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# Exact and limiting distributions in diagonal Pólya processes

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**Abstract** We investigate the Pólya process, which underlies an urn of white and blue balls growing in real time. A partial differential equation governs the evolution of the process. For urns with (forward or backward) diagonal ball addition matrix the partial differential equation is amenable to asymptotic solution. In the case of forward diagonal we find a solution via the method of characteristics; the numbers of white and blue balls, when scaled appropriately, converge in distribution to independent Gamma random variables. The method of characteristics becomes a bit too involved for the backward diagonal process, except in degenerate cases, where we have Poisson behavior. In nondegenerate cases, limits characterized implicitly by their recursive sequence of moments are found, via matrix formulation involving a Leonard pair.

**Keywords** Urns · Random structure · Stochastic process · Partial differential equation · Leonard pairs

## 1 Introduction

Associated with a Pólya urn is a process obtained by embedding in continuous time. In this paper we shall refer to this process as the Pólya process. The process was introduced in Athreya and Karlin (1968) to model the growth of an urn in discrete time according to certain rules. The Pólya process can be viewed as a transform. It was noted by Athreya and Karlin (1968) and others see for example Kotz, et al. (2000) that extracting information about the embedded discrete process from the Pólya process is fraught with difficulty in practice. Nevertheless, the Pólya process

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still stands as one of the best heuristics to approximate and understand the intricate Pólya urn models.

Our intent in this paper is to study the Pólya process in its own right for associated urns with diagonal structure, and obtain distributions. For a general urn of two colors, certain partial differential equations govern the progress. For urns with diagonal ball addition matrices, limit distributions are amenable via these equations. We are also optimistic that more general cases (not necessarily diagonal) may be within reach; this will be the subject of future work.

The plan of this paper is reflected in the organization of its sections. In Sect. 2 we describe the process, its associated ball addition matrix, and find its governing partial differential equation. In Sect. 3 we deal with the partial differential equation in the case of forward diagonal matrix, a case amenable to the method of characteristics. Independent Gamma limiting random variables are identified as the in-distribution limits for the (scaled) numbers of the white and blue balls. Except for simple degenerate processes, an application of the method of characteristics is rather involved in the case of backward diagonal matrix. In Sect. 4 the backward diagonal processes are discussed—the nondegenerate case is dealt with in Sect. 4.1, where a recursive structure for the asymptotic moments is found from the partial differential equation. In Sect. 4.2 the degenerate case is solved by the method of characteristics to obtain Poisson distributions. Sect. 5 concludes the paper with a discussion of the scope of this investigation, connections to the standard discrete Pólya urns, and the possible extension to $k$-color Pólya processes.

## 2 The Pólya process

The Pólya process is a renewal process with rewards. It comprises an increasing number of processes running in parallel. Generally, the various parallel processes may be dependent. The process grows out of a certain number of white and blue balls (thought to be contained in an urn). At time $t$, let the number of white balls be $W(t)$ and the number of blue balls be $B(t)$. Thus, initially we have $W_0 = W(0)$ white balls, and $B_0 = B(0)$ blue balls in the urn. Each ball generates a renewal after an independent Exp(1), an exponentially distributed random variable with parameter 1. We call a process evolving from a white ball a *white process*, and a process evolving from a blue ball a *blue process*. When a renewal occurs, a certain number of balls is added. That number depends on which colored process induced the renewal. It is assumed that ball additions take place instantaneously at the renewals. If a white process causes the renewal we add $a$ white balls and $b$ blue balls to the urn, and if a blue process causes the renewal we add $c$ white balls and $d$ blue balls to the urn.

It will organize the discussion to think of the ball addition scheme as a $2\times 2$ matrix $\mathbf{A}$, the rows of which are indexed by the color of the process inducing the renewal, and the columns of which are labeled by the colors of balls added to the urn:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We simply refer to this matrix as the *scheme*. Some of these processes were discussed on average in Mahmoud (2002, 2004). We investigate their full distribution here.

To understand the behavior over time, it is helpful to think via an analogy from a Markovian race (see Mahmoud 2002). At any stage, consider every ball in the urn as a runner in a race, whose running time is an independent Exp(1). The runners are grouped in teams by wearing shirts of the color of the process they represent. When a runner from the white team wins the race, we say the white team wins the race, then $a$ runners (balls) wearing white shirts and $b$ runners wearing blue shirts enter the race (are added to the urn). Alternatively, when a runner from the blue team wins the race, we say the blue team wins the race, then $c$ runners wearing white shirts and $d$ runners wearing blue shirts enter the race. Every new runner is endowed with an independent Exp(1) clock.

Another race among all the existing runners is immediately restarted. The collective process enjoys a memoryless property as it is induced by individuals based on the exponential distribution—if a runner has covered a certain fraction of the course in one race, the runner is not allowed to carry over the gain to the next race; the runner's remaining time to cover the rest of the course remains Exp(1), as a result of resetting the race.

Let Poi($\lambda$) denote a Poisson random variable with parameter $\lambda$. With each individual process (runner) existing at time $t$ we associate a random variable to represent the number of renewals it gives by time $t$. From basic properties of the Poisson process a runner entering the race by time $t' \le t$ gives Poi$(t-t')$ renewals by time $t$. We call the joint process $\mathbf{R}(t)$. That is, $\mathbf{R}(t) := (W(t), B(t))^T$. We can formulate two simultaneous incremental equations for the process $(W(t), B(t))^T$. Consider the process at time $t + \Delta t$, where $\Delta t$ is an infinitesimal increment of time. The number of white balls (given $\mathbf{R}(t)$) at time $t + \Delta t$ is what it was at time $t$, plus the number of white balls contributed by the various teams within the infinitesimal period $(t, t + \Delta t]$. Each member of the white team follows a Poisson process with parameter 1, and thus generates Poi$(\Delta t)$ renewals in an interval of length $\Delta t$. Likewise, each member of the blue team generates Poi$(\Delta t)$ renewals in an interval of length $\Delta t$. In turn, each newly born child in that interval may generate additional children by time $t + \Delta t$. Altogether the number of children generated by any new runner added in the period $(t, t + \Delta t]$ is $o_P((\Delta t)^2)$. Each renewal by a white process increases the white team by $a$ runners, and each renewal by a blue process increases the white team by $c$ runners. A similar argument holds for the blue team. We have

$$\begin{aligned}\mathbf{E}\Big[e^{uW(t+\Delta t)+vB(t+\Delta t)} \mid \mathbf{R}(t)\Big] = \mathbf{E}\Big[\exp\Big(\Big[W(t) + a\sum_{i=1}^{W(t)} X_i + c\sum_{j=1}^{B(t)} Y_i\Big]u \\ +\Big[B(t) + b\sum_{i=1}^{W(t)} X_i + d\sum_{j=1}^{B(t)} Y_i\Big]v \qquad (1) \\ +o_P\big((\Delta t)^2\big)\Big)\Big].\end{aligned}$$

For $X_1, \ldots, X_{W(t)}, Y_1, \ldots, Y_{B(t)}$ (conditionally) independent Poi$(\Delta t)$ random variables.

**Lemma 2.1** *The moment generating function* $\phi(t, u, v) := \mathbf{E}[\exp(uW(t)+vB(t))]$ *of the joint process satisfies*

$$\frac{\partial\phi}{\partial t} + \big(1 - \mathrm{e}^{au+bv}\big)\frac{\partial\phi}{\partial u} + \big(1 - \mathrm{e}^{cu+dv}\big)\frac{\partial\phi}{\partial v} = 0.$$

*Proof* Let $p_{km}(t) = P\big(W(t) = k, B(t) = m\big)$. Taking expectations of (Eq.2) and conditioning we see that

$$\begin{aligned}\phi(t + \Delta t, u, v) = \sum_{k,m} \mathbf{E}\Big[\mathrm{e}^{uk+vm} \exp\Big(\Big[a\sum_{i=1}^{k} X_i + c\sum_{j=1}^{m} Y_i\Big]u \\ +\Big[b\sum_{i=1}^{k} X_i + d\sum_{j=1}^{m} Y_i\Big]v \\ +o_P\big((\Delta t)^2\big)\Big) \,\Big|\, W(t) = k, B(t) = m\Big]p_{km}(t).\end{aligned}$$

The combined term $\mathbf{E}[e^{(au+bv)X_i}]$ is the moment generating function of $X_i$, evaluated at $au + bv$. With $X_i$ being $\mathrm{Poi}(\Delta t)$, the combined term is $\exp(\Delta t\ (\mathrm{e}^{au+bv} - 1))$. By independence of the $X_i$'s and $Y_j$'s and their identical distribution, we have

$$\begin{aligned}\phi(t + \Delta t, u, v) &= \sum_{k,m} \mathrm{e}^{uk+vm}\mathrm{e}^{k\Delta t(\mathrm{e}^{au+bv}-1)}\mathrm{e}^{m\Delta t(\mathrm{e}^{cu+dv}-1)} \\ &\quad\times\big(1 + o\big((\Delta t)^2\big)\big)p_{km}(t) \\ &= \sum_{k,m} \mathrm{e}^{uk+vm}\Big(\big[1 + k\Delta t\big(\mathrm{e}^{au+bv} - 1\big) + o\big((\Delta t)^2\big)\big] \\ &\quad\times\big[1 + m\Delta t\big(\mathrm{e}^{cu+dv} - 1\big) + o\big((\Delta t)^2\big)\big]\Big) \\ &\quad\times\big(1 + o\big((\Delta t)^2\big)\big)p_{km}(t) \\ &= \phi(t, u, v) + \Delta t\big(\mathrm{e}^{au+bv} - 1\big)\sum_{k,m} k\mathrm{e}^{ku+mv}p_{km}(t) \\ &\quad+\Delta t\big(\mathrm{e}^{cu+dv} - 1\big)\sum_{k,m} m\mathrm{e}^{ku+mv}p_{km}(t) + o\big((\Delta t)^2\big).\end{aligned}$$

The lemma follows upon reorganizing the expression above: move $\phi(t, u, v)$ to the left-hand side, divide by $\Delta t$ throughout, and take the limit, as $\Delta t \to 0$. □

## 3 The diagonal Pólya process

In this section we are concerned with the Pólya process with the (forward) diagonal urn scheme

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

This Yule-like process is relatively easy because any renewals from the white processes add only to the white processes, and the same applies to the blue processes.

And so, the numbers of white and blue processes are independent. The number of white balls in the urn depends only on $W_0$ and $a$, and independently, the number of blue balls depends only on $B_0$ and $d$. The joint distribution can be comprehended as a composition of two parallel schemes: the white processes with scheme

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

with $(W_0, 0)^{\mathrm{T}}$ initial conditions, and the blue processes with scheme

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}, \tag{3}$$

with $(0, B_0)^{\mathrm{T}}$ initial conditions.

Each of these two schemes is a one-color scheme. It is sufficient to study the white process (Eq.2) and the result for the blue process (Eq.3) will follow by symmetry. In the scheme (Eq.2) note that $\phi(t, u, v) = \mathbf{E}[e^{uW(t)}]$ does not depend on $v$, and subsequently we shall denote it by $\phi(t, u)$ for the rest of this section.

For the scheme (Eq.2) the partial differential equation in Lemma 2.1 can be expressed as

$$\frac{\partial \phi}{\partial t} + \left(1 - \mathrm{e}^{au}\right)\frac{\partial \phi}{\partial u} = 0,$$

which is to be solved under the initial condition $\phi(0, u) = \mathrm{e}^{W_0 u}$. We can handle this equation via the method of characteristics; see Levine (1997), for example.

**Lemma 3.1** *Under the initial condition $\phi(0, u) = \mathrm{e}^{W_0 u}$, the partial differential equation*

$$\frac{\partial \phi}{\partial t} + \left(1 - \mathrm{e}^{au}\right)\frac{\partial \phi}{\partial u} = 0,$$

*has the solution*

$$\phi(t, u) = \left(\frac{\mathrm{e}^{a(u-t)}}{\mathrm{e}^{a(u-t)} - \mathrm{e}^{au} + 1}\right)^{W_0/a}.$$

*Proof* We switch to new variables $(s, \tau)$, with constant $\tau = \tau(t, u)$ being the characteristic curves. Set

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1,$$

and

$$\frac{\mathrm{d}u}{\mathrm{d}s} = 1 - \mathrm{e}^{au}.$$

This transformation of variables entails

$$t = s, \qquad \text{and} \qquad u = s + \frac{1}{a}\ln(\mathrm{e}^{au} - 1) + \gamma,$$

where $\gamma$ is a constant of integration. The characteristic curves are then defined by the extra requirement that $u(0, \tau) = \tau$, giving

$$\tau = \frac{1}{a}\ln(e^{a\tau} - 1) + \gamma.$$

On characteristic curves we must have

$$\tau = u - t - \frac{1}{a}\ln(e^{au} - 1) + \frac{1}{a}\ln(e^{a\tau} - 1).$$

Under the initial condition $\phi(0, \tau) = e^{W_0\tau}$, the solution to the partial differential equation is

$$\begin{aligned}\phi(t, u) &= \exp\Big[\Big(u - t - \frac{1}{a}\ln(e^{au} - 1) + \frac{1}{a}\ln(e^{a\tau} - 1)\Big)W_0\Big] \\ &= \frac{e^{W_0(u-t)}(e^{a\tau} - 1)^{W_0/a}}{(e^{au} - 1)^{W_0/a}}.\end{aligned}$$

We can rewrite the latter equation in the form

$$(e^{au} - 1)\phi^{a/W_0} = e^{a(u-t)}(\phi^{a/W_0} - 1).$$

The solution as stated follows by solving algebraically for $\phi^{a/W_0}$, hence for $\phi$. □

**Proposition 3.1** *Starting with $W_0$ balls, the number of white balls in the Pólya process with the diagonal urn scheme*

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \tag{4}$$

*has mean*

$$\mathbf{E}\big[W(t)\big] = W_0 e^{at},$$

*and variance*

$$\mathbf{Var}\big[W(t)\big] = aW_0\big(e^{2at} - ae^{at}\big).$$

*Proof* Immediate by differentiation of the moment generating function in Lemma 3.1 (with respect to $u$) and evaluating at $u$=0. □

Note that $\phi(t, u)$, as a moment generating function, must stay positive. This imposes a restriction on $u$. The positivity of the denominator determines the range

$$0 \le t \le u - \frac{1}{a}\ln(e^{au} - 1),$$

and for every $t \ge 0$, there is a neighborhood of 0 on the $u$ scale for which the moment generating function exists. As we intend to develop asymptotics, as $t$ becomes very large, we must have $u$ restricted in a very small range. Specifically, we take $u = x/e^{at}$, where the scale $e^{at}$ was chosen because it is the leading asymptotic term in the standard deviation (as $t \to \infty$). It follows from Lemma 3.1 that

$$\mathbf{E}\Big[\exp\Big(\frac{W(t) - W_0 e^{at}}{e^{at}}\,x\Big)\Big] = \phi\Big(t, \frac{x}{e^{at}}\Big)e^{-W_0 x} \to \frac{e^{-W_0 x}}{(1 - ax)^{W_0/a}}.$$

The right-hand side is the moment generating function of Gamma$(\frac{W_0}{a}, a) - W_0$, and so

$$\frac{W(t)}{e^{at}} \xrightarrow{\mathcal{D}} \text{Gamma}\Big(\frac{W_0}{a}, a\Big).$$

We have established the main result of this section, summarized next.

**Theorem 3.1** *Let $(W(t), B(t))^T$ be the joint vector of the number of white and blue balls in the Pólya process with the diagonal urn scheme*

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

*If the process starts with $(W_0, B_0)^T$, then as $t \to \infty$,*

$$\begin{pmatrix} \dfrac{W(t)}{e^{at}} \\ \\ \dfrac{B(t)}{e^{dt}} \end{pmatrix} \xrightarrow{\mathcal{D}} \begin{pmatrix} \text{Gamma}\big(\frac{W_0}{a}, a\big) \\ \\ \text{Gamma}\big(\frac{B_0}{d}, d\big) \end{pmatrix},$$

*and the two gamma random variables in the limit are independent.*

*Remark* In the forward diagonal processes one may argue the existence of a limit via the martingale formulation in the classical branching process. However, such a proof is only existential, and will not identify the limit. We included this relatively simple case to show that the partial differential equation formulation can be amenable to explicit exact and asymptotic distributions.

## 4 The backward-diagonal Pólya process

In this section we handle Pólya processes with nonnegative integers on the backward-diagonal. That is, the processes we are dealing with has the scheme

$$\mathbf{A} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

When both backward diagonal entries are positive, the scheme is significantly more involved than the forward diagonal process, because a renewal from either color influences the other color, giving rise to strong dependence among the two colored processes. In the degenerate case, when $b$=0, or $c$=0 (but not both simultaneously zero), the case is degenerate, and can be handled by the method of characteristics yielding an exact solution of the partial differential equation. We solve the nondegenerate case only asymptotically.

This is a continuous analog of a Friedman-like process [see Friedman (1949)]. In the pure Friedman urn, $b = c$, but we relax this condition here.

4.1 The nondegenerate case

In the nondegenerate case $bc>0$, the differential equation of Lemma 2.1 simplifies to

$$\frac{\partial \phi}{\partial t} + \left(1 - \mathrm{e}^{bv}\right)\frac{\partial \phi}{\partial u} + \left(1 - \mathrm{e}^{cu}\right)\frac{\partial \phi}{\partial v} = 0.$$

We can formulate probabilistic limits via an identification of the moments. Take the derivative of the partial differential equation $i$ times with respect to $u$ and $j$ times with respect to $v$, then evaluate at $u=v=0$. Note that

$$h_{ij}(t) := \frac{\partial^{i+j}}{\partial u^i \partial v^j}\phi(t,0,0) = \mathbf{E}\left[W^i(t)B^j(t)\right].$$

Thus, the proposed procedure will produce a system of first order differential equations for the functions $h_{ij}(t)$. Namely,

$$\begin{aligned}\frac{\partial^{i+j}}{\partial u^i \partial v^j}\frac{\partial \phi}{\partial t}\bigg|_{u=v=0} &= \dot{h}_{ij}(t)\\ &= \sum_{k=0}^{j-1} b^{j-k}\binom{j}{k} h_{i+1,k}(t) + \sum_{k=0}^{i-1} c^{i-k}\binom{i}{k} h_{k,j+1}(t), \qquad (5)\end{aligned}$$

where the dot notation is for ordinary derivatives with respect to time, as usual. The *ordinary* differential equations for the moments can be solved by recognizing a pattern for $h_{i,j}$, when $i + j = m$. One obtains a linear system of simultaneous equations, with a tridiagonal matrix of coefficients:

$$\begin{pmatrix} \dot{h}_{m,0}(t) \\ \dot{h}_{m-1,1}(t) \\ \vdots \\ \dot{h}_{0,m}(t) \end{pmatrix} = \begin{pmatrix} 0 & mc & 0 & \dots & 0 & 0 \\ b & 0 & (m-1)c & \dots & 0 & 0 \\ 0 & 2b & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & (m-1)b & 0 & c \\ 0 & 0 & 0 & 0 & mb & 0 \end{pmatrix} \begin{pmatrix} h_{m,0}(t) \\ h_{m-1,1}(t) \\ \vdots \\ h_{0,m}(t) \end{pmatrix} + \mathbf{G}_m(t), \qquad (6)$$

where $\mathbf{G}_m(t)$ is a vector involving only lower order moments $h_{ij}(t)$, with $i+j < m$. We can write the latter linear system as

$$\dot{\mathbf{H}}_m(t) := \mathbf{K}_m \mathbf{H}_m(t) + \mathbf{G}_m(t),$$

where $\mathbf{H}_m(t)$ is the vector the derivative of which appears on the left-hand side of (Eq.6), and $\mathbf{K}_m$ is the $(m+1) \times (m+1)$ constant tridiagonal matrix on the right-hand side of (Eq.6). The solution of this linear system is given by

$$\mathbf{H}_m(t) = \mathrm{e}^{\mathbf{K}_m t}\mathbf{H}_m(0) + \mathrm{e}^{\mathbf{K}_m t}\int_0^t \mathrm{e}^{-\mathbf{K}_m s}\mathbf{G}_m(s)\,\mathrm{d}s. \qquad (7)$$

Note that the integral also gives a contribution of the leading asymptotic value $e^{\mathbf{K}_m t}$. This contribution, let us call it $\mathbf{F}_m(0)$, comes from the particular solution (the integral) at $s=0$. The term $\mathbf{F}_m(0)$ can be computed recursively, as we shall illustrate below on a few moments of low order.

In the sequel, the symbol $\mathbf{O}(g(t))$ stands for a vector all the components of which are $O(g(t))$ in the usual scalar sense. We shall show by induction on $m$ that this solution is

$$\mathbf{H}_m(t) = e^{\mathbf{K}_m t}\big(\mathbf{H}_m(0) - \mathbf{F}_m(0)\big) + \mathbf{O}(e^{(m-1)\sqrt{bc}\,t}); \tag{8}$$

here $\mathbf{H}_m(0) = (W_0^m, W_0^{m-1}B_0, \dots, B_0^m)^T$.

Before proceeding further let us fix some notation. Let $k = \sqrt{b/c}$, $\mathbf{D} = \mathbf{diag}(k, k^2, k^3, \dots, k^{m+1})$, and

$$\mathbf{L}_m = \begin{pmatrix} 0 & m & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & (m-1) & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & (m-2) & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \dots & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & (m-1) & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & m & 0 \end{pmatrix}.$$

It can be easily seen that $\mathbf{K}_m$ and $\sqrt{bc}\,\mathbf{L}_m$ are similar matrices, since $\mathbf{D}^{-1}\mathbf{K}_m\mathbf{D} = \sqrt{bc}\,\mathbf{L}_m$. Hence $\mathbf{K}_m$ and $\sqrt{bc}\,\mathbf{L}_m$ have the same eigenvalues. Therefore, it is enough to compute the eigenvalues of $\mathbf{L}_m$. Let $\mathbf{L}_m^* = \mathbf{diag}(m, m-2, \dots, -m)$. The result below follows from Terwilliger (2004). For additional discussion and further properties of Leonard pairs see Terwilliger (2000, 2002).

**Lemma 4.1** *The matrices $\mathbf{L}_m$ and $\mathbf{L}_m^*$ form a Leonard pair with the following properties:*

1. *$\mathbf{L}_m$ and $\mathbf{L}_m^*$ are similar and have the same eigenvalues.*
2. *There exists a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{L}_m\mathbf{P} = \mathbf{L}_m^*$.*
3. *Further, $\mathbf{P}^2 = 2^m\mathbf{I}$, and hence $\mathbf{P}^{-1} = \frac{1}{2^m}\mathbf{P}$.*
4. *The first row of $\mathbf{P}$ is the sequence of binomial coefficients $\binom{m}{0}, \binom{m}{1}, \cdots, \binom{m}{m}$.*
5. *The first column of $\mathbf{P}$ is all 1's.*

Hence, the eigenvalues of $\mathbf{K}_m$ are $m\sqrt{bc}$, $(m-2)\sqrt{bc}, \dots, -m\sqrt{bc}$. It is well known that the exponential of a matrix with distinct eigenvalues can be expressed as a linear combination of matrices called the idempotents associated with it (see Smiley 1965). So, in our case, there are matrices $\mathcal{E}_1, \dots, \mathcal{E}_k$ that admit the representation

$$\begin{aligned} e^{\mathbf{K}_m t} &= e^{m\sqrt{bc}\,t}\mathcal{E}_1 + e^{(m-2)\sqrt{bc}\,t}\mathcal{E}_2 + \dots + e^{-m\sqrt{bc}\,t}\mathcal{E}_{m+1} \\ &= e^{m\sqrt{bc}\,t}\mathcal{E}_1 + \mathbf{O}(e^{(m-2)\sqrt{bc}\,t}), \end{aligned}$$

where, for $i=1, \dots, k$, $\mathcal{E}_i$ has the following representation

$$\mathcal{E}_i = \frac{1}{\prod_{j\neq i}(\lambda_i - \lambda_j)} \prod_{j\neq i}(\mathbf{K}_m - \lambda_j\mathbf{I}).$$

Thus, if inductively (Eq.8) holds at $m-1$, the function $\mathbf{G}_m(s)$ in the integrand of (Eq.7) will be of the order $\mathbf{O}(e^{(m-1)\sqrt{bc}\,s})$. Upon substituting the integration limits after integration one term will be a constant. Together with the multiplier $e^{\mathbf{K}_m t}$ a term of the form $-\mathbf{F}_m(0)e^{\mathbf{K}_m t}$ appears in the solution. Then $e^{\mathbf{K}_m t}(\mathbf{H}_m(0)-\mathbf{F}_m(0))$ provides the leading term [of order $\mathbf{O}(e^{m\sqrt{bc}\,t})$], with all the other terms being $\mathbf{O}(e^{(m-1)\sqrt{bc}\,t})$, completing the induction.

According to Lemma 4.1, if we premultiply $\mathcal{E}_1$ by $(\mathbf{DP})^{-1}$ and postmultiply by $\mathbf{DP}$, we get

$$\begin{aligned}
(\mathbf{DP})^{-1}\mathcal{E}_1\mathbf{DP} &= \frac{1}{\prod_{j=2}^{m+1}(\lambda_1-\lambda_j)}\mathbf{P}^{-1}\mathbf{D}^{-1}(\mathbf{K}_m-\lambda_2\mathbf{I})\mathbf{DP}\\
&\quad\times\mathbf{P}^{-1}\mathbf{D}^{-1}(\mathbf{K}_m-\lambda_3\mathbf{I})\mathbf{DP}\\
&\quad\times\cdots\times\mathbf{P}^{-1}\mathbf{D}^{-1}(\mathbf{K}_m-\lambda_{m+1}\mathbf{I})\mathbf{DP}\\
&= \frac{1}{\prod_{j=2}^{m+1}(\lambda_1-\lambda_j)}\prod_{j=2}^{m+1}\big(\sqrt{bc}\,\mathbf{P}^{-1}\mathbf{L}_m\mathbf{P}-\lambda_j\mathbf{I}\big)\\
&= \frac{1}{\prod_{j=2}^{m+1}(\lambda_1-\lambda_j)}\prod_{j=2}^{m+1}\big(\sqrt{bc}\,\mathbf{L}_m^{*}-\lambda_j\mathbf{I}\big)\\
&= \frac{1}{\prod_{j=2}^{m+1}(\lambda_1-\lambda_j)}\prod_{j=2}^{m+1}\mathbf{diag}\big(\lambda_1-\lambda_j,\ldots,\lambda_{j-1}-\lambda_j,0,\\
&\qquad\lambda_{j+1}-\lambda_j,\ldots,\lambda_{m+1}-\lambda_j\big)\\
&= \begin{pmatrix}1&0&0&0&\cdots&0\\0&0&0&0&\cdots&0\\\vdots&\vdots&&\ddots&&\vdots\\0&0&&\cdots&&0\end{pmatrix}.
\end{aligned}$$

Subsequently, we determine $\mathcal{E}_1$ as

$$\mathcal{E}_1=\mathbf{DP}\begin{pmatrix}1&0&0&0&\cdots&0\\0&0&0&0&\cdots&0\\\vdots&\vdots&&\ddots&&\vdots\\0&0&&\cdots&&0\end{pmatrix}(\mathbf{DP})^{-1}.$$

We are mainly interested in finding the moments $h_{m,0}(t)=\mathbf{E}[W^m(t)]$, and $h_{0,m}(t)=\mathbf{E}[B^m(t)]$. To obtain asymptotic expressions for these it is enough to know the first and last rows of the idempotent matrix $\mathcal{E}_1$. This can be obtained quite easily by using Lemma 4.1 and the discussion above. It is given by the following statement.

**Lemma 4.2** *The first row of* $2^m\mathcal{E}_1$ *is given by* $\binom{m}{0},\frac{1}{k}\binom{m}{1},\frac{1}{k^2}\binom{m}{2},\ldots,\frac{1}{k^m}\binom{m}{m}$, *and its last row is given by* $k^m\binom{m}{0},k^{m-1}\binom{m}{1},k^{m-2}\binom{m}{2},\ldots,\binom{m}{m}$.

Let $\mathbf{J}$ be the $(m+1)$-component row vector $(1,0,0,\ldots,0)$. Hence, we have for large values of $t$,

$$\begin{aligned}
h_{m,0}(t) &= \mathbf{E}\big[W^m(t)\big] \\
&= \mathrm{e}^{m\sqrt{bc}\,t}\mathbf{J}\boldsymbol{\mathcal{E}}_1\begin{pmatrix} W_0^m \\ \vdots \\ B_0^m \end{pmatrix} - \mathrm{e}^{m\sqrt{bc}\,t}\mathbf{J}\boldsymbol{\mathcal{E}}_1\mathbf{F}_m(0) + \mathbf{O}(\mathrm{e}^{(m-1)\sqrt{bc}\,t}) \\
&= \frac{\mathrm{e}^{m\sqrt{bc}\,t}}{2^m}\Big(W_0 + \frac{B_0}{k}\Big)^m - \frac{1}{2^m}\Big[\binom{m}{0} f_{m,0}(0) + \frac{1}{k}\binom{m}{1} f_{m,1}(0) \\
&\quad + \cdots + \frac{1}{k^m}\binom{m}{m} f_{m,m}(0)\Big] + \mathbf{O}(\mathrm{e}^{(m-1)\sqrt{bc}\,t}),
\end{aligned} \tag{9}$$

where $f_{m,j}$, $j{=}0, \dots, m$, are the components of $\mathbf{F}_m(0)$. Similarly, we can work out the case of the blue process, via the last row of $\boldsymbol{\mathcal{E}}_1$.

The components $f_{m,j}$, $j{=}0, \dots, m$, can be computed recursively. For example, for $m{=}1$ we have $\mathbf{G}_1(0){=}\mathbf{0}$, and the system of differential equations is

$$\begin{pmatrix} \dot{h}_{10}(t) \\ \dot{h}_{01}(t) \end{pmatrix} = \begin{pmatrix} 0 & c \\ b & 0 \end{pmatrix} \begin{pmatrix} h_{10}(t) \\ h_{01}(t) \end{pmatrix}. \tag{10}$$

Thus,

$$\mathbf{E}\big[W(t)\big] = \frac{1}{2}\Big(W_0 + B_0\sqrt{\frac{c}{b}}\Big)\mathrm{e}^{\sqrt{bc}\,t} + \frac{1}{2}\Big(W_0 - B_0\sqrt{\frac{c}{b}}\Big)\mathrm{e}^{-\sqrt{bc}\,t}.$$

For $m{=}2$, we have

$$\begin{pmatrix} \dot{h}_{20}(t) \\ \dot{h}_{11}(t) \\ \dot{h}_{02}(t) \end{pmatrix} = \begin{pmatrix} 0 & 2c & 0 \\ b & 0 & c \\ 0 & 2b & 0 \end{pmatrix} \begin{pmatrix} h_{20}(t) \\ h_{11}(t) \\ h_{02}(t) \end{pmatrix} + \begin{pmatrix} c^2 h_{01}(t) \\ 0 \\ b^2 h_{10}(t) \end{pmatrix}.$$

Here $\mathbf{G}_2(t){=}\begin{pmatrix} c^2 h_{01}(t) \\ 0 \\ b^2 h_{10}(t) \end{pmatrix}$ is recursively determined from the first moments—the complete solution of (Eq.10). One finds an exact solution for the second moments. For brevity, we show only the asymptotic equivalent

$$\begin{aligned}
\mathbf{E}\big[W^2(t)\big] &= \Big[\frac{1}{4}\Big(W_0 + B_0\sqrt{\frac{c}{b}}\Big)^2 + \frac{c}{12}(W_0 + B_0) + \frac{1}{6}\Big(bW_0 + cB_0\Big)\sqrt{\frac{c}{b}}\,\Big]\mathrm{e}^{2\sqrt{bc}\,t} \\
&\quad + \mathbf{O}(\mathrm{e}^{\sqrt{bc}\,t}).
\end{aligned}$$

**Theorem 4.1** *Let $W(t)$ be the number of white balls in the Pólya process with the backward-diagonal urn scheme*

$$\mathbf{A} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

*If the process starts with $(W_0, B_0)^T$, then*

$$\frac{W(t)}{e^{\sqrt{bc}\,t}} \xrightarrow{\mathcal{D}} Z,$$

*where $Z$ is a random variable characterized by a recursive sequence of moments given in Eqs. 5, 6, 7, 8 and 9, and does depend on the initial conditions.*

A statement similar to that in Theorem 4.1 can be asserted for the blue balls.

### 4.2 The degenerate case

In a backward diagonal Pólya process, if one of the two entries on the backward diagonal is zero (but not both), the method of characteristics remains a transparent tool. Let us briefly look at the case $b>0$, and $c=0$. (Of course, the case $b=0$, and $c>0$ is symmetric.) It is evident that there is no change in the number of white balls. Therefore, $W(t) \equiv W_0$. The partial differential equation in Lemma 2.1 simplifies to

$$\frac{\partial \phi}{\partial t} + \left(1 - \mathrm{e}^{bv}\right)\frac{\partial \phi}{\partial u} = 0,$$

which has the solution

$$\phi(t, u, v) = \mathbf{E}\left[\mathrm{e}^{uW(t)+vB(t)}\right] = \mathrm{e}^{W_0 u}\mathrm{e}^{W_0 t(\mathrm{e}^{bv}-1)},$$

which is obtained from the constant characteristic curves

$$\tau = u - (1 - \mathrm{e}^{bv})t.$$

Indeed, $W(t) \equiv W_0$, and $B(t)$ has the moment generating function of a Poisson random variable:

$$B(t) = b\mathrm{Poi}(W_0 t).$$

*Remark* One can arrive at this observation directly from probabilistic considerations arguing the number of renewals of $W_0$ independent Poisson processes.

## 5 Discussion

We discussed a process of ball additions in real time to an urn. The Pólya process resembles in some aspects the growth of an urn (in discrete time) governed by a matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In the discrete urn process, at stage $n=1, 2, \ldots$ a ball is sampled at random from the urn, and returned to it after observing its color. Depending on the color of the ball withdrawn a number of extra balls is added to the urn according to the scheme **A**.

One of the earliest studies of discrete urn processes is Eggenberger and Pólya (1923) who modeled spreading phenomena such as contagion by an urn. The early models have been generalized in many directions. Modern reviews can be found in Kotz and Balakrishnan (1997), and in Mahmoud (2003). The Pólya–Eggenberger study and subsequent ones (Friedman 1949; Freedman 1965) on fixed schemata focused on adding a constant number of balls at each step. Few attempts have been made on schemes with nonconstant row sum, and mostly stayed at the average behavior. Perhaps Rosenblatt (1940) was the first to draw attention to these difficulties. The source of difficulty with nonconstant row sum is discussed in Kotz, et al. (2000), and reviewed in a broad sense in Mahmoud (2003).

Pólya urns progress in discrete time $n$. After $n$ draws, let $W_n$ and $B_n$ be the number of white and blue balls, respectively. In some sense these are equivalent to $W(t)$ and $B(t)$ in the Pólya process. Recall the race analogy for the latter. Let $t_n$ be the time of the n-th renewal. Whence $W(t_n)$ and $B(t_n)$ are, respectively, the numbers of white and blue runners in the Pólya process after $n$ races. By the independent identical distribution of running times, any of the runners is equally likely to win the race, that is,

$$P\{\text{white team wins the } (n+1)\text{st race} \mid \mathbf{R}(t_n)\} = \frac{W(t_n)}{W(t_n) + B(t_n)},$$

and $a$ white balls and $b$ blue balls will be added. Likewise,

$$P\{\text{blue team wins the } (n+1)\text{st race} \mid \mathbf{R}(t_n)\} = \frac{B(t_n)}{W(t_n) + B(t_n)},$$

and $c$ white balls and $d$ blue balls will be added. This constitutes a growth rule in the number of runners identical to that of the growth under random sampling from a Pólya urn with schema $\mathbf{A}$. In other words, $\mathbf{R}(t_n) \overset{\mathcal{D}}{=} \begin{pmatrix} W_n \\ B_n \end{pmatrix}$, if the two processes start with identical initial conditions.

However, $\mathbf{R}(t_n)$ is only a discretized form of a continuous renewal process with rewards, the renewals of which are the starting whistle of the races, and the rewards of which at every renewal are determined by $\mathbf{A}$. We can think of $\mathbf{R}(t_n)$, for $n{=}1, 2, \ldots$ as a series of snapshots in time of the continuous Pólya process at the moments when a renewal takes place in the process, with

$$\begin{pmatrix} W_n \\ B_n \end{pmatrix} \overset{\mathcal{D}}{=} \begin{pmatrix} W(t_n) \\ B(t_n) \end{pmatrix},$$

when $\begin{pmatrix} W_0 \\ B_0 \end{pmatrix} \overset{\mathcal{D}}{=} \begin{pmatrix} W(0) \\ B(0) \end{pmatrix}$.

This connection between the Pólya process and the Pólya urn provides a way, at least heuristically, to understand the discrete time urn growth. The Pólya process provides information at time $t$. The difficulty in inverting the result to the discrete domain is usually in determining $t_n$ in relation to $t$. The heuristic works well on average because $t_n$ is sharply concentrated. Thus our study of the diagonal Pólya process may provide an understanding of the corresponding discrete urn processes, in the more general case of nonconstant row sum, at least on average. We went much further with the Pólya process and obtained distributions, and it remains to be seen how they can be connected to the corresponding distributions in the discrete urn process.

To put the similarities in perspective, let us consider a standard example of Pólya urns. A classical urn is the well-studied Pólya–Eggenberger urn with the ball addition scheme $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It is well known that, as $n \to \infty$,

$$\mathbf{E}[W_n] \sim \frac{W_0}{W_0 + B_0} n;$$

see for example Johnson and Kotz (1977). Our result for the Pólya process with the same scheme is

$$\mathbf{E}\big[W(t)\big] = W_0 \mathrm{e}^t,$$

as $t \to \infty$ (see Proposition 3.1). The two systems have the same mean, under the interpretation that $t_n \sim \ln(n/(W_0 + B_0))$. Mahmoud (2004) gives a heuristic argument to discuss such connections (only on average), and mentions that some cases can be argued rigorously. This happens to be one of the cases for which we can device a rigorous proof.

While there are similarities between the Pólya process and the discrete urn process, there are also differences. For example, in the Pólya–Eggenberger urn with the scheme $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, by time $n$ in the discrete process, $W_n \le W_0 + n$, thus a limit (as $n \to \infty$) for $W_n/n$ must have support on [0, 1]; it is well known that the limit is Beta$(W_0, B_0)$. By contrast, in a Pólya process with the same scheme we have $W(t)/\mathrm{e}^t$ converging to a Gamma$(W_0, 1)$ random variable (as $t \to \infty$), with support on $[0, \infty)$. This happens because by time $t$, the number of renewals is not restricted and can be indefinitely large (though it can be demonstrated that it has high concentration around $\ln t$).

The $2\times 2$ Pólya process presented in this work can be generalized in many natural ways. For example, we can consider a $k$-color Pólya process arising on up to $k\ge 2$ colors of balls, with $k\times k$ schemes. One case that follows immediately from the work presented is a $k\times k$ Pólya–Eggenberger-like diagonal process

$$\begin{pmatrix} a_1 & 0 & 0 & 0 & \cdots & 0\\ 0 & a_2 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \ddots & & \vdots\\ 0 & 0 & & \cdots & & a_k \end{pmatrix},$$

with all $a_i\ge 0$, for $i=1, \ldots, k$ (not necessarily equal). The $k$ processes remain totally independent—let $X_i(t)$ be the number of balls of color $i$ by time $t$; without much extra effort we see that the limit joint distribution of $(\mathrm{e}^{-a_1 t} X_1(t), \ldots, \mathrm{e}^{-a_k t} X_k(t))$ is

$$\Big(\text{Gamma}\Big(\frac{X_1(0)}{a_1}, a_1\Big), \ldots, \text{Gamma}\Big(\frac{X_k(0)}{a_k}, a_k\Big)\Big).$$

We hope that our methods might generalize to other varieties of Pólya processes. For example, it may be possible to deal with a Bernard-Friedman-like scheme:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & a_1\\ 0 & 0 & 0 & 0 & \cdots & a_2 & 0\\ \vdots & \vdots & & & \ddots & & \vdots\\ a_k & 0 & & & \cdots & & 0 \end{pmatrix},$$

with all $a_i > 0$, for $i=1, \ldots, k$, although the method of characteristics for solving the governing partial differential equations may become significantly more involved. Many modifications and plausible offshoots are possible to emanate from this work, and we are optimistic that we and others can handle some of them.

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