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Joint distributions of numbers of runs of specified lengths in a sequence of Markov dependent multistate trials

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Abstract Let $Z_0, Z_1, \ldots, Z_n$ be a sequence of Markov dependent trials with state space $\Omega = \{F_1, \ldots, F_\lambda, S_1, \ldots, S_\nu\}$, where we regard $F_1, \ldots, F_\lambda$ as failures and $S_1, \ldots, S_\nu$ as successes. In this paper, we study the joint distribution of the numbers of S_i-runs of lengths k_{ij} $(i = 1, 2, \ldots, \nu,\ j = 1, 2, \ldots, r_i)$ based on four different enumeration schemes. We present formulae for the evaluation of the probability generating functions and the higher order moments of this distribution. In addition, when the underlying sequence is i.i.d. trials, the conditional distribution of the same run statistics, given the numbers of success and failure is investigated. We give further insights into the multivariate run-related problems arising from a sequence of the multistate trials. Besides, our results have potential applications to problems of various research areas and will come to prominence in the future.

Keywords Markov chain · Multistate trials · Runs · Moments · Enumeration schemes · Recursive scheme · Conditional distribution · Probability function · Probability generating function · Double generating function

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1 Introduction

The concept of runs has been used effectively in a wide range of areas such as reliability theory, start-up demonstration tests and statistical quality control (see Inoue, 2004; Chao et al., 1995; Balakrishnan et al. 1997; Shmueli and Cohen, 2000 and references therein). In a sequence of binary trials (success or failure), the distribution theory of success runs has been developed by many authors under various enumeration schemes. There are different ways of counting the number of success runs of length k in the literature (see Fu and Koutras, 1994; Balakrishnan and Koutras, 2002). It depends on the practical problem which way of counting should be adopted. The four best-known types of the ways of counting the number of success runs of length k are as follows:

(i) Type I enumeration scheme: the way of counting the number of non-overlapping and recurrent success runs of length k, in the sense of Feller's (1968) counting,
(ii) Type II enumeration scheme: the way of counting the number of success runs of length at least k, in the sense of Goldstein's (1990) counting (see Gibbons 1971),
(iii) Type III enumeration scheme: the way of counting the number of overlapping success runs of length k, in the sense of Ling's (1988) counting,
(iv) Type IV enumeration scheme: the way of counting the number of success runs of size exactly k, in the sense of Mood's (1940) counting.

It is natural to consider multivariate versions of run-related distribution. Let $Z_0, Z_1, \ldots, Z_n$ be a time homogeneous Markov chain defined on the state space $\Omega = \{F_1, \ldots, F_\lambda, S_1, \ldots, S_\nu\}$, where we regard $F_1, \ldots, F_\lambda$ as failures and $S_1, \ldots, S_\nu$ as successes. For $\boldsymbol{k}_i = (k_{i1}, k_{i2}, \ldots, k_{ir_i})$ and $\boldsymbol{\alpha}_i = (\alpha_{i1}, \ldots, \alpha_{ir_i})$ $i = 1, 2, \ldots, \nu$, let $\boldsymbol{N}(n, \boldsymbol{k}_i; \boldsymbol{\alpha}_i)$ be the r_i-dimensional random variables $(N(n, k_{i1}; \alpha_{i1}), \ldots, N(n, k_{ir_i}; \alpha_{ir_i}))$, where $N(n, k_{ij}; \alpha_{ij})$ represents the number of S_i-runs of length k_{ij} $(i = 1, 2, \ldots, \nu,\ j = 1, 2, \ldots, r_i)$ in $Z_0, Z_1, \ldots, Z_n$ by engaging Type α_{ij} (= I, II, III, IV) enumeration scheme. In the present paper we develop formulae for the derivation of the probability generating function (p.g.f.) and the higher order moments of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = (\boldsymbol{N}(n, \boldsymbol{k}_1; \boldsymbol{\alpha}_1), \ldots, \boldsymbol{N}(n, \boldsymbol{k}_\nu; \boldsymbol{\alpha}_\nu))$, where $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_\nu)$ and $\boldsymbol{k} = (\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_\nu)$.

We provide the perspectives on the multivariate run-related problems arising from the multistate trials. It will be of great value to consider the multivariate run-related distribution, since this distribution treated here is important not only for their theoretical interest but also for their applications to various areas.

In the case of binary trials, the distribution theory of runs has been developed very actively (see Aki and Hirano, 1988; Koutras and Alexandrou 1995; Antzoulakos and Chadjiconstantinidis, 2001; Godbole et al., 1997; Han and Aki, 1998). On the other hand, in the case of multistate trials, although the multivariate versions of run-related distributions are closely related to many important applications (see Balakrishnan and Koutras, 2002) and there is need to study such multivariate distributions, the development of the relevant distribution theory is very slow and insufficient. Shaughnessy (1981) and Bradley (1968) also stated this point (see Fu, 1996). In fact, many exact multivariate run-related distributions remain unknown. Moreover, the higher order moments of these distributions have never been examined. Therefore, it has become evident that a systematic study of multivariate distributions is required in order to tackle multivariate run-related problems. This is the main motivation for establishing formulae for the evaluation of the p.g.f. and the higher order moments of the multivariate run-related distribution.

In Sect. 2, when the underlying sequence is a sequence of Markov dependent multistate trials, we discuss the joint distribution of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = (\boldsymbol{N}(n, \boldsymbol{k}_1; \boldsymbol{\alpha}_1), \ldots, \boldsymbol{N}(n, \boldsymbol{k}_\nu; \boldsymbol{\alpha}_\nu))$. We present recursive schemes for the evaluation of the p.g.f. and the mixed $(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu)$-th moment about zero of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$, where $\boldsymbol{\beta}_i = (\beta_{i1}, \ldots, \beta_{ir_i}), i = 1, 2, \ldots, \nu$. The expression for the double generating function of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ is given. Sect. 3 studies the joint distribution of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ in the special case of i.i.d. trials. In Sect. 4, we investigate the conditional distribution of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$, given the numbers of success and failure in the i.i.d. trials. Finally, in Sect. 5, several interesting practical applications are discussed.

2 Markov dependent trials

Let $Z_0, Z_1, \ldots, Z_n$ be a time homogeneous Markov chain defined on the state space $\Omega = \{F_1, \ldots, F_\lambda, S_1, \ldots, S_\nu\}$. Assume that

$$p_{\omega_i \omega_j} = P(Z_t = \omega_j | Z_{t-1} = \omega_i) \quad \text{for } t \geq 1,\ \omega_i, \omega_j \in \Omega,\ i, j = 1, 2, \ldots, \lambda + \nu$$

and

$$\pi_{\omega_i} = P(Z_0 = \omega_i) \quad \text{for } \omega_i \in \Omega,\ i = 1, 2, \ldots, \lambda + \nu.$$

For $\alpha_{ij} = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$, $i = 1, 2, \ldots, \nu$, $j = 1, 2, \ldots, r_i$, we define

$$\mu(v; \alpha_{ij}) = \begin{cases} \left[\dfrac{v}{k_{ij}}\right] & \alpha_{ij} = \mathrm{I}, \\ I(v \geq k_{ij}) & \alpha_{ij} = \mathrm{II}, \\ \max\left\{0, v - (k_{ij} - 1)\right\} & \alpha_{ij} = \mathrm{III}, \\ I(v = k_{ij}) & \alpha_{ij} = \mathrm{IV}, \end{cases}$$

where

$$I(u) = \begin{cases} 1, & u \text{ is true}, \\ 0, & \text{otherwise}. \end{cases}$$

2.1 The probability generating functions

Let $P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x})$ and $P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} | Z_0 = \omega_i)$ be the probability function of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = (\boldsymbol{N}(n, \boldsymbol{k}_1; \boldsymbol{\alpha}_1), \ldots, \boldsymbol{N}(n, \boldsymbol{k}_\nu; \boldsymbol{\alpha}_\nu))$ and the conditional probability function of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ given $Z_0 = \omega_i$ ($\omega_i \in \Omega, i = 1, 2, \ldots, \lambda + \nu$), where $\boldsymbol{x} = (x_{11}, \ldots, x_{1r_1}, \ldots, x_{i1}, \ldots, x_{ir_i}, \ldots, x_{\nu 1}, \ldots, x_{\nu r_\nu})$. Then, the p.g.f. of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ and the conditional p.g.f. of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ given $Z_0 = \omega_i$ ($\omega_i \in \Omega$,

$i = 1, 2, \dots, \lambda + \nu$) will be denoted by $\phi_n(\boldsymbol{t}; \boldsymbol{\alpha})$ and $\phi_n^{(\omega_i)}(\boldsymbol{t}; \boldsymbol{\alpha})$, respectively, that is,

$$\phi_n(\boldsymbol{t}; \boldsymbol{\alpha}) = E\left[\boldsymbol{t}_1^{\boldsymbol{N}(n,\boldsymbol{k}_1;\boldsymbol{\alpha}_1)} \cdots \boldsymbol{t}_\nu^{\boldsymbol{N}(n,\boldsymbol{k}_\nu;\boldsymbol{\alpha}_\nu)}\right] = \sum_{\boldsymbol{x}} P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x}) t_{11}^{x_{11}} \cdots t_{\nu r_\nu}^{x_{\nu r_\nu}},$$

$$\begin{aligned}\phi_n^{(\omega_i)}(\boldsymbol{t}; \boldsymbol{\alpha}) &= E\left[\boldsymbol{t}_1^{\boldsymbol{N}(n,\boldsymbol{k}_1;\boldsymbol{\alpha}_1)} \cdots \boldsymbol{t}_\nu^{\boldsymbol{N}(n,\boldsymbol{k}_\nu;\boldsymbol{\alpha}_\nu)} | Z_0 = \omega_i\right] \\ &= \sum_{\boldsymbol{x}} P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} | Z_0 = \omega_i) t_{11}^{x_{11}} \cdots t_{\nu r_\nu}^{x_{\nu r_\nu}},\end{aligned}$$

where $\boldsymbol{t} = (\boldsymbol{t}_1, \boldsymbol{t}_2, \dots, \boldsymbol{t}_\nu), \boldsymbol{t}_i = (t_{i1}, t_{i2}, \dots, t_{ir_i})$ and $\boldsymbol{t}_i^{\boldsymbol{N}(n,\boldsymbol{k}_i;\boldsymbol{\alpha}_i)} = t_{i1}^{N(n,k_{i1};\alpha_{i1})} \cdots t_{ir_i}^{N(n,k_{ir_i};\alpha_{ir_i})}, \quad i = 1, 2, \dots, \nu.$

Theorem 2.1 *The p.g.f.* $\phi_n(\boldsymbol{t}; \boldsymbol{\alpha})$ *and the conditional p.g.f.'s* $\phi_n^{(F_i)}(\boldsymbol{t}; \boldsymbol{\alpha})$, $i = 1, 2, \dots, \lambda$, $\phi_n^{(S_i)}(\boldsymbol{t}; \boldsymbol{\alpha})$, $i = 1, 2, \dots, \nu$, *satisfy the following recursive relation*:

$$\phi_n(\boldsymbol{t}; \boldsymbol{\alpha}) = \sum_{m=1}^{\lambda} \pi_{F_m} \phi_n^{(F_m)}(\boldsymbol{t}; \boldsymbol{\alpha}) + \sum_{m=1}^{\nu} \pi_{S_m} \phi_n^{(S_m)}(\boldsymbol{t}; \boldsymbol{\alpha}), n \geq 0, \tag{1}$$

$$\phi_n^{(F_i)}(\boldsymbol{t}; \boldsymbol{\alpha}) = \sum_{m=1}^{\lambda} p_{F_i F_m} \phi_{n-1}^{(F_m)}(\boldsymbol{t}; \boldsymbol{\alpha}) + \sum_{m=1}^{\nu} p_{F_i S_m} \phi_{n-1}^{(S_m)}(\boldsymbol{t}; \boldsymbol{\alpha}), \tag{2}$$

$$n \geq 1, \ i = 1, 2, \dots, \lambda,$$

$$\phi_0^{(F_i)}(\boldsymbol{t}; \boldsymbol{\alpha}) = 1, \quad i = 1, 2, \dots, \lambda, \tag{3}$$

$$\begin{aligned}\phi_n^{(S_i)}(\boldsymbol{t}; \boldsymbol{\alpha}) = &\sum_{m=1}^{\lambda} \sum_{v=0}^{n-1} p_{S_i S_i}^{v} \, p_{S_i F_m} \, \boldsymbol{t}_i^{\boldsymbol{\mu}(v+1;\boldsymbol{\alpha}_i)} \, \phi_{n-v-1}^{(F_m)}(\boldsymbol{t}; \boldsymbol{\alpha}) \\ &+ \sum_{\substack{m=1,2,\dots,\nu \\ m \neq i}} \sum_{v=0}^{n-1} p_{S_i S_i}^{v} \, p_{S_i S_m} \, \boldsymbol{t}_i^{\boldsymbol{\mu}(v+1;\boldsymbol{\alpha}_i)} \, \phi_{n-v-1}^{(S_m)}(\boldsymbol{t}; \boldsymbol{\alpha}) \\ &+ p_{S_i S_i}^{n} \, \boldsymbol{t}_i^{\boldsymbol{\mu}(n+1;\boldsymbol{\alpha}_i)}, \quad n \geq 1, \ i = 1, 2, \dots, \nu,\end{aligned} \tag{4}$$

$$\phi_0^{(S_i)}(\boldsymbol{t}; \boldsymbol{\alpha}) = \boldsymbol{t}_i^{\boldsymbol{\mu}(1;\boldsymbol{\alpha}_i)}, \quad i = 1, 2, \dots, \nu, \tag{5}$$

where $\boldsymbol{t}_i^{\boldsymbol{\mu}(v;\boldsymbol{\alpha}_i)} = t_{i1}^{\mu(v;\alpha_{i1})} \cdots t_{ir_i}^{\mu(v;\alpha_{ir_i})}$.

Proof It is easy to check Eqs. (3) and (5). The proofs of (1) and (2) are easily completed by observing that

$$
\begin{aligned}
&E\left[\boldsymbol{t}_1^{\boldsymbol{N}(n,\boldsymbol{k}_1;\boldsymbol{\alpha}_1)}\cdots\boldsymbol{t}_\nu^{\boldsymbol{N}(n,\boldsymbol{k}_\nu;\boldsymbol{\alpha}_\nu)}\right]\\
&\quad=\sum_{m=1}^{\lambda+\nu}P(Z_0=\omega_m)\,E\left[\boldsymbol{t}_1^{\boldsymbol{N}(n,\boldsymbol{k}_1;\boldsymbol{\alpha}_1)}\cdots\boldsymbol{t}_\nu^{\boldsymbol{N}(n,\boldsymbol{k}_\nu;\boldsymbol{\alpha}_\nu)}\mid Z_0=\omega_m\right],\\
&E\left[\boldsymbol{t}_1^{\boldsymbol{N}(n,\boldsymbol{k}_1;\boldsymbol{\alpha}_1)}\cdots\boldsymbol{t}_\nu^{\boldsymbol{N}(n,\boldsymbol{k}_\nu;\boldsymbol{\alpha}_\nu)}\mid Z_0=F_i\right]\\
&\quad=\sum_{m=1}^{\lambda}P(Z_1=F_m\mid Z_0=F_i)\,E\left[\boldsymbol{t}_1^{\boldsymbol{N}(n,\boldsymbol{k}_1;\boldsymbol{\alpha}_1)}\cdots\boldsymbol{t}_\nu^{\boldsymbol{N}(n,\boldsymbol{k}_\nu;\boldsymbol{\alpha}_\nu)}\mid Z_1=F_m\right]\\
&\qquad+\sum_{m=1}^{\nu}P(Z_1=S_m\mid Z_0=F_i)\,E\left[\boldsymbol{t}_1^{\boldsymbol{N}(n,\boldsymbol{k}_1;\boldsymbol{\alpha}_1)}\cdots\boldsymbol{t}_\nu^{\boldsymbol{N}(n,\boldsymbol{k}_\nu;\boldsymbol{\alpha}_\nu)}\mid Z_1=S_m\right],\\
&\qquad i=1,2,\ldots,\lambda.
\end{aligned}
$$

Suppose that we have $Z_0=S_i$ $(i=1,2,\ldots,\nu)$. For $v=1,2,\ldots,n$, let $C_v^{S_iF_m}$ be the event that S_i occurs at trials $0,1,\ldots,v-1$ and the first F_m $(m=1,2,\ldots,\lambda)$ occurs at the v-th trial, for $v=1,2,\ldots,n$, let $C_v^{S_iS_m}$ be the event that S_i occurs at trials $0,1,\ldots,v-1$ and the first S_m $(m\neq i)$ occurs at the v-th trial, and let $C_{n+1}^{S_i}$ be the event that S_i occurs at trials $0,1,\ldots,n$ and neither the first F_m $(m=1,2,\ldots,\lambda)$ nor the first S_m $(m\neq i)$ occur in $Z_1,Z_2,\ldots,Z_n$, that is,

$$
\begin{aligned}
&C_v^{S_iF_m}=\{Z_0=Z_1=\cdots=Z_{v-1}=S_i,\ Z_v=F_m\},\\
&\quad v=1,\ldots,n,\ \ i=1,2,\ldots,\nu,\quad m=1,2,\ldots,\lambda,\\
&C_v^{S_iS_m}=\{Z_0=Z_1=\cdots=Z_{v-1}=S_i,\ Z_v=S_m\},\\
&\quad v=1,\ldots,n,\ \ i=1,2,\ldots,\nu,\quad m\neq i,\\
&C_{n+1}^{S_i}=\{Z_0=Z_1=\cdots=Z_{n-1}=Z_n=S_i\},\quad i=1,2,\ldots,\nu.
\end{aligned}
$$

Then, we immediately obtain

$$
\begin{aligned}
&E\left[\boldsymbol{t}_1^{\boldsymbol{N}(n,\boldsymbol{k}_1;\boldsymbol{\alpha}_1)}\cdots\boldsymbol{t}_\nu^{\boldsymbol{N}(n,\boldsymbol{k}_\nu;\boldsymbol{\alpha}_\nu)}\mid Z_0=S_i\right]\\
&\quad=\sum_{m=1}^{\lambda}\sum_{v=1}^{n}P(C_v^{S_iF_m}\mid Z_0=S_i)\,E\left[\boldsymbol{t}_1^{\boldsymbol{N}(n,\boldsymbol{k}_1;\boldsymbol{\alpha}_1)}\cdots\boldsymbol{t}_\nu^{\boldsymbol{N}(n,\boldsymbol{k}_\nu;\boldsymbol{\alpha}_\nu)}\mid C_v^{S_iF_m}\right]\\
&\qquad+\sum_{\substack{m=1,2,\ldots,\nu\\ m\neq i}}\ \sum_{v=1}^{n}P(C_v^{S_iS_m}\mid Z_0=S_i)\,E\left[\boldsymbol{t}_1^{\boldsymbol{N}(n,\boldsymbol{k}_1;\boldsymbol{\alpha}_1)}\cdots\boldsymbol{t}_\nu^{\boldsymbol{N}(n,\boldsymbol{k}_\nu;\boldsymbol{\alpha}_\nu)}\mid C_v^{S_iS_m}\right]\\
&\qquad+P(C_{n+1}^{S_i}\mid Z_0=S_i)\,E\left[\boldsymbol{t}_1^{\boldsymbol{N}(n,\boldsymbol{k}_1;\boldsymbol{\alpha}_1)}\cdots\boldsymbol{t}_\nu^{\boldsymbol{N}(n,\boldsymbol{k}_\nu;\boldsymbol{\alpha}_\nu)}\mid C_{n+1}^{S_i}\right],\ i=1,2,\ldots,\nu,
\end{aligned}
$$

which yields the Eq. (4) due to the homogeneity of Markov chain. The proof is completed. □

We will define the double generating functions $\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha})$ and $\Phi^{(\omega_i)}(\boldsymbol{t}, z; \boldsymbol{\alpha})$ ($\omega_i \in \Omega$, $i = 1, 2, \ldots, \lambda + \nu$) as

$$\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha}) = \sum_{n=0}^{\infty} \phi_n(\boldsymbol{t}; \boldsymbol{\alpha}) z^n,$$
$$\Phi^{(\omega_i)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) = \sum_{n=0}^{\infty} \phi_n^{(\omega_i)}(\boldsymbol{t}; \boldsymbol{\alpha}) z^n.$$

Using Theorem 2.1, we can obtain the following theorem.

Theorem 2.2 *The double generating functions* $\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha})$ *and* $\Phi^{(F_i)}(\boldsymbol{t}, z; \boldsymbol{\alpha})$, $i = 1, 2, \ldots, \lambda$, $\Phi^{(S_i)}(\boldsymbol{t}, z; \boldsymbol{\alpha})$, $i = 1, 2, \ldots, \nu$, *satisfy the following system of equations:*

$$\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha}) = \sum_{m=1}^{\lambda} \pi_{F_m} \Phi^{(F_m)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) + \sum_{m=1}^{\nu} \pi_{S_m} \Phi^{(S_m)}(\boldsymbol{t}, z; \boldsymbol{\alpha}), \tag{6}$$

$$\Phi^{(F_i)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) = 1 + \sum_{m=1}^{\lambda} p_{F_i F_m} z \Phi^{(F_m)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) + \sum_{m=1}^{\nu} p_{F_i S_m} z \Phi^{(S_m)}(\boldsymbol{t}, z; \boldsymbol{\alpha}), \tag{7}$$
$$i = 1, 2, \ldots, \lambda,$$

$$\begin{aligned}\Phi^{(S_i)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) = {} & P(\boldsymbol{t}_i, p_{S_i S_i} z; \boldsymbol{\alpha}_i) \\ & \times \left(1 + \sum_{m=1}^{\lambda} p_{S_i F_m} z \Phi^{(F_m)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \right. \\ & \left. + \sum_{m \neq i}^{\nu} p_{S_i S_m} z \Phi^{(S_m)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \right), \quad i = 1, 2, \ldots, \nu,\end{aligned} \tag{8}$$

where

$$P(\boldsymbol{t}_i, p_{S_i S_i} z; \boldsymbol{\alpha}_i) = \sum_{\upsilon=0}^{\infty} (p_{S_i S_i} z)^{\upsilon} \, \boldsymbol{t}_i^{\boldsymbol{\mu}(\upsilon+1; \boldsymbol{\alpha}_i)}, \quad i = 1, 2, \ldots, \nu. \tag{9}$$

The formula (6) yields

$$\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha}) = (\pi_{F_1}, \ldots, \pi_{F_\lambda}, \pi_{S_1}, \ldots, \pi_{S_\nu}) \begin{bmatrix} \Phi^{(F_1)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \\ \vdots \\ \Phi^{(F_\lambda)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \\ \Phi^{(S_1)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \\ \vdots \\ \Phi^{(S_\nu)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \end{bmatrix}. \tag{10}$$

Equations (7) and (8) can be expressed as

$$\begin{bmatrix} \Phi^{(F_1)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \\ \vdots \\ \Phi^{(F_\lambda)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \\ \Phi^{(S_1)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \\ \vdots \\ \Phi^{(S_\nu)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \end{bmatrix} = z\boldsymbol{A} \begin{bmatrix} \Phi^{(F_1)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \\ \vdots \\ \Phi^{(F_\lambda)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \\ \Phi^{(S_1)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \\ \vdots \\ \Phi^{(S_\nu)}(\boldsymbol{t}, z; \boldsymbol{\alpha}) \end{bmatrix} + \begin{bmatrix} 1 \\ \vdots \\ 1 \\ P(\boldsymbol{t}_1, p_{S_1S_1}z; \boldsymbol{\alpha}_1) \\ \vdots \\ P(\boldsymbol{t}_\nu, p_{S_\nu S_\nu}z; \boldsymbol{\alpha}_\nu) \end{bmatrix}, \tag{11}$$

where

$$\boldsymbol{A} = \left[\begin{array}{ccc|cccc} p_{F_1F_1} & \cdots & p_{F_1F_\lambda} & p_{F_1S_1} & p_{F_1S_2} & \cdots & p_{F_1S_\nu} \\ p_{F_2F_1} & \cdots & p_{F_2F_\lambda} & p_{F_2S_1} & p_{F_2S_2} & \cdots & p_{F_2S_\nu} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ p_{F_\lambda F_1} & \cdots & p_{F_\lambda F_\lambda} & p_{F_\lambda S_1} & p_{F_\lambda S_2} & \cdots & p_{F_\lambda S_\nu} \\ \hline a_{S_1F_1} & \cdots & a_{S_1F_\lambda} & 0 & b_{S_1S_2} & \cdots & b_{S_1S_\nu} \\ a_{S_2F_1} & \cdots & a_{S_2F_\lambda} & b_{S_2S_1} & 0 & \cdots & b_{S_1S_\nu} \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{S_\nu F_1} & \cdots & a_{S_\nu F_\lambda} & b_{S_\nu S_1} & b_{S_\nu S_2} & \cdots & 0 \end{array}\right]$$

and $a_{S_iF_j}$ $(1 \le i \le \nu, 1 \le j \le \lambda)$, $b_{S_iS_j}$ $(1 \le i, j \le \nu)$ are given by

$$a_{S_iF_j} = p_{S_iF_j} P(\boldsymbol{t}_i, p_{S_iS_i}z; \boldsymbol{\alpha}_i),$$
$$b_{S_iS_j} = \begin{cases} 0, & i = j, \\ p_{S_iS_j} P(\boldsymbol{t}_i, p_{S_iS_i}z; \boldsymbol{\alpha}_i), & i \ne j. \end{cases}$$

From Eqs. (10) and (11), we can proceed to derive a compact formula for the double generating function $\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha})$. The next theorem provides the details.

Theorem 2.3 *The double generating function* $\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha})$ *is given by*

$$\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha}) = (\pi_{F_1}, \ldots, \pi_{F_\lambda}, \pi_{S_1}, \ldots, \pi_{S_\nu})\ [\boldsymbol{I} - z\boldsymbol{A}]^{-1} \begin{bmatrix} \mathbf{1}_{\lambda\times 1} \\ P(\boldsymbol{t}_1, p_{S_1S_1}z; \boldsymbol{\alpha}_1) \\ \vdots \\ P(\boldsymbol{t}_\nu, p_{S_\nu S_\nu}z; \boldsymbol{\alpha}_\nu) \end{bmatrix},$$

where $\boldsymbol{I}$ *is the* $(\lambda + \nu) \times (\lambda + \nu)$ *identity matrix and* $P(\boldsymbol{t}_i, p_{S_iS_i}z; \boldsymbol{\alpha}_i)$, $i = 1, 2, \ldots, \nu$, *is given by in (9).*

Example 2.1 *Joint distribution of* $(N(n, k_{11}; \alpha_{11}), \ldots, N(n, k_{\nu 1}; \alpha_{\nu 1}))$. We treat the special case where $r_1 = \cdots = r_\nu = 1$. The double generating function is given by

$$\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha}) = (\pi_{F_1}, \ldots, \pi_{F_\lambda}, \pi_{S_1}, \ldots, \pi_{S_\nu})\ [\boldsymbol{I} - z\boldsymbol{A}]^{-1} \begin{bmatrix} \mathbf{1}_{\lambda\times 1} \\ P(t_{11}, p_{S_1S_1}z; \alpha_{11}) \\ \vdots \\ P(t_{\nu 1}, p_{S_\nu S_\nu}z; \alpha_{\nu 1}) \end{bmatrix},$$

where

$$P(t_{i1}, p_{S_iS_i}z; \alpha_{i1}) = \begin{cases} \dfrac{1-(p_{S_iS_i}z)^{k_{i1}}t_{i1}-(p_{S_iS_i}z)^{k_{i1}-1}(1-t_{i1})}{\left(1-p_{S_iS_i}z\right)(1-(p_{S_iS_i}z)^{k_{i1}}t_{i1})}, & \alpha_{i1}=\mathrm{I}, \\[2ex] \dfrac{1-(p_{S_iS_i}z)^{k_{i1}-1}(1-t_{i1})}{1-p_{S_iS_i}z}, & \alpha_{i1}=\mathrm{II}, \\[2ex] \dfrac{1-(p_{S_iS_i}z)^{k_{i1}-1}(1-t_{i1})-p_{S_iS_i}zt_{i1}}{(1-p_{S_iS_i}z)(1-p_{S_iS_i}zt_{i1})}, & \alpha_{i1}=\mathrm{III}, \\[2ex] \dfrac{1-(p_{S_iS_i}z)^{k_{i1}-1}(1-t_{i1})(1-p_{S_iS_i}z)}{1-p_{S_iS_i}z}, & \alpha_{i1}=\mathrm{IV}, \end{cases}$$

for $i=1,2,\ldots,\nu$.

In the case of $\lambda=1$, $\nu=2$, $r_1=r_2=1$, Balakrishnan and Koutras (2002) called the three distributions of the bivariate random variables $(N(n,k_{11};\mathrm{I}), N(n,k_{21};\mathrm{I}))$, $(N(n,k_{11};\mathrm{II}), N(n,k_{21};\mathrm{II}))$ and $(N(n,k_{11};\mathrm{III}), N(n,k_{21};\mathrm{III}))$ *Type* I, II *and* III *Markov-trinomial distributions of order* (k_{11},k_{21}), respectively. The results presented in this section are generalization of Markov-trinomial distributions.

2.2 Two enumeration schemes

(i)*The number of runs of type S_i*

We denote $R_n^{(S_i)}$ by the number of runs of type S_i $(i=1,2,\ldots,\nu)$. For example, the following sequence of length 23:

$$\underline{S_1}\ \underline{S_2S_2}\ \underline{S_1S_1S_1}\ F_1\ \underline{S_2S_2S_2}\ \underline{S_1S_1}\ \underline{S_2S_2S_2S_2}\ \underline{S_1S_1S_1S_1}\ F_1\ \underline{S_2}\ \underline{S_1}$$

contains 5 runs of type S_1, 4 runs of type S_2 ($R_{23}^{(S_1)}=5$ and $R_{23}^{(S_2)}=4$). By setting $r_1=\cdots=r_\nu=1$ and $\mu(v;\alpha_{i1})=I(v\geq 1)$ for $i=1,2,\ldots,\nu$ in Theorems 2.1–2.3, we can easily deal with this case.

(ii)*ℓ-Overlapping enumeration scheme ($\ell \geq 0$)*

Recently, Aki and Hirano (2000) introduced a generalized enumeration scheme which is called ℓ-overlapping counting (see Inoue and Aki, 2003 ; Antzoulakos, 2003). In Theorems 2.1–2.3, by setting

$$\mu(v;\alpha_{ij}) = \max\left\{0,\ \left[\frac{v-\ell_{ij}}{k_{ij}-\ell_{ij}}\right]\right\},$$

the results presented in Theorems 2.1–2.3 can be extended to cover this case easily.

2.3 Evaluation of moments

We denote by

$$\eta_n(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_\nu;\boldsymbol{\alpha}) = E\left[\prod_{i=1}^{\nu}(\boldsymbol{N}(n,\boldsymbol{k}_i;\boldsymbol{\alpha}_i))^{\boldsymbol{\beta}_i}\right] = E\left[\prod_{i=1}^{\nu}\prod_{j=1}^{r_i}\left(N(n,k_{ij};\alpha_{ij})\right)^{\beta_{ij}}\right]$$

the mixed $(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu)$-th moment about zero, where $\boldsymbol{\beta}_i = (\beta_{i1}, \ldots, \beta_{ir_i})$, $i = 1, 2, \ldots, \nu$. Replacing t_{ij} by $\mathrm{e}^{t_{ij}}$ $(i = 1, 2, \ldots, \nu,\ j = 1, 2, \ldots, r_i)$ in $\phi_n(\boldsymbol{t}; \boldsymbol{\alpha})$, $\phi_n^{(F_i)}(\boldsymbol{t}; \boldsymbol{\alpha})$ and $\phi_n^{(S_i)}(\boldsymbol{t}; \boldsymbol{\alpha})$, the moment generating functions of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = (\boldsymbol{N}(n, \boldsymbol{k}_1; \boldsymbol{\alpha}_1), \ldots, \boldsymbol{N}(n, \boldsymbol{k}_\nu; \boldsymbol{\alpha}_\nu))$ are obtained. Here, we write the moment generating functions as $\phi_n(\mathrm{e}^{\boldsymbol{t}}; \boldsymbol{\alpha})$, $\phi_n^{(F_i)}(\mathrm{e}^{\boldsymbol{t}}; \boldsymbol{\alpha})$ and $\phi_n^{(S_i)}(\mathrm{e}^{\boldsymbol{t}}; \boldsymbol{\alpha})$, respectively. We can get the mixed $(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu)$-th moments about zero as

$$\eta_n(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}) = \frac{\partial^{\beta_{11}+\cdots+\beta_{\nu r_\nu}}}{\partial t_{11}^{\beta_{11}} \cdots \partial t_{\nu r_\nu}^{\beta_{\nu r_\nu}}} \phi_n(e^{\boldsymbol{t}}; \boldsymbol{\alpha}) \Bigg|_{\boldsymbol{t}=\boldsymbol{0}},$$

$$\eta_n^{(F_i)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}) = \frac{\partial^{\beta_{11}+\cdots+\beta_{\nu r_\nu}}}{\partial t_{11}^{\beta_{11}} \cdots \partial t_{\nu r_\nu}^{\beta_{\nu r_\nu}}} \phi_n^{(F_i)}(e^{\boldsymbol{t}}; \boldsymbol{\alpha}) \Bigg|_{\boldsymbol{t}=\boldsymbol{0}}, \qquad i = 1, 2, \ldots, \lambda,$$

$$\eta_n^{(S_i)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}) = \frac{\partial^{\beta_{11}+\cdots+\beta_{\nu r_\nu}}}{\partial t_{11}^{\beta_{11}} \cdots \partial t_{\nu r_\nu}^{\beta_{\nu r_\nu}}} \phi_n^{(S_i)}(e^{\boldsymbol{t}}; \boldsymbol{\alpha}) \Bigg|_{\boldsymbol{t}=\boldsymbol{0}}, \qquad i = 1, 2, \ldots, \nu.$$

We have the next theorem from Theorem 2.1 directly.

Theorem 2.4 *The mixed* $(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu)$*-th moment about zero* $\eta_n(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha})$, $\eta_n^{(F_i)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}) i = 1, 2, \ldots, \lambda$ *and* $\eta_n^{(S_i)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}) i = 1, 2, \ldots, \nu$*satisfy the following recursive relation:*

$$\begin{aligned}
\eta_n(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}) &= \sum_{m=1}^{\lambda} \pi_{F_m} \eta_n^{(F_m)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}) \\
&\quad + \sum_{m=1}^{\nu} \pi_{S_m} \eta_n^{(S_m)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}), \quad n \geq 0, \\
\eta_n^{(F_i)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}) &= \sum_{m=1}^{\lambda} p_{F_i F_m} \eta_{n-1}^{(F_m)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}) \\
&\quad + \sum_{m=1}^{\nu} p_{F_i S_m} \eta_{n-1}^{(S_m)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}), \\
&\quad n \geq 1,\ i = 1, 2, \ldots, \lambda, \\
\eta_n^{(S_i)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}) & \\
&= \sum_{m=1}^{\lambda} \sum_{v=0}^{n-1} \sum_{\boldsymbol{b}_i=\boldsymbol{0}}^{\boldsymbol{\beta}_i} p_{S_i S_i}^{v} p_{S_i F_m} (\boldsymbol{\mu}(v+1; \boldsymbol{\alpha}_i))^{\boldsymbol{\beta}_i - \boldsymbol{b}_i} \\
&\quad \times \binom{\boldsymbol{\beta}_i}{\boldsymbol{b}_i} \eta_{n-v-1}^{(F_m)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{i-1}, \boldsymbol{b}_i, \boldsymbol{\beta}_{i+1}, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}), \\
&\quad + \sum_{m \neq i} \sum_{v=0}^{n-1} \sum_{\boldsymbol{b}_i=\boldsymbol{0}}^{\boldsymbol{\beta}_i} p_{S_i S_i}^{v} p_{S_i S_m} (\boldsymbol{\mu}(v+1; \boldsymbol{\alpha}_i))^{\boldsymbol{\beta}_i - \boldsymbol{b}_i}
\end{aligned}$$

$$\times \binom{\boldsymbol{\beta}_i}{\boldsymbol{b}_i} \eta_{n-v-1}^{(S_m)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{i-1}, \boldsymbol{b}_i, \boldsymbol{\beta}_{i+1}, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}),$$
$$+p_{S_i S_i}^n (\boldsymbol{\mu}(n+1; \boldsymbol{\alpha}_i))^{\boldsymbol{\beta}_i} \, \delta((\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{i-1}, \boldsymbol{\beta}_{i+1}, \ldots, \boldsymbol{\beta}_\nu), \, \mathbf{0}),$$
$$n \geq 1, \; i = 1, 2, \ldots, \nu,$$

with the initial conditions

$$\eta_0^{(F_i)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}) = \delta((\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu), \mathbf{0}), \qquad i = 1, 2, \ldots, \lambda,$$
$$\eta_0^{(S_i)}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_\nu; \boldsymbol{\alpha}) = (\boldsymbol{\mu}(1; \boldsymbol{\alpha}_i))^{\boldsymbol{\beta}_i} \, \delta((\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{i-1}, \boldsymbol{\beta}_{i+1}, \ldots, \boldsymbol{\beta}_\nu), \mathbf{0}),$$
$$i = 1, 2, \ldots, \nu,$$

where Kronecker delta $\delta(\boldsymbol{x}, \boldsymbol{y})$ equals one if $\boldsymbol{x} = \boldsymbol{y}$ and zero otherwise, $\boldsymbol{b}_i = (b_{i1}, \ldots, b_{ir_i})$,

$$\boldsymbol{\mu}(v; \boldsymbol{\alpha}_i)^{\boldsymbol{\beta}_i - \boldsymbol{b}_i} = \prod_{j=1}^{r_i} \mu(v; \alpha_{ij})^{\beta_{ij} - b_{ij}}, \quad \sum_{\boldsymbol{b}_i = \mathbf{0}}^{\boldsymbol{\beta}_i} = \sum_{b_{i1}=0}^{\beta_{i1}} \cdots \sum_{b_{ir_i}=0}^{\beta_{ir_i}} \quad \text{and}$$
$$\binom{\boldsymbol{\beta}_i}{\boldsymbol{b}_i} = \prod_{j=1}^{r_i} \binom{\beta_{ij}}{b_{ij}}.$$

In the case of $\lambda = \nu = 1$, $r_1 = 1$, Antzoulakos and Chadjiconstantinidis (2001) established recursive formulae for the evaluation of $E[(N(n, k_{11}; \mathrm{I}))^{\beta_{11}}]$, $E[(N(n, k_{11}; \mathrm{II}))^{\beta_{11}}]$ and $E[(N(n, k_{11}; \mathrm{III}))^{\beta_{11}}]$.

3 I.i.d. trials

Assume that $\Omega = \{F_1, S_1, \ldots, S_\nu\}$,

$$p_{\omega_i \omega_j} = P(Z_t = \omega_j | Z_{t-1} = \omega_i) = p_{\omega_j} \;\; \text{for } t \geq 1, \; \omega_i, \omega_j \in \Omega,$$
$$i, j = 1, 2, \ldots, \nu + 1$$

and

$$\pi_{F_1} = 1, \quad \pi_{S_1} = \cdots = \pi_{S_\nu} = 0.$$

Evidently, the underlying sequence reduces to the i.i.d. trials. In this case, the evaluation of the p.g.f. can be easily performed through the recurrence relation presented in Theorem 2.1. The double generating function $\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha})$ can be expressed in a more appealing form. More specifically, reduce the recurrences of Theorem 2.2 to the i.i.d. case ($p_{S_i S_j} \to p_{S_j}$, $p_{S_i F_j} \to p_{F_j}$) and replace all $\Phi^{(F_i)}(\boldsymbol{t}, z; \boldsymbol{\alpha})$, $i = 1, 2, \ldots, \lambda$ with $\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha})$. Then multiply by $-P(\boldsymbol{t}_i, p_{S_i} z; \boldsymbol{\alpha}_i)$ both side of (7) and add by parts to (8) and solve the resulting equation w.r.t. $\Phi^{(S_i)}(\boldsymbol{t}, z; \boldsymbol{\alpha})$, replace it to (7) and solve w.r.t. $\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha})$. Finally we get the following result.

Proposition 3.1 *The double generating function* $\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha})$ *is given by*

$$\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha}) = \frac{1}{1 - p_{F_1}z - \sum_{m=1}^{\nu} \frac{p_{S_m} z\, P(\boldsymbol{t}_m, p_{S_m} z; \boldsymbol{\alpha}_m)}{1 + p_{S_m} z\, P(\boldsymbol{t}_m, p_{S_m} z; \boldsymbol{\alpha}_m)}},$$

where $P(\boldsymbol{t}_i, p_{S_i} z; \boldsymbol{\alpha}_i)$, $i = 1, 2, \ldots, \nu$, *is given by (9) with* $p_{S_i S_i}$ *replaced by* p_{S_i} $(i = 1, 2, \ldots, \nu)$.

Example 3.1 *Joint distribution of* $(N(n, k_{11}; \alpha_{11}), \ldots, N(n, k_{\nu 1}; \alpha_{\nu 1}))$. We treat the special case where $r_1 = \cdots = r_\nu = 1$. The double generating function is given by

$$\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha}) = \frac{1}{1 - p_{F_1}z - \sum_{i=1}^{\nu} \frac{p_{S_i} z\, P(t_{i1}, p_{S_i} z; \alpha_{i1})}{1 + p_{S_i} z\, P(t_{i1}, p_{S_i} z; \alpha_{i1})}}, \tag{12}$$

where

$$\frac{p_{S_i} z P(t_{i1}, p_{S_i} z; \alpha_{i1})}{1 + p_{S_i} z P(t_{i1}, p_{S_i} z; \alpha_{i1})} = \begin{cases} \dfrac{p_{S_i} z - (p_{S_i} z)^{k_{i1}} + (p_{S_i} z)^{k_{i1}} t_{i1}(1 - p_{S_i} z)}{1 - (p_{S_i} z)^{k_{i1}}}, & \alpha_{i1} = \mathrm{I}, \\[2ex] \dfrac{p_{S_i} z - (p_{S_i} z)^{k_{i1}}(1 - t_{i1})}{1 - (p_{S_i} z)^{k_{i1}}(1 - t_{i1})}, & \alpha_{i1} = \mathrm{II}, \\[2ex] \dfrac{p_{S_i} z - (p_{S_i} z)^{k_{i1}}(1 - t_{i1}) - (p_{S_i} z)^2 t_{i1}}{1 - p_{S_i} z t_{i1} - (p_{S_i} z)^{k_{i1}}(1 - t_{i1})}, & \alpha_{i1} = \mathrm{III}, \\[2ex] \dfrac{p_{S_i} z - (p_{S_i} z)^{k_{i1}}(1 - t_{i1})(1 - p_{S_i} z)}{1 - (1 - t_{i1})(p_{S_i} z)^{k_{i1}}(1 - p_{S_i} z)}, & \alpha_{i1} = \mathrm{IV}, \end{cases}$$

for $i = 1, 2, \ldots, \nu$.

Remark 3.1 Inoue and Aki (2004) have also given the formulae for the double generating function with a different setup. In the case of $\nu = 2$, $r_1 = r_2 = 1$, Balakrishnan and Koutras (2002) derived the double generating functions of the joint distribution of $(N(n, k_{11}; \mathrm{I}), N(n, k_{21}; \mathrm{I}))$, $(N(n, k_{11}; \mathrm{II}), N(n, k_{21}; \mathrm{II}))$ and $(N(n, k_{11}; \mathrm{III}), N(n, k_{21}; \mathrm{III}))$. These joint distributions are called Type I, Type II and Type III trinomial distribution of order (k_{11}, k_{21}).

4 Conditional distributions

In this section, we consider the case where $\lambda = 1$. Let $Z_1, Z_2, \ldots, Z_n$ be a sequence of i.i.d. trials with failure F_1, successes $S_1, \ldots, S_\nu$ and the probabilities $P(Z_t = F_1) = p_{F_1}$, $P(Z_t = S_i) = p_{S_i}$ for $1 \leq t \leq n$, $i = 1, 2, \ldots, \nu$. We are going to investigate the conditional distribution of the run statistics $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = (\boldsymbol{N}(n, \boldsymbol{k}_1; \boldsymbol{\alpha}_1), \ldots, \boldsymbol{N}(n, \boldsymbol{k}_\nu; \boldsymbol{\alpha}_\nu))$, given the numbers $M_{n,i} = s_i$ $(0 \leq s_i \leq n)$ of successes S_i $(i = 1, 2, \ldots, \nu)$ in the n i.i.d. trials. Since $M_{n,i}$ is a sufficient statistic for p_{S_i} $(i = 1, 2, \ldots, \nu)$, the conditional distribution which we are searching for does not depend on p_{S_i} $(i = 1, 2, \ldots, \nu)$. In this section, we will use the notation $\Phi(\boldsymbol{t}, z, p_{S_1}, \ldots, p_{S_\nu}; \boldsymbol{\alpha})$ and $\phi_n(\boldsymbol{t}, p_{S_1}, \ldots, p_{S_\nu}; \boldsymbol{\alpha})$ instead of $\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha})$ and $\phi_n(\boldsymbol{t}; \boldsymbol{\alpha})$, respectively, that is,

$$\phi_n(\boldsymbol{t}, p_{S_1}, \ldots, p_{S_\nu}; \boldsymbol{\alpha}) = \sum_{\boldsymbol{x}} P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x}) t_{11}^{x_{11}} \cdots t_{\nu r_\nu}^{x_{\nu r_\nu}}, \tag{13}$$

$$\Phi(\boldsymbol{t}, z, p_{S_1}, \ldots, p_{S_\nu}; \boldsymbol{\alpha}) = \sum_{n=0}^{\infty} \phi_n(\boldsymbol{t}, p_{S_1}, \ldots, p_{S_\nu}; \boldsymbol{\alpha}) z^n. \tag{14}$$

We denote the probability generating function of the conditional probability function of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$, given that $M_{n,i} = s_i$ $(i = 1, 2, \ldots, \nu)$ by

$$\psi_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha}) = \sum_{\boldsymbol{x}} P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} \mid M_{n,1} = s_1, \ldots, M_{n,\nu} = s_\nu) \times t_{11}^{x_{11}} \cdots t_{\nu r_\nu}^{x_{\nu r_\nu}}, \tag{15}$$

where $\boldsymbol{s} = (s_1, \ldots, s_\nu)$.

In this subsection, an explicit formula for the double generating function of the quantity

$$a_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha}) = \frac{n!}{s_1! \cdots s_\nu! (n - s_1 - \cdots - s_\nu)!} \psi_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha})$$

is established. Each one of the two cases ((a) $p_{F_1} \neq 0$, (b) $p_{F_1} = 0$) is treated separately.

(a) The case $p_{F_1} \neq 0$

Theorem 4.1 *The generating function of* $a_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha})$ *takes on the form*

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{s_1, \ldots, s_\nu} a_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha}) u_1^{s_1} \cdots u_\nu^{s_\nu} z^n \\ &= \Phi\left(\boldsymbol{t}, (1 + \textstyle\sum_{i=1}^{\nu} u_i) z, \frac{u_1}{1+\sum_{i=1}^{\nu} u_i}, \ldots, \frac{u_\nu}{1+\sum_{i=1}^{\nu} u_i}; \boldsymbol{\alpha}\right). \end{aligned} \tag{16}$$

Proof Replacing $P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x})$ in (13) by

$$\begin{aligned} &P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x}) \\ &= \sum_{s_1, \ldots, s_\nu} P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} \mid M_{n,1} = s_1, \ldots, M_{n,\nu} = s_\nu) P \\ &\quad \times (M_{n,1} = s_1, \ldots, M_{n,\nu} = s_\nu) \\ &= \sum_{s_1, \ldots, s_\nu} \frac{n!}{s_1! \cdots s_\nu! (n - s_1 - \cdots - s_\nu)!} p_{F_1}^n \left(\frac{p_{S_1}}{p_{F_1}}\right)^{s_1} \cdots \left(\frac{p_{S_\nu}}{p_{F_1}}\right)^{s_\nu} \\ &\quad \times P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} \mid M_{n,1} = s_1, \ldots, M_{n,\nu} = s_\nu) \end{aligned}$$

and exploiting the expression (14), we have

$$\begin{aligned} \phi_n(\boldsymbol{t}, p_{S_1}, \ldots, p_{S_\nu}; \boldsymbol{\alpha}) = \sum_{s_1, \ldots, s_\nu} &\frac{n!}{s_1! \cdots s_\nu! (n - s_1 - \cdots - s_\nu)!} p_{F_1}^n \left(\frac{p_{S_1}}{p_{F_1}}\right)^{s_1} \\ &\cdots \left(\frac{p_{S_\nu}}{p_{F_1}}\right)^{s_\nu} \psi_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha}) \end{aligned}$$

or equivalently

$$\Phi(\boldsymbol{t}, z, p_{S_1}, \ldots, p_{S_\nu}; \boldsymbol{\alpha}) = \sum_{n=0}^{\infty} \sum_{s_1,\ldots,s_\nu} a_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha}) \left(\frac{p_{S_1}}{p_{F_1}}\right)^{s_1} \cdots \left(\frac{p_{S_\nu}}{p_{F_1}}\right)^{s_\nu} (p_{F_1} z)^n. \quad (17)$$

Setting $p_{S_i}/p_{F_1} = u_i$ $(i = 1, 2, \ldots, \nu)$ in (17), we get

$$\Phi\left(\boldsymbol{t}, z, \frac{u_1}{1+\sum_{i=1}^{\nu} u_i}, \ldots, \frac{u_\nu}{1+\sum_{i=1}^{\nu} u_i}; \boldsymbol{\alpha}\right)$$
$$= \sum_{n=0}^{\infty} \sum_{s_1,\ldots,s_\nu} a_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha}) u_1^{s_1} \cdots u_\nu^{s_\nu} \left(\frac{z}{1+\sum_{i=1}^{\nu} u_i}\right)^n,$$

which manifestly yields the desired result by replacing z by $(1 + \sum_{i=1}^{\nu} u_i)z$. □

From the representation of Theorem 4.1, the explicit formula for the generating function of $a_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha})$ is derived.

Theorem 4.2 *The generating function of $a_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha})$ is given by*

$$\Phi\left(\boldsymbol{t}, \left(1 + \sum_{i=1}^{\nu} u_i\right) z, \frac{u_1}{1+\sum_{i=1}^{\nu} u_i}, \ldots, \frac{u_\nu}{1+\sum_{i=1}^{\nu} u_i}; \boldsymbol{\alpha}\right) \quad (18)$$
$$= \frac{1}{1 - z - \sum_{i=1}^{\nu} \dfrac{u_i z P(\boldsymbol{t}_i, u_i z; \boldsymbol{\alpha}_i)}{1 + u_i z P(\boldsymbol{t}_i, u_i z; \boldsymbol{\alpha}_i)}},$$

where $P(\boldsymbol{t}_i, u_i z; \boldsymbol{\alpha}_i)$ is given by (9) with $p_{S_i S_i}$ replaced by $u_i (i = 1, 2, \ldots, \nu)$.

In the case of $\nu = 1$, $r_1 = 1$, Koutras and Alexandrou (1997) investigated the conditional distributions of run statistics $N(n, k_{11}; \mathrm{I})$, $N(n, k_{11}; \mathrm{I})$ and $N(n, k_{11}; \mathrm{III})$, given the number $M_{n,1} = s_1$ of success S_1 (see Balakrishnan and Koutras, 2002).

(b) The case $p_{F_1} = 0$

Theorem 4.3 *The generating function of $a_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha})$ takes on the form*

$$\sum_{n=0}^{\infty} \sum_{\substack{s_1,\ldots,s_\nu \\ s_1+\cdots+s_\nu=n}} a_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha}) u_1^{s_1} \cdots u_\nu^{s_\nu} z^n \quad (19)$$
$$= \Phi\left(\boldsymbol{t}, \sum_{i=1}^{\nu} u_i z, \frac{u_1}{\sum_{i=1}^{\nu} u_i}, \ldots, \frac{u_\nu}{\sum_{i=1}^{\nu} u_i}; \boldsymbol{\alpha}\right).$$

Proof Replacing $P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x})$ in (13) by

$$
\begin{aligned}
&P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x}) \\
&= \sum_{\substack{s_1,\ldots,s_\nu \\ s_1+\cdots+s_\nu=n}} P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} \mid M_{n,1} = s_1, \ldots, M_{n,\nu} = s_\nu) P \\
&\quad \times (M_{n,1} = s_1, \ldots, M_{n,\nu} = s_\nu) \\
&= \sum_{\substack{s_1,\ldots,s_\nu \\ s_1+\cdots+s_\nu=n}} \frac{n!}{s_1! \cdots s_\nu!} p_{S_1}^{s_1} \cdots p_{S_\nu}^{s_\nu} P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} \mid \\
&\quad \times M_{n,1} = s_1, \ldots, M_{n,\nu} = s_\nu),
\end{aligned}
$$

and exploiting the expression (14), we have

$$
\phi_n(\boldsymbol{t}, p_{S_1}, \ldots, p_{S_\nu}; \boldsymbol{\alpha}) = \sum_{\substack{s_1,\ldots,s_\nu \\ s_1+\cdots+s_\nu=n}} \frac{n!}{s_1! \cdots s_\nu!} p_{S_1}^{s_1} \cdots p_{S_\nu}^{s_\nu} \psi_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha})
$$

or equivalently

$$
\Phi(\boldsymbol{t}, z, p_{S_1}, \ldots, p_{S_\nu}; \boldsymbol{\alpha}) = \sum_{n=0}^{\infty} \sum_{\substack{s_1,\ldots,s_\nu \\ s_1+\cdots+s_\nu=n}} a_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha}) p_{S_1}^{s_1} \cdots p_{S_\nu}^{s_\nu} z^n. \tag{20}
$$

Setting $p_{S_i} = u_i / \sum_{i=1}^{\nu} u_i$ $(i = 1, 2, \ldots, \nu)$ in (20), we get

$$
\begin{aligned}
&\Phi\left(\boldsymbol{t}, z, \frac{u_1}{\sum_{i=1}^{\nu} u_i}, \ldots, \frac{u_\nu}{\sum_{i=1}^{\nu} u_i}; \boldsymbol{\alpha}\right) \\
&\qquad = \sum_{n=0}^{\infty} \sum_{\substack{s_1,\ldots,s_\nu \\ s_1+\cdots+s_\nu=n}} a_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha}) u_1^{s_1} \cdots u_\nu^{s_\nu} \left(\frac{z}{\sum_{i=1}^{\nu} u_i}\right)^n,
\end{aligned}
$$

which manifestly yields the desired result by replacing z by $\sum_{i=1}^{\nu} u_i z$. □

From the representation of Theorem 4.3, the explicit formula for the generating function of $a_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha})$ is derived.

Theorem 4.4 *The generating function of* $a_n(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\alpha})$ *is given by*

$$
\begin{aligned}
&\Phi\left(\boldsymbol{t}, \sum_{i=1}^{\nu} u_i z, \frac{u_1}{\sum_{i=1}^{\nu} u_i}, \ldots, \frac{u_\nu}{\sum_{i=1}^{\nu} u_i}; \boldsymbol{\alpha}\right) \\
&\qquad = \frac{1}{1 - \sum_{i=1}^{\nu} \dfrac{u_i z\, P(\boldsymbol{t}_i, u_i z; \boldsymbol{\alpha}_i)}{1 + u_i z\, P(\boldsymbol{t}_i, u_i z; \boldsymbol{\alpha}_i)}},
\end{aligned} \tag{21}
$$

where $P(\boldsymbol{t}_i, u_i z; \boldsymbol{\alpha}_i)$ *is given by* (9) *with* $p_{S_i S_i}$ *replaced by* $u_i (i = 1, 2, \ldots, \nu)$.

5 Applications

In this section, we consider the special case where $\lambda = 1$ and $r_1 = \cdots = r_\nu = 1$. Let $Z_1, Z_2, \ldots, Z_n$ be a sequence of i.i.d. trials with failure F_1, successes $S_1, \ldots, S_\nu$ and the probabilities $P(Z_t = F_1) = p_{F_1}$, $P(Z_t = S_i) = p_{S_i}$ for $1 \le t \le n$, $i = 1, 2, \ldots, \nu$. When no confusion is likely to arise, we will write simply k_i, α_i and t_i instead of k_{i1}, α_{i1} and t_{i1}, $i = 1, 2, \ldots, \nu$.

5.1 Multistate system

It is worth mentioning that the probability $P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{0})$ for $\alpha_i = \text{I, II, III}$, $i = 1, 2, \ldots, \nu$, is equal to the reliability of a consecutive k_1,$k_2, \ldots, k_\nu$-out-of-n:MFM system. According to Boutsikas and Koutras (2002), the consecutive k_1,$k_2, \ldots,$ k_ν-out-of-n:MFM system consists of n linearly arranged components and enter failure mode s whenever at least k_s consecutive components are failed in mode s ($s = 1, 2, \ldots, \nu$). Here, suppose we regard F_1 as a working state and $S_1, \ldots, S_\nu$ as failure modes. The exact reliability of the consecutive k_1,$k_2, \ldots, k_\nu$-out-of-n:MFM system with Markov dependent components can be evaluated through Theorems 2.1–2.3.

We will study the reliability of the system with i.i.d. components. Let $R(n, \boldsymbol{k})(= P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{0}))$ be the reliability of the system with i.i.d. components. Then the generating function of $R(n, \boldsymbol{k})$ is obtained by substituting $\boldsymbol{t} = \boldsymbol{0}$ in the formula (12), that is,

$$\sum_{n=0}^{\infty} R(n, \boldsymbol{k}) z^n = \Phi(\boldsymbol{0}, z; \boldsymbol{\alpha}) = \frac{1}{1 - p_{F_1} z - \sum_{i=1}^{\nu} \dfrac{p_{S_i} z - (p_{S_i} z)^{k_i}}{1 - (p_{S_i} z)^{k_i}}}. \tag{22}$$

In the following proposition, we derive a recursive scheme for the evaluation of $R(n, \boldsymbol{k})$.

Proposition 5.1 *The reliability $R(n, \boldsymbol{k})$ satisfies the following recursive relation:*

$$R(n, \boldsymbol{k}) = p_{F_1} R(n-1, \boldsymbol{k}) + \sum_{i=1}^{\nu} \sum_{v=0}^{[\frac{n-1}{k_i}]} p_{S_i}^{v k_i + 1} R(n - v k_i - 1, \boldsymbol{k}) \tag{23}$$

$$- \sum_{i=1}^{\nu} \sum_{v=1}^{[\frac{n}{k_i}]} p_{S_i}^{v k_i} R(n - v k_i, \boldsymbol{k}) \quad \textit{for } n \ge \min(k_1, \ldots, k_\nu),$$

$$R(n, \boldsymbol{k}) = 1 \quad \textit{for } n < \min(k_1, \ldots, k_\nu). \tag{24}$$

Proof From (22), we have

$$\sum_{n=0}^{\infty} R(n, \boldsymbol{k}) z^n = \left(p_{F_1} z + \sum_{i=1}^{\nu} \sum_{v=0}^{\infty} \left(p_{S_i} z - (p_{S_i} z)^{k_i} \right) (p_{S_i} z)^{v k_i} \right) \sum_{n=0}^{\infty} R(n, \boldsymbol{k}) z^n.$$

Equating the coefficients of z^n on the both sides of the above equation, we obtain Eq. (24). It is easy to check Eq. (24). The proof is completed. □

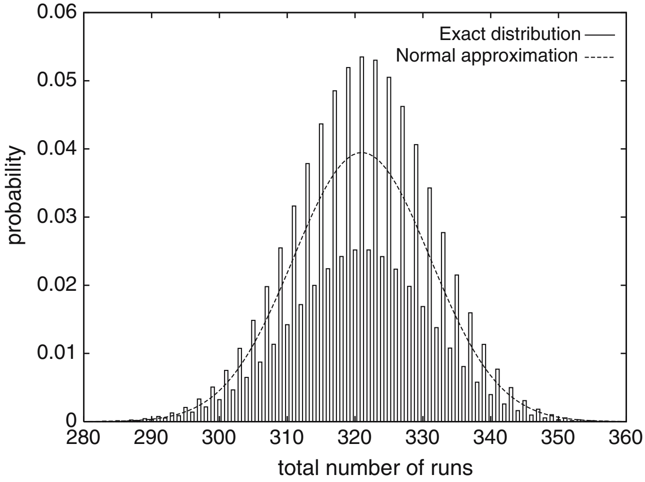

Fig. 1 The exact distribution of $R_{1000,800,200}$ and normal distribution $N(321, 102.18)$

Of course, we can evaluate the exact reliability $R(n, \boldsymbol{k})$ through the recurrence relation presented in Theorem 2.1. However, the recurrence relation presented in Proposition 5.1 is very simple and efficient.

5.2 Randomness tests

In this subsection, we consider the special case of $p_{F_1} = 0$. Let $R^{(S_i)}_{n,s_1,\dots,s_\nu}$ denote the number of runs of type S_i, given the numbers $M_{n,i} = s_i$ $(0 \le s_i \le n)$ of successes S_i $(i = 1, 2, \dots, \nu)$ in the n i.i.d. trials. Then the total number of runs $R_{n,s_1,\dots,s_\nu}$ is defined by

$$R_{n,s_1,\dots,s_\nu} = \sum_{i=1}^{\nu} R^{(S_i)}_{n,s_1,\dots,s_\nu}.$$

By setting $r_1 = \cdots = r_\nu = 1$, $t_1 = t_2 = \cdots = t_\nu = t$ and

$$P(\boldsymbol{t}_i, u_i z; \boldsymbol{\alpha}_i) = \frac{t}{1 - u_i z}, \qquad i = 1, 2, \dots, \nu,$$

in the formula (22), we can obtain the double generating function (w.r.t. $u_1, \dots, u_\nu$, z) of $n!/s_1! \cdots s_\nu! E[\, t^{R_{n,s_1,\dots,s_\nu}}\,]$ as follows:

$$\Phi(\underbrace{t, t, \dots, t}_{\nu}, \sum_{i=1}^{\nu} u_i z, \tfrac{u_1}{\sum_{i=1}^{\nu} u_i}, \dots, \tfrac{u_\nu}{\sum_{i=1}^{\nu} u_i}; \boldsymbol{\alpha}) = \frac{1}{1 - \sum_{i=1}^{\nu} \dfrac{u_i\, z\, t}{1 - u_i\, z\,(1-t)}}.$$

The exact distribution of $R_{n,s_1,\dots,s_\nu}$ is acquired by the expansion of the generating function in a multiple Taylor series of z, t and u_i $(i = 1, 2, \dots, \nu)$, which nowadays can be easily achieved by computer algebra systems. Furthermore, by

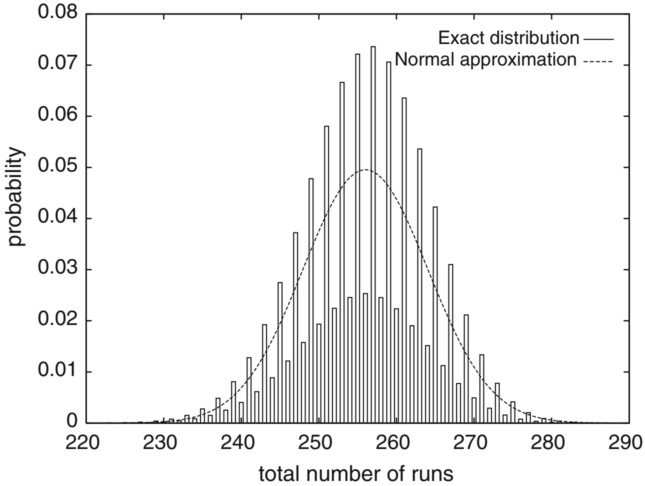

Fig. 2 The exact distribution of $R_{1000,850,150}$ and normal distribution $N(256, 64.83)$

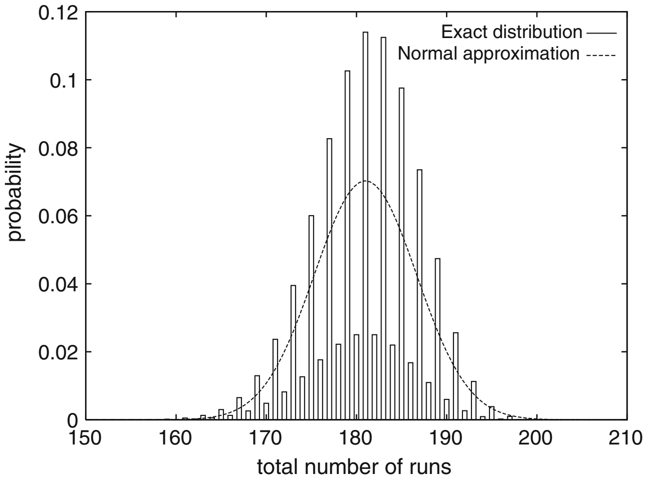

Fig. 3 The exact distribution of $R_{1000,900,100}$ and normal distribution $N(181, 32.25)$

making use of the derivatives of the above generating function up to the second order with respect to t, the mean and variance of $R_{n,s_1,\ldots,s_\nu}$ can be obtained. (The details can be worked out easily and are thus omitted here.) More specifically, we have the following result.

Proposition 5.2 *The mean and variance of* $R_{n,s_1,\ldots,s_\nu}$ *are given by*

$$E[R_{n,s_1,\ldots,s_\nu}] = \sum_{i=1}^{\nu} \frac{s_i(n - s_i + 1)}{n},$$

$$V[R_{n,s_1,\ldots,s_\nu}] = \sum_{i=1}^{\nu} \frac{s_i(n - s_i + 1)(s_i - 1)(n - s_i)}{n^2(n-1)} + \sum_{i \neq j} \frac{s_i s_j (s_i - 1)(s_j - 1)}{n^2(n-1)}.$$

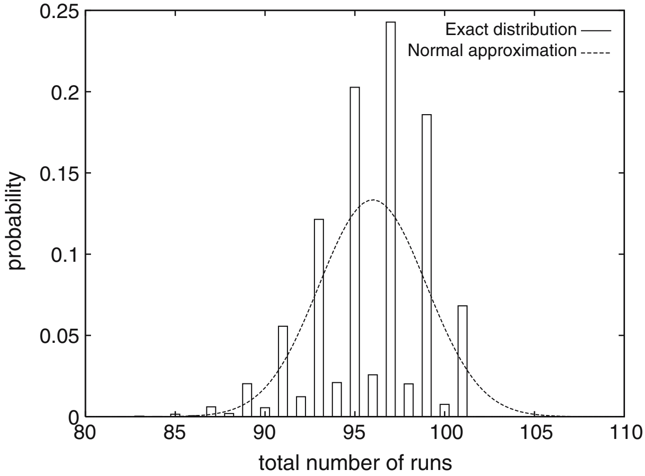

Fig. 4 The exact distribution of $R_{1000,950,50}$ and normal distribution $N(96, 8.94)$

The classical run tests was based on $R_{n,s_1,\ldots,s_\nu}$ (see Rubin *et al.,* 1990; Agin and Godbole, 1992; Shaughnessy, 1981). Schuster and Gu (1997) provide the recursive scheme for the evaluation of $R_{n,s_1,\ldots,s_\nu}$. In the case of $\nu = 2$, the randomness of the sequence is tested statistically by many authors (see Wald and Wolfowitz 1940; Mood 1940; Koutras and Alexandrou, 1997; Lou, 1997). It is well known that the distribution of R_{n,s_1,s_2}, when the ratio of s_1 to s_2 remains a positive constant while both numerator and denominator approach infinity, is approximated by the normal distribution with mean $E[R_{n,s_1,s_2}] = (2s_1s_2 + n)/n$ and variance $V[R_{n,s_1,s_2}] = 2s_1s_2(2s_1s_2 - n)/[n^2(n-1)]$ (see Wald and Wolfowitz, 1940 Bradley, 1968). In Figs. 1–4, however, there is significant differences between the exact and the approximate probabilities. Therefore, we think that the results presented in Figs. 1–4 highlight the importance of the exact distribution. As indicated by Bradley (1968), it rapidly becomes difficult to obtain the exact distributions of runs in a sequence of multistate trials, as the number n of trials increases (see Shaughnessy, 1981). Our results presented in this paper are useful for the numerical and symbolic calculations.

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