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A generalization of the Archimedean class of bivariate copulas

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Abstract We introduce and study a class of bivariate copulas depending on two univariate functions which generalizes the well-known Archimedean family. We provide several examples and some results about the concordance order.

Keywords Copula · Archimedean copula · Concordance order

1 Introduction

A *copula* is a function which joins or "couples" a multivariate distribution function to its one-dimensional marginal distribution functions (d.f.'s), in the sense of the following Theorem, due to A. Sklar, who introduced copulas (see Sklar, 1959, 1973). Here we recall it in the bivariate case.

Theorem 1.1 *If X and Y are random variables (r.v.'s) with unidimensional d.f.'s F and G, respectively, and joint d.f. H, then there exists a copula C (uniquely determined on* Range $F \times$ Range G*, and hence unique when X and Y are continuous) such that*

$$\forall (x, y) \in \overline{\mathbb{R}}^2 \qquad H(x, y) = C(F(x), G(y)). \tag{1}$$

Conversely, if C is a copula, and if F and G are univariate distribution functions, then the function H defined by Eq. (1) *is a bivariate distribution function with marginals F and G.*

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Copulas are of interest for statisticians because they are an important tool for building families of bivariate distributions with given marginals. Actually, Theorem 1.1 allows the use of copulas to solve a difficult problem, that of constructing bivariate d.f.'s, in two (easier) steps: the construction of the univariate marginal distributions and the construction of a copula. In the last few years there has been an increasing interest in the construction of new multivariate families of distribution functions and a large choice of d.f.'s has been put at the disposal of researchers and practitioners for the modelling and the simulation of multivariate data: see, e.g., the recent papers of Capéràa et al. (2000), Amblard and Girard (2002), Rodríguez-Lallena and Úbeda-Flores (2004), Durante (2005) and Durante et al. (2006). Extensive surveys of families of copulas can be also found in Joe (1997) and Nelsen (1999).

In particular, a very interesting class of copulas is the *Archimedean* one, whose investigation arose in the context of associative functions and probabilistic metric spaces (see Schweizer and Sklar, 1983; Alsina et al., 2006) and today has also many applications in the statistical context (Genest and MacKay, 1986; Nelsen, 1999; Avérous and Dortet–Bernadet, 2004; Müller and Scarsini, 2005). Moreover, Archimedean copulas are widely used in applications, especially in finance, insurance and actuarial science (see Frees and Valdez, 1998; Hennessy and Lapan, 2002), and in hydrology (see (Salvadori and De Michele, 2004), due to their simple form and nice properties, as well as to the easy procedures for choosing a particular member of a given family of Archimedean copulas to fit a data set (see Genest and Rivest, 1993; Wang and Wells, 2000).

In this paper, we introduce and study a class of bivariate copulas depending on two univariate functions which generalizes the well-known Archimedean family (Sect.3). We provide several examples (Sect.4) and some results about the concordance order (Sect. 5).

2 Preliminaries

We begin with the formal definition of copula (Sklar, 1959).

Definition 2.1 *A function* $C : [0,1]^2 \to [0,1]$ *is a copula if, for all* x, x', y, y' *in* $[0,1]$*, one has:*

$$C(x,0) = C(0,x) = 0, \quad C(x,1) = C(1,x) = x; \tag{2}$$

$$\begin{aligned} V_C\left([x,x'] \times [y,y']\right) := C(x',y') - C(x,y') \\ -C(x',y) + C(x,y) \geq 0, \end{aligned} \tag{3}$$

for $x \leq x'$, $y \leq y'$.

The copulas Π, M and W, given, respectively, by $\Pi(x,y) = xy$, $M(x,y) = \min\{x,y\}$ and $W(x,y) = \max\{x+y-1,0\}$, are of particular importance: for continuous r.v.'s X and Y, $C_{XY} = \Pi$ if, and only if, X and Y are independent, and $C_{XY} = M$ (resp. W) if, and only if, each of them is almost surely an increasing (resp. descreasing) function of the other one. For a complete monograph on copulas, we refer to Nelsen (1999).

Let us consider a function $\varphi : [0,1] \to [0,+\infty]$ that is continuous and decreasing (not necessarily strictly). The *pseudo-inverse* of the function φ is defined by

$$\varphi^{[-1]}(t) := \begin{cases} \varphi^{-1}(t), & t \in [0, \varphi(0)]\,, \\ 0, & t \in [\varphi(0), +\infty]\,. \end{cases}$$

We recall that, for all $t \in [0,1]$, one has

$$\varphi^{[-1]}\left(\varphi(t)\right) = t \tag{4}$$

and, for all $t \in [0,+\infty]$,

$$\varphi\left(\varphi^{[-1]}(t)\right) = \min\{t, \varphi(0)\}. \tag{5}$$

Finally, if φ is strictly decreasing with $\varphi(0) = +\infty$, then the pseudo-inverse coincides with the inverse, viz. $\varphi^{[-1]} = \varphi^{-1}$.

The following well-known result characterizes the class of *Archimedean copulas*.

Theorem 2.1 *Let $\varphi : [0,1] \to [0,+\infty]$ be a continuous and strictly decreasing function with $\varphi(1) = 0$. Let C_φ be the function given by*

$$C_\varphi(x,y) := \varphi^{[-1]}\left(\varphi(x) + \varphi(y)\right), \quad \textit{for all } x, y \in [0,1]\,. \tag{6}$$

Then C_φ is a copula if, and only if, φ is convex.

The function φ in Theorem 2.1 is an *additive generator* of C_φ. Notice that, by setting $h(t) := \exp\left(-\varphi(t)\right)$ for every $t \in [0,1]$, C_φ may be represented in the form

$$C_\varphi(x,y) = h^{[-1]}\left(h(x) \cdot h(y)\right) \quad \text{for all } x, y \in [0,1]\,. \tag{7}$$

This function h is a *multiplicative generator* of C_φ and Theorem 2.1 may be rephrased in the following (multiplicative) form.

Theorem 2.2 *The function C_φ defined by Eq.* (7) *is an (Archimedean) copula if, and only if, its multiplicative generator h is log-concave, viz. if, and only if, for every α, s and t in* $[0,1]$*, it satisfies the inequality*

$$h^\alpha(s)\, h^{1-\alpha}(t) \le h\left(\alpha s + (1-\alpha)t\right).$$

Neither the additive nor the multiplicative generator of an Archimedean copula is unique. In fact, if φ is an additive generator of C, then $\varphi_1 := k\,\varphi (k > 0)$ is also an additive generator for C. If h is a multiplicative generator of a copula D, then $h_1(t) := h(t^\alpha)$ $(\alpha > 0)$ is also a multiplicative generator for D.

In the sequel, we shall also need the well-known concepts of majorization and weak majorization and their characterizations (see Marshall and Olkin (1979) for more details).

Definition 2.2 *Let* $\mathbf{x} = (x_1, x_2)$ *and* $\mathbf{y} = (y_1, y_2)$ *be two points in* $\mathbb{R}^2$. *We say that* $\mathbf{x}$ *is weakly (super-)majorized by* $\mathbf{y}$ *if, and only if,*

$$\min\{x_1, x_2\} \geq \min\{y_1, y_2\} \quad \textit{and} \quad x_1 + x_2 \geq y_1 + y_2,$$

and one writes $\mathbf{x} \prec^w \mathbf{y}$.

We say that $\mathbf{x}$ *is majorized by* $\mathbf{y}$ *if, and only if,*

$$\min\{x_1, x_2\} \geq \min\{y_1, y_2\} \quad \textit{and} \quad x_1 + x_2 = y_1 + y_2,$$

and one writes $\mathbf{x} \prec \mathbf{y}$.

Theorem 2.3 *For any two vectors* $\mathbf{x} = (x_1, x_2)$ *and* $\mathbf{y} = (y_1, y_2)$ *in* $\mathbb{R}^2$ *the following statements are equivalent:*

(a) $\mathbf{x} \prec^w \mathbf{y}$;
(b) for every continuous, decreasing and convex function $g : \mathbb{R} \to \mathbb{R}$ *one has*

$$g(x_1) + g(x_2) \leq g(y_1) + g(y_2).$$

Theorem 2.4 *For any two vectors* $\mathbf{x} = (x_1, x_2)$ *and* $\mathbf{y} = (y_1, y_2)$ *in* $\mathbb{R}^2$ *the following statements are equivalent:*

(a) $\mathbf{x} \prec \mathbf{y}$;
(b) for every continuous convex function $g : \mathbb{R} \to \mathbb{R}$ *one has*

$$g(x_1) + g(x_2) \leq g(y_1) + g(y_2)$$

Theorems 2.3 and 2.4 can be found in Marshall and Olkin (1979) (Section 4.B).

3 The new family

We denote by Φ the class of all functions $\varphi : [0, 1] \to [0, +\infty]$ that are continuous and strictly decreasing, and by Ψ the class of all functions $\psi : [0, 1] \to [0, +\infty]$ that are continuous, decreasing and such that $\psi(1) = 0$. Moreover, we set $\Phi_0 := \Phi \cap \Psi$.

For all $(\varphi, \psi) \in \Phi \times \Psi$, we introduce the function $C_{\varphi,\psi} : [0, 1]^2 \to [0, 1]$ defined by

$$C_{\varphi,\psi}(x, y) := \varphi^{[-1]} \left(\varphi(x \wedge y) + \psi(x \vee y)\right), \tag{8}$$

where $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$. Notice that every such function is symmetric, i.e. $C_{\varphi,\psi}(x, y) = C_{\varphi,\psi}(y, x)$ for all $x, y \in [0, 1]$. By using Eq. (4) it is easily proved that, for all $x \in [0, 1]$,

$$C_{\varphi,\psi}(x, 1) = \varphi^{[-1]} \left(\varphi(x)\right) = x$$

and

$$0 \leq C_{\varphi,\psi}(x, 0) = \varphi^{[-1]} \left(\varphi(0) + \psi(x)\right) \leq \varphi^{[-1]} \left(\varphi(0)\right) = 0,$$

viz. $C_{\varphi,\psi}$ satisfies the boundary conditions (2).

Below we shall investigate under which conditions on φ and ψ, the function $C_{\varphi,\psi}$ defined by Eq. (8) is a copula.

Theorem 3.1 *Let φ and ψ belong to Φ and to Ψ, respectively, and let $C = C_{\varphi,\psi}$ be the function defined by Eq.* (8). *If φ is convex and $(\psi - \varphi)$ is increasing in* $[0, 1]$, *then C is a copula.*

Proof Since C satisfies Eq. (2), it suffices to show that C is 2-increasing, viz. that it satisfies Eq. (3). Let $R = [x_1, x_2] \times [y_1, y_2]$ be a rectangle contained in the unit square. We distinguish three cases.

If R is entirely contained in the triangular region $T^- := \{(x, y) \in [0, 1]^2 : x \geq y\}$, then

$$V_C(R) = \varphi^{[-1]}(\varphi(y_1) + \psi(x_1)) + \varphi^{[-1]}(\varphi(y_2) + \psi(x_2)) \\ -\varphi^{[-1]}(\varphi(y_2) + \psi(x_1)) - \varphi^{[-1]}(\varphi(y_1) + \psi(x_2)).$$

Set

$$s_1 := \varphi(y_1) + \psi(x_1), \quad s_2 := \varphi(y_2) + \psi(x_2),$$
$$t_1 := \varphi(y_2) + \psi(x_1), \quad t_2 := \varphi(y_1) + \psi(x_2).$$

Then $(t_1, t_2) \prec (s_1, s_2)$; moreover, $\varphi^{[-1]}$ is convex since it is the pseudo-inverse of the convex function φ. Therefore, Theorem 2.4 implies $V_C(R) \geq 0$.

Since C is symmetric, the same argument yields $V_C(R) \geq 0$ if the rectangle R is entirely contained in the triangular region $T^+ := \{(x, y) \in [0, 1]^2 : x \leq y\}$.

Next, consider the case in which the diagonal of R lies on the diagonal of the unit square, viz. $x_1 = y_1$ and $x_2 = y_2$. If $x_1 = 0$, then $V_C(R) = \varphi^{[-1]}(\varphi(x_2) + \psi(x_2)) \geq 0$. Assume then that $x_1 > 0$. One has

$$V_C(R) = \varphi^{[-1]}(\varphi(x_1) + \psi(x_1)) + \varphi^{[-1]}(\varphi(x_2) + \psi(x_2)) \\ -\varphi^{[-1]}(\varphi(x_1) + \psi(x_2)) - \varphi^{[-1]}(\varphi(x_1) + \psi(x_2)).$$

Now, set

$$s_1 := \varphi(x_1) + \psi(x_1), \quad s_2 := \varphi(x_2) + \psi(x_2), \quad t_1 := \varphi(x_1) + \psi(x_2) =: t_2.$$

Since $t \mapsto (\psi(t) - \varphi(t))$ is increasing in $[0, 1]$ one has $(t_1, t_2) \prec^w (s_1, s_2)$, and since $\varphi^{[-1]}$ is convex and decreasing, Theorem 2.3 ensures that $V_C(R) \geq 0$.

Finally, because every rectangle R in $[0, 1]^2$ can be decomposed into the union of at most three sub-rectangles R_i of the three previous types and, moreover, $V_C(R)$ is the sum of the values $V_C(R_i)$, it follows that $V_C(R) \geq 0$ when φ is convex and the function $(\psi - \varphi)$ is increasing in $[0, 1]$, as asserted.

Remark 1 Notice that, since $t \mapsto (\psi(t) - \varphi(t))$ is increasing, then, for all $t \in [0, 1]$, we have $\varphi(t) \geq \psi(t)$. In fact, if there existed $x_0 \in (0, 1)$ such that $\varphi(x_0) < \psi(x_0)$, then one would have

$$0 < \psi(x_0) - \varphi(x_0) \leq \psi(1) - \varphi(1) = -\varphi(1) \leq 0,$$

which is a contradiction.

Remark 2 Notice that, if (φ, ψ) is a pair of functions that generates a copula C of type (8), then, for any $c > 0$, also $(c\varphi, c\psi)$ generates C. Moreover, another pair generating a copula of type (8) is given by $(\varphi_\alpha, \psi_\alpha)$, where $\varphi_\alpha(t) := \varphi(t^\alpha)$ and $\psi_\alpha(t) := \psi(t^\alpha)$ for every $\alpha \in]0, 1]$.

Any result stated for additive generators can be readily translated into a corresponding one for multiplicative generators, in particular the preceding Theorem 3.1 can be easily reformulated.

Theorem 3.2 *Let h, k be two continuous and increasing functions from* $[0, 1]$ *into* $[0, 1]$ *such that $k(1) = 1$. If h is log-concave and $t \mapsto h(t)/k(t)$ is increasing, then*

$$C_{h,k}(x, y) := h^{[-1]}\left(h(x \wedge y) \cdot k(x \vee y)\right) \tag{9}$$

is a copula.

4 Examples

The most important family of copulas of type (8) is the Archimedean one. Specifically, given a function φ in Φ_0, $C_{\varphi,\varphi}$ is an Archimedean copula with additive generator φ. In particular, the copulas Π and W are of this type. Notice that, even when $\varphi \neq \psi$, $C_{\varphi,\psi}$ may be an Archimedean copula. Consider, for instance, $\varphi(t) := a - \ln t$ ($a \geq 0$) and $\psi(t) := -\ln t$, then $C_{\varphi,\psi} = \Pi$. In general, we do not know any conditions under which a copula $C_{\varphi,\psi}$ is Archimedean, when $\varphi \neq \psi$. Also the copula M is of type (8): it suffices to take $\psi = 0$ and $\varphi \in \Phi$. So, the family of copulas of type (8) is *comprehensive*, because it contains the copulas M, Π and W.

The following examples show that the class of copulas of type (8) includes many other well-known families.

4.1 Cuadras–Augé family

Given $\varphi(t) = -\ln t$ and $\psi(t) = -\ln t^{\alpha}$, $\alpha \in [0, 1]$, the corresponding copula of type (8), $C_{\varphi,\psi} = C_{\alpha}$, is given by

$$C_{\alpha}(x, y) : \begin{cases} xy^{\alpha}, & x \leq y, \\ x^{\alpha}y, & x \geq y, \end{cases}$$

viz. C_{α} is a member of the *Cuadras–Augé* family of copulas [see Cuadras and Augé (1981)]. In general, if C is a copula given by $C(x, y) := (x \wedge y) f(x \vee y)$, where f is a suitable function from [0, 1] into [0, 1] [see Durante (2005) for more details], then C is of type (8): it suffices to take $\varphi(t) := -\ln t$ and $\psi(t) = -\ln f(t)$.

4.2 MT-copulas

Let $\delta : [0, 1] \to [0, 1]$ be an increasing function such that $\delta(1) = 1$, $\delta(t) \leq t$ for all $t \in [0, 1]$ and $|\delta(t) - \delta(s)| \leq 2|t - s|$ for all $t, s \in [0, 1]$. Take $\varphi(t) = 1 - t$ and $\psi(t) = t - \delta(t)$. If ψ is decreasing, then the difference $(\psi - \varphi)$ is increasing and Theorem 3.1 ensures that the pair (φ, ψ) generates a copula $C_{\varphi,\psi} = C_{\delta}$ given by

$$C_{\delta}(x, y) := \max\{0, \delta(x \vee y) - |x - y|\} \quad \text{for all } x, y \in [0, 1]\,.$$

Thus C_δ is a member of the family of MT-copulas, characterized and studied in Durante et al. (2006).

4.3 Singular copulas

Now, we give two examples of *singular* copulas of type (8)(see Nelsen, 1999 for more details).

Example 1 Take $\varphi(t) = -\alpha t + \alpha$ ($\alpha \geq 1$) and $\psi(t) = 1 - t$. Then the pair $(\varphi, \psi) \in \Phi \times \Psi$ and satisfies the assumptions of Theorem 3.1. The corresponding copula, $C_{\varphi,\psi} = C_\alpha$, is given by

$$\begin{aligned} C_\alpha(x, y) &= \max\left\{0, x \wedge y - \frac{1}{\alpha}(1 - x \vee y)\right\} \\ &= \begin{cases} \dfrac{\alpha(x \wedge y) + (x \vee y) - 1}{\alpha}, & \alpha(x \wedge y) + (x \vee y) \geq 1; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The copula C_α has a probability mass $2/(\alpha + 1)$ uniformly distributed on the two segments connecting the point $(1/(\alpha + 1), 1/(\alpha + 1))$ with $(0, 1)$ and $(1, 0)$, respectively, and a probability mass $(\alpha - 1)/(\alpha + 1)$ uniformly distributed on the segment joining the point $(1/(\alpha + 1), 1/(\alpha + 1))$ to $(1, 1)$ (see also p. 57 of Nelsen, 1999). In particular, we obtain $C_1 = W$ and $C_\infty = M$. Notice that this class of copulas has been also used in De Schuymer et al. (2005).

Example 2 Take $\varphi(t) = 1 - t$ and, for every $\alpha \in [0, 1]$,

$$\psi(t) = \begin{cases} \alpha/2, & t \in [0, \alpha/2]\,; \\ \alpha - t, & t \in [\alpha/2, \alpha]\,; \\ 0, & t \in [\alpha, 1]\,. \end{cases}$$

Then, the pair (φ, ψ) is in $\Phi \times \Psi$ and satisfies the assumptions of Theorem 3.1. The corresponding copula, $C_{\varphi,\psi} = C_\alpha$, is given by

$$C_\alpha(x, y) := \begin{cases} \max\{0, x + y - \alpha\} & x, y \in [0, \alpha]\,; \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

Therefore, C_α spreads uniformly the mass on the two segments connecting, respectively, the points $(1, 1)$ with (α, α) and $(\alpha, 0)$ with $(0, \alpha)$. Notice that C_α is a member of the *Mayor-Torrens* family of copulas, which are also t-norms [see Klement (2000)].

4.4 Ordinal sum

It is known (see Schweizer and Sklar, 1983) that every copula can be represented as an ordinal sum of copulas, none of which has a non-trivial *idempotent element*. We recall that an element x in $[0, 1]$ is said to be *idempotent* for a copula C if, and only if, $C(x, x) = x$. Obviously, 0 and 1 are idempotent elements for every copula C. The set of idempotent elements of a copula of type (8) is characterized in the following.

Proposition 4.1 *Let C be a copula of type* (8) *generated by the pair* (φ, ψ)*. Then the set of idempotent elements of C is given by* $\{0\} \cup [a, 1]$*, where* $a := \inf\{t \in [0, 1] : \psi(t) = 0\}$.

Proof Given the copula C, let δ be its diagonal section given by $\delta(t) := C(t, t) = \varphi^{[-1]}(\varphi(t) + \psi(t))$. In particular, for all $t \in \,]0, 1[$ one has $\delta(t) < t$ if, and only if, $\min(\varphi(t) + \psi(t), \varphi(0)) > \varphi(t)$, which is equivalent to $\psi(t) > 0$. Since ψ is decreasing and $\psi(1) = 0$, then $\delta(t) < t$ if, and only if, t is in $]0, a[$ where $a := \inf\{t \in [0, 1] : \psi(t) = 0\}$.

Therefore, from the above result, we can derive that the only ordinal sum of copulas that can be expressed in the form (8) is the ordinal sum (C, M), where C is a suitable copula, with partition $\{[0, a], [a, 1]\}$. For example, the copula of Example 2 is the ordinal sum of (W, M) with respect to the partition $\{[0, \alpha], [\alpha, 1]\}$. Another example of ordinal sum is the following one.

Example 3 Take $\varphi(t) = -\ln t$ and, for every $\alpha \in \,]0, 1[$, $\psi(t) = -\ln(\min\{\alpha t, 1\})$. Then the pair $(\varphi, \psi) \in \Phi \times \Psi$ and satisfies the assumptions of Theorem 3.1. The corresponding copula, $C_{\varphi,\psi} = C_\alpha$, is given by

$$C_\alpha(x, y) := \begin{cases} \dfrac{xy}{\alpha} & (x, y) \in [0, \alpha]^2\,; \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

The copula C_α is the ordinal sum of (Π, M) with respect to the partition $\{[0, \alpha], [\alpha, 1]\}$. Moreover, C_α has a mass on the segment connecting the points $(1, 1)$ with (α, α), but has also an absolutely continuous component, because $\partial^2_{xy} C(x, y) \geq 0$ on $[0, \alpha]^2$.

4.5 Additive generators

Theorem 3.1 highlights the importance of finding generators in order to construct copulas of the type (8). To this purpose the following result, which is in the spirit of Theorem 4.3.8 of Nelsen (1999), provides useful conditions.

Theorem 4.1 *Let C be a copula of type* (8)*, generated by the pair* $(\varphi, \psi) \in \Phi \times \Psi$ *according to Theorem 3.1. Then, for almost all x and y in* $[0, 1]$*, one has*

$$\varphi'(x)\,\frac{\partial C}{\partial y}(x, y) = \psi'(y)\,\frac{\partial C}{\partial x}(x, y), \quad \text{if } x \leq y,$$

$$\psi'(x)\,\frac{\partial C}{\partial y}(x, y) = \varphi'(y)\,\frac{\partial C}{\partial x}(x, y), \quad \text{if } x > y.$$

Proof Since the functions φ and ψ are monotone, they are differentiable almost everywhere in $[0, 1]$. Assume $C(x, y) > 0$ at the point $(x, y) \in [0, 1]^2$ (if $C(x, y) = 0$ the proof is trivial). If $x \le y$, then $\varphi(C(x, y)) = \varphi(x) + \psi(y)$, so that the chain rule yields

$$\varphi'(C(x, y)) \frac{\partial C}{\partial x}(x, y) = \varphi'(x)$$
$$\varphi'(C(x, y)) \frac{\partial C}{\partial y}(x, y) = \psi'(y),$$

from which the assertion follows. In the case $x > y$, the proof is analogous.

Additive generators of Archimedean copulas can be combined together in order to construct copulas of the type (8). In fact, let φ and ψ belong to Φ_0; in view of Theorem 3.1, the convexity of φ and the condition that $(\psi - \varphi)$ be increasing ensure that $C_{\varphi,\psi}$ is a copula. Consider, for instance, the functions $\alpha(t) := 1-t$, $\beta(t) := -\ln t$ and $\gamma(t) := 1/t - 1$, which are, respectively, the additive generators of the Archimedean copulas W, Π and

$$\frac{\Pi}{\Sigma - \Pi}(x, y) := \frac{xy}{x + y - xy};$$

then one obtains the following copulas:

$$C_{\beta,\alpha}(x, y) = (x \wedge y) \exp\big((x \vee y) - 1\big),$$
$$C_{\gamma,\alpha}(x, y) = \frac{x \wedge y}{1 + (x \wedge y) - xy},$$
$$C_{\gamma,\beta}(x, y) = \frac{x \wedge y}{1 - (x \wedge y) \ln(x \vee y)}.$$

In general, for suitable choices of the generators, we can easily construct a two-parameters family of copulas: it suffices to take two families φ_α and ψ_β with the conditions described in Theorem 3.1.

5 Concordance order

In the class of copulas, we can introduce a partial order. Given two copulas C and D, D is said to be *more concordant* than C (one writes: $C \prec D$) if $C(x, y) \le D(x, y)$ for every x, y in $[0, 1]$ (see Joe,1997). In our class of copulas, the concordance order between two copulas is determined by the properties of their generators.

Theorem 5.1 *Let C and D be two copulas of type* (8) *generated, respectively, by the pairs (φ, ψ) and (γ, η). Let $\alpha := \varphi \circ \gamma^{[-1]}$ and $\beta := \psi \circ \eta^{[-1]}$. Then $C \prec D$ if, and only if,*

$$\alpha(a + b) \le \alpha(a) + \beta(b) \quad \textit{for all } \; a, b \in [0, +\infty]\,. \tag{10}$$

Proof Let x and y be in $[0, 1]$ and suppose, first, that $x \le y$. Then $C \prec D$ if, and only if,

$$\varphi^{[-1]}\left(\varphi(x) + \psi(y)\right) \le \gamma^{[-1]}\left(\gamma(x) + \eta(y)\right).$$

Let $\gamma(x) = a$ and $\eta(y) = b$, then the above inequality is equivalent to

$$\varphi^{[-1]}\left(\varphi \circ \gamma^{[-1]}(a) + \psi \circ \eta^{[-1]}(b)\right) \le \gamma^{[-1]}\left(a + b\right).$$

Applying the function γ to both sides, one has

$$\alpha^{[-1]}\left(\alpha(a) + \beta(b)\right) \ge a + b,$$

viz. condition (10).

If $x > y$, the proof can be completed by using the same arguments.

Notice that, if C and D are Archimedean copulas generated, respectively, by φ and γ, then $\alpha = \beta$ and condition (10) is equivalent to the subadditivity of α, as reported in Theorem 4.4.2 of Nelsen (1999).

Theorem 5.1 can be easily used to obtain conditions that ensures that a copula of our class has positive dependence (see Nelsen, 1999 for more details).

Corollary 5.1 *Let C be a copula of type* (8) *generated by* (φ, ψ). *Then*

(i) C is positively quadrant dependent (i.e. $C \succ \Pi$) if, and only if,

$$\varphi^{[-1]}(a + b) \ge \varphi^{[-1]}(a) \cdot \psi^{[-1]}(b) \quad \textit{for all } \ a, b \ge 0.$$

(ii) C is negatively quadrant dependent (i.e. $C \prec \Pi$) if, and only if,

$$\varphi\left(\exp[-(a + b)]\right) \le \varphi\left(\exp(-a)\right) + \psi(\exp(-b)) \quad \textit{for all } \ a, b \ge 0.$$

Proof The copula Π is generated by the pair $(-\ln t, -\ln t)$. Then, in view of Theorem 5.1, $C \succ \Pi$ if, and only if, for all $a, b \ge 0$

$$-\ln\left(\varphi^{[-1]}(a + b)\right) \le -\ln\left(\varphi^{[-1]}(a)\right) - \ln\left(\psi^{[-1]}(b)\right),$$

viz.

$$\varphi^{[-1]}(a + b) \ge \varphi^{[-1]}(a) \cdot \psi^{[-1]}(b).$$

The proof of the other case is analogous.

In two particular cases, the concordance order between two copulas of type (8) can be expressed in a form simpler than Eq.(10).

Corollary 5.2 *Let C and D be two copulas of type* (8) *generated, respectively, by the pairs* (φ, ψ) *and* (γ, η). *Let* $\alpha := \varphi \circ \gamma^{[-1]}$ *and* $\beta := \psi \circ \eta^{[-1]}$.

(a) If $\varphi = \gamma$ is a strictly decreasing function with $\varphi(0) = +\infty$ and if $\eta(0) = +\infty$, then $C \prec D$ if, and only if, $\psi(t) \ge \eta(t)$ for every $t \in [0, 1]$.

(b) If $\psi = \eta$ is a strictly decreasing function with $\psi(0) = +\infty$, then $C \prec D$ if, and only if, α is 1*–Lipschitz.*

Proof Since $\varphi = \gamma$ admits an inverse, one has $\alpha(t) = t$. Therefore (10) is equivalent to

$$b \leq \beta(b) = \psi \circ \eta^{[-1]}(b) \quad \text{for all } b \in [0, +\infty] .$$

Taking $b := \eta(t)$, one has $C \prec D$ if, and only if, $\psi(t) \geq \eta(t)$ for every $t \in [0, 1]$.

Analogously, for (b), since $\psi = \eta$ admits an inverse, one has $\beta(t) = t$, and (10) is equivalent to

$$\alpha(a + b) - \alpha(a) \leq b \quad \text{for all } a, b \in [0, +\infty] .$$

6 Concluding remarks

The concept of copula has an extension to the n-dimensional case ($n \geq 3$).

Definition 6.1 *A function $C : [0, 1]^n \to [0, 1]$ is an n-copula if, and only if, it satisfies the following conditions:*

(i) $C(\mathbf{x}) = 0$ if at least one coordinate of $\mathbf{x} \in [0, 1]^n$ is 0, and $C(\mathbf{x}) = x_i$ if all the coordinates of $\mathbf{x} \in [0, 1]^2$ are 1 except the i-th one;
(ii) C is n-increasing, viz., for all $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ in $[0, 1]^n$ with $a_i \leq b_i, i = 1, 2, \ldots, n$,

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C\left(x_{1_{i_1}}, \ldots, x_{n_{i_n}}\right) \geq 0,$$

where $x_{j_1} = a_j$ and $x_{j_2} = b_j$ for all $j \in \{1, 2, \ldots, n\}$.

Thus, it seems natural to construct the multivariate extension of this new family and a possible way is illustrated here.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a point in $[0, 1]^n$ and let $x_{(1)}, \ldots, x_{(n)}$ be the components of $\mathbf{x}$ rearranged in increasing order. Consider $\varphi_1 \in \Phi$ and $\varphi_2, \ldots, \varphi_n \in \Psi$ and define

$$C_{\varphi_1,\ldots,\varphi_n}(x_1, \ldots, x_n) := \varphi_1^{[-1]}\left(\varphi_1(x_{(1)}) + \varphi_2(x_{(2)}) + \cdots + \varphi_n(x_{(n)})\right) . \tag{11}$$

Notice that a multivariate Archimedean copula can be defined by taking $\varphi_1 = \varphi_2 = \cdots = \varphi_n$ in Eq.(11) and by supposing that $\varphi_1(1) = 0$ and $\varphi_1(0) = +\infty$.

It is easily shown that this function satisfies the boundary conditions for a multivariate copula; however, it is very hard to verify that it is n-increasing. In particular, the proof of Theorem 3.1 cannot be adapted to the case $n \geq 3$, because of the great number of cases that ought to be considered. We plan to deal with the multivariate case in a forthcoming paper.

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