AISM (2007) 59: 597–602
DOI 10.1007/s10463-006-0059-3

Sigeo Aki · Katuomi Hirano

On the waiting time for the first success run

Received: 28 February 2005 / Revised: 10 March 2006 /
Published online: 17 June 2006

Abstract Let k and m are positive integers with $k \geq m$. The probability generating function of the waiting time for the first occurrence of consecutive k successes in a sequence of m-th order Markov dependent trials is given as a function of the conditional probability generating functions of the waiting time for the first occurrence of consecutive m successes. This provides an efficient algorithm for obtaining the probability generating function when k is large. In particular, in the case of independent trials a simple relationship between the geometric distribution of order k and the geometric distribution of order $k-1$ is obtained.

Keywords Geometric distribution of order k · Probability generating function · Conditional expectation · Markov chain · Run · Discrete distribution

1 Introduction

The geometric distribution of order k is the distribution of the waiting time for the first occurrence of consecutive k successes in independent trials with success

This research was partially supported by the ISM Cooperative Research Program(2004-ISM-CRP-2006) and by a Grant-in-Aid for Scientific Research (C) of the JSPI (Grant Number 16500183)

S. Aki
Department of Mathematics, Faculty of Engineering, Kansai University, 3-3-35 Yamate-cho, Suita, Osaka 564-8680, Japan

K. Hirano (✉)
The Institute of Statistical Mathematics, Research Organization of Information and Systems, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan
E-mail: hirano@ism.ac.jp

K. Hirano
Department of Statistical Science, School of Multidisciplinary Sciences, The Graduate University for Advanced Studies, Tokyo, Japan

probability p. The exact probability function and recurrence relations for it have been derived by many authors using several approaches (see Feller, 1968, page 323 and Balakrishnan and Koutras, 2002).

Up to now, only a few papers have discussed relationships between distributions with different orders (see Aki and Hirano, 1994, 1995). In Sect. 2, we study the distribution of the waiting time for the first occurrence of consecutive k successes in a sequence of m-th order Markov dependent trials. The problem was studied by Aki, Balakrishnan and Mohanty (1996) by using both combinatorial methods and the method of conditional probability generating functions(p.g.f.'s). In the previous paper, a system of linear equations of conditional p.g.f.'s for the waiting time was given for m and $k(\geq m)$. The system was solved for the special case $m = 2$ in order to illustrate that it can be solved for small m. However, since the number of conditional p.g.f.'s increases as k becomes larger, explicit solution of the system of equations has not been derived for large k by using the usual method of conditional p.g.f.'s. We use another idea and provide the p.g.f. of the waiting time explicitly as a function of conditional p.g.f.'s of the waiting time for the first success run of length m. Based on the new result, we conclude that the number of different states for conditioning depends not on k but on m. This provides surprisingly good algorithm for obtaining the p.g.f. in case of large k.

2 Waiting time for the first success run in higher order Markov chain

In the present section, we study the distribution of the waiting time for the first "1"-run of length k in an m-th order Markov chain.

Let $X_1, X_2, \ldots$ be a {0,1}-valued m-th order Markov chain with initial probability

$$P((X_1, \ldots, X_m) = \boldsymbol{\alpha}) = p_i(\boldsymbol{\alpha}),$$

and transition probabilities

$$P(X_i = 1|(X_{i-m}, X_{i-m+1}, \ldots, X_{i-1}) = \boldsymbol{\alpha}) = p(\boldsymbol{\alpha}),$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \{0, 1\}^m$. We define $q(\boldsymbol{\alpha}) = 1 - p(\boldsymbol{\alpha})$. We consider only the case that $k \geq m$. Let τ_k be the waiting time for the first "1"-run of length k. For every $\boldsymbol{\alpha} \in \{0, 1\}^m$ and k, we consider the following conditional p.g.f.'s. Let $\boldsymbol{\alpha}_1 \equiv (1, \ldots, 1) \in \{0, 1\}^m$. At t_0 satisfying $m \leq t_0 \leq \tau_k$, the conditional expectation

$$E[t^{\tau_k - t_0}|(X_{t_0-m+1}, X_{t_0-m+2}, \ldots, X_{t_0}) = \boldsymbol{\alpha}]$$

does not depend on t_0 from the time-homogeneous m-th order Markovian property if $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}_1$. Hence, we may denote the conditional expectation as $\phi_k(t|\boldsymbol{\alpha})$. When $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1$, we use the notation $\phi_k(t|\boldsymbol{\alpha}_1)$ only if $t_0 = m$ or $X_{t_0-m} = 0$.

We consider first the case $k = m$ and derive $\phi_m(t|\boldsymbol{\alpha})$. On introducing the following two mappings :

$$f_0 : \{0, 1\}^m \ni \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \mapsto (\alpha_2, \ldots, \alpha_m, 0) \in \{0, 1\}^m$$

and

$$f_1 : \{0,1\}^m \ni \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \mapsto (\alpha_2, \ldots, \alpha_m, 1) \in \{0,1\}^m,$$

we may obtain $\{\phi_m(t|\boldsymbol{\alpha}); \boldsymbol{\alpha} \in \{0,1\}^m\}$ by solving the linear system of equations:

$$\begin{cases} \phi_m(t|(1,\ldots,1)) = 1 \\ \phi_m(t|\boldsymbol{\alpha}) = p(\boldsymbol{\alpha})t\phi_m(t|f_1(\boldsymbol{\alpha})) + q(\boldsymbol{\alpha})t\phi_m(t|f_0(\boldsymbol{\alpha})) \;\; \text{if } \boldsymbol{\alpha} \neq (1,\ldots,1). \end{cases}$$

Next, we study the general case of $k > m$.

Let $\boldsymbol{\alpha}_0 \equiv (1,\ldots,1,0) \in \{0,1\}^m$. We fix a t_0 satisfying $m \le t_0 \le \tau_k$ and consider the conditional p.g.f.

$$\phi_k(t|\boldsymbol{\alpha}_0) = E[t^{\tau_k - t_0}|(X_{t_0-m+1}, \ldots, X_{t_0}) = \boldsymbol{\alpha}_0].$$

Lemma 2.1 *For $k > m$,*

$$\phi_k(t|\boldsymbol{\alpha}_0) = \frac{p(\boldsymbol{\alpha}_1)t\phi_{k-1}(t|\boldsymbol{\alpha}_0)}{1 - q(\boldsymbol{\alpha}_1)t\phi_{k-1}(t|\boldsymbol{\alpha}_0)} \tag{1}$$

holds.

Proof Denote by $\tau_{k-1}(t_0)$ the time where the "1"-run of length $(k-1)$ occurs for the first time after t_0. Since $t_0 < \tau_{k-1}(t_0) < \tau_k$ holds with probability one, the following result holds true.

$$\begin{aligned} \phi_k(t|\boldsymbol{\alpha}_0) &= E[t^{\tau_k - t_0}|(X_{t_0-m+1}, \ldots, X_{t_0}) = \boldsymbol{\alpha}_0] \\ &= E[E[t^{\tau_k - \tau_{k-1}(t_0) + \tau_{k-1}(t_0) - t_0}|\tau_{k-1}(t_0), (X_{t_0-m+1}, \ldots, X_{t_0}) = \boldsymbol{\alpha}_0]| \\ &\quad\; (X_{t_0-m+1}, \ldots, X_{t_0}) = \boldsymbol{\alpha}_0] \\ &= E[t^{\tau_{k-1}(t_0) - t_0} E[t^{\tau_k - \tau_{k-1}(t_0)}|\tau_{k-1}(t_0)]|(X_{t_0-m+1}, \ldots, X_{t_0}) = \boldsymbol{\alpha}_0]. \end{aligned}$$

Since we see that

$$\begin{aligned} E[t^{\tau_k - \tau_{k-1}(t_0)}|\tau_{k-1}(t_0) = s] &= p(\boldsymbol{\alpha}_1)tE[t^{\tau_k - \tau_{k-1}(t_0)-1}|\tau_{k-1}(t_0) = s, X_{\tau_{k-1}(t_0)+1} = 1] \\ &\quad + q(\boldsymbol{\alpha}_1)tE[t^{\tau_k - \tau_{k-1}(t_0)-1}|\tau_{k-1}(t_0) = s, X_{\tau_{k-1}(t_0)+1} = 0] \\ &= p(\boldsymbol{\alpha}_1)t + q(\boldsymbol{\alpha}_1)t\phi_k(t|\boldsymbol{\alpha}_0), \end{aligned}$$

the formula does not depend on t_0. Therefore, we obtain

$$\phi_k(t|\boldsymbol{\alpha}_0) = \phi_{k-1}(t|\boldsymbol{\alpha}_0)(p(\boldsymbol{\alpha}_1)t + q(\boldsymbol{\alpha}_1)t\phi_k(t|\boldsymbol{\alpha}_0)). \tag{2}$$

This completes the proof.

By using (2) repeatedly, we have

$$\phi_k(t|\boldsymbol{\alpha}_0) = \phi_m(t|\boldsymbol{\alpha}_0) \prod_{j=1}^{k-m} (p(\boldsymbol{\alpha}_1)t + q(\boldsymbol{\alpha}_1)t\phi_{m+j}(t|\boldsymbol{\alpha}_0)). \tag{3}$$

In the case $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}_0$, following a similar procedure as in the derivation of (2) we obtain

$$\phi_k(t|\boldsymbol{\alpha}) = \phi_{k-1}(t|\boldsymbol{\alpha})(p(\boldsymbol{\alpha}_1)t + q(\boldsymbol{\alpha}_1)t\phi_k(t|\boldsymbol{\alpha}_0)). \tag{4}$$

By using (4) repeatedly, we may deduce

$$\phi_k(t|\boldsymbol{\alpha}) = \phi_m(t|\boldsymbol{\alpha}) \prod_{j=1}^{k-m} (p(\boldsymbol{\alpha}_1)t + q(\boldsymbol{\alpha}_1)t\phi_{m+j}(t|\boldsymbol{\alpha}_0)),$$

and comparing this equation with (3), we see that

$$\phi_k(t|\boldsymbol{\alpha}) = \frac{\phi_m(t|\boldsymbol{\alpha})}{\phi_m(t|\boldsymbol{\alpha}_0)}\phi_k(t|\boldsymbol{\alpha}_0), \quad \boldsymbol{\alpha} \neq \boldsymbol{\alpha}_0. \tag{5}$$

Therefore, if we know $\{\phi_m(t|\boldsymbol{\alpha}); \boldsymbol{\alpha} \in \{0, 1\}^m\}$ and $\phi_k(t|\boldsymbol{\alpha}_0)$, we can easily derive the other conditional p.g.f.'s $\phi_k(t|\boldsymbol{\alpha})$, $(\boldsymbol{\alpha} \neq \boldsymbol{\alpha}_0)$.

Thus, all we have to do is to obtain $\phi_k(t|\boldsymbol{\alpha}_0)$.

Let us define

$$F(s) \equiv \frac{p(\boldsymbol{\alpha}_1)ts}{1 - q(\boldsymbol{\alpha}_1)ts}$$

and denote by $F^{(\ell)}(s)$ the ℓ-th compositon of $F(s)$. Substituting $F(s)$ in s, we get

$$\begin{aligned} F(F(s)) &= \frac{p(\boldsymbol{\alpha}_1)t\frac{p(\boldsymbol{\alpha}_1)ts}{1-q(\boldsymbol{\alpha}_1)ts}}{1 - q(\boldsymbol{\alpha}_1)t\frac{p(\boldsymbol{\alpha}_1)ts}{1-q(\boldsymbol{\alpha}_1)ts}} \\ &= \frac{(p(\boldsymbol{\alpha}_1)t)^2 s}{1 - q(\boldsymbol{\alpha}_1)ts - q(\boldsymbol{\alpha}_1)tp(\boldsymbol{\alpha}_1)ts}. \end{aligned}$$

Generally, substituting repeatedly, we get

$$F^{(\ell)}(s) = \frac{(p(\boldsymbol{\alpha}_1)t)^\ell s}{1 - \sum_{j=1}^{\ell}(p(\boldsymbol{\alpha}_1)t)^{j-1}(q(\boldsymbol{\alpha}_1)t)s}.$$

Summarizing, we have the next lemma.

Lemma 2.2 *The ℓ-th compositon of $F(s)$ can be expressed as*

$$F^{(\ell)}(s) = \frac{(p(\boldsymbol{\alpha}_1)t)^\ell(1 - p(\boldsymbol{\alpha}_1)t)s}{1 - p(\boldsymbol{\alpha}_1)t - (q(\boldsymbol{\alpha}_1)t)(1 - (p(\boldsymbol{\alpha}_1)t)^\ell)s}.$$

Using Lemma 2.2, we can obtain the p.g.f. of the waiting time for the first occurrence of consecutive k successes in the m-th order Markov chain.

Theorem 2.1 *If $k > m$, the p.g.f. $\phi_k(t)$ of τ_k is given by*

$$\phi_k(t) = \frac{(p(\boldsymbol{\alpha}_1)t)^{k-m}(1 - p(\boldsymbol{\alpha}_1)t)\sum_{\boldsymbol{\alpha}\in\{0,1\}^m} p_i(\boldsymbol{\alpha})t^m\phi_m(t|\boldsymbol{\alpha})}{1 - p(\boldsymbol{\alpha}_1)t - (q(\boldsymbol{\alpha}_1)t)(1 - (p(\boldsymbol{\alpha}_1)t)^{k-m})\phi_m(t|\boldsymbol{\alpha}_0)}.$$

Proof Since

$$\phi_k(t) = \sum_{\boldsymbol{\alpha}\in\{0,1\}^m} p_i(\boldsymbol{\alpha})t^m\phi_k(t|\boldsymbol{\alpha}),$$

it suffices to obtain $\phi_k(t|\boldsymbol{\alpha})$.

If $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0$, by formula (1),

$$\phi_k(t|\boldsymbol{\alpha}_0) = F^{(k-m)}(\phi_m(t|\boldsymbol{\alpha}_0))$$

holds true, and hence we obtain, from Lemma 2.2,

$$\phi_k(t|\boldsymbol{\alpha}_0) = \frac{(p(\boldsymbol{\alpha}_1)t)^{k-m}(1-(p(\boldsymbol{\alpha}_1)t))\phi_m(t|\boldsymbol{\alpha}_0)}{1-p(\boldsymbol{\alpha}_1)t-(q(\boldsymbol{\alpha}_1)t)(1-(p(\boldsymbol{\alpha}_1)t)^{k-m})\phi_m(t|\boldsymbol{\alpha}_0)}. \tag{6}$$

If $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}_0$, we may obtain the result from (5) by substituting $\phi_m(t|\boldsymbol{\alpha})$ for $\phi_m(t|\boldsymbol{\alpha}_0)$ in the numerator of (6). This completes the proof.

Remark 1 As we have seen above, the p.g.f. $\phi_k(t)$ of τ_k is a function of $\phi_m(t|\boldsymbol{\alpha}_0)$ and $\phi_m(t|\boldsymbol{\alpha})$. This provides a surprisingly good algorithm for obtaining $\phi_k(t)$, especially when m is small and k is large.

As a special case, let $X_1, X_2, \ldots$ be independent identically distributed {0,1}-valued random variables with $P(X_i = 1) = p = 1-q$. For $k = 1, 2, 3, \ldots$, we denote by τ_k the waiting time for the first "1"-run of length k and by the p.g.f. of τ_k ($\tau_1 < \tau_2 < \cdots$). By definition, τ_1 follows the geometric distribution of order 1, $G_1(p)$. Then, similarly as Lemma 2.1, we obtain

$$\phi_k(t) = \frac{pt\phi_{k-1}(t)}{1-qt\phi_{k-1}(t)}.$$

Noting that

$$\phi_1(t) = \frac{pt}{1-qt},$$

we get

$$\phi_k(t) = \phi_1(t\phi_{k-1}(t)).$$

This leads us to the next theorem.

Theorem 2.2 *Let k be an integer greater than one and let N and $X_1, X_2, \ldots$ be independent random variables. Suppose that N follows the geometric distribution of order* 1 *and that for $i = 1, 2, \ldots$, X_i follows the geometric distribution of order $(k-1)$. Then, $X_1 + X_2 + \cdots + X_N + N$ follows the geometric distribution of order k.*

References

Aki, S., Balakrishnan, N., Mohanty, S. G. (1996). Sooner and later waiting time problems for success and failure runs in higher order Markov dependent trials. *Annals of the Institute of Statistical Mathematics*, 48, 773–787.

Aki, S., Hirano, K. (1994). Distributions of numbers of failures and suncesses until the first consecutive k successes. *Annals of the Institute of Statistical Mathematics*, 46, 193–202.

Aki, S., Hirano, K. (1995). Joint distributions of numbers of success-runs and failures until the first consecutive k successes. *Annals of the Institute of Statistical Mathematics*, 47, 225–235.

Balakrishnan, N., Koutras, M.V. (2002). *Runs and Scans with Applications*. New York: Wiley.

Feller, W. (1968). *An Introduction to Probability Theory and Its Applications* (Vol. 1, 3rd edition). New York: Wiley.