AISM (2006) 58: 609–618
DOI 10.1007/s10463-005-0020-x

N. Balakrishnan · Po Yang

Connections between the resolutions of general two-level factorial designs

Received: 28 February 2005 / Published online: 13 June 2006
© The Institute of Statistical Mathematics, Tokyo 2006

Abstract Indicator functions are new tools for studying two-level fractional factorial designs. This article discusses some properties of indicator functions. Using indicator functions, we study the connection between general two-level factorial designs of generalized resolutions.

Keywords Indicator function · Fractional word length · Generalized resolution · Resolution III^* design · Extended word length pattern

1 Introduction

Regular two-level factorial designs have been studied quite extensively. These designs have defining relations which can be used to study many properties of these designs. Non-regular designs have not been well studied since they have no defining relations. However, sometimes non-regular designs are more useful than regular designs since they need fewer runs; see, for example, Addelman (1961), Westlake (1965) and Draper (1985).

Pistone and Wynn (1996) introduced a method based on Gröbner bases, an area in computational commutative algebra, to study the identifiability problem in experimental designs. Subsequently, Fontana et al. (2000) introduced the indicator function as a tool to study fractional factorial designs without replicates, which was

N. Balakrishnan
Department of Mathematics and Statistics,
McMaster University, Hamilton L8S 4K1, ON, Canada
E-mail: bala@mcmail.cis.mcmaster.ca

Po Yang (✉)
Department of Mathematics and Statistics,
University of Prince Edward Island,
550 University Avenue, Charlottetown C1A 4P3,
PE, Canada
E-mail: pyang@upei.ca

subsequently extended to the case of replication by Ye (2003). Indicator functions are very useful when dealing with general two-level factorial designs. Applications of Gröbner bases to this and other areas of statistics have been discussed in detail by Pistone et al. (2001).

Deng and Tang (1999) and Tang and Deng (2000) defined the generalized resolution, that is, fractional resolution, and generalized aberration criteria for general two-level factorial designs. These criteria were redefined through indicator functions by Ye (2003) and Li et al. (2003). In this article, the definition of fractional resolutions by Li et al. (2003) is used and we will denote fractional resolutions by $N{\cdot}s$, where N is an integer and s is a fraction. A resolution $N^* \cdot s$ design is a resolution $N{\cdot}s$ design such that its indicator function contains no $(N+1)$-letter word.

Regular resolution III^* designs are regular resolution III designs in which no two-factor interactions are confounded with one another. This class of designs is valuable in composite designs and were first examined by Hartley (1959). Draper and Lin (1990) found that there is a connection between regular resolutions III^* and V designs; especially, they showed that resolution III^* m-factor regular designs can be converted into resolution V $(m-1)$-factor regular designs and, conversely, resolution V m-factor regular designs can be converted into resolution III^* $(m+1)$-factor regular designs, so that one can obtain resolution III^* designs through well-known resolution V designs. In this article, we extend these results to a more general case through indicator functions. We show that resolution III^* designs can be converted into designs with a resolution at least V; on the other hand, one can obtain regular resolution III^* designs from not only regular resolution V designs, but also regular designs with resolution greater than V which are known and have better properties from a practical point of view. More general results are also obtained. In particular, our results show that these generalizations are also true for non-regular designs.

In Sect. 2, we introduce indicator functions, generalized resolution and the extended word length pattern. Some properties of indicator functions and resolution $N^*{\cdot}s$ designs are discussed in Sect. 3. In Sect. 4, we provide a way to convert resolution $(2l-1)^*{\cdot}s$ designs to designs of resolution at least $(2l+1){\cdot}s$. A link between resolution $(2l-1){\cdot}s$ designs and resolution $2l\cdot s$ designs is also presented in this section. Finally, in Sect. 5, we show that resolution $III^*{\cdot}s$ designs can be obtained from designs with resolutions equal to or greater than V.

2 Indicator functions, fractional word length and extended word length pattern

Let D_{2^m} be the full two-level m-factor design, i.e., $D_{2^m}=\{x=(x_1, x_2, \ldots, x_m) \| x_i = 1 \text{ or } -1,\ i=1,2,\ldots,m\}$, $M=\{1,2,\ldots,m\}$, $L_{2^m} = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) | \alpha_i=1 \text{ or } 0\ \forall i \in M\}$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$, and $\| \alpha \| = \sum_{i=1}^m \alpha_i$. Then, $\| \alpha \|$ is the number of letters of x^α.

Let $\mathcal{F}$ be any two-level m-factor design such that for any $x \in \mathcal{F}$, $x \in D_{2^m}$, but x might be repeated in $\mathcal{F}$. The *indicator function* of $\mathcal{F}$ is a function $f(x)$ defined on D_{2^m} such that $f(x) = r_x$, where r_x is the number of appearances of the point x in design $\mathcal{F}$. In this article, we assume that $\mathcal{F}$ contains some but not all the points in D_{2^m}.

Fontana et al. (2000) and Ye (2003) showed that the indicator function $f(x)$ of $\mathcal{F}$ can be uniquely represented by a polynomial function

$$f(x) = \sum_{\alpha \in L_{2^m}} b_\alpha x^\alpha, \qquad (1)$$

where the coefficients $\{b_\alpha, \alpha \in L_{2^m}\}$ can be determined as $b_\alpha = \frac{1}{2^m} \sum_{x \in \mathcal{F}} x^\alpha$. In particular, $|b_\alpha/b_0| \leq 1$ and $b_0 = n/2^m$, where n is the total number of runs in $\mathcal{F}$.

Thus, given a design, we can find the coefficients of its indicator function. For a regular design, its indicator function is easy to find. For example, if $I=x_1x_4x_5$ and $I=x_2x_3x_4x_6$ are generators of a two-level 6-factor design, then we can easily check that the corresponding indicator function is $f(x) = \frac{1}{2^2}(1+x_1x_4x_5)(1+x_2x_3x_4x_6) = \frac{1}{4}(1 + x_1x_4x_5 + x_2x_3x_4x_6 + x_1x_2x_3x_5x_6)$.

Conversely, given an indicator function of a two-level factorial design $\mathcal{F}$, we can check whether it represents a regular design or not. Fontana et al. (2000) and Ye (2004) showed that $\mathcal{F}$ is a regular design (with or without replicates) if and only if $|\, b_\alpha/b_0 \,| = 1$ for all nonzero b_α in the indicator function of $\mathcal{F}$.

Example 1 An indicator function of a two-level 5-factor design $\mathcal{F}$ without replicate is $f(x) = \frac{1}{2} - \frac{1}{4}x_1x_2x_3 + \frac{1}{4}x_2x_3x_4 + \frac{1}{4}x_2x_3x_5 + \frac{1}{4}x_1x_2x_3x_4x_5$. Since there exists nonzero b_α such that $|b_\alpha/b_0| \neq 1$, $\mathcal{F}$ is not a regular design.

Li et al. (2003) extended the traditional definition of the word to non-regular designs by calling each term with a nonzero coefficient (except the constant) in the indicator function of a design a *word*. If x^α is a word, its *length* is defined as $||x^\alpha|| = ||\alpha|| + (1 - |b_\alpha/b_0|)$. Thus for regular designs, since $|b_\alpha/b_0|=1$, for each word x^α, its length is the number of letters of the word; for non-regular designs, the length of words may be fractional since $|b_\alpha/b_0|$ may be less than 1. In Example 1, the length of the word $x_1x_2x_3$ is 3.5. The *extended word length pattern* of $\mathcal{F}$ is defined to be $(f_1, \ldots, f_{1+(n-1)/n}, f_2, \ldots, f_{2+(n-1)/n}, \ldots, f_m, \ldots, f_{m+(n-1)/n})$, where $f_{i+j/n}$ is the number of length $(i + j/n)$ words. The *generalized resolution* is defined as the length of the shortest word. Thus, the resolution of the design in Example 1 is $III^* \cdot 5$. Given two extended word length patterns of two designs, the aberration criterion is defined by sequentially comparing the two extended word length patterns from the shortest-length word to the longest-length word.

In the following sections, $\mathcal{W}^\mathcal{F} = \{\alpha \in L_{2^m} \,|\, b_\alpha \neq 0$ and $\| \alpha \| \neq 0\}$, $\mathcal{W}_1^\mathcal{F} = \{\alpha \in \mathcal{W}^\mathcal{F} \,|\, \| \alpha \|$ is odd$\}$, and $\mathcal{W}_2^\mathcal{F} = \{\alpha \in \mathcal{W}^\mathcal{F} \,|\, \| \alpha \|$ is even$\}$.

3 Properties of indicator functions and resolution $N^* \cdot s$ designs

In this section, we will study some properties of indicator functions which will be used in the later sections. We will show that there is no $(2l + 1)$-factor design (without replicate) of resolution $(2l - 1)^* {\cdot} s$ when the run size of the design is not equal to 2^{2l}.

Proposition 1 *Assume that $\mathcal{F}$ is a two-level m-factor design and $f(x) = b_0 + b_\alpha x^\alpha$ is its indicator function. Then all the points in $\mathcal{F}$ have the same number of replicates $2b_0$ and $\mathcal{F}$ is a regular design.*

Proof For any $x \in D_{2^m}$, x^α can only be 1 or -1. Thus, $f(x)$ equals either $b_0 + b_\alpha$ or $b_0 - b_\alpha$. Since $\mathcal{F}$ contains some but not all the points in D_{2^m}, there exists at least one $x \in D_{2^m}$ such that $f(x)$=0. Thus, either $b_0 + b_\alpha$=0 or $b_0 - b_\alpha$=0. Consider $b_0 + b_\alpha$=0. Let $b_0 - b_\alpha = a \neq 0$. Then, any point x such that $f(x)$=$b_0 - b_\alpha$=a is in $\mathcal{F}$ and has the same number of replicates a. In this case, we have a=2b_0 and b_α=$-b_0$. When $b_0 - b_\alpha$=0, the proof follows similarly. In this case, a=2b_0 and b_α=b_0. Thus, $|b_\alpha/b_0|$=1, which shows $\mathcal{F}$ is a regular design.

Proposition 2 *Assume that $\mathcal{F}$ is a two-level m-factor design. For any $x \in \mathcal{F}$, if all the words in its indicator function are odd words, then the sum of the number of replicates of the points $x = (x_1, x_2, \ldots, x_m)$ and $-x = (-x_1, -x_2, \ldots, -x_m)$ is $2b_0$. If all the words in its indicator function are even words, then the points x and $-x$ have the same number of replicates.*

If there is no replicate and all the words in its indicator function are odd words, then either the point x or the point $-x$ is in $\mathcal{F}$ and $b_0 = 1/2$. In other words, if $b_0 \neq 1/2$, then there exists at least one $\alpha \in \mathcal{W}^{\mathcal{F}}$ such that $\| \alpha \|$ is even. In this case, there are at least two words in the indicator function.

Proof Assume that (1) is the indicator function of $\mathcal{F}$. Then for any $x \in \mathcal{F}$, $f(x) = b_0 + \sum_{\alpha \in \mathcal{W}^{\mathcal{F}}} b_\alpha x^\alpha$. If all the words are odd words, then $f(-x)$=$b_0 - \sum_{\alpha \in \mathcal{W}^{\mathcal{F}}} b_\alpha x^\alpha$. Thus $f(x) + f(-x)$=$2b_0$. If all the words are even words, the proof $f(x)$=$f(-x)$ follows similarly.

Assume that there is no replicate and all the words in its indicator function are odd words. Since $\#\mathcal{F} \neq 0, 2^m$, $0 < b_0 < 1$. It follows that $0 < 2b_0 < 2$. Note that $f(x) + f(-x)$=$2b_0$ and $f(x)$ and $f(-x)$ can only be 0 or 1, and we get $f(x) + f(-x) = 1$ for all $x \in D_{2^m}$. It then implies that $f(x) \neq f(-x)$, that is, either the point x or the point $-x$ is in $\mathcal{F}$ and b_0=1/2.

If $b_0 \neq 1/2$, by Proposition 1, there are at least two words in the indicator function.

Proposition 2 shows that a design with only odd words implies a half fraction. The result "If all the words in the indicator function are even words, then the points x and $-x$ have the same number of replicates" has also been given informally by Ching-Shui Cheng (a personal communication). He has shown that a design with only even words is a foldover of another design.

Example 1 shows that in the case of non-regular designs without replicate, if $b_0 = 1/2$ and $\#\mathcal{W}^{\mathcal{F}} \geq 2$, then it is possible that all the words in the indicator function are odd words.

Hartley (1959) pointed out that there is no regular $2^{5-2}_{III^*}$ design. Proposition 3 shows that this is also true in general.

Proposition 3 *Assume that $\mathcal{F}$ is a (2l+1)-factor resolution $(2l-1) \cdot s$ design without replicate. Then, it is not a resolution $(2l-1)^* \cdot s$ design if $\#\mathcal{F} \neq 2^{2l}$.*

Proof Since the design $\mathcal{F}$ has only $2l+1$ factors, it has no $(2l+2)$-letter word. If $\#\mathcal{F} \neq 2^{2l}$, then $b_0 \neq 1/2$. By Proposition 2, it must have a $(2l)$-letter word.

Example 1 shows that when $\mathcal{F}$ is a 5-factor design and $\#\mathcal{F} = 2^4$, there exists a design of resolution $III^* \cdot 5$.

4 Changing resolutions by converting a m-factor design into a $(m-1)$-factor design

In this section, we extend Theorems 1 and 2 of Draper and Lin (1990) to a more general case. We show that regular resolution III^* designs can be converted into designs with resolution at least V. A more general theorem will be proved in this section. As another special case of this theorem, a relation between designs of resolution $(2l-1)\cdot s$ and resolution $2l \cdot s$ is also presented. For this purpose, we will use the same transformations as Draper and Lin (1990) used in their work.

Assume that $x_1, x_2, \dots, x_m$ are the m factors that form a two-level factorial design $\mathcal{F}$ with the indicator function (1). Let x_k be any of the m factors and let $y_j = x_k x_j$, $j=1,2,\dots,m$ $(j \neq k)$. Then $y_1, y_2, \dots, y_{k-1}, y_{k+1}, \dots, y_m$ form a $(m-1)$-factor two-level factorial design $\hat{\mathcal{F}}$. We define $\alpha'_i = \alpha_i$, $i=1,2,\dots,m$ $(i \neq k)$. Then $x^\alpha = x_k^{\alpha_k}[\prod_{i\neq k}(x_k y_i)^{\alpha_i}] = x_k^{\|\alpha\|} y^{\alpha'}$, and $\| \alpha' \| = \| \alpha \|$ if $\alpha_k = 0$ and $\| \alpha \| -1$ if $\alpha_k = 1$.

Lemma 1 gives the indicator function of $\hat{\mathcal{F}}$. Lemma 2 provides the resolution of $\hat{\mathcal{F}}$ and follows from Lemma 1 and the above discussion directly.

Lemma 1 *Let $\mathcal{F}$ be a two-level m-factor design and $\hat{\mathcal{F}}$ be the corresponding $(m-1)$-factor design. If (1) is the indicator function of $\mathcal{F}$, then $g(y) = 2b_0 + 2\sum_{\alpha \in \mathcal{W}_2^{\mathcal{F}}} b_\alpha y^{\alpha'}$ is the indicator function of $\hat{\mathcal{F}}$.*

Proof From the above discussion, the indicator function of $\mathcal{F}$ can be written as $f(x) = b_0 + x_k \sum_{\alpha \in \mathcal{W}_1^{\mathcal{F}}} b_\alpha y^{\alpha'} + \sum_{\alpha \in \mathcal{W}_2^{\mathcal{F}}} b_\alpha y^{\alpha'}$. This can also be seen as the indicator function of the design with the factors $y_1, y_2, \dots, y_{k-1}, x_k, y_{k+1}, \dots, y_m$. When it is projected onto $y_1, y_2, \dots, y_{k-1}, y_{k+1}, \dots, y_m$, the resulting projected design is $\hat{\mathcal{F}}$. By Theorem 1 in Ye (2003), the indicator function of the projected design is $g(y) = 2b_0 + 2\sum_{\alpha \in \mathcal{W}_2^{\mathcal{F}}} b_\alpha y^{\alpha'}$.

Although $g(y)$ is only related to the even words in $f(x)$, the words in $g(y)$ can be odd words, that is, there may exist $\alpha \in \mathcal{W}_2^{\mathcal{F}}$ such that the length of its corresponding α' is odd. When there is only one even word in $f(x)$, $g(y)$ has only one word. By Proposition 1, the design is a regular design. When all the words in $f(x)$ are odd words, $g(y) = 2b_0$ for all $y \in D_{2^{m-1}}$, that is, $\hat{\mathcal{F}}$ is a full two-level $(m-1)$-factor design with $2b_0$ replicates for each point in $\hat{\mathcal{F}}$.

Lemma 2 *Let $\mathcal{F}$ be a two-level m-factor design and $\hat{\mathcal{F}}$ be the corresponding $(m-1)$-factor design. Assume that $2r$ is the number of letters in the shortest even word in (1). Let $A = \{\alpha \in \mathcal{W}_2^{\mathcal{F}} \mid \| \alpha \| = 2r\}$. Then, the resolution of $\hat{\mathcal{F}}$ is $(2r-1)\cdot s$ if there exists an $\alpha \in A$ such that $\alpha_k = 1$ and $2r \cdot t$ otherwise.*

To obtain the fractions s and t in Lemma 2, if there exists an $\alpha \in A$ such that $\alpha_k = 1$, and let $A_1 = \{\alpha \in A \mid \alpha_k = 1\}$, then $s = \min\{1 - |b_\alpha/b_0|,\ \alpha \in A_1\}$; otherwise, $t = \min\{1 - |b_\alpha/b_0|,\ \alpha \in A\}$.

Theorem 1 shows the relation between the resolutions of the original design $\mathcal{F}$ and the resolutions of the transformed design $\hat{\mathcal{F}}$.

Theorem 1 *Let $\mathcal{F}$ be a m-factor two-level fractional factorial design with the indicator function (1). Assume that* $2r$ *is the number of letters in the shortest even word in (1). Then regardless of what resolution of $\mathcal{F}$ is, $\mathcal{F}$ can always be converted into a* $(m-1)$*-factor design* $\hat{\mathcal{F}}$ *of resolution* $(2r-1)\cdot s$ *in the same number of runs. If there exists a* $k \in \{1, 2, \dots, m\}$ *such that for any* $\alpha \in A$, $\alpha_k \neq 1$*, then $\mathcal{F}$ can be converted into a* $(m-1)$*-factor design* $\hat{\mathcal{F}}$ *of resolution* $2r \cdot t$.

Proof Let $\alpha \in A$. Consider any k such that $\alpha_k=1$. By Lemma 2, we get the first result. The second result follows from Lemma 2 directly.

The following corollaries are obtained readily from Theorem 1. Corollary 1 is a generalization of Theorems 1 and 2 of Draper and Lin (1990), while Corollary 2 is an extension of Corollary 1 of Draper and Lin (1990).

Corollary 1 *A m-factor two-level fractional factorial design of resolution* $(2l-1)^* \cdot s_1$ *can be converted into a design of resolution at least* $(2l+1)\cdot s_2$ *in the same number of runs.*

Corollary 2 *Assume that a design of resolution* $III^* \cdot s_1$ *can be converted into a design of resolution* $V \cdot s_2$*. Then, if m is the maximum number of factors that can be accommodated in the design of resolution* $III^* \cdot s_1$*, then the maximum number of factors that can be accommodated in the design of resolution* $V \cdot s_2$ *with the same number of runs is at least* $m-1$.

Example 2 An indicator function of a 6-factor design is $f(x) = \frac{1}{4} + \frac{1}{8}x_1x_4x_5 + \frac{1}{8}x_2x_3x_6 - \frac{1}{8}x_1x_5x_6 - \frac{1}{8}x_2x_3x_4 - \frac{1}{8}x_2x_5x_6 - \frac{1}{8}x_1x_3x_6 - \frac{1}{8}x_2x_4x_5 - \frac{1}{8}x_1x_3x_4 + \frac{1}{4}x_1x_2x_3x_4x_5x_6$. This is a resolution $III^* \cdot 5$ design. Take, for example, k=6 (one can take any i, i=1, 2, ... , 6), that is, $y_i=x_6x_i$, i=1, 2, ... , 5. Since $f(x)$ only contains one even-letter word $x_1x_2x_3x_4x_5x_6$, by Corollary 1, $\mathcal{F}$ can be converted into a resolution V design and $g(y) = \frac{1}{2} + \frac{1}{2}y_1y_2y_3y_4y_5$.

Corollaries 3 and 4 provide connections between two-level designs of resolution $(2l-1)\cdot s$ and resolution $2l \cdot t$. In particular, when l=2, they show connections between two-level designs of resolution $III \cdot s$ and resolution $IV \cdot t$.

Corollary 3 *Let $\mathcal{F}$ be a two-level m-factor design of resolution* $(2l-1)\cdot s$*. If there is a 2l-letter word in the indicator function of $\mathcal{F}$ and there exists a* $k \in \{1, 2, \dots, m\}$ *such that for any* $\alpha \in A$, $\alpha_k \neq 1$*, then $\mathcal{F}$ can be converted into a* $(m-1)$*-factor design* $\hat{\mathcal{F}}$ *of resolution* $2l \cdot t$ *in the same number of runs.*

Corollary 4 *Let $\mathcal{F}$ be a two-level m-factor design of resolution* $2l \cdot t$*. Then, $\mathcal{F}$ can be converted into a* $(m-1)$*-factor design* $\hat{\mathcal{F}}$ *of resolution* $(2l-1)\cdot s$ *in the same number of runs.*

Example 3 An indicator function of a 7-factor regular design with generators $x_5=x_1x_2x_4$, $x_6=x_1x_3$, and $x_7=x_2x_3$ is $f(x) = \frac{1}{8}+\frac{1}{8}x_1x_3x_6+\frac{1}{8}x_2x_3x_7+\frac{1}{8}x_1x_2x_4x_5+\frac{1}{8}x_4x_5x_6x_7 + \frac{1}{8}x_1x_2x_6x_7 + \frac{1}{8}x_2x_3x_4x_5x_6 + \frac{1}{8}x_1x_3x_4x_5x_7$. This is a resolution III design. Since there exists a $k(=3)$ such that x_3 is not present in all the four-letter words, $\mathcal{F}$ can be converted into a 6-factor design of resolution IV and its indicator function is $g(y) = \frac{1}{4} + \frac{1}{4}y_1y_2y_4y_5 + \frac{1}{4}y_4y_5y_6y_7 + \frac{1}{4}y_1y_2y_6y_7$.

Example 4 An indicator function of a 7-factor design is $f(x) = \frac{3}{4} + \frac{1}{4}x_1x_3x_4x_7 + \frac{1}{4}x_1x_2x_4x_5 + \frac{1}{4}x_2x_3x_5x_7 + \frac{1}{2}x_2x_3x_4x_6x_7$. This is a resolution $4\frac{2}{3}$ design. If we take k=1, then $\mathcal{F}$ is converted into a 6-factor design of resolution III and its indicator function is $g(y) = \frac{3}{2} + \frac{1}{2}y_3y_4y_7 + \frac{1}{2}y_2y_4y_5 + \frac{1}{2}y_2y_3y_5y_7$.

5 Obtaining a resolution III^* design by converting a m-factor design into a $(m + 1)$-factor design

Assume that $y_i{=}(x_kx_l)x_i,\ i{=}1, 2, \dots, m,\ y_{m+1} = x_kx_l$. Then, $y_1, y_2, \dots, y_{m+1}$ form a $(m+1)$-factor two-level factorial design and $x_l{=}y_k,\ x_k{=}y_l$, and $x_i{=}y_ky_ly_i$, for any $i \neq k, l$. Since

$$x^{\alpha} = \left[\prod_{\substack{i=1\\ i\neq k,l}}^{m}(y_ky_ly_i)^{\alpha_i}\right] y_k^{\alpha_l}y_l^{\alpha_k} = \left[\prod_{\substack{i=1\\ i\neq k,l}}^{m}(y_i)^{\alpha_i}\right] y_k^{\|\alpha\|-\alpha_k}y_l^{\|\alpha\|-\alpha_l},$$

we define

$$\alpha_i' = \begin{cases} \alpha_i, & \text{if } 1\leq i\leq m \text{ and } i\neq k, l,\\ \|\alpha\| - \alpha_k (\text{mod } 2), & \text{if } i{=}k,\\ \|\alpha\| - \alpha_l (\text{mod } 2), & \text{if } i{=}l,\\ 0, & \text{if } i{=}m+1. \end{cases} \tag{2}$$

Then $x^{\alpha} = y^{\alpha'}$.

Lemma 3 gives the indicator function of the transformed design.

Lemma 3 *Let $\mathcal{F}$ be a two-level m-factor design and $\hat{\mathcal{F}}$ be the transformed $(m+1)$-factor design. If (1) is the indicator function of $\mathcal{F}$, then the indicator function of $\hat{\mathcal{F}}$ is $g(y) = \frac{1}{2}(\sum_{\alpha\in L_{2^m}} b_{\alpha}y^{\alpha'} + \sum_{\alpha\in L_{2^m}} b_{\alpha}y^{\alpha'+\varphi(mod\ 2)})$, where φ is a $1 \times (m + 1)$ vector such that k, l and the $(m + 1)$-th entries are 1 and all others are 0.*

Proof The indicator function of the $2^{(m+1)-1}$ design with the defining relation $y_{m+1} = y_ky_l$ is $f_0(y) = \frac{1}{2}(1+y_ky_ly_{m+1})$. From the above discussion, the right hand side of (1) can be written as $\sum_{\alpha\in L_{2^m}} b_{\alpha}y^{\alpha'}$. Thus, the indicator function of the design formed by the factors $y_1, y_2, \dots, y_m$ is $f_1(y) = \sum_{\alpha\in L_{2^m}} b_{\alpha}y^{\alpha'}$. Therefore, the indicator function of $\hat{\mathcal{F}}$ is $g(y) = f_0(y)f_1(y) = \frac{1}{2}(1 + y_ky_ly_{m+1})\sum_{\alpha\in L_{2^m}} b_{\alpha}y^{\alpha'} = \frac{1}{2}(\sum_{\alpha\in L_{2^m}} b_{\alpha}y^{\alpha'} + \sum_{\alpha\in L_{2^m}} b_{\alpha}y^{\alpha'+\varphi(\text{mod } 2)})$.

The following theorem is a generalization of the converse of Theorem 1 in Draper and Lin (1990). It shows that a regular resolution III^* design can be obtained from a design with resolution of not only V, but also greater than V. The result is also true for general two-level factorial designs.

Theorem 2 *Any m-factor two-level fractional factorial design $\mathcal{F}$ with resolution at least V can be converted into a $(m + 1)$-factor design $\hat{\mathcal{F}}$ of resolution III^*.*

Proof By (2),

$$\| \alpha' \| = \begin{cases} \| \alpha \| - 2, & \text{if } \| \alpha \| \text{ is odd, and } \alpha_k{=}1, \alpha_l{=}1, \\ \| \alpha \| + 2, & \text{if } \| \alpha \| \text{ is odd, and } \alpha_k{=}0, \alpha_l{=}0, \\ \| \alpha \|, & \text{otherwise,} \end{cases} \tag{3}$$

$$(\alpha' + \varphi)_i = \begin{cases} \alpha_i, & \text{if } 1 \le i \le m \text{ and } i \ne k, l, \\ \| \alpha \| - \alpha_k + 1 (\text{mod } 2), & \text{if } i{=}k, \\ \| \alpha \| - \alpha_l + 1 (\text{mod } 2), & \text{if } i{=}l, \\ 1, & \text{if } i{=}m+1, \end{cases}$$

and therefore

$$\| \alpha' + \varphi \| = \begin{cases} \| \alpha \| - 1, & \text{if } \| \alpha \| \text{ is even, and } \alpha_k{=}1, \alpha_l{=}1, \\ \| \alpha \| + 3, & \text{if } \| \alpha \| \text{ is even, and } \alpha_k{=}0, \alpha_l{=}0 \\ \| \alpha \| + 1, & \text{otherwise.} \end{cases} \tag{4}$$

By (3) and (4), for any $\alpha \in \mathcal{W}^{\mathcal{F}}$ such that $\| \alpha \| \ge 5$, $\| \alpha' \|$ and $\| \alpha' + \varphi \|$ are all at least 3 but not equal to 4. From Lemma 3, the indicator function of $\hat{\mathcal{F}}$ has a word $y^{\varphi} = y_k y_l y_{m+1}$ and its length is 3. Thus, the resolution of $\hat{\mathcal{F}}$ is III^*.

The following corollary is an extension of Corollary 2 of Draper and Lin (1990).

Corollary 5 *If $(m-1)$ is the maximum number of factors that can be accommodated in a design $\mathcal{F}$ of resolution $V \cdot s$, then the maximum number of factors that can be accommodated in a resolution III^* design with the same number of runs is at least m.*

Example 5 An indicator function of a two-level 11-factor resolution VII design is $f(x) = \frac{1}{4} + \frac{1}{4} x_2 x_3 x_5 x_6 x_7 x_{10} x_{11} + \frac{1}{4} x_1 x_3 x_4 x_5 x_8 x_9 x_{11} + \frac{1}{4} x_1 x_2 x_4 x_6 x_7 x_8 x_9 x_{10}$.

If we consider $k = 2$ and $l = 3$, then $x_2 x_3 x_5 x_6 x_7 x_{10} x_{11} = y_5 y_6 y_7 y_{10} y_{11}$, $x_1 x_3 x_4 x_5 x_8 x_9 x_{11} = y_1 y_2 y_4 y_5 y_8 y_9 y_{11}$, and $x_1 x_2 x_4 x_6 x_7 x_8 x_9 x_{10} = y_1 y_2 y_4 y_6 y_7 y_8 y_9 y_{10}$. Thus, by Lemma 3, the indicator function of the transformed design is $g(y) = \frac{1}{8} + \frac{1}{8} y_2 y_3 y_{12} + \frac{1}{8} y_5 y_6 y_7 y_{10} y_{11} + \frac{1}{8} y_1 y_2 y_4 y_5 y_8 y_9 y_{11} + \frac{1}{8} y_1 y_2 y_4 y_6 y_7 y_8 y_9 y_{10} + \frac{1}{8} y_2 y_3 y_5 y_6 y_7 y_{10} y_{11} y_{12} + \frac{1}{8} y_1 y_3 y_4 y_5 y_8 y_9 y_{11} y_{12} + \frac{1}{8} y_1 y_3 y_4 y_6 y_7 y_8 y_9 y_{10} y_{12}$. This is a resolution III^* design.

Example 6 An indicator function of a two-level 9-factor design is $f(x) = \frac{3}{4} + \frac{1}{4} x_1 x_3 x_4 x_6 x_7 + \frac{1}{4} x_2 x_3 x_5 x_6 x_9 + \frac{1}{4} x_1 x_2 x_4 x_5 x_7 x_9 + \frac{1}{2} x_1 x_2 x_5 x_6 x_8 x_9$.

If we consider $k{=}1$ and $l{=}6$, then $x_1 x_3 x_4 x_6 x_7 = y_3 y_4 y_7$, $x_2 x_3 x_5 x_6 x_9 = y_1 y_2 y_3 y_5 y_9$, $x_1 x_2 x_4 x_5 x_7 x_9 = y_1 y_2 y_4 y_5 y_7 y_9$, and $x_1 x_2 x_5 x_6 x_8 x_9 = y_1 y_2 y_5 y_6 y_8 y_9$. Thus, by Lemma 3, the indicator function of the transformed design is $g(y) = \frac{3}{8} + \frac{3}{8} y_1 y_6 y_{10} + \frac{1}{8} y_3 y_4 y_7 + \frac{1}{8} y_1 y_2 y_3 y_5 y_9 + \frac{1}{8} y_1 y_2 y_4 y_5 y_7 y_9 + \frac{1}{4} y_1 y_2 y_5 y_6 y_8 y_9 + \frac{1}{8} y_1 y_3 y_4 y_6 y_7 y_{10} + \frac{1}{8} y_2 y_3 y_5 y_6 y_9 y_{10} + \frac{1}{4} y_2 y_5 y_8 y_9 y_{10} + \frac{1}{8} y_2 y_4 y_5 y_6 y_7 y_9 y_{10}$. The word lengths of the 3-letter words $y_1 y_6 y_{10}$ and $y_3 y_4 y_7$ are 3 and $3\frac{2}{3}$, respectively. Thus, the resolution of the transformed design is III. Since there is no four-letter word in its indicator function, the transformed design is therefore of resolution III^*.

If we consider $k=4$ and $l=8$ (note that x_8 is not in any five-letter word), then $x_1x_3x_4x_6x_7 = y_1y_3y_6y_7y_8$, $x_2x_3x_5x_6x_9 = y_2y_3y_4y_5y_6y_8y_9$, $x_1x_2x_4x_5x_7x_9 = y_1y_2y_4y_5y_7y_9$, and $x_1x_2x_5x_6x_8x_9 = y_1y_2y_5y_6y_8y_9$. Thus, the indicator function of the transformed design is $g(y) = \frac{3}{8} + \frac{3}{8}y_4y_8y_{10} + \frac{1}{8}y_1y_3y_6y_7y_8 + \frac{1}{4}y_1y_2y_5y_6y_8y_9 + \frac{1}{8}y_1y_2y_4y_5y_7y_9 + \frac{1}{8}y_1y_3y_4y_6y_7y_{10} + \frac{1}{8}y_2y_3y_5y_6y_9y_{10} + \frac{1}{4}y_1y_2y_4y_5y_6y_9y_{10} + \frac{1}{8}y_2y_3y_4y_5y_6y_8y_9 + \frac{1}{8}y_1y_2y_5y_7y_8y_9y_{10}$. There is only one three-letter word $y_4y_8y_{10}$ and its word length is 3. Note that there is no four-letter word in the indicator function and hence this is also a resolution III^* design. However, if we compare the two transformed designs by the minimum aberration criteria, the second design is better since it has only one three-letter word.

Hence, when we choose k or l, it is better to choose the one whose factor is not contained in any five-letter word.

6 Conclusions

In this article, we have studied some properties of indicator functions. Using indicator functions, we have showed that there are connections between designs of resolution III^* and designs of resolution greater than or equal to V. These have generalized the results of Draper and Lin (1990) and are also true for general two-level factorial designs. Theorem 4.1 in Sect. 4 is a more general result. As a special case of the theorem, we have also provided a connection between resolution $(2l-1)\cdot s$ designs and resolution $2l \cdot s$ designs.

Acknowledgements The first author thanks the Natural Sciences and Engineering Research Council of Canada for funding this research, while the second author acknowledges the partial support received from her Ontario Graduate Scholarship.

References

Addelman, S. (1961). Irregular fractions of the 2^n factorial experiments. *Technometrics*, *3*, 479–496.

Cheng, S. W., Ye, K. Q. (2004). Geometric isomorphism and minimum aberration for factorial designs with quantitative factors. *The Annals of Statistics*, *32*, 2168–2185.

Deng, L-Y., Tang, B. (1999). Generalized resolution and minimum aberration criteria for Plackett-Burman and other non-regular factorial designs. *Statistica Sinica*, *9*, 1071–1082.

Draper, N. R. (1985). Small composite designs. *Technometrics*, *27*, 173–180.

Draper, N. R., Lin, D. K. J. (1990). Connections between two-level designs of resolutions III^* and V. *Technometrics*, *32*, 283–288.

Fontana, R., Pistone, G., Rogantin, M. P. (2000). Classification of two-level factorial fractions. *Journal of Statistical Planning and Inference*, *87*, 149–172.

Hartley, H. O. (1959). Smallest composite designs for quadratic response surfaces. *Biometrics*, *15*, 611–624.

Li, W., Lin, D. K. J., Ye, K. Q. (2003). Optimal foldover plans for non-regular orthogonal designs. *Technometrics*, *45*, 347–351.

Pistone, G., Wynn, H. P. (1996). Generalised confounding with Gröbner bases. *Biometrika*, *83*, 653–666.

Pistone, G., Riccomagno, E., Wynn, H. P. (2001). *Algebraic Statistics: Computational Commutative Algebra in Statistics*. Chapman & Hall/CRC, London.

Tang, B., Deng, L-Y. (2000). Minimum G_2-aberration and nonregular fractional factorial designs. *The Annals of Statistics 27*, 1914–1926.

Westlake, W. J. (1965). Composite designs based on irregular fractions of factorials. *Biometrics*, *21*, 324–336.
Ye, K. Q. (2003). Indicator function and its application in two-level factorial designs. *The Annals of Statistics*, *31*, 984–994.
Ye, K. Q. (2004). A note on regular fractional factorial designs. *Statistica Sinica*, *14*, 1069–1074.