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Joint distributions of runs in a sequence of higher-order two-state Markov trials

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Abstract Consider a time homogeneous $\{0,\ 1\}$-valued m-dependent Markov chain $\{X_{-m+1+n}, n \geqslant 0\}$. In this paper, we study the joint probability distribution of number of 0-runs of length $k_0 (k_0 \geqslant m)$ and number of 1-runs of length $k_1 (k_1 \geqslant m)$ in n trials. We study the joint distributions based on five popular counting schemes of runs. The main tool used to obtain the probability generating function of the joint distribution is the conditional probability generating function method. Further a compact method for the evaluation of exact joint distribution is developed. For higher-order two-state Markov chain, these joint distributions are new in the literature of distributions of run statistics. We use these distributions to derive some waiting time distributions.

Keywords Runs · Joint distribution · m-dependent Markov Bernoulli trials · Conditional probability generating function

1 Introduction

Distribution theory related to run statistics has become a very active area in recent years. Particularly in the last decade it has been extensively studied by many researchers (Aki and Hirano 1994, 1995; Aki et al. 1996; Antzoulakos 1999; Balakrishnan 1997; Fu and Koutras 1994; Han and Aki 1999, 2000a,b,c; Inoue and Aki 2002, 2003; Koutras and Alexandrou 1995; Uchida 1998; and references therein.).

At early stages run statistics were defined with respect to Bernoulli trials (BT). Several generalizations to usual BT such as non-identical BT, Markov dependent BT (MBT), higher-order MBT, Binary sequences of order k, Binary sequences of

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order (k, r), multi-state trials, Markov dependent multi-state trials etc have been considered in the literature. Further depending on the requirements of different statistical problems, several new counting schemes of runs and hence types of runs were introduced. Some of the most commonly used types of runs and corresponding run statistics are as follows.

- $N_{n,k}$: the number of non-overlapping consecutive k successes until n trials in the sense of Feller's (1968) counting.
- $M_{n,k}$: the number of overlapping consecutive k successes until n trials in the sense of Ling's (1988) counting.
- $G_{n,k}$: the number of success-runs of size greater than or equal to k until n trials.
- $E_{n,k}$: the number of success runs of exact length k followed by failure until n trials in the sense of Mood's (1940) counting.
- $X_{n,k,\ell}$: the number of ℓ-overlapping $(0 \leqslant \ell \leqslant k-1)$ success runs of length k in n trials according to Aki and Hirano (2000) i.e. the number of success runs of length k which may have overlapping part of length ℓ with the previous success run of length k that has been counted.

To obtain the distributions of run statistics, method of combinatorics was used at early stages. But in case of generalized sequence of trials this method becomes complicated. Hence some new methods, such as conditional probability generating function method (pgf), Markov chain imbedding method, induced Markov chain method, exponential family approach etc were developed and used to obtain the distribution of run statistics.

The main tool used for obtaining the distributions in this paper is conditional pgf method. This method is not new in the literature of distributions of run statistics. It can be used in two different ways. In the first way, condition is put on the next event (see Hirano and Aki 1993; Uchida 1998) and in the second way the condition is put on the next trial from each stage (see Aki 1997; Aki et al. 1996; Ebneshahrashoob and Sobel 1990; Han and Aki 2000b; Uchida and Aki 1995).

Higher-order MBT have been considered by Schwager (1983), Uchida (1998) and Han and Aki (2000b) to study distribution of $X_{n,k}$, the number of success runs of length k in n trials. While Aki et al. (1996) studied distribution of sooner and later waiting time problems in case of higher-order MBT by using pgf method and combinatorial method.

The joint distribution of runs and patterns have been studied by Fu (1996) in case of multi-state trials by introducing forward and backward principle for method of Markov chain imbedding. Doi and Yamamoto (1998), Shinde and Kotwal (2006) also studied joint distribution of run statistics in case of multi-state trials. Han and Aki (1999) introduced a concept of Markov chain embeddable variable of multinomial type and returnable type to study joint distribution of runs in case of multi-state trials, under the first four counting schemes introduced above.

In this paper we consider a sequence of m-dependent MBT. Let X^i_{n,k_i} $(i = 0, 1)$ be the number of i-runs of length k_i in n trials. In Sect. 2 we derive pgf of distribution of $\left(X^0_{n,k_0}, X^1_{n,k_1}\right)$ under the above-mentioned five counting schemes, by using conditional pgf method. Using the pgfs derived in Sect. 2, we develop an algorithm to get exact probability distribution in Sect. 3. The formula for exact probability is interpreted in terms of joint distribution of $\left(X^0_{n,k_0}, X^1_{n,k_1}\right)$ and $\left(X^0_{n-1,k_0}, X^1_{n-1,k_1}\right)$. As an application of these distributions, in Sect. 4 we obtain distribution of sooner and later waiting time problems. In Sect. 5, we evaluate the exact probability distributions of $\left(X^0_{n,k_0}, X^1_{n,k_1}\right)$ under all the counting schemes and distributions of some waiting time random variables.

2 Distribution of $\left(X^0_{n,k_0}, X^1_{n,k_1}\right)$

Let $X_{-m+1}, X_{-m+2}, \ldots, X_0, X_1, \ldots, X_n$ be a time homogeneous, m-dependent sequence of MBT with transition probabilities,

$$P(X_i = 1 \,|X_{i-1} = x_m, X_{i-2} = x_{m-1}, \ldots, X_{i-m} = x_1) = p_{x_1x_2\ldots x_m},$$

$$P(X_i = 0 \,|X_{i-1} = x_m, X_{i-2} = x_{m-1}, \ldots, X_{i-m} = x_1) = q_{x_1x_2\ldots x_m},$$

$$\text{for} \quad i = 1, 2, \ldots, n, \text{ and } x_1, x_2, \ldots, x_m = 0, 1.$$

where $q_{x_1x_2\ldots x_m} = 1 - p_{x_1x_2\ldots x_m}$ and $0 < p_{x_1x_2\ldots x_m} < 1 \quad \forall x_1, x_2, \ldots, x_m = 0, 1.$

Let the initial probability distribution be,

$$P(X_{-m+1} = x_1, X_{-m+2} = x_2, \ldots, X_0 = x_m) = \pi_{x_1x_2\ldots x_m}, \quad x_1, x_2, \ldots, x_m = 0, 1.$$

Here we obtain the joint distribution of $\left(X^0_{n,k_0}, X^1_{n,k_1}\right)$ by converting the sequence of m-dependent MBT to the sequence of one-dependent Markov trials. This conversion is based on the transformation of m-digit binary number to a unique equivalent decimal number as used by Uchida (1998). Uchida (1998) studied the distribution of $X^1_{n,k}$, the number of success runs of length k in n trials in case of sequence of m-dependent MBT under first four counting schemes by using method of conditional pgfs.

First we introduce the notations necessary for the conversion of sequence of m-dependent MBT to the sequence of one-dependent Markov trials and describe the method in short.

Let N^2_m be the set of all possible outcomes of sequence of length m i.e. $N^2_m = \{x_1x_2\ldots x_m \mid x_1, x_2, \ldots, x_m = 0, 1, \}$. It consists of 2^m elements. Treating every element $x_1x_2\ldots x_m$ of N^2_m as a m-digit binary number, transform it into unique equivalent decimal number by using a function $g : N^2_m \rightarrow N^{10}_m$ such that $g(x_1x_2\ldots x_m) = \sum_{i=1}^{m-1} 2^{m-i+1}\, x_i$, for all $x_1x_2\ldots x_m \in N^2_m$, where $N^{10}_m = \{0, 1, 2, \ldots, 2^m - 1, \}$. Let $f_i (i = 0, 1)$ be the mapping from N^{10}_m to N^{10}_m such that $f_i(x) = 2x + i (\bmod 2^m)$. If x is equivalent decimal of binary number $x_1x_2\ldots x_m$ then $f_i(x)(i = 0, 1)$ calculates the decimal equivalent of binary number $x_2\ldots x_m\, i$.

For example, for m=3, we have $N^2_3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$ and $N^{10}_3 = \{0, 1, 2, \ldots, 7, \}$. Then the decimal equivalent of 3-digit binary number 110 is 6. Transition from 110 to state 1(0) results in new state 101(100), i.e. the decimal equivalent of the binary number 101(100) is calculated by $f_1(6)$ $(f_0(6))$.

The sequence $\{X_i, i \geqslant -m + 1\}$ of m-dependent MBT is thus converted into one-dependent $\{0, 1, 2, \ldots, 2^m - 1, \}$-valued sequence $\{Z_i, i \geqslant 0\}$ of Markov trials with the transition probability matrix $P = \left(\left(p_{xy}\right)\right)$ of order $2^m \times 2^m$ where

$$P_{xy} = \begin{cases} p_x & \text{if } y = f_1(x) \\ q_x & \text{if } y = f_0(x) \qquad \forall\, x \in N^{10}_m \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Let $\phi_{n,\alpha}(t_0, t_1)$ $(\alpha = N, G, M, E, X)$ be the *pgf* of distribution of $\left(X^0_{n,k_0,\alpha}, X^1_{n,k_1,\alpha}\right)$ where $X^i_{n,k_i,\alpha}(i = 0, 1)$ denotes the number of i-runs of length k_i in n MBT under the counting scheme corresponding to α, where α is defined as follows. (see Han & Aki 1998).

$$\alpha = \begin{cases} N & \text{for non-overlapping counting scheme,} \\ G & \text{for atleast k counting scheme,} \\ M & \text{for overlapping counting scheme,} \\ E & \text{for exactly k counting scheme,} \\ X & \text{for generalized } \ell - \text{overlapping counting scheme.} \end{cases}$$

For a non-negative integer $c \leqslant n$, assume that we have observed until $(n-c)^{th}$ trial (i.e. $X_{-m+1}, X_{-m+2}, \ldots, X_0, X_1, \ldots, X_{n-c}$).

Let $\phi^i_{c,\alpha}(t_0, t_1)(\alpha = N, G, M, E, X \text{ and } i = 1, 2, \ldots, 2^m - 2)$ be the pgf of joint distribution of number of 0-runs of length k_0 and number of 1-runs of length k_1 in $X_{n-c+1}, \ldots, X_n$ given that $X_{n-c} = x_m, X_{n-c-1} = x_{m-1}, \ldots, X_{n-c-m+1} = x_1$ and $g(x_1 x_2 \ldots x_m) = i$.

Observe that for $x_1 = x_2 = \cdots = x_m = 1(0)$, $g(x_1 x_2 \ldots x_m) = 2^m - 1$ (0). Hence we define $\phi^{(2^m-1,j)}_{c,\alpha}(t_0, t_1)$ $\left(\phi^{(0,j)}_{c,\alpha}(t_0, t_1)\right)$ as pgf of joint distribution of number of 0-runs of length k_0 and number of 1-runs of length k_1 in $X_{n-c+1}, \ldots, X_n$ given that $X_{n-c} = X_{n-c-1} = \cdots = X_{n-c-m+1} = 1(0)$ and currently at $(n-c)^{th}$trial 1-run (0-run) of length j has occurred, $j = 1, 2, \ldots, k_1$ $(j = 1, 2, \ldots, k_0)$.

Assuming $\pi_{00\ldots0} = 1$, for $\alpha = N, G, M, E, X$, we have,

$$\phi_{n,\alpha}(t_0, t_1) = \phi^{(0,0)}_{n,\alpha}(t_0, t_1) \tag{2}$$

Also we have, for $\alpha = N, G, M, E, X$,

$$\phi^{(i,j)}_{0,\alpha}(t_0, t_1) = 1 \quad \forall(i, j) \tag{3}$$

and

$$\phi^i_{0,\alpha}(t_0, t_1) = 1 \quad i = 1, 2, \ldots, 2^m - 2.$$

Now conditioning on the next trial from every stage, in general, we get the following recurrent relations of conditional pgfs of $\left(X^0_{n,k_0,\alpha}, X^1_{n,k_1,\alpha}\right)$ for $c = 1, 2, \ldots, n$ and $\alpha = N, G, M, E, X$.

$$\begin{aligned} \phi^{(0,0)}_{n,\alpha}(t_0, t_1) &= p_0\, \phi^{f_1(0)}_{n-1,\alpha}(t_0, t_1) + q_0 \phi^{(0,1)}_{n-1,\alpha}(t_0, t_1), \\ \phi^i_{c,\alpha}(t_0, t_1) &= p_i\, \phi^{f_1(i)}_{c-1,\alpha}(t_0, t_1) + q_i \phi^{f_0(i)}_{c-1,\alpha}(t_0, t_1), \\ & \qquad i = 1, 2, \ldots, 2^{m-1} - 2, 2^{m-1} + 1, \ldots, 2^m - 2; \end{aligned}$$

$$\phi^i_{c,\alpha}(t_0, t_1) = \begin{cases} p_i\, \phi^{(2^m-1,m)}_{c-1,\alpha}(t_0, t_1) + q_i \phi^{f_0(i)}_{c-1,\alpha}(t_0, t_1) & \text{if } i = 2^{m-1} - 1 \\ p_i\, \phi^{f_1(i)}_{c-1,\alpha}(t_0, t_1) + q_i \phi^{(0,m)}_{c-1,\alpha}(t_0, t_1) & \text{if } i = 2^{m-1} \end{cases},$$

$$\phi^{(0,j)}_{c,\alpha}(t_0, t_1) = p_0\, \phi^{f_1(0)}_{c-1,\alpha}(t_0, t_1) + q_0 \phi^{(0,j+1)}_{c-1,\alpha}(t_0, t_1), \quad j = 1, 2, \ldots, k_0 - 2$$

and

$$\phi_{c,\alpha}^{(0,k_0-1)}(t_0,t_1) = \begin{cases} p_0\,\phi_{c-1,\alpha}^{f_1(0)}(t_0,t_1) + q_0 t_0 \phi_{c-1,\alpha}^{(0,k_0)}(t_0,t_1) \\ \quad \text{if } \alpha = N, G, M, X \text{ and } c = 1, 2, \ldots, n \\ \quad\quad \text{or if } \alpha = E \text{ and } c = 1. \\ p_0\,\phi_{c-1,\alpha}^{f_1(0)}(t_0,t_1) + q_0 \phi_{c-1,\alpha}^{(0,k_0)}(t_0,t_1) \\ \quad \text{if } \alpha = E \text{ and } c = 2, \ldots, n. \end{cases}.$$

Similarly for $i = 2^m - 1$,

$$\phi_{c,\alpha}^{(i,j)}(t_0,t_1) = p_i\,\phi_{c-1,\alpha}^{(i,j+1)}(t_0,t_1) + q_i \phi_{c-1,\alpha}^{f_0(i)}(t_0,t_1), \quad j = 1, 2, \ldots, k_1 - 2,$$

$$\phi_{c,\alpha}^{(i,k_1-1)}(t_0,t_1) = \begin{cases} p_i t_1\,\phi_{c-1,\alpha}^{(i,k_1)}(t_0,t_1) + q_i \phi_{c-1,\alpha}^{f_0(i)}(t_0,t_1) \\ \quad \text{if } \alpha = N, G, M, X \text{ and } c = 1, 2, \ldots, n \\ \quad\quad \text{or if } \alpha = E \text{ and } c = 1. \\ p_i\,\phi_{c-1,\alpha}^{(i,k_1)}(t_0,t_1) + q_i \phi_{c-1,\alpha}^{f_0(i)}(t_0,t_1) \\ \quad \text{if } \alpha = E \text{ and } c = 2, \ldots, n. \end{cases},$$

$$\phi_{c,\alpha}^{(0,k_0)}(t_0,t_1) = \begin{cases} p_0\,\phi_{c-1,\alpha}^{f_1(0)}(t_0,t_1) + q_0 \phi_{c-1,\alpha}^{(0,1)}(t_0,t_1) & \text{if } \alpha = N \\ p_0\,\phi_{c-1,\alpha}^{f_1(0)}(t_0,t_1) + q_0 t_0 \phi_{c-1,\alpha}^{(0,k_0)}(t_0,t_1) & \text{if } \alpha = M \\ p_0\,\phi_{c-1,\alpha}^{f_1(0)}(t_0,t_1) + q_0 \phi_{c-1,\alpha}^{(0,k_0)}(t_0,t_1) & \text{if } \alpha = G \\ p_0\,\phi_{c-1,\alpha}^{f_1(0)}(t_0,t_1) + q_0 t_0 \phi_{c-1,\alpha}^{(0,\ell_0+1)}(t_0,t_1) & \text{if } \alpha = X \\ p_0 t_0 \phi_{c-1,\alpha}^{f_1(0)}(t_0,t_1) + q_0 \phi_{c-1,\alpha}^{(0,k_0+1)}(t_0,t_1) & \text{if } \alpha = E \end{cases}.$$

For $i = 2^m - 1$,

$$\phi_{c,\alpha}^{(i,k_1)}(t_0,t_1) = \begin{cases} p_i\,\phi_{c-1,\alpha}^{(i,1)}(t_0,t_1) + q_i \phi_{c-1,\alpha}^{f_0(i)}(t_0,t_1) & \text{if } \alpha = N \\ p_i t_1\,\phi_{c-1,\alpha}^{(i,k_1)}(t_0,t_1) + q_i \phi_{c-1,\alpha}^{f_0(i)}(t_0,t_1) & \text{if } \alpha = M \\ p_i\,\phi_{c-1,\alpha}^{(i,k_1)}(t_0,t_1) + q_i \phi_{c-1,\alpha}^{f_0(i)}(t_0,t_1) & \text{if } \alpha = G \\ p_i\,\phi_{c-1,\alpha}^{(i,\ell_1+1)}(t_0,t_1) + q_i \phi_{c-1,\alpha}^{f_0(i)}(t_0,t_1) & \text{if } \alpha = X \\ p_i\,\phi_{c-1,\alpha}^{(i,k_1+1)}(t_0,t_1) + q_i t_1 \phi_{c-1,\alpha}^{f_0(i)}(t_0,t_1) & \text{if } \alpha = E \end{cases},$$

where for $\alpha = E, \phi_{c,\alpha}^{(2^m-1,k_1+1)}(t_0,t_1)\big(\phi_{c,\alpha}^{(0,k_0+1)}(t_0,t_1)\big)$ is the pgf of joint distribution of number of 0-runs of length k_0 and number of 1-runs of length k_1 in $X_{n-c+1}, \ldots, X_n$ given that $X_{n-c} = X_{n-c-1} = \cdots = X_{n-c-m+1} = 1(0)$ and currently at $(n-c)^{th}$ trial length of 1-run (0-run) is more than $k_1(k_0)$. Therefore the recurrent relations of the conditional pgfs $\phi_{c,E}^{(2^m-1,k_1+1)}(t_0,t_1)$ and $\phi_{c,E}^{(0,k_0+1)}(t_0,t_1)$ are as follows.

$$\phi_{c,E}^{(0,k_0+1)}(t_0,t_1) = p_0\,\phi_{c-1,E}^{f_1(0)}(t_0,t_1) + q_0 \phi_{c-1,E}^{(0,k_0+1)}(t_0,t_1)$$

and for $i = 2^m - 1$,

$$\phi_{c,E}^{(i,k_1+1)}(t_0,t_1) = p_i\,\phi_{c-1,E}^{(i,k_1+1)}(t_0,t_1) + q_i \phi_{c-1,E}^{f_0(i)}(t_0,t_1).$$

The above set of recurrent relations of conditional pgfs for each α can be written as follows.

$$\underline{\phi}_{c,\alpha}(t_0,t_1)=\begin{cases}\left(A_\alpha+B_{0,\alpha}t_0+B_{1,\alpha}\,t_1\right)\,\underline{\phi}_{c-1,\alpha}(t_0,t_1)\\ \quad\text{if}\,\alpha=N,\,M,\,G,\,X\text{ and }c=1,2,\ldots,n\\ \quad\quad\text{or if}\,\alpha=E\text{ and }c=2,\ldots,n.\\ \left(A_\alpha^{(n)}+B_{0,\alpha}^{(n)}t_0+B_{1,\alpha}^{(n)}\,t_1\right)\,\underline{\phi}_{c-1,\alpha}(t_0,t_1)\quad\text{if}\,\alpha=E\text{ and }c=1.\end{cases}\tag{4}$$

where

$$\underline{\phi}_{c,\alpha}(t_0,t_1)$$

$$=\begin{cases}\left(\phi_{c,\alpha}^{(0,k_0)}\ \phi_{c,\alpha}^{(0,k_0-1)}\ldots\phi_{c,\alpha}^{(0,0)}\phi_{c,\alpha}^{1}\,\phi_{c,\alpha}^{2}\ \ldots\phi_{c,\alpha}^{2^m-2}\ \phi_{c,\alpha}^{(2^m-1,1)}\ \phi_{c,\alpha}^{(2^m-1,2)}\ldots\ \phi_{c,\alpha}^{(2^m-1,k_1)}\right)'\\ \quad\text{if}\,\alpha=N,X\\ \left(\phi_{c,\alpha}^{(0,k_0)}\ \phi_{c,\alpha}^{(0,k_0-1)}\ \ldots\ \phi_{c,\alpha}^{(0,0)}\ \phi_{c,\alpha}^{1}\ \phi_{c,\alpha}^{2}\ \ldots\ \phi_{c,\alpha}^{2^m-2}\ \phi_{c,\alpha}^{(2^m-1,m)}\ \phi_{c,\alpha}^{(2^m-1,m+1)}\ \ldots\ \phi_{c,\alpha}^{(2^m-1,k_1)}\right)'\\ \quad\text{if}\,\alpha=M,G\\ \left(\phi_{c,\alpha}^{(0,k_0+1)}\ \phi_{c,\alpha}^{(0,k_0)}\ \ldots\ \phi_{c,\alpha}^{(0,0)}\ \phi_{c,\alpha}^{1}\ \phi_{c,\alpha}^{2}\ \ldots\ \phi_{c,\alpha}^{2^m-2}\ \phi_{c,\alpha}^{(2^m-1,m)}\ \phi_{c,\alpha}^{(2^m-1,m+1)}\ \ldots\ \phi_{c,\alpha}^{(2^m-1,k_1+1)}\right)'\\ \quad\text{if}\,\alpha=E.\end{cases}$$

and $\left(A_\alpha+B_{0,\alpha}+B_{1,\alpha}\right)$ for $\alpha=N,M,G,X,E$ and $\left(A_E^{(n)}+B_{0,E}^{(n)}+B_{1,E}^{(n)}\right)$ are the square matrices of order equal to size of $\underline{\phi}_{c,\alpha}(t_0,t_1)$. The i^{th} row of the matrix $\left(A_\alpha+B_{0,\alpha}+B_{1,\alpha},\right)$ corresponds to the recurrent relation of i^{th} element of $\underline{\phi}_{c,\alpha}(t_0,t_1)$ with elements of $\underline{\phi}_{c-1,\alpha}(t_0,t_1)$ and its elements are the coefficients of elements of $\underline{\phi}_{c-1,\alpha}(t_0,t_1)$.

i.e. the order of $\left(A_\alpha+B_{0,\alpha}+B_{1,\alpha}\right)=\begin{cases}2^m+k_0+k_1-1 & \text{if}\,\alpha=N,X\\ 2^m+k_0+k_1-m & \text{if}\,\alpha=M,G\\ 2^m+k_0+k_1-m+2 & \text{if}\,\alpha=E.\end{cases}$

From Eqs. (2) and (3), we get for each $\alpha=N,M,G,X,E$,

$$\phi_{n,\alpha}(t_0,t_1)=\left[0\ 0\cdots0\ 1\ 0\cdots0\right]\,\underline{\phi}_{n,\alpha}(t_0,t_1)=\underline{p}'\,\underline{\phi}_{n,\alpha}(t_0,t_1)$$

and $\underline{\phi}_{0,\alpha}(t_0,t_1)=\underline{1}$,

where $\underline{1}$ is column vector with all its elements equal to one.

Using Eq. (4) recurrently we get,

$$\underline{\phi}_{n,\alpha}(t_0,t_1)=\begin{cases}\left(A_\alpha+B_{0,\alpha}\,t_0+B_{1,\alpha}\,t_1\right)^n\,\underline{1} & \text{if}\,\alpha=N,M,G,X,\\ \left(A_\alpha+B_{0,\alpha}\,t_0+B_{1,\alpha}\,t_1\right)^{n-1}\left(A_\alpha^{(n)}+B_{0,\alpha}^{(n)}\,t_0+B_{1,\alpha}^{(n)}\,t_1\right)\underline{1} & \text{if}\,\alpha=E.\end{cases}$$

Hence the pgf of distribution of $\left(X^0_{n,k_0,\alpha}, X^1_{n,k_1,\alpha}\right)$ is given by,

$$\phi_{n,\alpha}(t_0,t_1) = \begin{cases} \underline{p}'\left(A_\alpha + B_{0,\alpha}\,t_0 + B_{1,\alpha}\,t_1\right)^n \underline{1} & \text{if } \alpha = N, M, G, X, \\ \underline{p}'\left(A_\alpha + B_{0,\alpha}\,t_0 + B_{1,\alpha}\,t_1\right)^{n-1}\left(A^{(n)}_\alpha + B^{(n)}_{0,\alpha}\,t_0 + B^{(n)}_{1,\alpha}\,t_1\right)\underline{1} \\ \qquad \text{if } \alpha = E. \end{cases}$$

Let $D_\alpha = A_\alpha + B_{0,\alpha} + B_{1,\alpha}$ $(\alpha = N, M, G, X, E)$ and $D_\alpha = \left(\left(d_{ij}\right)\right)$. We define the matrix $E_{x,y} = \left(\left(e_{ij}\right)\right)$ of same order as that of D_α such that,

$$e_{ij} = \begin{cases} 1 & \text{if } i = x\,,\ j = y \\ 0 & \text{otherwise} \end{cases}.$$

Also let '$\cdot *$' be the binary operator such that for two matrices $A = \left(\left(a_{ij}\right)\right)$ and $B = \left(\left(b_{ij}\right)\right)$ of equal order, $A \cdot * B = \left(\left(a_{ij}b_{ij}\right)\right)$. Now we specify the matrix $D_\alpha(\alpha = N, M, G, X, E)$ explicitly and separate the matrices A_α, $B_{0,\alpha}$ and $B_{1,\alpha}$ from it.

The elements d_{ij} $(i, j = -k_0, -k_0+1, \ldots, 0, 1, 2, \ldots, 2^m + k_1 - 2)$ of the matrix D_X are as follows.

$$d_{ij} = \begin{cases} p_0 & \text{if } i = -k_0, -(k_0-1), \ldots, -1, 0 \text{ and } j = 1 \\ q_0 & \text{if } i = -(k_0-1), -(k_0-2), \ldots, -1, 0 \text{ and } j = i-1 \\ q_0 & \text{if } i = -k_0 \text{ and } j = -(\ell_0+1) \\ p_i & \text{if } i = 1, 2, \ldots, 2^{m-1}-2, 2^{m-1}, \ldots, 2^m-2 \text{ and } j = f_1(i) \\ p_i & \text{if } i = 2^{m-1}-1 \text{ and } j = 2^m+m-2 \\ q_i & \text{if } i = 1, 2, \ldots, 2^{m-1}-1, 2^{m-1}+1, \ldots, 2^m-2 \text{ and } j = f_0(i) \\ q_i & \text{if } i = 2^{m-1} \text{ and } j = -m \\ q_{2^m-1} & \text{if } i = 2^m-1, 2^m, \ldots, 2^m+k_1-2 \text{ and } j = 2^m-2 \\ p_{2^m-1} & \text{if } i = 2^m-1, 2^m, \ldots, 2^m+k_1-3 \text{ and } j = i+1 \\ p_{2^m-1} & \text{if } i = 2^m+k_1-2 \text{ and } j = 2^m+\ell_1-1 \\ 0 & \text{otherwise} \end{cases}$$

The matrices A_X, $B_{0,X}$ and $B_{1,X}$ can be obtained from D_X by using the relations

$$B_{0,X} = D_X \cdot * \left(\sum_{i=1}^{2^m+k_0+k_1-1} E_{i,1}\right),$$

$$B_{1,X} = D_X \cdot * \left(\sum_{i=1}^{2^m+k_0+k_1-1} E_{i,2^m+k_0+k_1-1}\right)$$

$$\text{and } A_X = D_X - B_{0,X} - B_{1,X}.$$

As non-overlapping counting scheme is particular case of generalized ℓ-overlapping counting scheme with $\ell = 0$, the joint distribution of $\left(X^0_{n,k_0,N}, X^1_{n,k_1,N}\right)$ is same as the distribution of $\left(X^0_{n,k_0,X}, X^1_{n,k_1,X}\right)$ with $\ell_0 = 0$ and $\ell_1 = 0$.

The elements d_{ij} $(i, j = -k_0, -k_0+1, \ldots, 0, 1, 2, \ldots, 2^m + k_1 - m - 1)$ of the matrix D_M are as follows.

$$d_{ij} = \begin{cases} p_0 & \text{if } i = -k_0, -(k_0-1), \ldots, -1, 0 \text{ and } j = 1 \\ q_0 & \text{if } i = -(k_0-1), -(k_0-2), \ldots, -1, 0 \text{ and } j = i-1 \\ q_0 & \text{if } i = -k_0 \text{ and } j = i \\ p_i & \text{if } i = 1, 2, \ldots, 2^m - 2 \text{ and } j = f_1(i) \\ q_i & \text{if } i = 1, 2, \ldots, 2^{m-1}-1, 2^{m-1}+1, \ldots, 2^m - 2 \text{ and } j = f_0(i) \\ q_i & \text{if } i = 2^{m-1} \text{ and } j = -m \\ q_{2^m-1} & \text{if } i = 2^m - 1, 2^m, \ldots, 2^m + k_1 - m - 1 \text{ and } j = 2^m - 2 \\ p_{2^m-1} & \text{if } i = 2^m - 1, 2^m, \ldots, 2^m + k_1 - m - 2 \text{ and } j = i+1 \\ p_{2^m-1} & \text{if } i = 2^m + k_1 - m - 1 \text{ and } j = i \\ 0 & \text{otherwise} \end{cases}$$

The matrices A_M, $B_{0,M}$ and $B_{1,M}$ are specified by the relations,

$$B_{0,M} = D_M \cdot * \left(\sum_{i=1}^{2^m+k_0+k_1-m} E_{i,1} \right),$$

$$B_{1,M} = D_M \cdot * \left(\sum_{i=1}^{2^m+k_0+k_1-m} E_{i,2^m+k_0+k_1-m} \right)$$

$$\text{and } A_M = D_M - B_{0,M} - B_{1,M}.$$

We have for $\alpha = G$, $D_\alpha = D_M$ and

$$B_{0,G} = D_G \cdot * E_{2,1},$$

$$B_{1,G} = D_G \cdot * E_{2^m+k_0+k_1-m-1,\, 2^m+k_0+k_1-m},$$

$$\text{and } A_G = D_G - B_{0,G} - B_{1,G}.$$

The elements d_{ij} $(i, j = -(k_0+1), -k_0, \ldots, 0, 1, 2, \ldots, 2^m + k_1 - m)$ of the matrix D_E are as follows.

$$d_{ij} = \begin{cases} p_0 & \text{if } i = -(k_0+1), -k_0, \ldots, -1, 0 \text{ and } j = 1 \\ q_0 & \text{if } i = -k_0, -(k_0-1), \ldots, -1, 0 \text{ and } j = i-1 \\ q_0 & \text{if } i = -(k_0+1) \text{ and } j = i \\ p_i & \text{if } i = 1, 2, \ldots, 2^m - 2 \text{ and } j = f_1(i) \\ q_i & \text{if } i = 1, 2, \ldots, 2^{m-1}-1, 2^{m-1}+1, \ldots, 2^m - 2 \text{ and } j = f_0(i) \\ q_i & \text{if } i = 2^{m-1} \text{ and } j = -m \\ p_{2^m-1} & \text{if } i = 2^m - 1, 2^m, \ldots, 2^m + k_1 - m - 1 \text{ and } j = i+1 \\ p_{2^m-1} & \text{if } i = 2^m + k_1 - m \text{ and } j = i \\ q_{2^m-1} & \text{if } i = 2^m - 1, 2^m, \ldots, 2^m + k_1 - m \text{ and } j = 2^m - 2 \\ 0 & \text{otherwise} \end{cases}$$

The matrices A_E, $B_{0,E}$ and $B_{1,E}$ can be specified by the following relations.

$$B_{0,E} = D_E \cdot * E_{2,k_0+3},$$

$$B_{1,E} = D_E \cdot * E_{2^m+k_0+k_1-m+1,2^m+k_0}$$

$$\text{and } A_E = D_E - B_{0,E} - B_{1,E}.$$

We also have $D_E = A_E^{(n)} + B_{0,E}^{(n)} + B_{1,E}^{(n)}$ and the matrices $B_{0,E}^{(n)}$, $B_{1,E}^{(n)}$ and $A_E^{(n)}$ are as follows.

$$\begin{aligned} B_{0,E}^{(n)} &= D_E \cdot * \left(E_{2,k_0+3} + E_{3,2}\right), \\ B_{1,E}^{(n)} &= D_E \cdot * \left(E_{2^m+k_0+k_1-m+1,2^m+k_0} + E_{2^m+k_0+k_1-m,2^m+k_0+k_0-m+1}\right) \\ \text{and } A_E^{(n)} &= D_E - B_{0,E}^{(n)} - B_{1,E}^{(n)}. \end{aligned}$$

3 Exact distribution of $\left(X_{n,k_0}^0, X_{n,k_1}^1\right)$

From the previous section, we have the pgf of $\left(X_{n,k_0,\alpha}^0, X_{n,k_1,\alpha}^1\right)$ as follows.

$$\phi_{n,\alpha}(t_0,t_1) = \begin{cases} \underline{p}' \left(A_\alpha + B_{0,\alpha}\, t_0 + B_{1,\alpha}\, t_1\right)^n \underline{1} & \text{if } \alpha = N, M, G, X \\ \underline{p}' \left(A_\alpha + B_{0,\alpha}\, t_0 + B_{1,\alpha}\, t_1\right)^{n-1} \left(A_\alpha^{(n)} + B_{0,\alpha}^{(n)}\, t_0 + B_{1,\alpha}^{(n)}\, t_1\right) \underline{1} \\ \qquad \text{if } \alpha = E. \end{cases} \tag{5}$$

The above form of pgfs in general involves matrix polynomial $(A + B_0t_0 + B_1t_1)^n$ of order n. Hence the joint probability distribution can be obtained by expanding the polynomial with respect to t_0 and t_1.

That is,

$$\begin{aligned} &P\left(X_{n,k_0,\alpha}^0 = x_0, X_{n,k_1,\alpha}^1 = x_1\right) \\ &\quad = \text{coefficient of } t_0^{x_0} t_1^{x_1} \ in\ \phi_{n,\alpha}(t_0,t_1), \quad (x_0,x_1) \in S_{n,\alpha} \end{aligned} \tag{6}$$

where

$$S_{n,\alpha} = \{(x_0,x_1) \mid 0 \leqslant x_i \leqslant \beta_n(i,\alpha)\ ,\ i = 0, 1\}$$

and

$$\beta_n(i,\alpha) = \begin{cases} \left[\frac{n}{k_i}\right] & \text{if } \alpha = N \\ n - k_i + 1 & \text{if } \alpha = M \\ \left[\frac{n+1}{k_i+1}\right] & \text{if } \alpha = G, E \\ \left[\frac{n-k_i}{k_i-\ell_i}\right] & \text{if } \alpha = X \end{cases}, \quad i = 0,\ 1.$$

The following Lemma gives the recurrent relations of the coefficient matrices of $t_0^{x_0} t_1^{x_1}$ in the expansion of the matrix polynomial $(A + B_0t_0 + B_1t_1)^n$ for $(x_0,x_1) \in S_n$ where $S_n = \{(x_0,x_1) \mid 0 \leqslant x_0 + x_1 \leqslant n\}$.

Lemma 3.1 *Let $C_n(x_0,x_1)$ be the coefficient matrix of $t_0^{x_0} t_1^{x_1}$ in the expansion of the matrix polynomial $(A + B_0t_0 + B_1t_1)^n$. Then $C_n(x_0,x_1)$ satisfies the recurrent relation*

$$\begin{aligned} C_n(x_0,x_1) = {} & C_{n-1}(x_0,x_1)\, A + C_{n-1}(x_0-1,x_1)\, B_0 \\ & + C_{n-1}(x_0,x_1-1)\, B_1, \quad (x_0,x_1) \in S_n \end{aligned} \tag{7}$$

with $C_1(0,0) = A$, $C_1(1,0) = B_0$ and $C_1(0,1) = B_1$.

Proof Obviously for n=1, we have

$$C_1(0,0) = A, \quad C_1(1,0) = B_0 \text{ and } C_1(0,1) = B_1.$$

and for n=2, $C_2(x_0, x_1)$ satisfies (Eq.8) for $(x_0, x_1) \in S_2$.

Assume that Eq. (8) is true for some $r(2 \leqslant r < n,)$. Hence we have,

$$(A + B_0\, t_0 + B_1\, t_1)^r = \sum_{(x_0,x_1)\in S_r} C_r(x_0, x_1)\; t_0^{x_0}\, t_1^{x_1},$$

Then

$$\begin{aligned}
&(A + B_0\, t_0 + B_1\, t_1)^{r+1} \\
&\quad = (A + B_0\, t_0 + B_1\, t_1)^r\, (A + B_0\, t_0 + B_1\, t_1) \\
&\quad = \left(\sum_{(x_0,x_1)\in S_r} C_r(x_0, x_1)\; t_0^{x_0}\, t_1^{x_1} \right) (A + B_0\, t_0 + B_1\, t_1) \\
&\quad = \sum_{(x_0,x_1)\in S_{r+1}} \{C_r(x_0, x_1)\, A + C_r(x_0 - 1, x_1)\, B_0 + C_r(x_0, x_1 - 1)\, B_1\}\; t_0^{x_0}\, t_1^{x_1}.
\end{aligned}$$

Comparing the coefficients in $(A + B_0\, t_0 + B_1\, t_1)^{r+1}$, we get (3.3).

Theorem 3.1 *The exact probability distribution of* $\left(X^0_{n,k_0,\alpha}, X^1_{n,k_1,\alpha}\right)$ *is given by,*

$$P\left(X^0_{n,k_0,\alpha} = x_0, X^1_{n,k_1,\alpha} = x_1\right) = \begin{cases} \underline{p}' C_{n,\alpha}(x_0, x_1)\; \underline{1} & \text{if } \alpha = N, M, G, X \\ \underline{p}' \Big\{ C_{n-1,\alpha}(x_0, x_1) A^{(n)}_{\alpha} + C_{n-1,\alpha}(x_0 - 1, x_1)\; B^{(n)}_{0,\alpha} & \\ \qquad + C_{n-1,\alpha}(x_0, x_1 - 1)\; B^{(n)}_{1,\alpha} \Big\}\; \underline{1} & \text{if } \alpha = E \end{cases} \tag{8}$$

where $C_{n,\alpha}(x_0, x_1)$ *is the coefficient matrix of* $t_0^{x_0} t_1^{x_1}$ *in the expansion of matrix polynomial* $\left(A_\alpha + B_{0,\alpha} t_0 + B_{1,\alpha}\, t_1\right)^n$ *and it satisfies the relation,*

$$\begin{aligned}
C_{n,\alpha}(x_0, x_1) = C_{n-1,\alpha}(x_0, x_1)\; A_\alpha + C_{n-1,\alpha}(x_0 - 1, x_1)\; B_{0,\alpha} \\
+ C_{n-1,\alpha}(x_0, x_1 - 1)\; B_{1,\alpha}, \quad (x_0, x_1) \in S_{n,\alpha}
\end{aligned}$$

with $C_{1,\alpha}(0,0) = A_\alpha$, $C_{1,\alpha}(1,0) = B_{0,\alpha}$ *and* $C_{1,\alpha}(0,1) = B_{1,\alpha}$.

Proof The proof follows from Eqs. (5), (6) and Lemma 3.1.

Let the double pgf of $\left(X^0_{n,k_0,\alpha}, X^1_{n,k_1,\alpha}\right)$ be $\Phi_\alpha(t_0, t_1, z,)$ $(\alpha = N, M, G, X)$. i.e.

$$\begin{aligned}
\Phi_\alpha(t_0, t_1, z) &= \sum_{n=0}^{\infty} \phi_{n,\alpha}(t_0, t_1,)\, z^n \\
&= \sum_{n=0}^{\infty} \sum_{\underline{x}\in S_{n,\alpha}} P\left(X^0_{n,k_0,\alpha} = x_0, X^1_{n,k_1,\alpha} = x_1\right)\; t_0^{x_0}\, t_1^{x_1}\, z^n.
\end{aligned}$$

Simplifying the above expression we get the double pgf of $\left(X^0_{n,k_0,\alpha}, X^1_{n,k_1,\alpha}\right)$ $(\alpha = N, M, G, X)$ as,

$$\Phi_\alpha(t_0, t_1, z) = \underline{p}' \left[I - z\left(A_\alpha + B_{0,\alpha}\, t_0 + B_{1,\alpha}\, t_1\right)\right]^{-1} \underline{1}, \quad 0 < z < 1, \tag{9}$$

where I is the identity matrix.

The expected number of i-runs of length k_i in n trials is,

$$E\left(X^i_{n,k_i,\alpha}\right) = \frac{\partial}{\partial t_i} \phi_{n,\alpha}(t_0, t_1) \Big|_{t_0=1, t_1=1} \quad i = 0, 1.$$

On simplifying this expression we have,

$$E(X^i_{n,k_i,\alpha}) = \underline{p}' \sum_{j=1}^{n} (A_\alpha + B_{0,\alpha} + B_{1,\alpha})^{j-1} B_{i,\alpha}\, \underline{1}, \quad i = 0, 1. \tag{10}$$

Remark 3.1 The exact probability distribution of $\left(X^0_{n,k_0,\alpha}, X^1_{n,k_1,\alpha}\right)(\alpha = N, M, G, X)$ given in (8) can be expressed as,

$$\begin{aligned} P&\left(X^0_{n,k_0,\alpha} = x_0, X^1_{n,k_1,\alpha} = x_1\right) \\ &= \underline{p}' \left\{C_{n-1,\alpha}(x_0, x_1)\, A_\alpha + C_{n-1,\alpha}(x_0 - 1, x_1)\, B_{0,\alpha} \right. \\ &\quad \left. + C_{n-1,\alpha}(x_0, x_1 - 1)\, B_{1,\alpha}\right\} \underline{1} \end{aligned}$$

The components of the above expression can be interpreted as,

$$\begin{aligned} &\underline{p}'\left\{C_{n-1,\alpha}(x_0, x_1) A_\alpha\right\} \underline{1} \\ &\quad = P\left(X^0_{n,k_0,\alpha} = x_0, X^1_{n,k_1,\alpha} = x_1; X^0_{n-1,k_0,\alpha} = x_0, X^1_{n-1,k_1,\alpha} = x_1\right) \\ &\underline{p}'\left\{C_{n-1,\alpha}(x_0 - 1, x_1) B_{0,\alpha}\right\} \underline{1} \\ &\quad = P\left(X^0_{n,k_0,\alpha} = x_0, X^1_{n,k_1,\alpha} = x_1; X^0_{n-1,k_0,\alpha} = x_0 - 1, X^1_{n-1,k_1,\alpha} = x_1\right) \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\underline{p}'\left\{C_{n-1,\alpha}(x_0, x_1 - 1) B_{1,\alpha}\right\} \underline{1} \\ &\quad = P\left(X^0_{n,k_0,\alpha} = x_0, X^1_{n,k_1,\alpha} = x_1; X^0_{n-1,k_0,\alpha} = x_0, X^1_{n-1,k_1,\alpha} = x_1 - 1\right) \end{aligned}$$

so that

$$\begin{aligned} &P\left(X^0_{n,k_0,\alpha} = x_0, X^1_{n,k_1,\alpha} = x_1\right) \\ &\quad = P\left(X^0_{n,k_0,\alpha} = x_0, X^1_{n,k_1,\alpha} = x_1; X^0_{n-1,k_0,\alpha} = x_0, X^1_{n-1,k_1,\alpha} = x_1\right) \\ &\qquad + P\left(X^0_{n,k_0,\alpha} = x_0, X^1_{n,k_1,\alpha} = x_1; X^0_{n-1,k_0,\alpha} = x_0 - 1, X^1_{n-1,k_1,\alpha} = x_1\right) \\ &\qquad + P\left(X^0_{n,k_0,\alpha} = x_0, X^1_{n,k_1,\alpha} = x_1; X^0_{n-1,k_0,\alpha} = x_0, X^1_{n-1,k_1,\alpha} = x_1 - 1\right) \end{aligned} \tag{12}$$

The above interpretation is useful for deriving different waiting time distributions.

4 Waiting time distributions

In this section we obtain distributions of different waiting time random variables. Let F_i be the event that i-run of length k_i occur for the first time (i=0, 1) and $W_S(W_L)$ be the waiting time for sooner (later) occurring event between F_0 and F_1. Let W_S^i $\left(W_L^i\right)$ (i=0, 1) be the waiting time for sooner (later) occurring event between F_0 and F_1 given that F_i is the sooner (later) event and $W_{r_i}^i$ (i=0, 1) is the waiting time for r_i^{th} occurrence of i-run of length k_i.

Then we have,

$$P(W_a = r) = P\left(W_a^0 = r\right) + P\left(W_a^1 = r\right) \quad a = S, L. \tag{13}$$

Let $\psi_S(t)$, $\psi_L(t)$,$\psi_S^i(t)$, $\psi_L^i(t)$ and $\psi_{r_i}^i(t)$be the pgf of W_S, W_L,W_S^i , W_L^i and $W_{r_i}^i$ respectively.

The pgfs of sooner and later waiting time distributions for success and failure runs in case of second order Markov dependent trials have been obtained by Aki et al. (1996) using probability generating function method. In case of general m-order Markov dependent trials, they have given the system of equations of 2^m conditional pgfs of waiting time to be solved with respect to conditional pgf.

We note that the distribution of W_S is same under all the five counting schemes while distribution of W_L is same for non-overlapping, overlapping, at least k, and generalized ℓ-overlapping counting schemes and is different for the exactly k counting scheme. In the next two subsections we obtain pgf of sooner and later waiting time distribution by using $\phi_n(t_0, t_1) = \underline{p}'\,(A + B_0 t_0 + B_1\, t_1)^n\, \underline{1}$, the general form of pgf of the distribution of $\left(X_{n,k_0}^0, X_{n,k_1}^1\right)$.

4.1 Sooner waiting time distribution

The $P\left(W_S^0 = r\right)$ can be written as,

$$P\left(W_S^0 = r\right) = P\left(X_{r,k_0}^0 = 1, X_{r,k_1}^1 = 0;\, X_{r-1,k_0}^0 = 0, X_{r-1,k_1}^1 = 0\right)$$

Using interpretations in (3.7), we get,

$$\begin{aligned} P\left(W_S^0 = r\right) &= \underline{p}' C_{r-1}(0, 0)\, B_0\, \underline{1} \\ &= \underline{p}' A^{r-1}\, B_0\, \underline{1} \end{aligned}$$

Similarly,

$$P\left(W_S^1 = r\right) = \underline{p}' A^{r-1}\, B_1\, \underline{1}.$$

Hence the pgf of W_S^i ($i = 0, 1$) is,

$$\psi_S^i(t) = t\, \underline{p}'\, (I - A\, t)^{-1} B_i\, \underline{1} \quad i = 0, 1.$$

Using Eq. 13, we obtain the pgf of W_S as,

$$\psi_S(t) = t\, \underline{p}'\, (I - A\, t)^{-1} (B_0 + B_1)\, \underline{1}.$$

The exact probability distribution of random variable W_S is given by,

$$P(W_S = r) = \underline{p}' A^{r-1}\, (B_0 + B_1)\, \underline{1} \quad r \geqslant \min(k_0, k_1). \tag{14}$$

4.2 Later waiting time distribution

Observe that $W_L^0 \overset{d}{=} W_S^1 + W_1^0 \mid (F_1$ has occured)

$$\text{and } W_L^1 \overset{d}{=} W_S^0 + W_1^1 \mid (F_0 \text{ has occured})$$

Distribution of $W_{r_i}^i (i = 0,\ 1)$can be obtained by using marginal distributions of $X_{n,k_0}^0$and X_{n,k_1}^1. The pgf of marginal distribution of $X_{n,k_1}^i (i = 0,\ 1)$ is given by,

$$\phi_n(t_0,\ 1) = \underline{p}'\left((A + B_1) + B_0 t_0\right)^n \underline{1}$$
$$\text{and } \phi_n(1, t_1) = \underline{p}'\left((A + B_0) + B_1 t_1\right)^n \underline{1}.$$

The probability distribution of $W_{r_i}^i$can be obtained by using the fact that,

$$P\left(W_{r_i}^i = n\right) = P\left(X_{n,k_i}^i = r_i;\ X_{n-1,k_i}^i = r_i - 1\right) \quad i = 0,\ 1.$$

Using the interpretation in (11) in case of marginal distribution of $X_{n,k_i}^i (i = 0,\ 1)$ and proceeding as in Sect. 4.1 we have,

$$P\left(W_1^0 = n\right) = \underline{p}'(A + B_1)^{n-1} B_0 \underline{1}$$
$$\text{and } P\left(W_1^1 = n\right) = \underline{p}'(A + B_0)^{n-1} B_1\, \underline{1}.$$

Hence we have pgfs of $W_1^0$and W_1^1as,

$$\psi_1^0(t) = t\ \underline{p}'\left(I - (A + B_1)t\right)^{-1} B_0 \underline{1}$$
$$\text{and } \psi_1^1(t) = t\ \underline{p}'\left(I - (A + B_0)t\right)^{-1} B_1 \underline{1}.$$

Then we get the pgf of W_L as,

$$\psi_L(t) = t^2 \underline{p}'\ \left\{(I - At)^{-1} B_1 (I - (A + B_1)t)^{-1} B_0\right\}\ \underline{1}$$
$$+\ t^2 \underline{p}'\ \left\{(I - At)^{-1} B_0 (I - (A + B_0)t)^{-1} B_1\right\}\ \underline{1}$$

Collecting the coefficients of t^rin $\psi_L(t)$, we have,

$$P(W_L = r) = \underline{p}' \left\{ \sum_{i=k_1}^{r-k_0} A^{i-1} B_1 (A + B_1)^{r-i-1} B_0 + \sum_{i=k_0}^{r-k_1} A^{i-1} B_0 (A + B_0)^{r-i-1} B_1 \right\} \underline{1}, \quad r \geqslant k_0 + k_1. \quad (15)$$

Remark 4.1 The formula Eqs. (14) and (15) can be used to obtain the sooner and later waiting time distributions under the counting scheme α, ($\alpha = N, M, G, X$) by replacing A, $B_0$and B_1 by $A_\alpha B_{0,\alpha}$ and $B_{1,\alpha}$ respectively. In the Mood's way of counting success runs (failure runs), the runs of exact length $k_1 (k_0)$ followed and preceded by failure (success), except the first run which may not preceded by an failure (success) or the last run which may not be followed by failure (success) are counted. But in case of waiting time of the $F_1(F_0)$, one has to wait for the success (failure) run of exact length $k_1(k_0)$ which is not followed by failure (success). Hence the formulae for the calculation of the sooner and later waiting time distributions in case of Mood's way of counting runs are not same as in Eqs. (14) and (15). The formula for the exact probability distribution of W_S is,

$$P(W_S=r)=\underline{p}'\left(A_E^{(n)}\right)^{r-1}\left(\left(B_{0,E}^{(n)}-B_{0,E}\right)+\left(B_{1,E}^{(n)}-B_{1,E}\right)\right)\underline{1},\quad r\geqslant \min(k_0,k_1)$$

and the formula for the exact probability distribution of W_L is,

$$\begin{aligned}P(W_L=r)=\underline{p}'&\left\{\sum_{i=k_1}^{r-k_0}\left(A_E-B_{0,E}^{(n)}+B_{0,E}\right)^i\right.\\&\left.\times B_{1,E}(A_E-B_{0,E}^{(n)}+B_{0,E}+B_{1,E})^{r-i-2}\left(B_{0,E}^{(n)}-B_{0,E}\right)\right\}\underline{1}\\&+\underline{p}'\left\{\sum_{i=k_0}^{r-k_1}\left(A_E-B_{1,E}^{(n)}+B_{1,E}\right)^i\right.\\&\left.\times B_{0,E}(A_E-B_{1,E}^{(n)}+B_{1,E}+B_{0,E})^{r-i-2}\left(B_{1,E}^{(n)}-B_{1,E}\right)\right\}\underline{1},\\&r\geqslant k_0+k_1.\end{aligned}$$

Remark 4.2 The sooner (later) waiting time distribution can be obtained in general for sooner (later) occurring event between r_i^{th}occurrence of i-run of length $k_i(i=0,1)$.

5 Numerical study

We consider the 4-dependent sequence of MBT. Let for the transformed $\{0,1,2,\ldots,15\}$-valued Markov chain $\{Z_i, i\geqslant 0\}$, $\pi_0=1$ and the transition probabilities be,

$$\begin{aligned}&\left(p_0\ p_1\ p_2\ p_3\ p_4\ p_5\ p_6\ p_7\ p_8\ p_9\ p_{10}\ p_{11}\ p_{12}\ p_{13}\ p_{14}\ p_{15}\right)\\&=\left(0.1\ 0.8\ 0.5\ 0.6\ 0.4\ 0.55\ 0.65\ 0.85\ 0.2\ 0.3\ 0.5\ 0.8\ 0.4\ 0.7\ 0.75\ 0.9\right)\end{aligned}$$

Let $k_0=5$ and $k_1=6$. In the following Tables 1, 2, 3, 4, we present the joint distribution of $\left(X_{25,5,\alpha}^0, X_{25,6,\alpha}^1\right)$ for $\alpha=N,G,X,E$. The distribution of $\left(X_{25,5,\alpha}^0, X_{25,6,\alpha}^1\right)$ for $\alpha=M$ is shown in the three dimensional plot in Fig. 1. Figure 2 gives the plot of sooner waiting time (W_S) distribution, which is common under all schemes. In Fig. 3, we have plotted distribution of later waiting time(W_L), which is common for $\alpha=N,M,G$ and X and differs for $\alpha=E$.

6 Discussion

The distributions derived in this paper are new in the literature of run statistics. The method of conditional pgfs, used in this paper to obtain all the distributions is

Table 1 Distribution of $\left(X^0_{25,5,N}, X^1_{25,6,N}\right)$

$X^0_{25,5,N}$	$X^1_{25,6,N}$					Sum
	0	1	2	3	4	
0	0.01685	0.05871	0.09833	0.07747	0.01023	0.26159
1	0.04328	0.09169	0.08703	0.01807		0.24006
2	0.07667	0.08814	0.02908			0.19389
3	0.10144	0.04230				0.14374
4	0.08893					0.08893
5	0.07179					0.07179
Sum	0.39896	0.28084	0.21444	0.09553	0.01023	1.00000

Table 2 Distribution of $\left(X^0_{25,5,G}, X^1_{25,6,G}\right)$

$X^0_{25,5,G}$	$X^1_{25,6,G}$				Sum
	0	1	2	3	
0	0.01685	0.17803	0.06523	0.00147	0.26159
1	0.28539	0.30711	0.02461	1.98E-05	0.61713
2	0.09427	0.02430	0.00021		0.11878
3	0.00244	6.33E-05			0.00250
4	1.24E-06				0.00000
Sum	0.39896	0.50950	0.09005	0.00149	1.00000

Table 3 Distribution of $\left(X^0_{25,5,E}, X^1_{25,6,E}\right)$

$X^0_{25,5,E}$	$X^1_{25,6,E}$				Sum
	0	1	2	3	
0	0.81692	0.08209	0.00380	0.00007	0.90287
1	0.08366	0.00961	0.00041	1.73E-05	0.09369
2	0.00302	0.00034	0.00002		0.00338
3	0.00005	4.77E-06			0.00006
4	2.60E-07				2.60E-07
Sum	0.90365	0.09204	0.00422	0.00009	1.00000

Table 4 Distribution of $\left(X^0_{25,5,X}, X^1_{25,6,X}\right)$ with ℓ_0=3 and ℓ_1=2

$X^0_{25,5,X}$	$X^1_{25,6,X}$						Sum
	0	1	2	3	4	5	
0	0.01685	0.03871	0.06108	0.06764	0.05252	0.02478	0.26159
1	0.01636	0.02642	0.03080	0.02455	0.01165		0.10978
2	0.02190	0.02861	0.02794	0.01789	0.00455		0.10089
3	0.02747	0.02844	0.02354	0.01165			0.09109
4	0.03274	0.02617	0.01755	0.00455			0.08100
5	0.03701	0.02245	0.01164				0.07110
6	0.03987	0.01720	0.00455				0.06162
7	0.04071	0.01164					0.05234
8	0.04569	0.00455					0.05023
9	0.02977						0.02977
10	0.01880						0.01880
11	0.07179						0.07179
Sum	0.39896	0.20418	0.17709	0.12627	0.06872	0.02478	1.00000

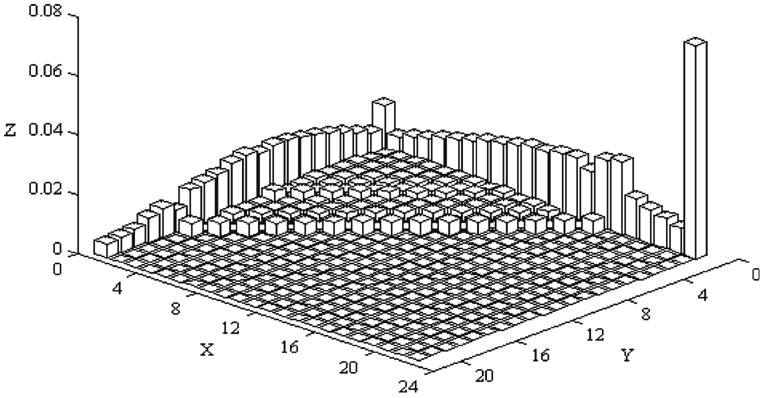

Fig. 1 Distribution of $\left(X^0_{25,5,M}, X^1_{25,5,M}\right)$

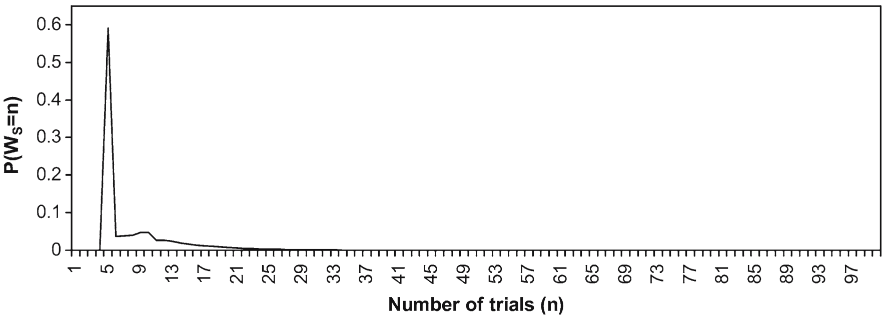

Fig. 2 Sooner waiting time distribution for $\alpha = N, M, G, E, X$

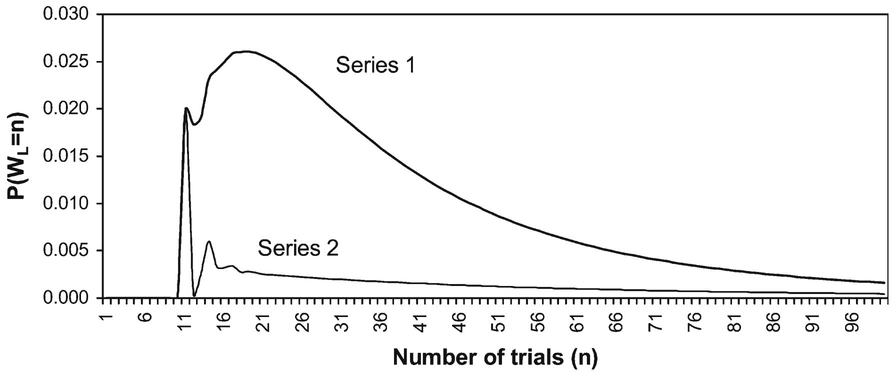

Fig. 3 Later waiting time distribution: series 1 for $\alpha = N, M, G, X$; Series 2 for $\alpha = E$

already well-established method. But here we have implemented this method such that the recurrent relations of conditional pgfs can be expressed in a matrix form. Recurrent use of this representation gives the pgf of the distribution of random variables under study in the compact form. We have demonstrated the use of simple algorithm given for the evaluation of exact probability distribution from the derived form of pgf.

Apart from the distribution of $\left(X^0_{n,k_0}, X^1_{n,k_1}\right)$, under the five counting schemes, one may consider the joint distribution of $\left(X^0_{n,k_0}, X^1_{n,k_1}\right)$ where the 0-runs and 1-runs are counted using two different counting schemes. We briefly discuss one example of this type of random variable as follows.

Example 6.1 Consider the random variable $\left(X^0_{n,k_0,N}, X^1_{n,k_1,G}\right)$. Then along with the recurrent relations of $\phi_c^{(0,i)}(t_0,t_1)$, i=0, 1, ..., k_0-1, $\phi_c^j(t_0,t_1)$, j=1, 2, ..., 2^m-2 and $\phi_c^{(2^m-1,i)}(t_0,t_1)$, $i=m, m+1, \ldots, k_1-1$(these recurrent relations are same for $\alpha = N, M, G, X$), we have to use the recurrent relations of $\phi_{c,N}^{(0,k_0)}(t_0,t_1)$ and $\phi_{c,G}^{(2^m-1,k_1)}(t_0,t_1)$ from Sect. 2. This set of recurrent relations can be written as,

$$\underline{\phi}_c(t_0,t_1) = (A + B_0t_0 + B_1t_1)\,\underline{\phi}_{c-1}(t_0,t_1), \quad \text{for } c = 1,\ 2,\ \ldots,\ n$$

where $\underline{\phi}_c(t_0,t_1) = \left(\phi_c^{(0,k_0)}, \phi_c^{(0,k_0-1)} \ldots \phi_c^{(0,0)} \phi_c^1, \phi_c^2 \ldots \phi_c^{2^m-2}\, \phi_c^{(2^m-1,m)}\, \phi_c^{(2^m-1,m+1)} \ldots \phi_c^{(2^m-1,k_1)}\right)'$ and $(A + B_0t_0 + B_1t_1)$ is matrix of order $2^m + k_0 + k_1 - m$.

This gives the pgf of random variable $\left(X^0_{n,k_0,N}, X^1_{n,k_1,G}\right)$ as,

$$\phi_n(t_0,t_1) = \underline{p}'(A + B_0t_0 + B_1t_1)^n \underline{1},$$

where $B_0 = q_0E_{2,1}$, $B_1 = B_{1,G}$ and $A = \left(D_G - E_{1,1} \cdot *D_G + q_0E_{1,k_0}\right) - B_0 - B_1$

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