Ann. Inst. Statist. Math.
Vol. 57, No. 4, 781–788 (2005)

MULTIVARIATE VERSIONS OF BLOMQVIST'S BETA AND SPEARMAN'S FOOTRULE

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(Received October 28, 2003; revised May 28, 2004)

Abstract. In this paper we define multivariate versions of the medial correlation coefficient and the rank correlation coefficient Spearman's footrule in terms of copulas. We also present corresponding results for the sample statistic and provide a comparison of lower bounds among different measures of multivariate association.

Key words and phrases: Blomqvist's beta, copula, measures of multivariate association, Spearman's footrule.

1. Introduction

If X and Y are continuous random variables with medians $\widetilde{x}$ and $\widetilde{y}$, respectively, then the population version of the medial correlation coefficient (also known as Blomqvist's beta) is given by $\beta_{X,Y} = \Pr[(X-\widetilde{x})(Y-\widetilde{y}) > 0] - \Pr[(X-\widetilde{x})(Y-\widetilde{y}) < 0]$: see Blomqvist (1950). When H denotes the joint distribution function of X and Y, it readily follows that $\beta_{X,Y} = 4H(\widetilde{x},\widetilde{y}) - 1$. Another known nonparametric measure of association is the coefficient known as "Spearman's footrule"

$$f_S = 1 - \frac{3}{n^2-1}\sum_{i=1}^{n} |p_i - q_i|,$$

where p_i and q_i denote the ranks of n observed values of two variates X and Y: see Spearman (1906).

The purpose of this paper is to define and study multivariate versions of the medial correlation coefficient and the coefficient Spearman's footrule.

Since our results involve the concept of a copula we review this notion—for a complete survey, see Nelsen (1999). Let $n \geq 2$ be a natural number. A (n-dimensional) *copula* (briefly n-copula) is the restriction to $[0,1]^n$ of a continuous multivariate distribution function whose margins are uniform on $[0,1]$. The importance of copulas in statistics is described in the following result: Let $X_1, X_2, \dots, X_n$ be n random variables with joint distribution function H and respective one-dimensional marginal distribution functions $F_1, F_2, \dots, F_n$. Then there exists an n-copula C such that $H(x_1, x_2, \dots, x_n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n))$ for all $(x_1, x_2, \dots, x_n)$ in $[-\infty,\infty]^n$. Let $\boldsymbol{u} = (u_1, u_2, \dots, u_n)$ be in $[0,1]^n$, and let Π^n denote the n-copula of independent continuous random variables, i.e., $\Pi^n(\boldsymbol{u}) = u_1u_2\cdots u_n$. Any n-copula C satisfies the following inequalities: $W^n(\boldsymbol{u}) = \max(\sum_{i=1}^n u_i - n + 1, 0) \leq C(\boldsymbol{u}) \leq \min(u_1, u_2, \dots, u_n) = M^n(\boldsymbol{u})$. M^n is an n-copula for all $n \geq 2$, but not W^n (except if $n = 2$). Let $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$ and

$\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, and let $\boldsymbol{X} > \boldsymbol{x}$ denote the component-wise inequality $X_i > x_i$, $i = 1, 2, \ldots, n$. If $\boldsymbol{U}$ is a vector of uniform $[0,1]$ random variables with n-copula C, $\overline{C}$ denotes the survival function, $\overline{C}(\boldsymbol{u}) = \Pr[\boldsymbol{U} > \boldsymbol{u}]$; and $\hat{C}$ denotes the survival copula, $\hat{C}(\boldsymbol{u}) = \Pr[\mathbf{1} - \boldsymbol{U} \leq \boldsymbol{u}]$. $\hat{C}$ is always an n-copula; however $\overline{C}$ never is. Let δ_C denote the *diagonal section* of an n-copula C, i.e., $\delta_C(t) = C(t, t, \ldots, t)$, $t \in [0,1]$; and $\delta_C^{(-1)}$ denotes the *cadlag* inverse of δ_C, i.e., $\delta_C^{(-1)}(t) = \sup\{u \in [0,1] \mid \delta_C(u) \leq t\}$, $t \in [0,1]$.

2. Concordance and measures of association

Two observations (x_1, y_1) and (x_2, y_2) from a pair of continuous random variables are *concordant* if $x_1 < x_2$ and $y_1 < y_2$, or $x_1 > x_2$ and $y_1 > y_2$; and they are *discordant* if $x_1 < x_2$ and $y_1 > y_2$, or $x_1 > x_2$ and $y_1 < y_2$. Let (X_1, Y_1) and (X_2, Y_2) be two independent vectors of continuous random variables with common margins and respective 2-copulas C_1 and C_2. Let Q_2 denote the difference between the probabilities of concordance and discordance of (X_1, Y_1) and (X_2, Y_2), i.e., $Q_2 = \Pr[(X_1 - X_2)(Y_1 - Y_2) > 0] - \Pr[(X_1 - X_2)(Y_1 - Y_2) < 0]$. Then $Q_2 = Q_2(C_1, C_2) = 4\int_{[0,1]^2} C_2(u,v)\mathrm{d}C_1(u,v) - 1$ (see Nelsen (1999) for details).

Observe that Blomqvist proposed the measure using a random vector and the population medians rather than two random vectors in the expression for Q_2. Moreover, if C denotes the copula of the pair (X, Y), then $\beta_{X,Y} = \beta_C = 4C(1/2, 1/2) - 1$. Note also that the population version of Spearman's footrule—denoted by $\varphi_{X,Y}$ or φ_C—is given by $\varphi_{X,Y} = \varphi_C = 1 - 3\int_{[0,1]^2} |x - y|\mathrm{d}C(x,y) = (3Q_2(C, M^2) - 1)/2$: see Nelsen (1999).

In higher dimensions, two observations $\boldsymbol{x}$ and $\boldsymbol{y}$ from a vector $\boldsymbol{X}$ of continuous random variables are concordant if for all $i \neq j$, (x_i, x_j) and (y_i, y_j) are concordant; however, discordance does not generalize. Nelsen (2002) presents the probability of concordance in terms of n-copulas: Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be independent vectors of continuous random variables with common univariate margins and n-copulas C_1 and C_2, respectively, and let Q'_n denote the probability of concordance between $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, i.e., $Q'_n = \Pr[\boldsymbol{X}_1 > \boldsymbol{X}_2] + \Pr[\boldsymbol{X}_1 < \boldsymbol{X}_2]$. Then $Q'_n = Q'_n(C_1, C_2) = \int_{[0,1]^n} C_2(\boldsymbol{u})\mathrm{d}C_1(\boldsymbol{u}) + \int_{[0,1]^n} C_1(\boldsymbol{u})\mathrm{d}C_2(\boldsymbol{u})$.

Q_n is defined as a linear function of Q'_n in the following manner:

$$Q_n(C_1, C_2) = \frac{2^{n-1}Q'_n(C_1, C_2) - 1}{2^{n-1} - 1}, \tag{2.1}$$

so that $Q_n(M^n, M^n) = 1$ and $Q_n(\Pi^n, \Pi^n) = 0$.

In the literature we can find measures of multivariate association which are based upon the probability of concordance expressed in terms of the n-copula C associated with a continuous random vector: For example, $\tau_{n,C} = Q_n(C, C)$ and $\rho_{n,C} = (n+1)(2^{n-1} - 1)Q_n(C, \Pi^n)/[2^n - (n+1)]$ (see Nelsen (1996, 2002) for more details). $\tau_{n,C}$ and $\rho_{n,C}$ are generalizations of the well-known Kendall's tau and Spearman's rho, respectively: see Nelsen (1999).

We finish this section with some notation. If Ω is a measure of multivariate association, let $\Omega_{av,C}$ denote the average of the $\binom{n}{2}$ pairwise bivariate measures.

3. A multivariate version of Blomqvist's beta

Let H be a continuous n-variate distribution and let $\boldsymbol{X}$ have distribution H. If we define a multivariate version of Blomqvist's beta, denoted by $\beta_{n,H}$, such that $\beta_{n,H} = 0$

when H is the distribution of independent random variables, and $\beta_{n,H} = 1$ for perfect positive dependence, as in the bivariate case, and based on the probability of concordance using the population medians, i.e., $\Pr[\boldsymbol{X} < \widetilde{\boldsymbol{x}} \text{ or } \boldsymbol{X} > \widetilde{\boldsymbol{x}}]$, where $\widetilde{\boldsymbol{x}} = (\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n)$ denotes the respective medians, then we have

$$\beta_{n,H} = \frac{2^{n-1}\Pr[\boldsymbol{X} < \widetilde{\boldsymbol{x}} \text{ or } \boldsymbol{X} > \widetilde{\boldsymbol{x}}] - 1}{2^{n-1} - 1}. \tag{3.1}$$

In the following result, we express the multivariate version of the medial correlation coefficient given in (3.1) in terms of n-copulas.

THEOREM 3.1. *Let $\boldsymbol{U}$ be a vector of uniform* $[0,1]$ *random variables with n-copula C. Then the measure defined by* (3.1) *is given by*

$$\beta_{n,C} = \frac{2^{n-1}[C(\mathbf{1/2}) + \hat{C}(\mathbf{1/2})] - 1}{2^{n-1} - 1}.$$

PROOF. Since $\overline{C}(\mathbf{1/2}) = \Pr[\boldsymbol{U} > \mathbf{1/2}] = \Pr[\mathbf{1} - \boldsymbol{U} \leq \mathbf{1/2}] = \hat{C}(\mathbf{1/2})$, we have the following chain of equalities:

$$\begin{aligned}
\beta_{n,C} &= \frac{2^{n-1}\Pr[\boldsymbol{U} < \mathbf{1/2} \text{ or } \boldsymbol{U} > \mathbf{1/2}] - 1}{2^{n-1} - 1} = \frac{2^{n-1}(\Pr[\boldsymbol{U} < \mathbf{1/2}] + \Pr[\boldsymbol{U} > \mathbf{1/2}]) - 1}{2^{n-1} - 1} \\
&= \frac{2^{n-1}[C(\mathbf{1/2}) + \overline{C}(\mathbf{1/2})] - 1}{2^{n-1} - 1} = \frac{2^{n-1}[C(\mathbf{1/2}) + \hat{C}(\mathbf{1/2})] - 1}{2^{n-1} - 1},
\end{aligned}$$

which completes the proof.

Note that the upper bound for $\beta_{n,C}$ is 1 (which, for instance, can be attained when $C = M^n$). In the following theorem we show that the lower bound $-1/(2^{n-1} - 1)$—in the case that $\Pr[\boldsymbol{X} < \widetilde{\boldsymbol{x}} \text{ or } \boldsymbol{X} > \widetilde{\boldsymbol{x}}] = 0$—is best-possible.

THEOREM 3.2. *Let $\boldsymbol{U} = (U_1, U_2, \dots, U_n)$ be a vector of uniform* $[0,1]$ *random variables with n-copula C. If at least one of the* 2*-margins of C is W^2 then $\beta_{n,C} = -1/(2^{n-1} - 1)$.*

PROOF. Let $i, j \in \{1, 2, \dots, n\}$ such that $i \neq j$, and suppose that the 2-copula associated with the pair of random variables (U_i, U_j) is W^2. Then $U_i = 1 - U_j$; so that if $U_i > 1/2$ then $U_j < 1/2$ or else if $U_i < 1/2$ then $U_j > 1/2$, and hence $\Pr[\boldsymbol{U} < \mathbf{1/2}$ or $\boldsymbol{U} > \mathbf{1/2}] = 0$. Therefore $\beta_{n,C} = -1/(2^{n-1} - 1)$, which completes the proof.

Let (X, Y, Z) be a random vector with 3-copula C, and let $\beta_{X,Y}$, $\beta_{X,Z}$ and $\beta_{Y,Z}$ denote the Blomqvist's beta of the three bivariate margins of C. The following result shows that $\beta_{3,C}$ coincides with the average pairwise Blomqvist's beta.

THEOREM 3.3. *Let (X, Y, Z) be a vector of uniform* $[0,1]$ *random variables with* 3*-copula C. Then $\beta_{3,C} = (\beta_{X,Y} + \beta_{X,Z} + \beta_{Y,Z})/3$.*

PROOF. Since we have $\hat{C}(1/2, 1/2, 1/2) = 3/2 - 2 + C(1, 1/2, 1/2) + C(1/2, 1, 1/2) + C(1/2, 1/2, 1) - C(1/2, 1/2, 1/2)$, then $\beta_{3,C} = (4[C(1/2, 1/2, 1/2) + \hat{C}(1/2, 1/2, 1/2)] -$

$1)/3 = (4/3)[C(1,1/2,1/2)+C(1/2,1,1/2)+C(1/2,1/2,1)]-1 = (\beta_{X,Y}+\beta_{X,Z}+\beta_{Y,Z})/3$, as required.

Example 1. Let C be the n-copula given by

$$C(\boldsymbol{u}) = W^2(u_1,u_2)u_3\cdots u_n, \qquad \boldsymbol{u}=(u_1,u_2,\ldots,u_n)\in[0,1]^n. \tag{3.2}$$

C is an n-copula such that only one bivariate margin is W^2 and the rest Π^2. From Theorem 3.2, we have that $\beta_{n,C} = -1/(2^{n-1}-1)$; however, $\beta_{av,C} = -2/[n(n-1)]$, and, by induction, it is easy to prove that $\beta_{av,C} < \beta_{n,C}$ for all $n \geq 4$.

In some sense, the version $\beta_{n,C}$ 'can improve' to that of $\beta_{av,C}$, as the following example shows.

Example 2. Let C_λ be the n-copula given by

$$C_\lambda(\boldsymbol{u}) = \left(\prod_{i=1}^n u_i\right)\left[1+\lambda\prod_{i=1}^n(1-u_i)\right], \qquad \boldsymbol{u}=(u_1,u_2,\ldots,u_n)\in[0,1]^n, \tag{3.3}$$

with λ in $[0,1]$. C_λ belongs to the Farlie-Gumbel-Morgenstern family of n-copulas (see Nelsen (1999) for more details). Since $\hat{C}_\lambda(\boldsymbol{u}) = (\prod_{i=1}^n u_i)[1+(-1)^n\lambda\prod_{i=1}^n(1-u_i)]$, $\boldsymbol{u}\in[0,1]^n$, then $\beta_{n,C_\lambda} = \lambda(1+(-1)^n)/[2^n(2^{n-1}-1)]$, and $\beta_{av,C_\lambda} = 0$. Observe that all the bivariate margins of C_λ are Π^2, however $\beta_{n,C_\lambda} > 0$ for all even natural number $n \geq 4$. Note also that $\tau_{n,C_\lambda} = \lambda(1+(-1)^n)/[3^n(2^{n-1}-1)]$ and $\rho_{n,C_\lambda} = \lambda(n+1)(1+(-1)^n)/(2\cdot 3^n[2^n-(n+1)])$, while that $\tau_{av,C_\lambda} = \rho_{av,C_\lambda} = 0$.

Nelsen (2002) defines a multivariate version of the medial correlation coefficient in the following manner: $\beta'_{n,C} = (2^nC(\mathbf{1/2})-1)/(2^{n-1}-1)$ for any n-copula C. We note that $\beta'_{n,C}$ is a 'particular case' of $\beta_{n,C}$. For instance, suppose that the distribution function of a random vector $\boldsymbol{X}$ is *radially symmetric*, i.e., for any vector $\boldsymbol{U}$ of uniform $[0,1]$ random variables, we have that $\Pr[\boldsymbol{U}\leq\boldsymbol{u}] = \Pr[\boldsymbol{U} > \mathbf{1}-\boldsymbol{u}]$, $\boldsymbol{u}\in[0,1]^n$ (see Nelsen (1993) for details), or $C(\boldsymbol{u}) = \hat{C}(\boldsymbol{u})$ where C is the n-copula associated with $\boldsymbol{U}$; whence $\beta'_{n,C} = \beta_{n,C}$.

Recently, it has been proved that the pointwise best-possible bounds on the set of 2-copulas and a given value of Blomqvist's beta are 2-copulas (see Nelsen and Úbeda-Flores (2004) for more details). It has been also shown that the best-possible bounds on the set of n-copulas C such that $C(\mathbf{1/2}) = \theta$ (for appropriate θ in $[0,1]$) are not n-copulas: see Rodríguez-Lallena and Úbeda-Flores (2004). This suggests that we can not generalize in a same manner the study of the best-possible bounds on sets of n-copulas when the value of the multivariate version of Blomqvist's beta is known, as can be done in the bivariate case.

4. A multivariate version of Spearman's footrule

We now define a multivariate analog population version of the Spearman's footrule based on the probability of concordance. This multivariate version will be denoted as $\varphi_{\boldsymbol{X}}$ (or $\varphi_{n,C}$, where C is the n-copula associated with a vector $\boldsymbol{X}$). If we require that this version should be of the form $\varphi_{n,C} = aQ_n(C,M^n)+b$, with $a,b\in\mathbb{R}$, and such that $\varphi_{n,\Pi^n} = 0$ and $\varphi_{n,M^n} = 1$, as in the bivariate case, then we have to define $\varphi_{n,C}$ as follows:

$$\varphi_{\boldsymbol{X}} = \varphi_{n,C} = \frac{(n+1)(2^{n-1}-1)}{2^{n-1}(n-1)} Q_n(C, M^n) - \frac{2^n-(n+1)}{2^{n-1}(n-1)}. \tag{4.1}$$

We provide the expression for the measure defined by (4.1) in terms of the diagonal sections of C and $\hat{C}$, but first we need a preliminary lemma.

LEMMA 4.1. *Let C be an n-copula. Then $\int_{[0,1]^n} M^n(\boldsymbol{u})\mathrm{d}C(\boldsymbol{u}) = \int_0^1 \delta_{\hat{C}}(t)\mathrm{d}t$ and $\int_{[0,1]^n} C(\boldsymbol{u})\mathrm{d}M^n(\boldsymbol{u}) = \int_0^1 \delta_C(t)\mathrm{d}t$.*

PROOF. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two vectors of uniform $[0,1]$ random variables with respective n-copulas C and M^n. Then

$$\begin{aligned}
\int_{[0,1]^n} M^n(\boldsymbol{u})\mathrm{d}C(\boldsymbol{u}) &= E[M^n(\boldsymbol{X})] = \int_0^1 t\mathrm{d}\Pr[M^n(\boldsymbol{X}) \le t](t) \\
&= 1 - \int_0^1 \Pr[M^n(\boldsymbol{X}) \le t]\mathrm{d}t \\
&= \int_0^1 \Pr[\boldsymbol{X} > \boldsymbol{t}]\mathrm{d}t = \int_0^1 \delta_{\hat{C}}(1-t)\mathrm{d}t = \int_0^1 \delta_{\hat{C}}(t)\mathrm{d}t.
\end{aligned}$$

On the other hand, and using a similar argument, we have

$$\begin{aligned}
\int_{[0,1]^n} C(\boldsymbol{u})\mathrm{d}M^n(\boldsymbol{u}) &= 1 - \int_0^1 \Pr[C(\boldsymbol{Y}) \le t]\mathrm{d}t = 1 - \int_0^1 \Pr[\delta_C(Y_1) \le t]\mathrm{d}t \\
&= 1 - \int_0^1 \Pr[Y_1 \le \delta_C^{(-1)}(t)]\mathrm{d}t = 1 - \int_0^1 \delta_C^{(-1)}(t)\mathrm{d}t = \int_0^1 \delta_C(t)\mathrm{d}t,
\end{aligned}$$

which completes the proof.

THEOREM 4.1. *Let C be an n-copula and let $\varphi_{n,C}$ be the Spearman's footrule coefficient defined by* (4.1). *Then*

$$\varphi_{n,C} = \frac{n+1}{n-1}\int_0^1 [\delta_C(t) + \delta_{\hat{C}}(t)]\mathrm{d}t - \frac{2}{n-1}. \tag{4.2}$$

PROOF. From expressions (2.1) and (4.1), and using Lemma 4.1, we have the following chain of equalities:

$$\begin{aligned}
\varphi_{n,C} &= \frac{(n+1)(2^{n-1}-1)}{2^{n-1}(n-1)}\left(\frac{2^{n-1}Q_n'(C,M^n)-1}{2^{n-1}-1}\right) - \frac{2^n-(n+1)}{2^{n-1}(n-1)} \\
&= \frac{n+1}{2^{n-1}(n-1)}\left[2^{n-1}\left(\int_{[0,1]^n} C(\boldsymbol{u})\mathrm{d}M^n(\boldsymbol{u}) + \int_{[0,1]^n} M^n(\boldsymbol{u})\mathrm{d}C(\boldsymbol{u})\right) - 1\right] \\
&\quad - \frac{2^n-(n+1)}{2^{n-1}(n-1)} \\
&= \frac{n+1}{n-1}\int_0^1 [\delta_C(t) + \delta_{\hat{C}}(t)]\mathrm{d}t - \frac{2}{n-1},
\end{aligned}$$

as claimed.

Observe that for a pair of continuous random variables with 2-copula C we have $\varphi_{2,C} = 6\int_0^1 \delta_C(t)\mathrm{d}t - 2$, since $\delta_{\hat{C}}(t) = 2t - 1 + \delta_C(1-t)$ for each t in $[0,1]$.

The upper bound for $\varphi_{n,C}$ is 1 (which is attained only in the case $C = M^n$). In the next result we provide a lower bound for $\varphi_{n,C}$.

THEOREM 4.2. *Let C be an n-copula. Then $\varphi_{n,C} \geq -1/n$.*

PROOF. Since $W^n \leq C$ for any n-copula C, we have that $\delta_C(t) \geq \delta_{W^n}(t) = \max(nt - n + 1, 0)$ for all t in $[0,1]$. Since $\hat{C}$ is also an n-copula, from (4.2) we obtain that

$$\varphi_{n,C} \geq \frac{2(n+1)}{n-1}\int_0^1 \max(nt - n + 1, 0)\mathrm{d}t - \frac{2}{n-1} = -\frac{1}{n},$$

which completes the proof.

Whereas the lower bound in Theorem 4.2 is best-possible when $n = 2$ (for instance, when $C = W^2$), the bound may well fail to be best possible for $n \geq 3$ since the Fréchet lower bound is not a distribution function for these cases.

We now show that the coefficient $\varphi_{3,C}$ for a vector (X,Y,Z) of continuous random variables with 3-copula C can be written as the arithmetic mean of the Spearman's footrule of the three bivariate margins (φ_{XY}, φ_{XZ} and φ_{YZ}) of C.

THEOREM 4.3. *Let (X,Y,Z) be a vector of uniform $[0,1]$ random variables with 3-copula C. Then $\varphi_{3,C} = (\varphi_{XY} + \varphi_{XZ} + \varphi_{YZ})/3$.*

PROOF. Since $\delta_{\hat{C}}(t) = 3t - 2 + C(1-t, 1-t, 1) + C(1-t, 1, 1-t) + C(1, 1-t, 1-t) - \delta_C(t)$, from (4.2) it is easy to obtain that $\varphi_{3,C} = 2\int_0^1 [C(t,t,1) + C(t,1,t) + C(1,t,t)]\mathrm{d}t - 2$. On the other hand, $\varphi_{XY} = 6\int_0^1 C(t,t,1)\mathrm{d}t - 2$, $\varphi_{XZ} = 6\int_0^1 C(t,1,t)\mathrm{d}t - 2$ and $\varphi_{YZ} = 6\int_0^1 C(1,t,t)\mathrm{d}t - 2$, whence the result follows.

Example 3. Consider the n-copula C given by (3.2). Then, we have that $\varphi_{av,C} = -1/[n(n-1)]$. On the other hand, since $\delta_{\hat{C}}(t) = \delta_C(t) = \max(2t-1, 0)t^{n-2}$, $t \in [0,1]$, from (4.2) we obtain that $\varphi_{n,C} = (n + 1 - 2^n)/(2^{n-2}n(n-1)^2)$. By induction, it is easy to prove that $\varphi_{av,C} < \varphi_{n,C}$ for all $n \geq 4$.

Example 4. Let C_λ be the n-copula given by (3.3). Then, after some calculations, we have that $\varphi_{n,C_\lambda} = [(1 + (-1)^n)\lambda(n+1)(n!)^2]/[(n-1)(2n+1)!]$. Observe also that $\varphi_{av,C_\lambda} = 0$.

Let $m \geq 2$ be a natural number. If $X_i = (X_{i1}, X_{i2}, \ldots, X_{im})$, $i = 1, 2, \ldots, n$, is a random sample of size n from a continuous distribution function, then

$$f_{m,S} = 1 - \frac{m+1}{m-1} \cdot \frac{\sum_{i=1}^n R_i}{n^2 - 1}$$

is the sample version of (4.2), where R_i is the range (maximum minus minimum) of the ranks of the variables in the i-th observation. Note that, unlike the population version, if, for example, the rankings are identical for all variables except reversed for one variable, then $f_{m,S}$ will be near $-.5$ while the pairwise average will be near zero.

5. A comparison of lower bounds among measures of multivariate association

Nelsen (1996) shows that a lower bound for $\tau_{n,C}$ is $-1/(2^{n-1}-1)$. The next result proves that this bound is also best-possible for $\tau_{n,C}$.

THEOREM 5.1. *Let $\boldsymbol{U} = (U_1, U_2, \ldots, U_n)$ be a vector of uniform $[0,1]$ random variables with n-copula C. If at least one of the 2-margins of C is W^2 then $\tau_{n,C} = -1/(2^{n-1}-1)$.*

PROOF. Let $(u_1, u_2, \ldots, u_n)$ and $(u'_1, u'_2, \ldots, u'_n)$ be two observations of $\boldsymbol{U}$. Let $i, j \in \{1, 2, \ldots, n\}$ such that $i \neq j$, and suppose that the 2-copula associated with the pair of random variables (U_i, U_j) is W^2. Then the pairs (u_i, u_j) and (u'_i, u'_j) satisfy that $u_i < u'_i$ and $u_j > u'_j$, or $u_i > u'_i$ and $u_j < u'_j$; so that $2\int_{[0,1]^n} C(\boldsymbol{u})\mathrm{d}C(\boldsymbol{u}) = \Pr[\boldsymbol{X} < \boldsymbol{Y}$ or $\boldsymbol{X} > \boldsymbol{Y}] = 0$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are two independent random vectors each with distribution function H; and hence $\tau_{n,C} = -1/(2^{n-1}-1)$.

As Nelsen (1996) points out, a lower bound for $\rho_{n,C}$ is $(2^n-(n+1)!)/(n![2^n-(n+1)])$, but this bound may well fail to be best-possible. We have a similar situation with the lower bound for $\varphi_{n,C}$—recall Theorem 4.2. The following example shows that we probably do not have a similar result for $\rho_{n,C}$ and $\varphi_{n,C}$ to those of Theorems 3.2 and 5.1 for $\beta_{n,C}$ and $\tau_{n,C}$, respectively.

Example 5. Consider the n-copula C given by (3.2). Then, after some algebra, we have that $\rho_{n,C} = -(n+1)/[3(2^n-n-1)]$. Note that this value is greater than the bound given by Nelsen for all $n \geq 4$. We have a similar situation with the value for $\varphi_{n,C}$ (given in Example 3) and the bound given in Theorem 4.2.

It is still an open problem to know the best-possible lower bounds for $\rho_{n,C}$ and $\varphi_{n,C}$.

Acknowledgements

This work was partially supported by the Ministerio de Ciencia y Tecnología (Spain) and FEDER, under research project BFM2003-06522. The author wishes to thank Roger B. Nelsen and the Department of Mathematical Sciences of the Lewis & Clark College (Portland, OR) where this work was carried out in part, and two anonymous referees for their helpful comments.

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