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Classification of three-word indicator functions of two-level factorial designs

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Abstract Indicator functions have been in the literature for several years, and yet only a few of their properties have been examined. In this paper, we study some properties of indicator functions of two-level fractional factorial designs. For example, we show that there is no indicator function with only two words, and also classify all indicator functions with only three words. The results imply that there is no valuable non-regular design with only three or less words in its indicator function.

Keywords Fractional factorial design · Indicator function · Word · Non-regular design · Foldover design ·

1 Introduction

Pistone and Wynn (1996) found that Gröbner bases, an area in computational commutative algebra, have applications in experimental designs. They studied the identifiability problem in experimental designs by representing factorial designs as solutions of a set of polynomial equations. Based on this idea, Fontana et al. (2000) and Ye (2003) showed that two-level fractional factorial designs can be represented by indicator functions, which have been proved to be powerful tools for studying factorial designs, especially, non-regular designs since these designs have no defining relations like regular designs; see, for example, Li et al., (2003) and Ye (2004).

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Subsequently, Pistone and Rogantin (2003) and Cheng and Ye (2004) established that general fractional factorial designs, three or higher levels or multilevel, can also be represented by indicator functions. The applications of Gröbner bases in other area of statistics can be found, for example, in Pistone et al. (2001). In this paper, we only study indicator functions of two-level factorial designs.

Denote by D_{2^m} the full two-level m-factor design, i.e.,

$$D_{2^m} = \{x = \{x_1, x_2, \dots, x_m\} \mid x_i = 1 \text{ or } -1,\ i = 1, 2, \dots, m\}.$$

Let $\mathcal{F}$ be any two-level m-factor design such that for any $x \in \mathcal{F}$, $x \in D_{2^m}$, but x might be repeated in $\mathcal{F}$. The *indicator function* of $\mathcal{F}$ is a function $f(x)$ defined on D_{2^m} such that

$$f(x) = \begin{cases} r_x & \text{if } x \in \mathcal{F} \\ 0 & \text{if } x \notin \mathcal{F}, \end{cases}$$

where r_x is the number of appearances of the point x in factorial design $\mathcal{F}$. Let

$$L_{2^m} = \{\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_m\} \mid \alpha_i = 1 \text{ or } 0,\ i = 1, 2, \dots, m\}$$

and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$. Fontana et al. (2000) and Ye (2003) established that the indicator function $f(x)$ of $\mathcal{F}$ can be uniquely represented by a polynomial function

$$f(x) = \sum_{\alpha \in L_{2^m}} b_\alpha x^\alpha. \tag{1}$$

Each term except for constant in $f(x)$ is called a *word* (Li et al.,, 2003). A word which contains even (or odd) letters is called an *even-letter (or odd-letter) word.* Note that each word implies an (partial) alias relationship (see details, Li et al.,, 2003). The coefficients of $f(x)$ can be calculated by the formula

$$b_\alpha = \frac{1}{2^m} \sum_{x \in \mathcal{F}} x^\alpha. \tag{2}$$

In particular, $b_0 = n/2^m$, where n is the total number of runs. Thus, for an indicator function of an un-replicated design, $0 < b_0 \leq 1$ since $n \neq 0$. b_0 also provides the information of the run size since $n = 2^m b_0$. For example, if Eq. (1) is the indicator function of an un-replicated two-level factorial design, then, $b_0 = 1$ if and only if $n = 2^m$, that is, $b_0 = 1$ if and only if the design is a full two-level m-factor design; in this case, by the uniqueness of the indicator function, $f(x) \equiv 1$ for any $x \in D_{2^m}$; thus, when there are words in the indicator function of an un-replicated two-level factorial design, $0 < b_0 < 1$; moreover, $b_0 = 1/2$ if and only if $n = 2^{m-1}$; that is, $b_0 = 1/2$ if and only if the run size of the design is 2^{m-1}, in this case, we call the design a *half fraction* comparing to the full two-level m-factor design.

The coefficients of indicator functions satisfy $|b_\alpha/b_0| \leq 1$ for any $\alpha \in L_{2^m}$ such that $b_\alpha \neq 0$. Fontana et al. (2000) and Ye (2004) showed that a two-level factorial design is a regular design if and only if $|b_\alpha/b_0| = 1$ for any $\alpha \in L_{2^m}$ such that $b_\alpha \neq 0$. Note that when there are words in the indicator function of an un-replicated two-level factorial design, $0 < b_0 < 1$, therefore, we have $|b_\alpha| < 1$ for any $\alpha \in L_{2^m}$ such that $b_\alpha \neq 0$.

For example (Fontana et al. 2000), an indicator function of an un-replicated two-level 5-factor design $\mathcal{F}$ is: $f(x) = (1/2) - (1/4)x_1x_2x_3 - (1/4)x_1x_2x_3x_4 - (1/4)x_1x_2x_3x_5 + (1/4)x_1x_2x_3x_4x_5$. Note that the absolute value of the coefficient of each non-constant term is $(1/4)$, which is smaller than $b_0 = (1/2)$, and so, this design is a non-regular design. There are four words in the indicator function $f(x)$: $x_1x_2x_3$, $x_1x_2x_3x_4$, $x_1x_2x_3x_5$, and $x_1x_2x_3x_4x_5$. The word $x_1x_2x_3x_4$, for example, implies that x_1x_2, x_1x_3, x_1x_4 are partially aliased with x_3x_4, x_2x_4, x_2x_3, respectively. Since $b_0 = (1/2)$, the number of runs of this design is 16 and, therefore, this design is a half fraction.

It is known that there is no regular design with only two words in their indicator functions, but there are regular designs with one or three words. Balakrishnan and Yang (2006) recently showed that indicator functions with only one word must be regular designs or replicates of regular designs. Theoretically, one might be interested to know whether there exist non-regular designs with only two or three words in their indicator functions, and if they do exist, what forms do those indicator functions have. Practically, a design which has less alias relationships and has a smaller run size is desired and we hope to know if such non-regular designs exist. In this paper, we show that indicator functions with some special character must represent half fractions. We also study some properties of indicator functions with only one odd or even-letter word. After we prove there is no indicator function with only two words, we classify all indicator functions with only three words. The results show that among the designs with three or less words in their indicator functions, there is no smaller non-regular fraction which has the same or less alias relationships compared to regular designs.

In Sect. 2, we show that indicator functions with all the words containing the same factor must represent half fractions. Indicator functions with more than two words but only one odd or even-letter word are studied in Sect. 3. In Sect. 4, we establish that there is no indicator function with only two words. Indicator functions with more than two words but only two even-letter words are also considered in this section. We prove that the indicator functions with only three words must have one or three even-letter words and provide the forms of indicator functions for each case in Sect. 5. In this paper, we only consider indicator functions of un-replicated designs except Lemma 3.1 and Theorem 4.1.

Define a *foldover* of a factorial design as the procedure of adding a new fraction in which signs are reversed on one or more factors of the original design. The resulting design is called a *foldover design*. In this paper, we call a set of factors whose signs are reversed in the foldover design a *foldover plan* (Li et al., 2003). Denote $\parallel \alpha \parallel = \sum_{i=1}^{m} \alpha_i$. Then, $\parallel \alpha \parallel$ is the number of letters of x^α. Denote

$$\Omega = \{\alpha \in L_{2^m} \mid b_\alpha \neq 0 \text{ and } ||\alpha|| \neq 0\},$$
$$\Omega_1 = \{\alpha \in L_{2^m} \mid b_\alpha \neq 0 \text{ and } ||\alpha|| \text{ is 0 or even}\}$$

and

$$\Omega_2 = \{\alpha \in L_{2^m} \mid b_\alpha \neq 0 \text{ and } ||\alpha|| \text{ is odd}\}.$$

2 Indicator functions which represent half fractions

There are no indicator functions of regular designs with all the words containing the same factor or all the words are odd-letter words. However, some indicator functions of non-regular designs do have these forms; see Fontana et al. (2000). In this section, we show that indicator functions with these characters must represent a half fraction.

Proposition 2.1 *Assume that Eq. (1) is an indicator function of a two-level factorial design. If there exists a foldover plan such that the indicator function $g(x)$ of the foldover design contains no words, then this design is a half fraction.*

Proof The constant of $g(x)$ is $2b_0$ (see Li et al., 2003). If $g(x)$ does not contain any word, then $g(x) = 2b_0$. Since $0 < b_0 < 1$ and $g(x)$ must be an integer for any $x \in D_{2^m}$, we get $2b_0 = 1$, i.e., $b_0 = (1/2)$.

Corollary 2.1 *Assume that Eq. (1) is an indicator function of a two-level factorial design. If there exists a main factor which is contained in all the words, then this design is a half fraction.*

Proof The result follows if we choose the foldover plan as reversing the sign of the factor which is contained in all the words in $f(x)$.

Corollary 2.2 below is obtained from Proposition 2.1 by reversing the signs of all the factors. This result was obtained by Balakrishnan and Yang (2006) using a different argument.

Corollary 2.2 *Assume that Eq. (1) is an indicator function of a two-level factorial design. If all the words in the indicator function are odd-letter words, then the design is a half fraction.*

3 Indicator functions with one even or odd word

Lemma 3.1 *Assume that Eq. (1) is the indicator function of $\mathcal{F}$. Then the points $x = \{x_1, x_2, \ldots, x_m\}$ and $-x = \{-x_1, -x_2, \ldots, -x_m\}$ have different numbers of replicates if and only if $\sum_{\alpha\in\Omega_2} b_\alpha x^\alpha \neq 0$. Moreover,*

$$\sum_{\alpha\in\Omega_1} b_\alpha = \frac{1}{2}(f(1, 1, \ldots, 1) + f(-1, -1, \ldots, -1)) \tag{3}$$

and

$$\sum_{\alpha\in\Omega_2} b_\alpha = \frac{1}{2}(f(1, 1, \ldots, 1) - f(-1, -1, \ldots, -1)). \tag{4}$$

Proof Eq. (1) can be written as

$$f(x) = \sum_{\alpha\in\Omega_1} b_\alpha x^\alpha + \sum_{\alpha\in\Omega_2} b_\alpha x^\alpha.$$

So

$$f(-x) = \sum_{\alpha\in\Omega_1} b_\alpha x^\alpha - \sum_{\alpha\in\Omega_2} b_\alpha x^\alpha.$$

Thus $\sum_{\alpha\in\Omega_2} b_\alpha x^\alpha \neq 0$ if and only if $f(x) \neq f(-x)$ and if and only if x and $-x$ have different numbers of replicates. Note that

$$f(1, 1, \ldots, 1) = \sum_{\alpha\in\Omega_1} b_\alpha + \sum_{\alpha\in\Omega_2} b_\alpha \tag{5}$$

and

$$f(-1, -1, \ldots, -1) = \sum_{\alpha\in\Omega_1} b_\alpha - \sum_{\alpha\in\Omega_2} b_\alpha, \tag{6}$$

from which we readily obtain Eqs. (3) and (4).

Proposition 3.1 *Assume that Eq. (1) is the indicator function of a two-level factorial design. Then,* $\sum_{\alpha\in\Omega_1} b_\alpha = \frac{1}{2}$ *and*

$$\sum_{\alpha\in\Omega_2} b_\alpha = \begin{cases} \frac{1}{2} & \textit{if } (1, 1, \ldots, 1)\in\mathcal{F},\ (-1, -1, \ldots, -1) \notin \mathcal{F} \\ -\frac{1}{2} & \textit{if } (1, 1, \ldots, 1) \notin \mathcal{F},\ (-1, -1, \ldots, -1)\in\mathcal{F} \end{cases}$$

if and only if $\sum_{\alpha\in\Omega_2} b_\alpha \neq 0$*, and*

$$\sum_{\alpha\in\Omega_1} b_\alpha = \begin{cases} 1 & \textit{if } (1, 1, \ldots, 1),\ (-1, -1, \ldots, -1)\in\mathcal{F} \\ 0 & \textit{if } (1, 1, \ldots, 1),\ (-1, -1, \ldots, -1) \notin \mathcal{F} \end{cases}$$

if and only if $\sum_{\alpha\in\Omega_2} b_\alpha = 0$.

Proof By Eqs. (5) and (6), $\sum_{\alpha\in\Omega_2} b_\alpha \neq 0$ if and only if $f(1, 1, \ldots, 1)\neq f(-1, -1, \ldots, -1)$. Note that $f(x)$ can only be 0 or 1, and so $f(1, 1, \ldots, 1)\neq f(-1, -1, \ldots, -1)$ if and only if $f(1, 1, \ldots, 1) + f(-1, -1, \ldots, -1) = 1$ and

$$f(1, \ldots, 1) - f(-1, \ldots, -1) = \begin{cases} 1 & \text{if } (1, \ldots, 1)\in\mathcal{F},\ (-1, \ldots, -1) \notin \mathcal{F} \\ -1 & \text{if } (1, \ldots, 1) \notin \mathcal{F},\ (-1, \ldots, -1) \in \mathcal{F}. \end{cases}$$

By Lemma 3.1, we obtain the first result.

On the other hand, by Eqs. (5) and (6), $\sum_{\alpha\in\Omega_2} b_\alpha = 0$ if and only if $f(1, 1, \ldots, 1) = f(-1, -1, \ldots, -1)$, i.e., if and only if

$$f(1, \ldots, 1) + f(-1, \ldots, -1) = \begin{cases} 2 & \text{if } (1, \ldots, 1),\ (-1, \ldots, -1)\in\mathcal{F} \\ 0 & \text{if } (1, \ldots, 1),\ (-1, \ldots, -1) \notin \mathcal{F}. \end{cases}$$

By Lemma 3.1, we obtain the second result.

Corollary 3.1 *If there is only one odd-letter word* x^α *in the indicator function of a two-level factorial design, then either* $(1, 1, \ldots, 1)$ *or* $(-1, -1, \ldots, -1)$ *is in* $\mathcal{F}$ *and*

$$b_\alpha = \begin{cases} \frac{1}{2} & \textit{if } (1, 1, \ldots, 1)\in\mathcal{F},\ (-1, -1, \ldots, -1) \notin \mathcal{F} \\ -\frac{1}{2} & \textit{if } (1, 1, \ldots, 1) \notin \mathcal{F},\ (-1, -1, \ldots, -1)\in\mathcal{F}. \end{cases} \tag{7}$$

Proposition 3.2 *If there is only one even-letter word x^α in the indicator function of a two-level factorial design, then*

$$b_0 = b_\alpha = \frac{1}{4} \textit{ or}$$
$$b_0 = -3b_\alpha = \frac{3}{4} \Longleftrightarrow \textit{either } (1, 1, \ldots, 1) \textit{ or } (-1, -1, \ldots, -1) \textit{ is in } \mathcal{F},$$

$$b_0 = b_\alpha = \frac{1}{2}, b_0 = \frac{1}{3}b_\alpha = \frac{1}{4} \textit{ or}$$
$$b_0 = 3b_\alpha = \frac{3}{4} \Longleftrightarrow (1, 1, \ldots, 1),\ (-1, -1, \ldots, -1) \in \mathcal{F}$$

and

$$b_0 = -b_\alpha = \frac{1}{4}, \frac{1}{2} \textit{ or } \frac{3}{4} \Longleftrightarrow (1, 1, \ldots, 1),\ (-1, -1, \ldots, -1) \notin \mathcal{F}.$$

Proof The indicator function of the foldover design obtained by reversing the signs on all the factors is $g(x) = 2b_0 + 2b_\alpha x^\alpha$. There exist y and z such that $g(y) = 2b_0 + 2b_\alpha$ and $g(z) = 2b_0 - 2b_\alpha$. So, $b_0 = (1/4)(g(y) + g(z))$. Since $g(x)$ can be 0, 1 or 2, $b_0 = (1/4)$, $(1/2)$ or $3/4$.

Note that $b_\alpha \neq 0$, we get

$$b_0 = b_\alpha = \frac{1}{4} \text{ or } b_0 = -3b_\alpha = \frac{3}{4} \Longleftrightarrow b_0 + b_\alpha = \frac{1}{2},$$
$$b_0 = b_\alpha = \frac{1}{2}, b_0 = \frac{1}{3}b_\alpha = \frac{1}{4} \text{ or } b_0 = 3b_\alpha = \frac{3}{4} \Longleftrightarrow b_0 + b_\alpha = 1$$

and

$$b_0 = -b_\alpha = \frac{1}{4}, \frac{1}{2} \text{ or } \frac{3}{4} \Longleftrightarrow b_0 + b_\alpha = 0.$$

By Proposition 3.1, we obtain the results.

4 Indicator functions which contain two special words

In this section, we prove that when all the runs in a design have the same number of replicates, the indicator function of this design can not contain only two words. To prove this, we need Remark 1 below.

Remark 1 Let x^α and x^β be two different words. Then, we can choose a point $y \in D_{2^m}$ such that $y^\alpha = y^\beta = \pm 1$ or $y^\alpha = -y^\beta = \pm 1$. This point can be chosen as follows:

(a) When all the factors in x^α are also in x^β: assume that x_i is in x^α and x_j is in x^β but not in x^α. We can choose a point y such that its ith entry is ± 1 and other entries are 1 so that $y^\alpha = y^\beta = \pm 1$ or its ith entry is ± 1, jth entry is -1 and other entries are 1 so that $y^\alpha = -y^\beta = \pm 1$.

(b) When there exists a factor x_i which is in x^α but not in x^β and a factor x_j which is in x^β but not in x^α: We can choose a point y such that its ith and jth entries are ± 1 and other entries are 1 so that $y^\alpha = y^\beta = \pm 1$ or its ith entry is ± 1, jth entry is ∓ 1 and other entries are 1 so that $y^\alpha = -y^\beta = \pm 1$.

Now, we are ready to prove Theorem 4.1.

Theorem 4.1 *There is no two-level factorial design such that all the points in it have the same number of replicates and its indicator function has only two words.*

Proof Assume that $f(x)$ is the indicator function of a design $\mathcal{F}_1$ which does not have replicates. Let $\mathcal{F}_2$ be the design which contains the same points as $\mathcal{F}_1$ and each point has n replicates. Then, by using the formula in Eq. (2), it is easy to check that the indicator function of $\mathcal{F}_2$ is $nf(x)$. Thus, if $f(x)$ can not contain only two words, the indicator function of $\mathcal{F}_2$ also can not contain only two words.

Now we establish that $f(x)$ can not have just two words.

Assume that there exists a design such that its indicator function is $f(x) = b_0 + b_\alpha x^\alpha + b_\beta x^\beta$. By Remark 1, we can choose a point y such that $y^\alpha = y^\beta = 1$ and a point z such that $z^\alpha = z^\beta = -1$. Then, $f(y) = b_0 + b_\alpha + b_\beta$ and $f(z) = b_0 - b_\alpha - b_\beta$, and thus, $b_0 = (1/2)(f(y) + f(z))$, which can only be 0, $(1/2)$, or 1. But b_0 can not be 0 or 1, and so $b_0 = (1/2)$. We can also choose another point h such that $h^\alpha = -h^\beta = 1$, in which case, we get $b_\beta = (1/2)(f(y) - f(h))$, which can be $\pm(1/2)$. Similarly, $b_\alpha = \pm(1/2)$. Since x^α and x^β can only be 1 and -1, $f(x)$ can never be an integer.

Remark 2 Balakrishnan and Yang (2006) showed that if $b_0 \neq (1/2)$, then there are at least two words in the indicator function. By Theorem 4.1, if $b_0 \neq (1/2)$, then there are at least three words in the indicator function.

Lemma 4.1 *If there exists a foldover plan of a two-level factorial design such that the indicator function of the foldover design has only two words, then this design is a half fraction. Moreover, if the two words are x^α and x^β, then $b_\alpha = \pm(1/4)$ and $b_\beta = \pm(1/4)$.*

Proof Assume that Eq. (1) is the indicator function of the original design and the indicator function of the foldover design is $g(x) = 2b_0 + 2b_\alpha x^\alpha + 2b_\beta x^\beta$. Then, we can choose $y, z \in D_{2^m}$ such that $g(y) = 2b_0 + 2b_\alpha + 2b_\beta$ and $g(z) = 2b_0 - 2b_\alpha - 2b_\beta$. So, $b_0 = (1/4)(g(y) + g(z))$. Since $g(x)$ can only be 0, 1, or 2, b_0 can only be $(1/4)$, $(1/2)$, or $3/4$. We can also choose $h \in D_{2^m}$ such that $g(h) = 2b_0 - 2b_\alpha + 2b_\beta$. So $b_\alpha = (1/4)(g(y) - g(h))$, which can be $\pm(1/4)$ and $\pm(1/2)$. Similarly, $b_\beta = \pm(1/4), \pm(1/2)$.

(a) When $b_0 = (1/4)$. If $|b_\alpha| = |b_\beta| = (1/2)$, then $g(x) = (1/2) \pm x^\alpha \pm x^\beta$, which can not be an integer. If $|b_\alpha| = |b_\beta| = (1/4)$, then $g(x) = (1/2) \pm (1/2)x^\alpha \pm (1/2)x^\beta$, which can not also be an integer. If one of b_α and b_β, say b_α, is such that $|b_\alpha| = (1/2)$ and the other one $|b_\beta| = (1/4)$, then $g(x) = (1/2) \pm x^\alpha \pm (1/2)x^\beta$, which may be negative for some points in D_{2^m}. Thus, $b_0 \neq (1/4)$.
(b) When $b_0 = (1/2)$. If $|b_\alpha| = |b_\beta| = (1/2)$, then $g(x) = 1 \pm x^\alpha \pm x^\beta$, which may be negative. If one of b_α and b_β, say b_α, is such that $|b_\alpha| = (1/2)$ and the other one $|b_\beta| = (1/4)$, then $g(x) = 1 \pm x^\alpha \pm (1/2)x^\beta$, which can never be an integer. If $|b_\alpha| = |b_\beta| = (1/4)$, then $g(x) = 1 \pm (1/2)(x^\alpha) \pm (1/2)(x^\beta)$, which is always an integer between 0 and 2. Thus, when $b_0 = (1/2)$, $|b_\alpha| = |b_\beta| = (1/4)$.

(c) When $b_0 = 3/4$. If $|b_\alpha| = |b_\beta| = (1/2)$ or $(1/4)$, then we can similarly show that $g(x)$ can not be an integer. If one of b_α and b_β, say b_α, such that $|b_\alpha| = (1/2)$ and the other one $|b_\beta| = (1/4)$, then $g(x) = (3/2) \pm x^\alpha \pm (1/2)(x^\beta)$, which may equal 3 for some points in D_{2^m}. Thus, $b_0 \neq 3/4$.

Theorem 4.2 below provides the coefficients of the two even-letter words in more detail if there are two even-letter words in the indicator function.

Theorem 4.2 *If the indicator function of a two-level factorial design $\mathcal{F}$ has more than two words but only two of them are even-letter words, say x^α and x^β, then this design must be a half fraction and*

$$\begin{cases} b_\alpha = -b_\beta = \pm\frac{1}{4} & \textit{if } (1, 1, \ldots, 1) \textit{ or } (-1, -1, \ldots, -1) \in \mathcal{F} \\ b_\alpha = b_\beta = \frac{1}{4} & \textit{if } (1, 1, \ldots, 1), (-1, -1, \ldots, -1) \in \mathcal{F} \\ b_\alpha = b_\beta = -\frac{1}{4} & \textit{if } (1, 1, \ldots, 1), (-1, -1, \ldots, -1) \notin \mathcal{F}. \end{cases} \tag{8}$$

Proof By assumption, the indicator function of the foldover design obtained by reversing the signs of all the factors is $g(x) = 2b_0 + 2b_\alpha x^\alpha + 2b_\beta x^\beta$. By Lemma 4.1, $b_0 = (1/2)$.

(a) If $\sum_{\alpha \in \Omega_2} b_\alpha \neq 0$, by Proposition 3.1, $b_0 + b_\alpha + b_\beta = (1/2)$. Since $b_0 = (1/2)$, $b_\alpha + b_\beta = 0$. By Lemma 4.1, $b_\alpha = -b_\beta = \pm(1/4)$.
(b) If $\sum_{\alpha \in \Omega_2} b_\alpha = 0$, by Proposition 3.1, $b_0 + b_\alpha + b_\beta = 1$ or 0.
If $b_0 + b_\alpha + b_\beta = 1$, then $b_\alpha + b_\beta = \frac{1}{2}$. By Lemma 4.1, $b_\alpha = b_\beta = (1/4)$.
If $b_0 + b_\alpha + b_\beta = 0$, then $b_\alpha + b_\beta = -\frac{1}{2}$. By Lemma 4.1, $b_\alpha = b_\beta = -(1/4)$.

When there are two odd-letter words, say x^α and x^β, in the indicator function, it is hard to say about their coefficients when either $(1, 1, \ldots, 1)$ or $(-1, -1, \ldots, -1)$ is in $\mathcal{F}$, but when both $(1, 1, \ldots, 1)$ and $(-1, -1, \ldots, -1)$ are either in $\mathcal{F}$ or not in $\mathcal{F}$, the sum of the two coefficients is equal to 0 by Proposition 3.1. Thus, $b_\alpha = -b_\beta$.

5 Indicator functions with only three words

In this section, we discuss indicator functions with only three words and give the classification of the indicator functions.

Assume that $f(x) = b_0 + b_\alpha x^\alpha + b_\beta x^\beta + b_\gamma x^\gamma$ is the indicator function of an un-replicated two-level factorial design $\mathcal{F}$. By Remark 1, given $\alpha, \beta \in \Omega$, there exists a point $x \in D_{2^m}$ such that x^α and x^β have either the same sign or different signs. Given α, β and x, i.e., given x^α and x^β, x^γ is either 1 or -1. The following claims will be used later in this section.

Claim 1 Given the indicator function $f(x)$, if there exist $y, z \in D_{2^m}$ such that $y^\alpha = z^\alpha$ and $y^\beta = z^\beta$, but $y^\gamma \neq z^\gamma$, then $b_\gamma = 1 \pm (1/2)$.

By assumptions, $f(y) - f(z) = b_\gamma(y^\gamma - z^\gamma)$, which yields $b_\gamma = \pm(1/2)$.

Claim 2 There is no indicator function which has the form

$$f(x) = \frac{1}{2} + bx^\alpha - bx^\beta \pm \frac{1}{2}x^\gamma. \tag{9}$$

When x^α and x^β have different signs, $f(x) = 1 \pm 2b$ or $\pm 2b$. For $f(x)$ to be 0 or 1, $|b|$ has to equal (1/2). In this case, $|b_\alpha/b_0| = |b_\beta/b_0| = |b_\gamma/b_0| = 1$, and thus the design is a regular design. But, then $|b_\alpha| = |b_\beta| = |b_\gamma| = b_0$ has to equal (1/4), which provides a contradiction.

Now, we are ready to establish the following theorem.

Theorem 5.1 *Assume that $f(x) = b_0 + b_\alpha x^\alpha + b_\beta x^\beta + b_\gamma x^\gamma$ is the indicator function of a two-level factorial design $\mathcal{F}$. Then, either one or all of the three words are even-letter words and $\mathcal{F}$ is either a $\frac{1}{4}$ fraction or a $\frac{3}{4}$ fraction. More specifically,*

1. *When there is only one even-letter word, say x^γ, then,*
   (a) *if $\mathcal{F}$ is a $\frac{1}{4}$ fraction, then $f(x)$ has the forms:*

$$f(x) = \begin{cases} \frac{1}{4} + \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta + \frac{1}{4}x^\gamma & \text{if } (1,\ldots,1)\in\mathcal{F}, (-1,\ldots,-1) \notin \mathcal{F} \\ \frac{1}{4} - \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta + \frac{1}{4}x^\gamma & \text{if } (1,\ldots,1) \notin \mathcal{F}, (-1,\ldots,-1)\in\mathcal{F} \\ \frac{1}{4} \pm \frac{1}{4}x^\alpha \mp \frac{1}{4}x^\beta - \frac{1}{4}x^\gamma & \text{if } (1,\ldots,1), (-1,\ldots,-1) \notin \mathcal{F}; \end{cases} \tag{10}$$

   (b) *if $\mathcal{F}$ is a $\frac{3}{4}$ fraction, then $f(x)$ has the forms:*

$$f(x) = \begin{cases} \frac{3}{4} + \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta - \frac{1}{4}x^\gamma & \text{if } (1,\ldots,1)\in\mathcal{F}, (-1,\ldots,-1) \notin \mathcal{F} \\ \frac{3}{4} - \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta - \frac{1}{4}x^\gamma & \text{if } (1,\ldots,1) \notin \mathcal{F}, (-1,\ldots,-1)\in\mathcal{F} \\ \frac{3}{4} \mp \frac{1}{4}x^\alpha \pm \frac{1}{4}x^\beta + \frac{1}{4}x^\gamma & \text{if } (1,\ldots,1), (-1,\ldots,-1)\in\mathcal{F}. \end{cases} \tag{11}$$

2. *When all the words are even-letter words, then,*
   (a) *if $\mathcal{F}$ is a $\frac{1}{4}$ fraction, then $f(x)$ has the forms:*

$$f(x) = \begin{cases} \frac{1}{4} + \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta + \frac{1}{4}x^\gamma & \text{if } (1,\ldots,1), (-1,\ldots,-1)\in\mathcal{F} \\ \frac{1}{4} + \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta - \frac{1}{4}x^\gamma & \text{if } (1,\ldots,1), (-1,\ldots,-1) \notin \mathcal{F}; \end{cases} \tag{12}$$

   (b) *if $\mathcal{F}$ is a $\frac{3}{4}$ fraction, then $f(x)$ has the forms:*

$$f(x) = \begin{cases} \frac{3}{4} - \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta + \frac{1}{4}x^\gamma & \text{if } (1,\ldots,1), (-1,\ldots,-1)\in\mathcal{F} \\ \frac{3}{4} - \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta - \frac{1}{4}x^\gamma & \text{if } (1,\ldots,1), (-1,\ldots,-1) \notin \mathcal{F}. \end{cases} \tag{13}$$

Proof (1) If all the three words are odd-letter words, then, by Corollary 2.2, $b_0 = \frac{1}{2}$. By Proposition 3.1, either $(1, 1, \ldots, 1)$ or $(-1, -1, \ldots, -1)$ is in $\mathcal{F}$ and

$$f(x) = \begin{cases} \frac{1}{2} + b_\alpha x^\alpha + b_\beta x^\beta + (\frac{1}{2} - b_\alpha - b_\beta)x^\gamma & \text{if } (1, 1, \ldots, 1)\in\mathcal{F} \\ \frac{1}{2} + b_\alpha x^\alpha + b_\beta x^\beta + (-\frac{1}{2} - b_\alpha - b_\beta)x^\gamma & \text{if } (-1, -1, \ldots, -1) \in \mathcal{F}. \end{cases}$$

(a) When $f(x) = (1/2)+b_\alpha x^\alpha+b_\beta x^\beta+((1/2)-b_\alpha-b_\beta)x^\gamma$: since there exists a point x such that $x^\alpha = -x^\beta = 1$, if $x^\gamma = 1$, then, $f(x) = 1-2b_\beta$, which yields $b_\beta = (1/2)$. So $f(x) = (1/2)+b_\alpha x^\alpha+(1/2)(x^\beta)-b_\alpha x^\gamma$, which is of the form Eq. (9), and by *Claim 2*, this is impossible. If $x^\gamma = -1$, then $f(x) = 2b_\alpha$, which yields $b_\alpha = (1/2)$. So $f(x) = (1/2)+(1/2)(x^\alpha)+b_\beta x^\beta - b_\beta x^\gamma$. This is again of the form Eq. (9).

(b) When $f(x) = (1/2) + b_\alpha x^\alpha + b_\beta x^\beta + (-(1/2) - b_\alpha - b_\beta)x^\gamma$: there exists a point x such that $x^\alpha = -x^\beta = 1$. If $x^\gamma = 1$, then $f(x) = -2b_\beta$, which yields $b_\beta = -(1/2)$. So $f(x) = (1/2) + b_\alpha x^\alpha - (1/2)(x^\beta) - b_\alpha x^\gamma$, which is of the form Eq. (9), and by *Claim 2*, this is impossible. If $x^\gamma = -1$, then $f(x) = 1 + 2b_\alpha$, which yields $b_\alpha = -1/2$. So $f(x) = (1/2) - (1/2)(x^\alpha) + b_\beta x^\beta - b_\beta x^\gamma$. This is also impossible by *Claim 2*.

(2) If there are two even-letter words, say x^α and x^β, in the three words, then by Corollary 3.1 and Theorem 4.2, we obtain

$$f(x) = \begin{cases} \frac{1}{2} + \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta + \frac{1}{2}x^\gamma & \text{if } (1, \ldots, 1) \in \mathcal{F}, (-1, \ldots, -1) \notin \mathcal{F} \\ \frac{1}{2} + \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta - \frac{1}{2}x^\gamma & \text{if } (1, \ldots, 1) \notin \mathcal{F}, (-1, \ldots, -1) \in \mathcal{F}, \end{cases}$$

which is of the form Eq. (9). By *Claim 2*, this is impossible.

(3) If there is one even-letter word, say x^γ, in the three words, then by Proposition 3.2, $b_0 = 1/4, (1/2)$ or $3/4$. However, if $b_0 = (1/2)$, by Proposition 3.2, $|b_\gamma| = b_0 = (1/2)$ and either both $(1, 1, \ldots, 1)$ and $(-1, -1, \ldots, -1)$ are in $\mathcal{F}$ or both $(1, 1, \ldots, 1)$ and $(-1, -1, \ldots, -1)$ are not in $\mathcal{F}$; thus, by Proposition 3.1, $f(x)$ is of the form Eq. (9), which is impossible by *Claim 2*. Thus, b_0 can only be $(1/4)$ or $3/4$.
Note that if $f(x)$ has three words with constant b_0, then, by Corollary 3.5 of Fontana et al. (2000), the indicator function $g(x)$ of its complementary fraction also contains the same three words and its constant is $1 - b_0$. Thus, we only need to consider the case $b_0 = (1/4)$ and Eq. (11) can be obtained directly from Eq. (10) and Corollary 3.5 of Fontana et al. (2000).
Now, when $b_0 = (1/4)$, by Proposition 3.1 and Proposition 3.2, $f(x)$ has the following possible forms:

(a) $f(x) = \frac{1}{4} + b_\alpha x^\alpha + (\frac{1}{2} - b_\alpha)x^\beta + \frac{1}{4}x^\gamma$ if $(1, 1, \ldots, 1) \in \mathcal{F}$, $(-1, -1, \ldots, -1) \notin \mathcal{F}$.
Considering various cases, we have the table below:

Case no.	x^α	x^β	x^γ	$f(x)$	$b_\alpha, b_\beta, b_\gamma$
1	1	1	1	1	
			-1	$\frac{1}{2}$	Impossible
2	-1	-1	1	0	
			-1	$-\frac{1}{2}$	Impossible
3	1	-1	1	$2b_\alpha$	$b_\alpha = \frac{1}{2}$
			-1	$2b_\alpha - \frac{1}{2}$	$b_\alpha = \frac{1}{4}, \frac{3}{4}$
4	-1	1	1	$1 - 2b_\alpha$	$b_\alpha = \frac{1}{2}$
			-1	$\frac{1}{2} - 2b_\alpha$	$b_\alpha = \frac{1}{4}, -\frac{1}{4}$

Since $b_\gamma \neq \pm\frac{1}{2}$, by *Claim 1*, for each case, $x^\gamma \equiv 1$ or -1. From *Case 4*, $b_\alpha = (1/2)$, $(1/4)$ and $-(1/4)$. If $b_\alpha = (1/2)$, then $b_\beta = (1/2) - b_\alpha = 0$, which is a contradiction. If $b_\alpha = -(1/4)$, then in *Case 3*, $f(x)$ can not be an integer. If $b_\alpha = (1/4)$, then $f(x)$ in the other three cases can be 0 or 1. Thus $b_\alpha = (1/4)$. This gives the first form of Eq. (10).

(b) $f(x) = \frac{1}{4} + b_\alpha x^\alpha + (-\frac{1}{2} - b_\alpha)x^\beta + \frac{1}{4}x^\gamma$ if $(1, 1, \dots, 1) \notin \mathcal{F}, (-1, -1, \dots, -1) \in \mathcal{F}$.
Similar to the discussion for (i), we get $b_\alpha = -(1/4)$. This gives the second form of Eq. (10).

(c) $f(x) = \frac{1}{4} + b_\alpha x^\alpha - b_\alpha x^\beta + \frac{3}{4}x^\gamma$ if $(1, 1, \dots, 1), (-1, -1, \dots, -1) \in \mathcal{F}$.
When $x^\alpha = -x^\beta = 1$, $f(x) = 1 + 2b_\alpha$ or $-\frac{1}{2} + 2b_\alpha$, for $f(x)$ to be 0 or 1, $b_\alpha = -(1/2), (1/4)$ or $3/4$; when $x^\alpha = -x^\beta = -1$, $f(x) = 1 - 2b_\alpha$ or $-(1/2) - 2b_\alpha$, which yields $b_\alpha = (1/2), -(1/4)$ or $-3/4$, a contradiction. Thus, $f(x)$ can not be this form.

(d) $f(x) = \frac{1}{4} + b_\alpha x^\alpha - b_\alpha x^\beta - \frac{1}{4}x^\gamma$ if $(1, 1, \dots, 1), (-1, -1, \dots, -1) \notin \mathcal{F}$.
Similar to the discussion for (i), we get $b_\alpha = \pm(1/4)$. This gives the third form of Eq. (10).

(4) If all the three words are even-letter words, then $\sum_{\alpha \in \Omega_2} b_\alpha = 0$. By Proposition 3.1, $f(x)$ has the following two possible forms:

(a) $f(x) = b_0 + b_\alpha x^\alpha + b_\beta x^\beta + (1 - b_0 - b_\alpha - b_\beta)x^\gamma$, if $(1, 1, \dots, 1), (-1, -1, \dots, -1) \in \mathcal{F}$.
Considering various cases, we have the table below:

Case no.	x^α	x^β	x^γ	$f(x)$	$b_\alpha, b_\beta, b_\gamma$
1	1	1	1	1	
			-1	$2(b_0 + b_\alpha + b_\beta) - 1$	$b_0 + b_\alpha + b_\beta = \frac{1}{2}, 1$
2	-1	-1	1	$1 - 2(b_\alpha + b_\beta)$	$b_\alpha + b_\beta = 0, \frac{1}{2}$
			-1	$-1 + 2b_0$	$b_0 = \frac{1}{2}$
3	1	-1	1	$1 - 2b_\beta$	$b_\beta = \frac{1}{2}$
			-1	$2(b_0 + b_\alpha) - 1$	$b_0 + b_\alpha = \frac{1}{2}, 1$
4	-1	1	1	$1 - 2b_\alpha$	$b_\alpha = \frac{1}{2}$
			-1	$2(b_0 + b_\beta) - 1$	$b_0 + b_\beta = \frac{1}{2}, 1$

(i) To show that $b_0 \neq (1/2)$: assume that $b_0 = (1/2)$. Then for *Case 4*, if $b_\alpha = (1/2)$, then $f(x)$ is of the form Eq. (9), which is impossible; if $b_0 + b_\beta = (1/2)$, then $b_\beta = 0$, a contradiction; if $b_0 + b_\beta = 1$, then $b_\beta = 1/2$ and $f(x)$ is again of the form Eq. (9), which is impossible. Thus, in *Case 2*, that is, when $x^\alpha = x^\beta = -1$, x^γ must be 1, which needs

$$b_\alpha + b_\beta = 0 \tag{14}$$

or

$$b_\alpha + b_\beta = \frac{1}{2}. \tag{15}$$

(ii) To show that $b_\beta \neq (1/2)$: Assume that $b_\beta = (1/2)$. Then, since $b_0 \neq 0$, $b_\alpha + b_\beta \neq (1/2)$. So $b_\alpha + b_\beta = 0$ by Eqs. (14) and (15) and, thus, $b_\alpha = -(1/2)$. Therefore, for Case 4, we have $b_0 + b_\beta = (1/2)$ or 1, which yields $b_0 = 0$ or $(1/2)$, respectively, a contradiction. Thus, in *Case 3*, that is, when $x^\alpha = -x^\beta = 1$, x^γ must be -1, which needs

$$b_0 + b_\alpha = \frac{1}{2} \tag{16}$$

or

$$b_0 + b_\alpha = 1. \tag{17}$$

(iii) To show that $b_\alpha \neq (1/2)$: assume that $b_\alpha = (1/2)$. Then, by Eqs. (16) and (17), $b_0 = 0$ or $1/2$, which is impossible. Thus, in *Case 4*, that is, when $x^\alpha = -x^\beta = -1$, x^γ must be -1, which needs

$$b_0 + b_\beta = \frac{1}{2} \tag{18}$$

or

$$b_0 + b_\beta = 1. \tag{19}$$

(iv) By (1), Eqs. (14) and (15), we know that $b_0 + b_\alpha + b_\beta$ can not be $(1/2)$ or 1. Thus, in *Case 1*, that is, when $x^\alpha = x^\beta = 1$, x^γ must be 1 and $f(x) = 1$.

Now, by Eqs. (14), (16) and (19), we get $b_0 = 3/4$, $b_\alpha = -b_\beta = -b_\gamma = -1/4$, which gives the first form of Eq. (13). By Eqs. (14), (17) and (18), we get $b_0 = 3/4$, $b_\alpha = -b_\beta = b_\gamma = (1/4)$, which also gives the first form of Eq. (13). By Eqs. (15), (16) and (18), we get $b_0 = b_\alpha = b_\beta = b_\gamma = (1/4)$, which gives the first form of Eq. (12). By Eqs. (15), (17) and (19), we get $b_0 = 3/4$, $b_\alpha = b_\beta = -b_\gamma = (1/4)$, which again gives the first form of (13). All other combinations of Eqs. (14) or (15), (16) or (17), and (18) or (19) lead to the solutions with b_0 equalling $(1/2)$, which are contradictions.

(b) $f(x) = b_0 + b_\alpha x^\alpha + b_\beta x^\beta + (-b_0 - b_\alpha - b_\beta)x^\gamma$ if $(1, 1, \ldots, 1), (-1, -1, \ldots, -1) \notin \mathcal{F}$.

Considering various cases, we have the table below:

Case no.	x^α	x^β	x^γ	$f(x)$	$b_\alpha, b_\beta, b_\gamma$
1	1	1	1	0	
			-1	$2(b_0 + b_\alpha + b_\beta)$	$b_0 + b_\alpha + b_\beta = 0, \frac{1}{2}$
2	-1	-1	1	$-2(b_\alpha + b_\beta)$	$b_\alpha + b_\beta = 0, -\frac{1}{2}$
			-1	$2b_0$	$b_0 = \frac{1}{2}$
3	1	-1	1	$-2b_\beta$	$b_\beta = -\frac{1}{2}$
			-1	$2(b_0 + b_\alpha)$	$b_0 + b_\alpha = 0, \frac{1}{2}$
4	-1	1	1	$-2b_\alpha$	$b_\alpha = -\frac{1}{2}$
			-1	$2(b_0 + b_\beta)$	$b_0 + b_\beta = 0, \frac{1}{2}$

1. To show that $b_0 \neq (1/2)$: assume that $b_0 = (1/2)$. For *Case 4*, if $b_\alpha = -(1/2)$, then, $f(x)$ is of the form Eq. (9), which is impossible; if $b_0 + b_\beta = 0$, then $b_\beta = -(1/2)$ and $f(x)$ also has the form Eq. (9), which is again impossible; $b_0 + b_\beta$ can not be $(1/2)$, since, then, $b_\beta = 0$. Thus, when $x^\alpha = x^\beta = -1$, x^γ must be 1, which needs

$$b_\alpha + b_\beta = 0 \tag{20}$$

or

$$b_\alpha + b_\beta = -\frac{1}{2}. \tag{21}$$

2. To show that $b_\beta \neq -(1/2)$: Assume that $b_\beta = -1/2$. Then $b_\alpha + b_\beta \neq -(1/2)$. So $b_\alpha + b_\beta = 0$ by Eqs. (20) and (21) and, thus, $b_\alpha = (1/2)$. Therefore, for *Case 4*, we have $b_0 + b_\beta = 0$ or $(1/2)$, which yields $b_0 = (1/2)$ or 1, respectively, a contradiction. Thus, when $x^\alpha = -x^\beta = 1$, x^γ must be -1, which needs

$$b_0 + b_\alpha = 0 \tag{22}$$

or

$$b_0 + b_\alpha = \frac{1}{2}. \tag{23}$$

3. To show that $b_\alpha \neq -(1/2)$: Assume that $b_\alpha = -(1/2)$. Then, by Eqs. (22) and (23), $b_0 = (1/2)$ or 1, which is impossible. Thus, when $x^\alpha = -x^\beta = -1$, x^γ must be -1, which needs

$$b_0 + b_\beta = 0 \tag{24}$$

or

$$b_0 + b_\beta = \frac{1}{2}. \tag{25}$$

4. By (1), Eqs. (20) and (21), we know that $b_0 + b_\alpha + b_\beta$ can not be 0 or $\frac{1}{2}$. Thus, when $x^\alpha = x^\beta = 1$, x^γ must be 1 and $f(x) = 0$.
Now, by Eqs. (20), (22) and (25), we get $b_0 = -b_\alpha = b_\beta = -b_\gamma = (1/4)$, which gives the second form of Eq. (12). By Eqs. (20), (23) and (24), we get $b_0 = b_\alpha = -b_\beta = -b_\gamma = (1/4)$, which also gives the second form of Eq. (12). By Eqs. (21), (22) and (24), we get $b_0 = -b_\alpha = -b_\beta = b_\gamma = (1/4)$, which again gives the second form of Eq. (12). By Eqs. (21), (23) and (25), we get $b_0 = 3/4$, $b_\alpha = b_\beta = b_\gamma = -(1/4)$, which gives the second form of Eq. (13). All other combinations of Eqs. (20) or (21), (22) or (23), and (24) or (25) lead to solutions with b_0 equalling $(1/2)$ or 0, which are contradictions.

From Theorem 5.1, one can see that when $b_0 = (1/4)$, the corresponding designs are regular designs; when $b_0 = 3/4$, the corresponding designs are non-regular designs. The non-regular designs are complementary fractions of the corresponding regular designs. Given a regular design with three words in its indicator function; there is no non-regular design which has less number of runs and the same three words in its indicator function; therefore, there is no non-regular design which has smaller run size and has the same or less alias relationships; in other words, among the designs with only three or less words in their indicator functions, regular designs are optimal designs.

6 Conclusions

In this paper, we have studied indicator functions with a few words. After investigating some properties of indicator functions, we have obtained that there is no two-level factorial design with only two words in its indicator function and provided the possible forms of the indictor functions with three words. We conclude that there is no valuable non-regular design with three or less words in its indicator function.

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