol{X}}(x) = \hat{F}_{\boldsymbol{q}_1}(x) + o_p(1) = h_1 \circ \hat{F}(x) + o_p(1).$$

Hence,

$$T = \int_0^1 h_2 \circ \hat{G} \circ \hat{F}^{-1} \circ h_1^{-1}(p)\mathrm{d}p + o_p(1).$$

Defining

$$\phi(S) = h_2 \circ S \circ h_1^{-1},$$

we have from Theorem 2 and the functional δ method of Gill (1989),

$$\sqrt{M}(\phi(\hat{\Lambda}) - \phi(\Lambda)) \xrightarrow{d} h_2' \circ \Lambda \circ h_1^{-1} \times \mathbb{Z}_{\Lambda,\ \mathrm{GRSS}} \circ h_1^{-1}.$$

Note that $\int_0^1 h_2 \circ \hat{G} \circ \hat{F}^{-1} \circ h_1^{-1}(p)\mathrm{d}p = \int \phi(\hat{\Lambda})(p)\mathrm{d}p$. By another application of the functional δ method with $\psi(S) = \int_0^1 S(p)\mathrm{d}p$, we get the desired result. □

Proof of Corollary 5 For BRSS, $h_1(x) = h_2(x) = x$. The rest follows directly from Theorem 3. □

Lemma 4 *Suppose that*

$$\sqrt{M}((\hat{F},\ \hat{G}) - (F,\ G)) \xrightarrow{d} \left(\frac{\mathbb{W}_F}{h_1' \circ F},\ \frac{\mathbb{W}_G}{h_2' \circ G}\right).$$

Let $\hat{F}^*$ and $\hat{G}^*$ be the bootstrap versions of F and G, respectively. Let $\phi : B_1 \to B_2$ is compactly differentiable at $(F,\ G)$, and let $\psi : B_2 \to \mathbb{R}$ be measurable and continuous in a subset of B_2. Then,

$$L^*\left(\psi\left(\sqrt{M}\left(\phi(F^*,\ G^*) - \phi(\hat{F},\ \hat{G})\right)\right)\right)$$
$$\xrightarrow{P} L\left(\psi\left(d\phi(F,\ G)\left(\frac{\mathbb{W}_F}{h_1' \circ F},\ \frac{\mathbb{W}_G}{h_2' \circ G}\right)\right)\right),$$

where L and L^* denote, respectively, the "law" of the original samples $\boldsymbol{X}$, $\boldsymbol{Y}$ and the bootstrapped samples $\boldsymbol{X}^*$, $\boldsymbol{Y}^*$.

Proof of Lemma 4 By the Skorohod–Dudley–Wichura (SDW) representation theorem (see, for example Billingsley, 1968), we can construct a sequence $(\hat{F}', \hat{G}') \stackrel{\mathrm{d}}{=} (\hat{F}, \hat{G})$ with

$$\sqrt{M}((\hat{F}, \hat{G}) - (F, G)) \stackrel{\|\cdot\|_\infty}{\rightarrow} (U', V') \text{ a.s.}$$

where $W' \stackrel{\mathrm{d}}{=} W$. Let $\hat{F}'^*, \hat{G}'^*$ be the bootstrap versions based on $\hat{F}', \hat{G}'$. Then,

$$\sqrt{M}((\hat{F}'^*, \hat{G}'^*) - (\hat{F}', \hat{G}')) \stackrel{d}{\rightarrow} W^* \stackrel{\mathrm{d}}{=} W'.$$

By SDW again, we can get $\hat{F}'^{*'} \stackrel{\mathrm{d}}{=} \hat{F}'^*$ and $\hat{G}'^{*'} \stackrel{\mathrm{d}}{=} \hat{G}'^*$ such that

$$\sqrt{M}((\hat{F}'^{*'} - \hat{G}'^{*'}) - (\hat{F}', \hat{G}')) \stackrel{\|\cdot\|_\infty}{\rightarrow} W^{*'} \stackrel{\mathrm{d}}{=} W^* \text{ a.s.}$$

Hence,

$$\sqrt{M}(\hat{F}'^{*'}, \hat{G}'^{*'}) - (F, G) \stackrel{\|\cdot\|_\infty}{\rightarrow} W^{*'} + W'$$

and

$$\sqrt{M}((\hat{F}', \hat{G}') - (F, G)) \stackrel{\|\cdot\|_\infty}{\rightarrow} W'.$$

Hence,

$$\begin{aligned}
&\sqrt{M}\left\{\phi(\hat{F}'^{*'}, \hat{G}'^{*'}) - \phi(\hat{F}', \hat{G}')\right\} \\
&\quad = \sqrt{M}\left\{\phi(\hat{F}'^{*'}, \hat{G}'^{*'}) - \phi(F, G)\right\} - \sqrt{M}\left\{\phi(\hat{F}', \hat{G}') - \phi(F, G)\right\} \\
&\quad \stackrel{\|\cdot\|_\infty}{\rightarrow} d\phi(F, G)(W^{*'} + W') - d\phi(F, G)(W') \\
&\quad = d\phi(F, G)(W^{*'}) \text{ a.s.}
\end{aligned}$$

Since ψ is continuous at $d\phi(F, G)(W)$ a.s.,

$$\psi\left[\sqrt{m}\left\{\phi(\hat{F}'^{*'}, \hat{G}'^{*'}) - \phi(\hat{F}', \hat{G}')\right\}\right] \stackrel{\|\cdot\|_\infty}{\rightarrow} \psi\left[d\phi(F, G)(W^{*'})\right] \text{ a.s.}$$

Since

$$\psi[\sqrt{M}\{\phi(\hat{F}'^{*'}, \hat{G}'^{*'}) - \phi(\hat{F}', \hat{G}')\}] \stackrel{\mathrm{d}}{=} \psi[\sqrt{M}\{\phi(\hat{F}'^*, \hat{G}'^*) - \phi(\hat{F}', \hat{G}')\}],$$

we have

$$\psi[\sqrt{M}\{\phi(\hat{F}'^*, \hat{G}'^*) - \phi(\hat{F}', \hat{G}')\}] \stackrel{d}{\rightarrow} \psi[d\phi(F, G)(W)]$$

Hence,

$$d\langle L^*[\psi[\sqrt{M}\{\phi(\hat{F}'^*, \hat{G}'^*) - \phi(\hat{F}', \hat{G}')\}]], L[\psi\{d\phi(F, G)(W)\}]\rangle \stackrel{\text{a.s.}}{\rightarrow} 0.$$

Since $(\hat{F}, \hat{G}) \stackrel{\mathrm{d}}{=} (\hat{F}, \hat{G})$, we have

$$d\langle L^*[\psi[\sqrt{M}\{\phi(\hat{F}^*, \hat{G}^*) - \phi(\hat{F}, \hat{G})\}]], L[\psi\{d\phi(F, G)(W)\}]\rangle \stackrel{P}{\rightarrow} 0.$$

□

Theorem 6 *Let $g \circ G^{-1}(\cdot)$ be continuous and bounded away from zero on $[a_2,\ b_2]$. Then, as $m \to \infty$,*

$$\sqrt{M}(\hat{\Delta}^* - \hat{\Delta}) \xrightarrow{d} \mathbb{Z}_\Delta,$$

where $\mathbb{Z}_\Delta$ is as in Theorem 1.

Proof of Theorem 6 Follows from Lemma 4 and Theorem 1 using $\phi(F,\ G) = G^{-1} \circ F$ and $\psi \equiv$ identity . □

Theorem 7 *As $m \to \infty$,*

$$\sqrt{M}(\hat{\Lambda}^* - \hat{\Lambda}) \xrightarrow{d} \mathbb{Z}_\Lambda,$$

where $\mathbb{Z}_\Lambda$ is as in Theorem 2.

Proof of Theorem 7 Follows from Lemma 4 and Theorem 2 using $\phi(F,\ G) = G \circ F^{-1}$ and $\psi \equiv$ identity .

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