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ON TESTING THE DILATION ORDER AND HNBUE ALTERNATIVES

F. BELZUNCE*, J. F. PINAR** AND J. M. RUIZ*

Dpto. Estadística e Investigación Operativa, Facultad de Matematicas, Campus de Espinardo, Universidad de Murcia, 30100 Espinardo (Murcia), Spain, e-mail: belzunce@um.es; jfpinar@um.es; jmruizgo@um.es

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Abstract. In this paper we develop a new family of tests for the dilation order based in a characterization of the dilation order. This family of tests statistics can be used for testing the exponentiality against HNBUE (HNWUE) alternatives. Asymptotic distributional results are given for both families of tests. For the HNBUE (HNWUE) we also derive the exact distribution under the null hypothesis.

Key words and phrases: Asymptotic normality, dilation order, harmonic new better [worst] than used, normalized spacings, order statistics, Pitman's asymptotic efficiency.

1. Introduction and motivation

In the literature several orderings of distributions have been defined to describe when one random variable is more dispersed than another random variable, examples of such orderings are the dispersive order (see Lewis and Thompson (1981) and Shaked and Shanthikumar (1994)) and the right spread order (see Fernández-Ponce *et al.* (1998) and Shaked and Shanthikumar (1998), Section 2.B.).

Hickey (1986) gives a more general concept of dispersive ordering. A random variable Y is more dispersed in dilation than a random variable X (denoted by $X \leq_{\text{dil}} Y$) if

$$E[\varphi(X - E[X])] \leq E[\varphi(Y - E[Y])],$$

for every convex function φ, provided previous expectations exist. Taking $\varphi(x) = x^2$, we get $X \leq_{\text{dil}} Y \Rightarrow \text{Var}(X) \leq \text{Var}(Y)$; therefore the dilation order is a stronger notion to compare the dispersion of two random variables than to compare the numerical values of the variances (which is less informative). It is clear from the definition of dilation order that it is location-free.

However this is a weaker notion than the dispersive and right spread orders, in fact we have the following relationships (see Fernández-Ponce *et al.* (1998) and Fagiouli *et al.* (1999)):

$$\leq_{\text{disp}} \Rightarrow \leq_{\text{rs}} \Rightarrow \leq_{\text{dil}} .$$

For a study of the previous orderings, relations with other orderings and applications, see Shaked and Shanthikumar (1994) and the references therein.

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An interesting property of the dilation order is the characterization of the HNBUE (HNWUE) aging classes. It can be seen (see Shaked and Shanthikumar (1994), p. 68) that

$$\text{(1.1)} \qquad X \text{ is HNBUE (HNWUE) if, and only if, } X \leq_{\text{dil}} (\geq_{\text{dil}}) Y,$$

where Y is an exponential random variable with $E[Y] = E[X]$. Let us recall that the exponential random variable has the non-aging property.

Applications of HNBUE (HNWUE) aging classes in reliability theory, including bounds for the survival function, can be seen in Klefsjö (1981, 1982*a*), Basu and Ebrahimi (1986), Basu and Kirmani (1986), Klefsjö (1986), Pal (1988), Pérez-Ocón and Gámiz-Pérez (1995*a*, 1995*b*), Cai (1995) and Pellerey (2000), which shows the importance of HNBUE (HNWUE) aging notion in reliability theory.

In order to verify the notions of dilation order and HNBUE (HNWUE) aging classes several tests and graphical techniques have been developed, as can be seen in Hollander and Proschan (1975, 1980), Klefsjö (1982*b*, 1983*b*), Basu and Ebrahimi (1985), Singh and Kochar (1986), Aly (1990), Hendi *et al.* (1998), Jammalamadaka and Lee (1998), Belzunce *et al.* (2000) and Klar (2000).

The purpose of this paper is to develop a family of tests for dilation order and HNBUE (HNWUE) alternatives.

For this purpose, in Section 2, we address the problem of testing the dilation order. Later in Section 3 we develop the family of tests for HNBUE (HNWUE) alternatives, which is based on the family developed in Section 2 from characterization (1.1). For both families we give the asymptotic distribution and we prove the consistency of our proposed tests. We also compare these tests with other existing tests in terms of Pitman's asymptotic efficiency. For the family developed in Section 3 we also give the exact distribution under the null hypothesis. Applications to some data sets are given in Section 4.

Throughout this paper "increasing" means "nondecreasing".

2. A family of tests for dilation order

Let X and Y be two random variables, with distribution functions F and G respectively. In this section we consider the problem of testing the null hypothesis

$$H_0 : X =_{\text{dil}} Y$$

vs the alternative

$$H_1 : X \lneqq_{\text{dil}} Y,$$

given random samples of X and Y.

This problem has been considered by Aly (1990) and Belzunce *et al.* (2000). Aly (1990) considers the problem of testing the dispersive order, but, as can be seen in Belzunce *et al.* (2000), his test is consistent against the dilation order.

For the dilation order Fagiouli *et al.* (1999) and Belzunce *et al.* (2000) have proved, independently, the following characterization:

$X \leq_{\text{dil}} Y$ if, and only if,

$$\text{(2.1)} \qquad \int_0^p F^{-1}_{X-E(X)}(t)dt \geq \int_0^p F^{-1}_{Y-E(Y)}(t)dt, \qquad \text{for all} \quad p \in (0,1),$$

where $F^{-1}(p) = \inf\{x : F(x) \geq p\}$.

Therefore if $X \leq_{\text{dil}} Y$, we have that

$$g(s) = \int_0^s F^{-1}(t)dt - sE[X] - \int_0^s G^{-1}(t)dt + sE[Y] \geq 0,$$

so $h(p) = \int_0^p g(s)ds$ is an increasing function in $p \in (0,1)$. Then

$$\Delta_{\text{dil}}^{\alpha}(X,Y) = \int_0^1 (h(p) - h(\alpha p))dp \geq 0, \qquad \text{for all} \quad \alpha \in (0,1).$$

Clearly, $\Delta_{\text{dil}}^{\alpha}(X,Y)$ is a family of measures of deviation from H_0 to H_1, since $\Delta_{\text{dil}}^{\alpha}(X,Y) = 0$ if $X =_{\text{dil}} Y$ and $\Delta_{\text{dil}}^{\alpha}(X,Y) > 0$ if $X \lneqq_{\text{dil}} Y$.

As can be easily shown,

$$\Delta_{\text{dil}}^{\alpha}(X,Y) = \int_0^1 J_{\alpha}(p)g(p)dp,$$

where

$$J_{\alpha}(p) = \begin{cases} p\left(\frac{1}{\alpha} - 1\right) & 0 \leq p \leq \alpha, \\ 1 - p & \alpha \leq p \leq 1. \end{cases}$$

Replacing the distribution functions of X and Y by their empirical versions we obtain a family of statistics depending on a parameter $\alpha \in (0,1)$, which can be used for testing the dilation order.

If $X_{(1)}, \ldots, X_{(n)}$ and $Y_{(1)}, \ldots, Y_{(m)}$ are ordered samples of X and Y respectively, our family of tests can be written, after some algebraic manipulations, in the form

$$\widehat{\Delta}_{\text{dil}}^{\alpha}(X,Y) = \frac{1}{n}\sum_{i=1}^{n}\left(c_{i,n}^{\alpha} - \frac{1-\alpha^2}{6}\right)X_{(i)} - \frac{1}{m}\sum_{i=1}^{m}\left(c_{i,m}^{\alpha} - \frac{1-\alpha^2}{6}\right)Y_{(i)},$$

where

$$c_{i,r}^{\alpha} = \begin{cases} \left(\frac{1}{\alpha} - 1\right)\frac{3r^2\alpha - 3i^2 + 3i - 1}{6r^2} & i < k+1, \\ \frac{3\alpha(r-k-1)^2 + (k-r\alpha)^3 + \alpha(3r-3k-2)}{6r^2\alpha} & i = k+1, \\ \frac{3(r-i)^2 + 3(r-i) + 1}{6r^2} & i > k+1, \end{cases} \tag{2.2}$$

with $\frac{k}{r} \leq \alpha < \frac{k+1}{r}$.

Next we study the asymptotic distribution of $\widehat{\Delta}_{\text{dil}}^{\alpha}(X,Y)$, that follows from Theorems 2 and 3 from Stigler (1974).

Let us denote

$$\sigma^2(P_{\alpha}, F) = \iint_{\mathbb{R}^2} P_{\alpha}(F(x))P_{\alpha}(F(y))[F(\min(x,y)) - F(x)F(y)]dxdy, \tag{2.3}$$

and

$$\mu(P_{\alpha}, F) = \int_{\mathbb{R}} xP_{\alpha}(F(x))dF(x), \tag{2.4}$$

where

$$P_\alpha(p) = \begin{cases} \frac{1}{2}(\frac{1}{\alpha} - 1)(\alpha - p^2) - \frac{1-\alpha^2}{6} & p \leq \alpha, \\ \frac{1}{2}(1-p)^2 - \frac{1-\alpha^2}{6} & p > \alpha. \end{cases} \tag{2.5}$$

Then we get the following result.

THEOREM 2.1. *Let X and Y be two continuous nonnegative random variables, with distribution functions F and G respectively, such that $E[X^2]$, $E[Y^2] < +\infty$ and $\sigma^2(P_\alpha, F)$, $\sigma^2(P_\alpha, G) > 0$. Let*

$$\sigma^2(n,m) = \frac{m\sigma^2(P_\alpha, F) + n\sigma^2(P_\alpha, G)}{n+m},$$

and let $X_{(1)}, \ldots, X_{(n)}$ and $Y_{(1)}, \ldots, Y_{(m)}$ be the ordered samples from random samples of sizes n and m from X and Y respectively. Then

$$\left(\frac{nm}{n+m}\right)^{1/2} \left(\frac{\widehat{\Delta}^\alpha_{\mathrm{dil}}(X,Y) - (\mu(P_\alpha, F) - \mu(P_\alpha, G))}{\sigma(n,m)}\right) \xrightarrow{L} N(0,1), \tag{2.6}$$

if $\min(n,m) \to +\infty$ and $(m,n) \in D_\lambda := \{(m,n) \mid \lambda \leq m/(n+m) \leq 1-\lambda\}$ for some $\lambda \in (0, 1/2]$.

In practice $\sigma^2(n,m)$ is unknown, but it can be replaced by the consistent estimator

$$\widehat{\sigma}^2(n,m) = \frac{m\sigma^2(F_n) + n\sigma^2(G_m)}{n+m}, \tag{2.7}$$

where

$$\sigma^2(F_n) = \sum_{i=1}^{n-1}\sum_{j=i}^{n} \delta_{ij} P_\alpha\left(\frac{i}{n}\right) P_\alpha\left(\frac{j}{n}\right)\left(\frac{i}{n} - \frac{ij}{n^2}\right)(X_{(j+1)} - X_{(j)})(X_{(i+1)} - X_{(i)})$$

and

$$\sigma^2(G_m) = \sum_{i=1}^{m-1}\sum_{j=i}^{m} \delta_{ij} P_\alpha\left(\frac{i}{m}\right) P_\alpha\left(\frac{j}{m}\right)\left(\frac{i}{m} - \frac{ij}{m^2}\right)(Y_{(j+1)} - Y_{(j)})(Y_{(i+1)} - Y_{(i)}),$$

with $\delta_{ii} = 1$ and $\delta_{ij} = 2$ if $i \neq j$.

From (2.6) and (2.7) an asymptotically distribution free test for testing H_0 against H_1 can be given. We reject H_0 if $(\frac{nm}{n+m})^{1/2}\widehat{\Delta}^\alpha_{\mathrm{dil}}(X,Y)/\widehat{\sigma}(n,m) > z_q$, where z_q is the $(1-q)$-quantile of the standard normal distribution function.

The consistency of our tests follows if we show that

$$\Delta^\alpha_{\mathrm{dil}}(X,Y) = \mu(P_\alpha, F) - \mu(P_\alpha, G). \tag{2.8}$$

It is not difficult to see that

$$\int_0^1 J_\alpha(p)\left(\int_0^p F^{-1}(s)ds - pE[X]\right)dp$$
$$= \int_0^{F^{-1}(\alpha)} \frac{x}{2}\left(\frac{1}{\alpha} - 1\right)\left(\alpha - F^2(x)\right)dF(x)$$

Table 1. Maximum ARE of $\hat{\Delta}^{\alpha}_{\mathrm{dil}}(X,Y)$.

Test	Marzec & Marzec	Wilcoxon	Savage	W	S	$V_{0\cdot 6,N}$
Exponential	1.170	1.475	1.106	1.348	1.311	1.320
LFR	2.282	3.295	0.618	1.339	0.732	0.584
Makehann	1.329	2.033	0.678	0.949	0.650	0.581

Table 2. Minimum ARE of $\hat{\Delta}^{\alpha}_{\mathrm{dil}}(X,Y)$.

Test	Marzec & Marzec	Wilcoxon	Savage	W	S	$V_{0\cdot 6,N}$
Exponential	0.944	1.189	0.892	1.087	1.057	1.065
LFR	1.885	2.723	0.511	1.107	0.605	0.483
Makehann	1.302	1.991	0.664	0.929	0.636	0.569

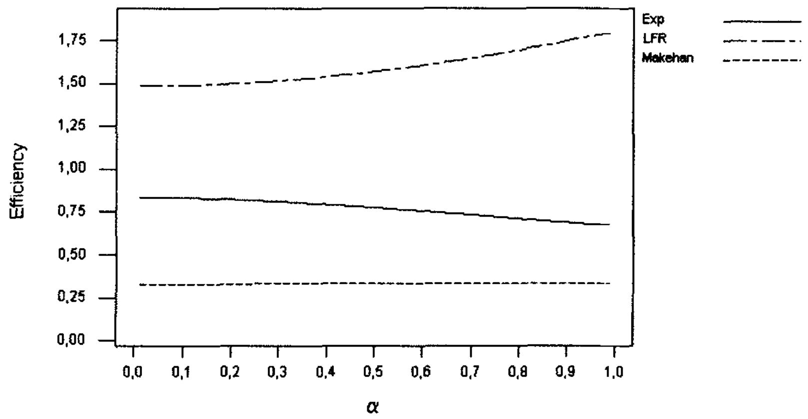

Fig. 1. Efficiency of $\hat{\Delta}^{\alpha}_{\mathrm{dil}}(X,Y)$ as a function of α, for translated exponential, LFR and Makehann models.

$$+\int_{F^{-1}(\alpha)}^{\infty}\frac{x}{2}(1-F(x))^2\,dF(x)-\frac{1-\alpha^2}{6}E[X]$$
$$=\int_0^{\infty}xP_\alpha(F(x))dF(x),$$

and the same holds for G, which proves (2.8).

2.1 *Asymptotic relative efficiency*

In order to see the performance of our test, we have compared Pitman's asymptotic efficiency of our test with that of Marzec and Marzec (1991) test, Wilcoxon (1945) test, Savage (1956) test and W, S, $V_{0\cdot 6,N}$ tests proposed by Kochar (1979, 1981) and Bagai and Kochar (1986). We have considered that Y follows an exponential distribution with unit mean and three possible alternatives for X, with distribution functions

$$1-\exp\{-(1+\theta)x\}, \quad 1-\exp\left\{-\left(x+\frac{\theta x^2}{2}\right)\right\} \quad \text{and}$$
$$1-\exp\{-[x+\theta(x+\exp(-x)-1)]\},$$

which correspond to a translated exponential, a linear failure rate (LFR) and a Makehann distribution respectively.

The asymptotic relative efficiencies (ARE) of our test with respect to the previous ones are functions of $\alpha \in (0,1)$; we give in Table 1 the maximum ARE and in Table 2 the minimum ARE. These tables show that our test is quite efficient with respect to Marzec and Marzec (1991), Wilcoxon (1945) and Kochar (1979) tests, and in particular for the translated exponential distribution (for a complete description of the efficiencies for all values of α, see Fig. 1).

3. A family of tests for HNBUE [HNWUE] aging class

The ideas developed in previous section can be used to develop a family of tests for HNBUE (HNWUE) alternatives.

In this section we consider a random sample of a random variable X, with distribution function F, and we want to test the null hypothesis

$$H_0 : X \text{ is exponential } (F(t) = 1-\exp(-\lambda t), \lambda \text{ unspecified})$$

against the alternative

$$H_1 : X \text{ is HNBUE [HNWUE] but it is not exponential.}$$

If Y follows an exponential distribution with mean $E[X]$, by characterization (1.1) we get that $\Delta^{\alpha}_{\mathrm{HNBUE}}(X) \equiv \Delta^{\alpha}_{\mathrm{dil}}(X,Y)$ is a family of measures of deviation from H_0 to H_1. It is easy to see that

$$\Delta^{\alpha}_{\mathrm{HNBUE}}(X) = \int_0^1 J_\alpha(p)\left(\int_0^p F^{-1}(t)dt - pE[X]\right)dp + \frac{1-\alpha}{6}\left(\frac{5}{3}-\frac{5\alpha}{6}+\frac{(1-\alpha)^2}{\alpha}\ln(1-\alpha)\right)E[X].$$

Our family of tests is based on the empirical counterpart of this measure, which can be written in the form of linear combination of order statistics as

$$\widehat{\Delta}^{\alpha}_{\mathrm{HNBUE}}(X) = \frac{1}{n}\sum_{i=1}^{n}\left(c^{\alpha}_{i,n} + \frac{1-\alpha}{6}\left[\frac{2}{3}-\frac{11\alpha}{6}+\frac{(1-\alpha)^2}{\alpha}\ln(1-\alpha)\right]\right)X_{(i)}, \tag{3.1}$$

where $c^{\alpha}_{i,n}$ is as in (2.2).

In order to make our tests scale invariant, we take the statistic

$$\Omega^{\alpha}_{\mathrm{HNBUE}}(X) \equiv \frac{\widehat{\Delta}^{\alpha}_{\mathrm{HNBUE}}(X)}{\overline{X}},$$

where $\overline{X}$ is the sample mean.

This family of statistics can be written in terms of the normalized spacings, that is, $D_i \equiv (n-i+1)(X_{(i)} - X_{(i-1)})$, in the way

$$\Omega^\alpha_{\text{HNBUE}}(X) = \frac{\sum_{i=1}^n e^\alpha_{i,n} D_i}{\sum_{i=1}^n D_i},$$

where

$$e^\alpha_{i,n} = \frac{1}{n-i+1}\sum_{j=i}^n c^\alpha_{j,n} + \frac{1-\alpha}{6}\left[\frac{2}{3} - \frac{11\alpha}{6} + \frac{(1-\alpha)^2}{\alpha}\ln(1-\alpha)\right].$$

Next we study the exact distribution of $\Omega^\alpha_{\text{HNBUE}}(X)$.

Since the statistics are scale invariant, we can take $\lambda = \frac{1}{2}$ in Theorem 2.4 of Box (1954) to obtain the exact distribution of $\Omega^\alpha_{\text{HNBUE}}(X)$ under the null hypothesis.

THEOREM 3.1. *Let X be a random variable with distribution function given by $F(t) = 1 - \exp(-t/2)$ for $t > 0$, and let $X_{(1)}, \ldots, X_{(n)}$ be the ordered sample for a random sample from X. Then*

$$P(\Omega^\alpha_{\text{HNBUE}}(X) \le x) = 1 - \sum_{i=1}^n \prod_{j=1, j\ne i}^n \left(\frac{e^\alpha_{i,n} - x}{e^\alpha_{i,n} - e^\alpha_{j,n}}\right) I(x, e^\alpha_{i,n})$$

for $e^\alpha_{i,n} \ne e^\alpha_{j,n}$, for all $i \ne j$, for fixed n, where $I(x,y) = 1$ if $x < y$ and $I(x,y) = 0$ if $x \ge y$.

Now we study the asymptotic distribution using Theorems 2 and 3 from Stigler (1974). First we infer that of $\widehat{\Delta}^\alpha_{\text{HNBUE}}(X)$ from (3.1).

THEOREM 3.2. *Let X be a continuous nonnegative random variable with distribution function F, such that $E[X^2] < +\infty$. Let $\mu(Q_\alpha, F)$ and $\sigma^2(Q_\alpha, F)$ be as in Section 2, where*

$$Q_\alpha(p) = \begin{cases} \frac{1}{2}(\frac{1}{\alpha} - 1)(\alpha - p^2) + \frac{1-\alpha}{6}[\frac{2}{3} - \frac{11\alpha}{6} + \frac{(1-\alpha)^2}{\alpha}\ln(1-\alpha)] & p \le \alpha, \\ \frac{1}{2}(1-p)^2 + \frac{1-\alpha}{6}[\frac{2}{3} - \frac{11\alpha}{6} + \frac{(1-\alpha)^2}{\alpha}\ln(1-\alpha)] & p > \alpha, \end{cases} \tag{3.2}$$

and suppose $\sigma^2(Q_\alpha, F) > 0$. Then

$$\sqrt{n}\left(\frac{\widehat{\Delta}^\alpha_{\text{HNBUE}}(X) - \mu(Q_\alpha, F)}{\sigma(Q_\alpha, F)}\right) \xrightarrow{L} N(0,1).$$

Now we use the previous result and Slutsky's theorem to get the limiting distribution of $\Omega^\alpha_{\text{HNBUE}}(X)$.

THEOREM 3.3. *Under the conditions of Theorem 3.2 it follows that*

$$\sqrt{n}\left(\Omega^\alpha_{\text{HNBUE}}(X) - \frac{\mu(Q_\alpha, F)}{E[X]}\right) \xrightarrow{L} N(\mu^*, (\sigma^*)^2),$$

where $\mu^* = \mu(Q^*_\alpha, F)/E[X]$, $(\sigma^*)^2 = \sigma^2(Q^*_\alpha, F)/E[X]$ and $Q^*_\alpha(p) = Q_\alpha(p) - \mu(Q_\alpha, F)/E[X]$.

Finally we derive the asymptotic distribution of $\Omega^{\alpha}_{\mathrm{HNBUE}}(X)$ under the null hypothesis. Due to scale invariance, we can take F to be exponential with parameter $\lambda = 1$, and then we get $\mu(Q_\alpha, F) = 0$ and $\sigma^2(Q_\alpha, F) = \frac{1}{6480}(\frac{1}{\alpha} - 1)^2[1080\alpha - 2744\alpha^2 + 2492\alpha^3 - 767\alpha^4 + 60(18 - 22\alpha + 11\alpha^2)(1-\alpha)^2 \ln(1-\alpha) - 180(1-\alpha)^4 \ln^2(1-\alpha)]$. Therefore under H_0 we have that

$$\left(\frac{n}{\sigma^2(Q_\alpha, F)}\right)^{1/2} \Omega^{\alpha}_{\mathrm{HNBUE}}(X) \tag{3.3}$$

converge towards a standard normal distribution.

Thus for high values of n we can reject the null hypothesis if (3.3) is greater than z_q [less than z_{1-q}].

Since $\mu(Q_\alpha, F) = 0$ when F is exponential, in order to prove the consistency of our test we only need to show that $\mu(Q_\alpha, F) >$ [<]0 when X is HNBUE [HNWUE] but it is not exponential, and this follows from the equality $\Delta^{\alpha}_{\mathrm{HNBUE}}(X) = \mu(Q_\alpha, F)$, obtained in a similar way to (2.8).

3.1 *Asymptotic relative efficiency*

As in Section 2 we have calculated our test's ARE with respect to some existing tests for aging classes, namely:

- A_2 and V for IFR [DFR] class, given by Klefsjö (1983*a*) and Proschan and Pyke (1967), respectively.
- J_n, U_n and S_n for NBU [NWU] class, given by Hollander and Proschan (1972), Ahmad (1975) and Deshpande and Kochar (1983), respectively.
- B for IFRA [DFRA] class, given by Klefsjö (1983*a*).
- V^* and $V_n(k)$ for DMRL [IMRL] class, given by Hollander and Proschan (1975) and Bandyopadhyay and Basu (1990), respectively.
- K^* for HNBUE [HNWUE] class, given by Hollander and Proschan (1975, 1980) and Klefsjö (1983*b*).

The alternatives we have considered in this case are LFR, Makehan, Weibull and Gamma models. LFR and Makehan models are given in Subsection 2.1; Weibull and

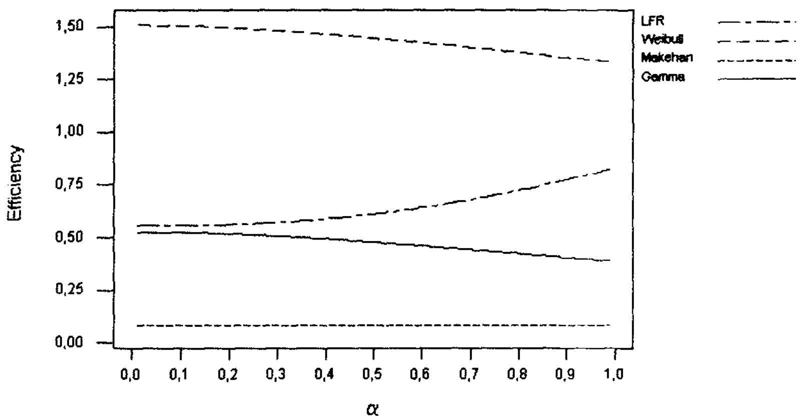

Fig. 2. Efficiency of $\Delta^{\alpha}_{\mathrm{HNBUE}}(X)$ as a function of α, for LFR, Weibull, Makehann and Gamma models.

Table 3. Maximum ARE of $\hat{\Delta}^{\alpha}_{\mathrm{HNBUE}}(X)$.

Test	LFR	Makehan	Weibull	Gamma
A_2	2.255	1.420	2.072	2.673
V	1.454	1.325	1.397	*
J_n	2.436	1.243	1.118	*
U_n	2.684	1.487	2.062	*
S_n	21.077	16.899	2.579	*
B	3.246	1.420	1.199	1.053
V^*	1.002	1.420	2.153	3.735
$V_n(0.01)$	1.105	1.001	1.105	1.283
$V_n(0.05)$	1.100	1.030	1.242	1.547
K^*	1.096	0.998	1.047	1.175

Table 4. Minimum ARE of $\hat{\Delta}^{\alpha}_{\mathrm{HNBUE}}(X)$.

Test	LFR	Makehan	Weibull	Gamma
A_2	1.524	1.340	1.838	1.986
V	0.983	1.250	1.240	*
J_n	1.646	1.173	0.992	*
U_n	1.814	1.403	1.830	*
S_n	14.245	15.944	2.288	*
B	2.194	1.340	1.064	0.782
V^*	0.677	1.340	1.911	2.775
$V_n(0.01)$	0.747	0.945	0.981	0.953
$V_n(0.05)$	0.743	0.972	1.102	1.149
K^*	0.741	0.941	0.929	0.873

Gamma distribution functions are given by $1 - \exp(-x^{\theta})$ and $\frac{1}{\Gamma(\theta)} \int_0^x t^{\theta-1} \exp(-t)dt$, respectively.

As we can see from Table 3, we can choose values of $\alpha \in (0,1)$ so that our test performs well against all the other tests. Table 4 shows that it remains quite efficient against most of the other tests even at worst, specifically for Makehan and Weibull alternatives (a complete description for all values α, can be seen in Fig. 2).

4. Applications to some data sets

In order to apply the previous to some data sets, we need to know in what situations the assumptions of the hypothesis are reasonable. For example for the dilation order test, we need to assume that $X \leq_{\mathrm{dil}} Y$, a similar comment holds for the HNBUE [HNWUE] test.

From characterization (2.1), we should check whether the inequality

$$D_X(p) \geq D_Y(p), \qquad \text{for all} \quad p \in [0,1],$$

holds, where $D_X(p) = \int_0^p F^{-1}(t)dt - E[X]p$.

Following Barlow *et al.* ((1972), pp. 235–237), a nonparametric estimator of $D_X(p)$, given a random sample $X_1, X_2, \ldots, X_n$ of X, is given by interpolation of the points $(0,0)$ and $(k/n, (\sum_{i=1}^{k} X_{(i)} - k\overline{X})/n)$ for $k = 1, \ldots, n$. Then comparing the two nonparametric estimators of $D_X(p)$ and $D_Y(p)$, we can have some empirical evidence about the assumption $X \leq_{\text{dil}} Y$.

For the HNBUE hypothesis a similar argument can be done. In fact Aly (1992) proposes a so called HNBUE plot, based on the characterization

$$X \text{ is HNBUE [HNWUE]} \Leftrightarrow \Delta(p) = \phi(p) + \exp\{-F_X^{-1}(p)/E[X]\} - 1 \geq (\leq) 0 \quad \text{for all} \quad p \in (0,1),$$

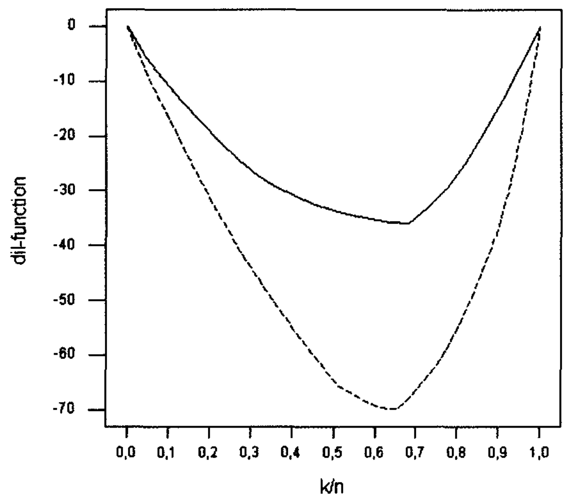

Fig. 3. Nonparametric estimators of $D_X(p)$ and $D_Y(p)$ for Data set I.

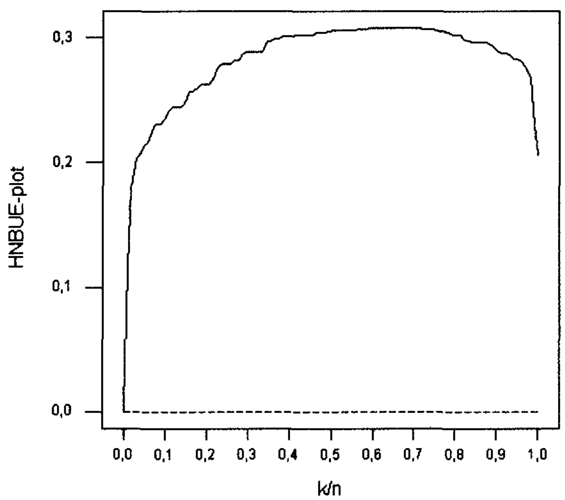

Fig. 4. Nonparametric estimator of $\Delta(p)$ for Data set II.

where $\phi(p) = \int_0^p F^{-1}(t)dt/E[X]$.

Analogously a nonparametric estimator for $\Delta(p)$, can be obtained by interpolation of the points $(0,0)$ and $(k/n, \sum_{i=1}^{k}(1-(i-1)/n)(X_{(i)} - X_{(i-1)})/\overline{X} + \exp\{-X_{(k)}/\overline{X}\}+1)$ for $k = 1, \ldots, n$.

• *Data set* I. The data set we have considered to test equality in dilation order consists of two groups of survival times of RFM strain male mice (Hoel (1972)). The first group lived in a conventional laboratory environment, while the second did in a germ free environment. Mice died because of thymic lymphoma.

First we plot the nonparametric estimators of $D_X(p)$ and $D_Y(p)$ (here X and Y are the random variables representing the survival times of a RFM strain male mouse in a conventional laboratory environment and in a germ free environment, respectively). The plot (see Fig. 3) suggests that we can assume that $X \leq_{\rm dil} Y$.

We carried out the test for different values of $\alpha \in (0,1)$. Results are given in Table 5; the p-value was calculated using the asymptotic distribution of $(\frac{nm}{n+m})^{1/2}\widehat{\Delta}^{\alpha}_{\rm dil}(X,Y)/\widehat{\sigma}(n,m)$.

According to results, we come to the conclusion that $X \lneqq_{\rm dil} Y$, and the rejection of the equality is more clear for higher values of α.

• *Data set* II. Next we apply the test developed in Section 3 to X = "Fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second" (Engelhardt *et al.* (1981)). Again we first plot the nonparametric

Table 5.

α	$(\frac{nm}{n+m})^{1/2}\widehat{\Delta}^{\alpha}_{\rm dil}(X,Y)/\widehat{\sigma}(n,m)$	p-value
0.1	2.3425	0.0096
0.2	2.3683	0.0089
0.3	2.4084	0.0080
0.4	2.4660	0.0068
0.5	2.5456	0.0055
0.6	2.6464	0.0041
0.7	2.7772	0.0027
0.8	2.8996	0.0019
0.9	3.0129	0.0013

Table 6.

α	$(\frac{n}{\sigma^2(Q_\alpha,F)})^{1/2}\Omega^{\alpha}_{\rm HNBUE}(X)$	p-value
0.1	3.6758	0.00012
0.2	3.6274	0.00014
0.3	3.5691	0.00018
0.4	3.5292	0.00021
0.5	3.5346	0.00020
0.6	3.5885	0.00017
0.7	3.6718	0.00012
0.8	3.7678	0.00008
0.9	3.8422	0.00006

estimator of $\Delta(p)$. In this case the plot (see Fig. 4) suggests that X is HNBUE.

In this case we test the hypothesis

$$H_0 : X \text{ is exponential } (F(t) = 1 - \exp(-\lambda t), \lambda \text{ unspecified})$$

against the alternative

$$H_1 : X \text{ is HNBUE and is not exponential.}$$

In Table 6 we summarize the results where the p-value are computed by the approximation to the normal distribution. As we can see our test accepts clearly the HNBUE alternative for all values of α, and more clearly for low and high values of α.

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