Ann. Inst. Statist. Math.
Vol. 57, No. 3, 531–552 (2005)

COMPARISON OF THE CUSCORE, GLRT AND CUSUM CONTROL CHARTS FOR DETECTING A DYNAMIC MEAN CHANGE

DONG HAN[1] AND FUGEE TSUNG[2]

[1] *Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200030, China*
[2] *Department of Industrial Engineering and Engineering Management, Hong Kong University of Science and Technology, Kowloon, Hong Kong, China*

(Received March 11, 2004; revised August 16, 2004)

Abstract. Although statistical process control (SPC) techniques have been focused mostly on detecting constant mean shifts, dynamic and time-varying process changes frequently occur in the monitoring of feedback-controlled and autocorrelated processes. In this research, the performances of cumulative score (Cuscore), generalized likelihood ratio test (GLRT), and cumulative sum (CUSUM) charts in detecting a dynamic mean change that finally approaches a steady-state value are compared. Theoretical results in average run length (ARL) comparison are provided. From the theoretical study we find that, when the steady-state value is greater or less than a critical value, $R\delta/2 + \delta/2$, the Cuscore and CUSUM charts have a different performance in detecting the mean change. We prove also that the GLRT has the best performance among the three charts in detecting any mean change for which the steady-state value is not equal to δ or δR, when the in-control ARL is large.

Key words and phrases: Statistical process control, change point detection, average run length.

1. Introduction

The importance of statistical process control (SPC) techniques in quality improvement is well recognized in industry. Currently most competitive manufacturing companies are implementing SPC in various applications. Although SPC techniques are popular, the current methods have been focused mostly on monitoring and detection of constant shifts in the mean. SPC methods for detecting dynamic mean changes, i.e., non-constant time-varying shifts in the mean, have not been thoroughly studied. The detection of dynamic mean changes is particularly important in monitoring autocorrelated or feedback-controlled processes where dynamic patterns in the mean shifts are usually observed. The main purpose of this paper is to present some theoretical results on the performance of conventional monitoring and detection methods under the situation with a dynamic mean change that approaches a stable value (i.e., a steady state). Two motivated examples on autocorrelated processes and feedback-controlled processes are illustrated in Section 2. In Section 3, the cumulative score (Cuscore), generalized likelihood ratio test (GLRT), and cumulative sum (CUSUM) charts, with corresponding notation are briefly outlined. The estimations and comparisons of the average run lengths (ARLs) of the three tests in detecting dynamic mean changes when the in-control ARL is large, are given in Section 4. Section 5 contains some numerical simulation results which illustrate the detection performance of the three tests. The proofs of the theorems in this paper are shown in Section 6. Conclusions and problems for further study are discussed in the last section.

2. Motivating examples

2.1 *Processes with feedback control*

Many sophisticated manufacturing processes are equipped with feedback control for short-term variation reduction. However, as discussed in Box and Luceño (1997), for long-term process improvement, SPC techniques are still needed to detect the out-of-control problems related to assignable causes. Currently most SPC charting techniques treat the feedback-controlled process essentially as a black box. They are applied primarily to the process output after feedback control and are often ineffective as the dynamic information contained in the control schemes is ignored. Keats *et al.* (1996) gave a comprehensive illustration of monitoring output deviation from target for three commonly used feedback controllers: Proportional-Integral-Derivative (PID), Proportional-Integral (PI), and Integral (I) controllers. Tsung and Tsui (2003) further indicated that although the mean shift in the original process is constant, the actual mean shift in the output of the feedback-controlled process changes over time depending on the feedback control compensation.

Here we use a throttle position sensor (TPS) assembly process as an example that was used to demonstrate the applicability and efficiency of a monitoring strategy by Tsung *et al.* (1999). The TPS is a potentiometer, mounted on the throttle body of a vehicle, which detects the opening of the throttle plate and sends this information to the power train control module. The process outputs are the rotor-end play readings (i.e., the distances between the potentiometer arms and the substrates) collected from a TPS assembly process of an automotive supplier. The variability of the rotor-end play may cause mal-function of the TPS. For example, large rotor-end play will cause open circuit conditions, while small rotor-end play will cause "sticking" of the TPS when the potentiometer arm inside comes in contact with the underlying substrate. Thus, to reduce process variation, it is necessary to control the rotor-end play by adjusting the screw heights of the press via feedback control such as MMSE. It is also important to monitor the feedback controlled process using SPC for long-term process improvement.

The output of the MMSE feedback-controlled process, i.e, the rotor-end play readings in this case, can be represented as

$$e_k = a_k + \mu \widetilde{f}_k,$$

where a_k's are independent and identically distributed (i.i.d.) normal variables with mean 0 and variance 1, μ is the magnitude of the mean shift fault of the process and $\widetilde{f}_k$ is the fault signature of the f_k, where f_k is magnitude of the mean shift fault occurring in the original data. It should be noted that if a step mean shift fault occurs in the original process, the resulting mean shift in the output will not be a step function but a time-dependent function since the form of the fault signature $\widetilde{f}_k$ depends heavily on the process dynamics, the feedback control scheme, and the parameters of the disturbance model. The research results in this paper will provide some guidelines in the efficient use of control charts for dynamic mean shifts in feedback-controlled processes.

2.2 *Monitoring of autocorrelated processes*

As automated sampling technology develops and high volume production processes become more common, the need to monitor autocorrelated process data will increase. To deal with autocorrelation in data, different approaches have been proposed in the literature. See, for example, Vasilopoulos and Stamboulis (1978), Alwan and Roberts (1988), Montgomery and Mastrangelo (1991), and Yashchin (1993). One increasingly popular scheme proposed by Alwan and Roberts (1988) is to model the autocorrelated processes using time series models. If the model is adequate, the model residuals (specifically, one-step-ahead prediction errors) are approximately uncorrelated, and therefore, conventional control charts can be applied to the residuals.

A brief review how to generate the uncorrelated residuals was given in Apley and Shi (1999). The residuals can also be represented as

$$e_k = a_k + \mu \widetilde{f}_k,$$

which is the same as the MMSE controlled process output. If a constant mean shift occurs in the original process, the resulting mean shift in the residuals will not be a step function but a dynamic function where the form of the fault signature $\widetilde{f}_k$ depends on the time series model and its parameters (see Wardell *et al.* (1992)).

Hu and Roan (1996) investigated the fault signature for the residuals of different first-order autoregressive-moving-average (ARMA$(1,1)$) processes. In their paper, three zones in the ARMA$(1,1)$ parameter space are determined by the stability conditions, the value of the first transient shift and steady state shift in each zone. In zone 1, the process outputs consist of a steady-state mean shift of increased magnitude with several steps delay. In zone 2, the process outputs consist of a spike followed by a steady state shift of a smaller magnitude. In zone 3, the process outputs are decaying, oscillating and reaching a steady-state value. The run length performance of Shewhart charts in these three zones has been studied in detail (Tsung and Tsui (2003)). In order to make use of the fault signature information to detect the mean shifts, the LRT, GLRT and Cuscore charts have been developed.

Using the likelihood ratio statistic, Siegmund and Venkatraman (1995) proposed a CUSUM-like control chart called LRT (the likelihood ratio test) which does not depend on a reference value δ which, for the CUSUM chart, is the magnitude of a shift in the process mean to be detected quickly. Their simulation results show that the LRT is better than the CUSUM control chart in detecting a mean shift which is larger or smaller than δ and is only slightly inferior in detecting mean shift of size δ. Apley and Shi (1994, 1999) developed the GLRT (the generalized likelihood ratio test) for monitoring autocorrelated process and found that the GLRT performance is far superior to either a CUSUM or a Shewhart chart on the residuals for various models. The Cuscore chart, which is based on the principle of Fisher's efficient score statistic (1925), was further developed by Bagshaw and Johnson (1977), Box and Ramírez (1992), Box and Luceño (1997), Ramírez (1998), Luceño (1999), and Shu *et al.* (2002).

As can be seen in the literature, most work focus on a study of the performance of various control charts in detecting constant mean shifts, and the study is mainly based on the numerical simulation of the average run length (ARL). Though the theoretical approximations of ARLs of the LRT and CUSUM chart in detecting a constant mean shift have been done by Siegmund and Venkatraman (1995) and Wu (1994), their methods are not efficient in estimating the ARLs of the Cuscore, GLGT and CUSUM tests in detecting a dynamic mean change $\{\mu_i\}$. In this paper, we shall not only present a different approach to estimate the ARLs of the three tests but also compare the performance of the three test both in theoretical estimation and numerical simulation in detecting a dynamic mean change which finally goes to a steady-state value.

3. The Cuscore, GLRT and CUSUM tests

The controlled outputs or the residuals can be seen as a general model. Let X_i $(i = 1, 2, \ldots)$ be the i-th observation on an i.i.d. process and its distribution be a normal with $N(\mu_i, \sigma^2)$. Suppose that at some time period τ, which is usually called a change point, the distribution of X_i changes from $N(\mu_0, \sigma^2)$ to $N(\mu_{\tau+k}, \sigma^2)$ $(k = 0, 1, 2, \ldots)$, in other words, from time period τ onwards the mean of X_i undergoes a change of size $\mu_i - \mu_0$, where we assume that μ_0 and σ are known and without loss of generality $\mu_0 = 0$ and $\sigma = 1$. Let $\delta > 0$ be a reference value which is related to the mean shift magnitude of particular interest. The first time (stopping time) outside the control limit $c > 0$ for the one-sided standard CUSUM chart is of the form:

$$T_C(c) = \inf\left\{n \geq 1 : \max_{1\leq k\leq n}\left[\sum_{i=n-k+1}^{n}(X_i - \delta/2)\right] \geq c\right\}.$$

The stopping time for the one-sided Cuscore chart is defined as follows (see Luceño (1999)):

$$T_{SO}(c) = \inf\left\{n \geq 1 : \max_{1\leq k\leq n}\left[\sum_{i=n-k+1}^{n} r_i(X_i - \delta r_i/2)\right] \geq c\right\},$$

where r_i is the value of a known signal which may badly disturb the process mean. The $\{r_i\}$ can also be called as reference pattern of the mean change. Obviously, $T_C(c) = T_{SO}(c)$ when $r_i \equiv 1$. That is, the CUSUM chart is the special example of the Cuscore chart. Note that the reference pattern $\{r_i\}$ is usually taken as the fault signature $\{\widetilde{f}_i\}$ in the residual-based charts (see Luceño (1999)).

Motivated by the works done by Siegmund and Venkatraman (1995) and Apley and Shi (1999), we consider the stopping time for the one-sided GLRT as follows:

$$T_{GL}(c) = \inf\left\{n \geq 1 : \max_{1\leq k\leq n}\left[\left(\sum_{i=n-k+1}^{n} r_i^2\right)^{-1/2}\sum_{i=n-k+1}^{n} r_i X_i\right] \geq c\right\},$$

where the reference pattern $\{r_i\}$ depends on the change point time τ, having the form

$$r_i = \begin{cases} 1 & \text{at} \quad i < \tau \\ \widetilde{r}_{i-\tau+1} & \text{at} \quad i \geq \tau. \end{cases}$$

Note that the GLRT here is slightly different from that one given by Apley and Shi (1999). It may be said that Apley and Shi's model stresses on detecting the "beginning" (or "past") change pattern, i.e. $r_1, r_2, \ldots$, but our version emphasizes the "end" (or "current") change pattern, i.e. $\ldots, r_{n-1}, r_n$. When $r_i \equiv 1$, the GLRT is the same as the LRT control chart proposed and studied by Siegmund and Venkatraman (1995). The LRT chart can be written as

$$T_L(c) = \inf\left\{n \geq 1 : \max_{1\leq k\leq n}\left[k^{-1/2}\sum_{i=n-k+1}^{n} X_i\right] \geq c\right\}.$$

To compare the performance of the three charts in detecting a dynamic mean change $\{\mu_i\}$, we need some of corresponding notation. Let $P(\cdot)$ and $E(\cdot)$ denote the probability and expectation operators when there is no change. Denote $P_{\tau\mu_i}(\cdot)$ and $E_{\tau\mu_i}(\cdot)$ as the probability and expectation when the change point is at τ and the mean change value is $\{\mu_i\}$. When $\mu_i \equiv \mu$, μ is usually called the mean shift value. The two most commonly used operating measures in SPC are the in-control average run length (ARL_0) and the out-of-control average run length (ARL_μ), defined by

$$ARL_0(T) = E(T), \qquad ARL_{\tau\mu_i}(T) = E_{\tau\mu_i}(T).$$

Here T is a stopping time (or the alarming time) outside a control limit with a detecting procedure. When $\mu_i \equiv \mu$ and $\tau = 1$, we denote $E_{\tau\mu_i}(T)$ by $ARL_\mu(T)$.

Usually, comparisons of control chart performance are made by designing the charts to have a common ARL_0 and then comparing the $ARL_{\tau\mu_i}$'s of the control charts for a given change μ_i and a change point τ. The chart with the smaller $ARL_{\tau\mu_i}$ is considered to have better performance.

It often occurs in practice that the mean change may finally approach a stable value. Thus, we assume that a mean change $\{\mu_i\}$ and a reference pattern $\{r_i\}$ considered in this paper satisfy that there exist two positive numbers μ and R such that

$$\mu = \lim_{i\to\infty} \mu_i, \qquad \sum_{i=1}^{\infty} |r_i - R| < \infty. \tag{3.1}$$

By (3.1) we know that $\sum_{i=k}^{\infty} |r_i - R| \to 0$ as $k \to \infty$. The two number μ and R can be called, respectively, a steady-stable value of the mean change and steady-stable value of the reference pattern. Two examples for $\{r_i\}$ are given in the following (see Hu and Roan (1996) and Box and Luceño (1997)).

Example 1. $r_i = R + b(\frac{1}{2})^{i-1}$, where $R > 0$, and b are two constants.

Example 2. $r_i = R + b[\sin(i\theta)](\frac{3}{4})^{i-1}$, where $R > 0$, b and θ are three constants.

4. Comparison of the Cuscore, GLRT and CUSUM tests

To compare the performance of the Cuscore, GLRT and CUSUM tests in detecting a dynamic mean change we shall first give an approximation of $ARL_{\tau\mu_i}$s of the three tests under a condition that the ARL_0 is large enough. The reason to assume that $ARL_0 \to \infty$, is not only for us to obtain the theoretical approximations of $ARL_{\tau\mu_i}$s but also is due to a consideration of practicality. In fact, if a false alarm obtained in detecting the mean changes can make a big loss in practice, an alternative approach is to take ARL_0 large to avoid or reduce the loss. Obviously, $ARL_0 \to \infty$ means the corresponding control limit $c \to \infty$.

Denote by $\Phi(\cdot)$ the distribution function of the standard normal distribution. Let $K = \int_0^\infty x\psi^2(x)dx$, where $\psi(x) = 2x^{-2}\exp\{-2\sum_{n=1}^\infty \Phi(-x\sqrt{n}/2)/n\}$. We give now the approximation of ARL_0s and $ARL_{\tau\mu_i}$s for the Cuscore, GLRT and CUSUM charts in the following two theorems.

THEOREM 4.1. *For any small ϵ $(0 < \epsilon < R/2)$, there are three positive constants M_1, M_2 and L such that*

$$M_1 \frac{e^{\delta c R_1/R_2}}{c} \le ARL_0(T_{SO}) \le M_2(c)^{3/2} e^{\delta c R_2/R_1}, \tag{4.1}$$

$$M_1 \frac{e^{\delta c}}{c} \le ARL_0(T_C) \le M_2(c)^{3/2} e^{\delta c} \tag{4.2}$$

and

$$\sqrt{2\pi} \frac{e^{[c(1-2\epsilon/R)]^2/2}}{Kc} \le ARL_0(T_{GL}) \le Lce^{c^2/2} \tag{4.3}$$

for large c, where $R_1 = R - \epsilon > 0$, $R_2 = R + \epsilon$, $M_1 < 1$, $M_2 > 2d\sqrt{\pi \delta R_2/R_1}$ and $L > \sqrt{2\pi}$.

THEOREM 4.2. *For any small ϵ and large c*

$$M_1 \frac{e^{\delta c R_1'/R_2}}{c} \le ARL_{\tau\mu_i}(T_{SO}) \le M_2'(c)^{3/2} e^{\delta c R_2'/R_1} \tag{4.4}$$

for $0 < \mu < \delta R/2$,

$$M_1 \frac{e^{\delta c(1-2\mu/\delta-\epsilon)}}{c} \le ARL_{\tau\mu_i}(T_C) \le M_2'(c)^{3/2} e^{\delta c(1-2\mu/\delta-\epsilon)} \tag{4.5}$$

for $0 < \mu < \delta/2$,

$$ARL_{\tau\mu_i}(T_{SO}) \sim \frac{c}{(\mu - \delta R/2)R} \tag{4.6}$$

for $\mu > \delta R/2$,

$$ARL_{\tau\mu_i}(T_C) \sim \frac{c}{(\mu - \delta/2)} \tag{4.7}$$

for $\mu > \delta/2$,

$$ARL_{\tau\mu_i}(T_{GL}) \sim \frac{c^2}{\mu^2} \tag{4.8}$$

for $\mu > 0$, where $R_1' = (R - 2\mu/\delta - \epsilon) > 0$, $R_2' = (R - 2\mu/\delta + \epsilon)$, $M_2' > 2d\sqrt{\pi\delta R_2'/R_1}$ and $x \sim y$ means that $x/y \to 1$.

We can see from (4.6), (4.7) and (4.8) that the detecting performance of the GLRT is more robust than that of the Cuscore and CUSUM charts in the sense that the ARLs of the GLRT do not depend on the steady-stable value of the reference pattern or reference value but the Cuscore and CUSUM charts do.

The comparison of the Cuscore, GLRT and CUSUM tests are given in the following theorems.

THEOREM 4.3. *Let $0 < R < 1$. For the Cuscore and CUSUM tests, if $ARL_0(T_{SO}) = ARL_0(T_C) \to \infty$, then*

(a$_1$) *$ARL_{\tau\mu_i}(T_{SO}) < ARL_{\tau\mu_i}(T_C)$ for $\mu < \delta(R+1)/2$, $\mu \neq \delta/2$ and $\mu \neq \delta R/2$;*
(b$_1$) *$ARL_{\tau\mu_i}(T_{SO}) \sim ARL_{\tau\mu_i}(T_C)$ for $\mu = \delta(R+1)/2$;*
(c$_1$) *$ARL_{\tau\mu_i}(T_{SO}) > ARL_{\tau\mu_i}(T_C)$ for $\mu > \delta(R+1)/2$.*

By Theorem 4.3 we know that if $R < 1$, then the performance of the Cuscore chart is better than the CUSUM chart with the small steady-state value μ, and the CUSUM chart is better than the Cuscore chart with the large μ when the ARL_0 is large.

THEOREM 4.4. *Let $R > 1$. For the Cuscore and CUSUM tests, if $ARL_0(T_{SO}) = ARL_0(T_C) \to \infty$, then*
(a$_2$) *$ARL_{\tau\mu_i}(T_{SO}) > ARL_{\tau\mu_i}(T_C)$ for $\mu < \delta(R+1)/2$, $\mu \neq \delta/2$ and $\mu \neq \delta R/2$;*
(b$_2$) *$ARL_{\tau\mu_i}(T_{SO}) \sim ARL_{\tau\mu_i}(T_C)$ for $\mu = \delta(R+1)/2$;*
(c$_2$) *$ARL_{\tau\mu_i}(T_{SO}) < ARL_{\tau\mu_i}(T_C)$ for $\mu > \delta(R+1)/2$.*

It follows from Theorem 4.4 that the performance of the Cuscore and CUSUM charts with $R > 1$ is on the contrary in case of $R < 1$.

THEOREM 4.5. *For the Cuscore, GLRT and CUSUM tests, as $ARL_0(T_{SO}) = ARL_0(T_C) = ARL_0(T_{GL}) \to \infty$,*
(a$_3$) *$ARL_{\tau\mu_i}(T_{SO}) > ARL_{\tau\mu_i}(T_{GL})$ for $\mu \neq \delta R$;*
(b$_3$) *$ARL_{\tau\mu_i}(T_C) > ARL_{\tau\mu_i}(T_{GL})$ for $\mu \neq \delta$;*
(c$_3$) *$ARL_{\tau\mu_i}(T_{SO}) \sim ARL_{\tau\mu_i}(T_{GL})$ for $\mu = \delta R$;*
(d$_3$) *$ARL_{\tau\mu_i}(T_C) \sim ARL_{\tau\mu_i}(T_{GL})$ for $\mu = \delta$.*

Theorem 4.5 tell us that when $ARL_0 \to \infty$, the GLRT has the best performance among the three tests except the steady-state value μ is equal to δ or δR.

COROLLARY 4.1. *As $ARL_0(T_{SO}) = ARL_0(T_C) \to \infty$,*

$$ARL_{\tau\mu_i}(T_{SO}) < ARL_{\tau\mu_i}(T_C)$$

for $\mu = \delta R$ and $R \neq 1$,

$$ARL_{\tau\mu_i}(T_{SO}) > ARL_{\tau\mu_i}(T_C)$$

for $\mu = \delta$ and $R \neq 1$, and

$$ARL_{\tau\mu_i}(T_{SO}) \sim ARL_{\tau\mu_i}(T_C)$$

for $\mu = \delta$ and $R = 1$.

By Corollary 4.1 we see that the CUSUM and Cuscore charts have the best performance if the steady-stable value of a dynamic mean change is equal to δ and δR, respectively.

Remark 1. The values $R\delta/2$ and $\delta/2$ can be seen as two handicaps respectively for the Cuscore and CUSUM charts. It follows from Theorem 4.3 and Theorem 4.4 that the sum $R\delta/2 + \delta/2$ is just a critical value. When the steady-stable value is great or less than the critical value, the Cuscore and CUSUM charts have a different performance in detecting the mean change.

Remark 2. It has been shown by Moustakides (1986) and Ritov (1990) that the performance in detecting the mean shift of the one-sided CUSUM control chart with the reference value δ is optimal if the real mean shift is δ. The similar results for the CUSUM chart with the steady-stable value δ, can be seen in (d_3) of Theorem 4.5.

Remark 3. The above results are only dependent on the three values μ, R and δ, since both patterns of the dynamic mean change $\{\mu_i\}$ and the reference value $\{r_i\}$ has neither influence on the detecting performance of the three tests when the $ARL_0 \to \infty$ or $c \to \infty$.

5. Numerical illustration

In this section we show some simulation results of ARL's of the Cuscore, GLRT and CUSUM charts, that is, the ARL's of the upward stopping times. The numerical results of ARL's were obtained based on 10000-repetition experiment. Tables 1 and 2 illustrate the simulation results in detecting two types of the dynamics mean changes with change point $\tau = 1$. In the last three rows of Tables 1 and 2, r_k is the reference pattern, δ is the reference value and c denotes various values of the control limit. The dynamic mean change $\{\mu p_k\}$ is listed in the first column of Tables 1 and 2, where $\{p_k\}$ denotes the pattern of the mean change, which is taken as $p_k = 1/4 + 3/2(1/2)^k$ in Table 1 and $p_k = 5/4 - 1/4(1/2)^{k-1}$ in Table 2.

As can be seen from a comparison of the simulation results in Tables 1 and 2 that: (1) The simulation results are basically consistent with the theoretical results of Theorems 4.3 and 4.4. In fact, by Theorems 4.3 and 4.4 we know that, when the steady-state value is greater or less than the critical value, $\delta(1 + R)/2$, the Cuscore and CUSUM charts have a contrary performance in detecting the mean change. There are two critical values, $(1 + 1/4)/2 = 0.625$ and $(1 + 5/4)/2 = 1.125$ in the numerical examples since the steady-stable value, R, of the reference pattern, $\{r_i\}$, is equal to 1/4 or 5/4. Note that $p_k = 1/4 + 3/2(1/2)^k \to 1/4$, and for large k, $\mu p_k < 0.625$ for $\mu \le 2$ and $\mu p_k > 0.625$

Table 1. Comparison of ARL's of three control charts with $ARL_0 \approx 870$, ($p_k = \frac{1}{4} + \frac{3}{2}(\frac{1}{2})^k$).

$\{\mu p_k\}$	Cuscore	CUSUM	Cuscore	GLRT	GLRT
0	870	870	870	868	870
$\{0.1p_k\}$	605	711	725	833	836
$\{0.25p_k\}$	361	526	559	679	682
$\{0.5p_k\}$	180	322	367	371	377
$\{0.75p_k\}$	103	202	237	206	214
$\{1p_k\}$	68.9	128	160	126	135
$\{1.25p_k\}$	48.3	84.6	110	81.2	89.2
$\{1.5p_k\}$	35.8	55.9	77.9	54.6	61.9
$\{2p_k\}$	21.6	25.6	40.2	24.2	31.6
$\{3p_k\}$	9.17	6.28	13.5	6.05	7.51
$\{4p_k\}$	4.27	2.62	6.09	1.76	1.94
r_k	$\frac{1}{4} + \frac{3}{2}(\frac{1}{2})^k$		$\frac{5}{4} - \frac{1}{2}(\frac{1}{2})^k$	$\frac{1}{4} + \frac{3}{2}(\frac{1}{2})^k$	$\frac{5}{4} - \frac{1}{2}(\frac{1}{2})^k$
c	3.172	5.624	5, 11	3.675	3.675
δ	1	1	1		

Table 2. Comparison of ARL's of three control charts with $ARL_0 \approx 870$, ($p_k = 5/4 - 1/4(1/2)^{k-1}$).

$\{\mu p_k\}$	Cuscore	CUSUM	Cuscore	GLRT	GLRT
0	870	870	870	871	871
$\{0.1p_k\}$	184	325	368	377	380
$\{0.25p_k\}$	54.5	94.2	113	93.9	96.6
$\{0.5p_k\}$	21.5	24.0	27.8	28.9	29.7
$\{0.75p_k\}$	12.6	11.9	12.3	14.8	14.8
$\{1p_k\}$	8.33	7.84	7.63	9.52	9.30
$\{1.25p_k\}$	5.90	5.89	5.55	6.84	6.54
$\{1.5p_k\}$	4.43	4.75	4.41	5.24	4.98
$\{2p_k\}$	2.85	3.52	3.22	3.47	3.30
$\{3p_k\}$	1.69	2.39	2.21	1.99	1.95
$\{4p_k\}$	1.20	1.96	1.87	1.37	1.37
r_k	$\frac{1}{4}+\frac{3}{2}(\frac{1}{2})^k$		$\frac{5}{4}-\frac{1}{2}(\frac{1}{2})^k$	$\frac{1}{4}+\frac{3}{2}(\frac{1}{2})^k$	$\frac{5}{4}-\frac{1}{2}(\frac{1}{2})^k$
c	3.172	5.624	5.11	3.675	3.675
δ	1	1	1		

for $\mu \geq 3$. We can see that the critical value, 0.625, is just between $\{2p_k\}$ and $\{3p_k\}$ in Table 1 since the performance of the Cuscore chart with $r_k = 1/4 + 3/2(1/2)^k$, $k \geq 1$, is better than the CUSUM chart for $\mu p_k < 0.625$ and $\mu \leq 2$, and bad than the CUSUM chart for $\mu p_k > 0.625$ and $\mu \geq 3$. For $p_k = 5/4 - 1/4(1/2)^{k-1}$ in Table 2, the two critical values, 0.625 and 1.125 are between $\{0.5p_k\}$ and $\{0.75p_k\}$ and $\{0.75p_k\}$ and $\{1p_k\}$, respectively. (2) The performance of the three tests depends not only on the pattern of the mean change but also on the reference value and the reference pattern. (3) The GLRT is more robust than the Cuscore chart since the influence of the reference pattern $\{r_i\}$ on the detecting performance of the GLRT is less than that of the Cuscore chart. (4) The simulation values, $ARL_{\tau\mu_i}$s of the GLRT are not always less than that of the other two charts since the $ARL_0 (\approx 870)$ is not large enough. This shows that the condition, $ARL_0 \to \infty$, is necessary for the results of Theorem 4.5.

6. Proofs of the theorems

PROOF OF THEOREM 4.1. For small $\epsilon > 0$, it follows from (3.1) that there exits a large natural number k_0 such that

$$R_1^2 < \frac{1}{k} \sum_{i=n-k+1}^{n} r_i^2 < R_2^2$$

for all $n \geq k_0$ and $1 \leq k \leq n - k_0$, where $R_1 = R - \epsilon > 0$ and $R_2 = R + \epsilon$. Let

$$A_m = \left\{ \sum_{i=n-k+1}^{n} r_i(X_i - \delta r_i/2) < c, 1 \leq k \leq n, 1 \leq n \leq m \right\}$$

for $m \leq k_0$,

$$B_{m,k_0} = \left\{ \sum_{i=n-k+1}^{n} r_i(X_i - \delta r_i/2) < c, 1 \leq k \leq n - k_0, k_0 < n \leq m \right\},$$

$$B_{m,k_0}(\epsilon) = \left\{ \frac{\sum_{i=n-k+1}^{n} r_i X_i}{[\sum_{i=n-k+1}^{n} r_i^2]^{1/2}} < \frac{c}{R_1\sqrt{k}} + \frac{\delta R_2 \sqrt{k}}{2}, 1 \le k \le n - k_0, k_0 < n \le m \right\},$$

and

$$C_{m,k_0} = \left\{ \sum_{i=n-k+1}^{n} r_i(X_i - \delta r_i/2) < c, n - k_0 + 1 \le k \le n, k_0 < n \le m \right\}$$

for $m > k_0$. Then,

$$\{T_{SO} > m\} = A_{k_0} B_{m,k_0} C_{m,k_0},$$

for $m > k_0$. Note that the set B_{m,k_0} can be rewritten as

$$B_{m,k_0} = \left\{ \frac{\sum_{i=n-k+1}^{n} r_i X_i}{[\sum_{i=n-k+1}^{n} r_i^2]^{1/2}} < \frac{c}{\sqrt{\sum_{i=n-k+1}^{n} r_i^2}} + \frac{\delta \sqrt{\sum_{i=n-k+1}^{n} r_i^2}}{2}, \right.$$
$$\left. 1 \le k \le n - k_0, k_0 < n \le m \right\}.$$

Obviously, $B_{m,k_0}(\epsilon) \supset B_{m,k_0}$. Since the function, $\frac{c}{R_1\sqrt{x}} + \frac{\delta R_2}{2}\sqrt{x}$ $(x > 0)$, attains its minimum value $\sqrt{2\delta c R_2/R_1}$ at $x = 2c/(\delta R_1 R_2)$, it follows that

$$\begin{aligned} P(B_{m,k_0}(\epsilon)) &\le P\left(\frac{\sum_{i=k_0+1}^{m} r_i X_i}{[\sum_{i=k_0+1}^{m} r_i^2]^{1/2}} < \sqrt{2\delta c R_2/R_1} \right) \\ &= \Phi(\sqrt{2\delta c R_2/R_1}) \end{aligned}$$

for $m = k_0 + dc$, where $d = 2/(\delta R_1 R_2)$ and $\Phi(\cdot)$ denotes the distribution function of the standard normal distribution. We further have

$$P(B_{m,k_0}(\epsilon)) \le P(B_{k_0+jdc,k_0}(\epsilon))$$

for $k_0 + jdc \le m < k_0 + (j+1)dc$ and

$$\begin{aligned} P(B_{k_0+jdc,k_0}(\epsilon)) &\le P\left(\frac{\sum_{i=k_0+(l-1)dc+1}^{k_0+ldc} r_i X_i}{[\sum_{i=k_0+(l-1)dc+1}^{k_0+ldc} r_i^2]^{1/2}} < \sqrt{2\delta c R_2/R_1}, 1 \le l \le j \right) \\ &= [\Phi(\sqrt{2\delta c R_2/R_1})]^j, \end{aligned}$$

where the last equality holds since the events

$$\left\{ \frac{\sum_{i=k_0+(l-1)dc+1}^{k_0+ldc} r_i X_i}{[\sum_{i=k_0+(l-1)dc+1}^{k_0+ldc} r_i^2]^{1/2}} \right\},$$

$1 \le l \le j$, are independent mutually. Hence,

$$ARL_0(T_{SO}) = \sum_{m=1}^{k_0} P(A_m) + \sum_{m=k_0+1}^{\infty} P(A_{k_0} B_{m,k_0} C_{m,k_0}) \tag{6.1}$$

$$\begin{aligned}
&\leq \sum_{m=1}^{k_0} P(A_m) + \sum_{m=k_0+1}^{\infty} P(B_{m,k_0}(\epsilon)) \\
&\leq k_0 + dc + \sum_{j=1}^{\infty} \sum_{m=k_0+jdc}^{k_0+(j+1)dc-1} P(B_{m,k_0}(\epsilon)) \\
&\leq k_0 + dc + dc\sum_{j=1}^{\infty}[\Phi(\sqrt{2\delta c R_2/R_1})]^j \\
&= k_0 + dc + dc\frac{\Phi(\sqrt{2\delta c R_2/R_1})}{1-\Phi(\sqrt{2\delta c R_2/R_1})} \\
&\sim k_0 + dc + 2dc\sqrt{\pi\delta c R_2/R_1}e^{\delta c R_2/R_1}
\end{aligned}$$

as $c \to \infty$, since $\Phi(x) \to 1 - e^{-x^2/2}/(\sqrt{2\pi}x)$ as $x \to \infty$. Thus, there exits a constant M_2 such that $M_2 > 2d\sqrt{\pi\delta R_2/R_1}$ and

$$ARL_0(T_{SO}) \leq M_2(c)^{3/2}e^{\delta c R_2/R_1}$$

for large c. This proves the upward inequality of (4.1).

Let

$$B'_{m,k_0}(\epsilon) = \left\{ \frac{\sum_{i=n-k+1}^{n} r_i X_i}{[\sum_{i=n-k+1}^{n} r_i^2]^{1/2}} < \frac{c}{R_2\sqrt{k}} + \frac{\delta R_1\sqrt{k}}{2}, \quad 1 \leq k \leq n-k_0, k_0 < n \leq m \right\}$$

and

$$C_{m,k_0}(\epsilon) = \left\{ \frac{\sum_{i=n-k+1}^{n} r_i X_i}{[\sum_{i=n-k+1}^{n} r_i^2]^{1/2}} < \frac{c}{R_2\sqrt{k}} + \frac{\delta R_1\sqrt{k}}{2}, \quad n-k_0+1 \leq k \leq n, k_0 < n \leq m \right\},$$

for $m > k_0$. Obviously, $B'_{m,k_0}(\epsilon) \subset B_{m,k_0}$ and $C_{m,k_0}(\epsilon) \subset C_{m,k_0}$ for $m > k_0$.

Since the function, $\frac{c}{R_2\sqrt{x}} + \frac{\delta R_1}{2}\sqrt{x}$ $(x > 0)$, attains its minimum value $\sqrt{2\delta c R_1/R_2}$ at $x = 2c/(\delta R_1 R_2)$ and there exists two positive numbers $a < 1$ and $b > 1$ such that

$$\frac{c}{R_2\sqrt{x}} + \frac{\delta R_1}{2}\sqrt{x} \geq 2\sqrt{\delta c R_1/R_2}$$

for $0 < x \leq ac$ and $x \geq bc$, it follows that

$$C_{m,k_0}(\epsilon) \supset \left\{ \left[\sum_{i=n-k+1}^{n} r_i^2\right]^{-1/2} \sum_{i=n-k+1}^{n} r_i X_i < \sqrt{2\delta c R_1/R_2}, n-k_0+1 \leq k \leq n, \quad k_0 < n \leq m \right\}$$

and $B'_{m,k_0}(\epsilon) \supset D_{m,k_0}(\epsilon)E_{m,k_0}(\epsilon)F_{m,k_0}(\epsilon)$ for $m > k_0$, where

$$D_{m,k_0}(\epsilon) = \left\{ \left[\sum_{i=n-k+1}^{n} r_i^2 \right]^{-1/2} \sum_{i=n-k+1}^{n} r_i X_i < 2\sqrt{\delta c R_1/R_2}, \right.$$

$$\left. 1 \le k \le \min\{ac, n\} k_0 < n \le m \right\},$$

$$E_{m,k_0}(\epsilon) = \left\{ \left[\sum_{i=n-k+1}^{n} r_i^2 \right]^{-1/2} \sum_{i=n-k+1}^{n} r_i X_i < \sqrt{2\delta c R_1/R_2}, \right.$$

$$\left. \min\{ac, n\} + 1 \le k \le \min\{bc - 1, n\} k_0 < n \le m \right\},$$

and

$$F_{m,k_0}(\epsilon) \left\{ \left[\sum_{i=n-k+1}^{n} r_i^2 \right]^{-1/2} \sum_{i=n-k+1}^{n} r_i X_i < 2\sqrt{\delta c R_1/R_2}, \right.$$

$$\left. \min\{bc - 1, n\} + 1 \le k \le n, k_0 < n \le m \right\}.$$

Let $M - k_0 = e^{\delta c R_1/R_2}/c$. By Lemma 1 in Lai (1974) we have

$$\begin{aligned}
P(C_{M,k_0}) &\ge P(C_{M,k_0}(\epsilon)) \\
&\ge [\Phi(\sqrt{2\delta c R_1/R_2})]^{k_0(M-k_0)} \\
&\sim \left[1 - \frac{e^{-\delta c R_1/R_2}}{\sqrt{2\pi}\sqrt{2\delta c R_1/R_2}} \right]^{k_0 e^{\delta c R_1/R_2}/c} \\
&\sim \left(1 - o\left(\frac{1}{c\sqrt{c}} \right) \right)
\end{aligned}$$

and

$$\begin{aligned}
P(B_{M,k_0}) &\ge P(B'_{M,k_0}(\epsilon)) \\
&\ge P(D_{M,k_0}(\epsilon))P(E_{M,k_0}(\epsilon))P(F_{M,k_0}(\epsilon)) \\
&\ge [\Phi(2\sqrt{\delta c R_1/R_2})]^{(M-k_0)^2} [\Phi(\sqrt{2\delta c R_1/R_2})]^{(b-a)c(M-k_0)} \\
&\sim \left[1 - \frac{e^{-2\delta c R_1/R_2}}{2\sqrt{2\pi}\sqrt{\delta c R_1/R_2}} \right]^{e^{2\delta c R_1/R_2}/c^2} \left[1 - \frac{e^{-\delta c R_1/R_2}}{2\sqrt{\pi \delta c R_1/R_2}} \right]^{(b-a)e^{\delta c R_1/R_2}} \\
&\sim \left(1 - o\left(\frac{1}{\sqrt{c}} \right) \right)
\end{aligned}$$

as $c \to \infty$. By Lemma 1 in Lai (1974) again we have $P(T_{SO} > m) \ge P(A_{k_0})P(B_{m,k_0})P(C_{m,k_0})$ for $m > k_0$. Obviously, $P(A_m) \to 1$ for $m \le k_0$ as $c \to \infty$.

Thus, there exits a positive constant $M_1 < 1$ such that

$$(6.2) \qquad \begin{aligned} ARL_0(T_{SO}) &\geq \sum_{m=1}^{k_0} P(A_m) + \sum_{m=k_0+1}^{\infty} P(A_{k_0})P(B_{m,k_0})P(C_{m,k_0}) \\ &\geq \sum_{m=1}^{k_0} P(A_m) + (M-k_0)P(A_{k_0})P(B_{M,k_0})P(C_{M,k_0}) \\ &\sim k_0 + e^{\delta c R_1/R_2}/c\left(1 - o\left(\frac{1}{\sqrt{c}}\right)\right) \\ &\geq M_1 e^{\delta c R_1/R_2}/c \end{aligned}$$

for large c. This is the downward inequality of (4.1).

Taking $R_1 = R_2 = 1$ in (4.1) we can obtain (4.2) since $T_C(c) = T_{SO}(c)$ when $r_i \equiv 1$.

Next we estimate $ARL_0(T_{GL})$. Since

$$\begin{aligned} P(T_{GL} > m) &= P\left(\left(\sum_{i=n-k+1}^{n} r_i^2\right)^{-1/2} \sum_{i=n-k+1}^{n} r_i X_i < c, 1 \leq k \leq n, 1 \leq n \leq m\right) \\ &\leq P(X_k < c, 1 \leq k \leq m) = [\Phi(c)]^m \end{aligned}$$

and $1 - \Phi(x) \sim \frac{e^{-x^2/2}}{x}(1 - O(1/x))$ as $x \to \infty$, it follows that

$$ARL_0(T_{GL}) \leq \sum_{m=1}^{\infty} [\Phi(c)]^m \leq Lce^{c^2/2}$$

for large c, where L is a constant such that $L > \sqrt{2\pi}$.

To prove the downward inequality of $ARL_0(T_{GL})$, let

$$G_m = \left\{\left(\sum_{i=n-k+1}^{n} r_i^2\right)^{-1/2} \sum_{i=n-k+1}^{n} r_i X_i < c, 1 \leq k \leq n, 1 \leq n \leq m\right\}$$

for $m \leq k_0$,

$$H_{m,k_0} = \left\{\left(\sum_{i=n-k+1}^{n} r_i^2\right)^{-1/2} \sum_{i=n-k+1}^{n} r_i X_i < c, 1 \leq k \leq n-k_0, k_0 < n \leq m\right\}$$

and

$$I_{m,k_0} = \left\{\left(\sum_{i=n-k+1}^{n} r_i^2\right)^{-1/2} \sum_{i=n-k+1}^{n} r_i X_i < c, n-k_0+1 \leq k \leq n, k_0 < n \leq m\right\}$$

for $m > k_0$. The set H_{m,k_0} can be rewritten by

$$\begin{aligned} H_{m,k_0} = \Bigg\{ &\sum_{i=n-k+1}^{n} X_i/\sqrt{k} < c\left(k^{-1} \sum_{i=n-k+1}^{n} r_i^2\right)^{1/2} /R - Y_{n-k+1}(n)/R\sqrt{k}, \\ &\qquad 1 \leq k \leq n-k_0, k_0 < n \leq m \Bigg\}, \end{aligned}$$

where, $Y_{n-k+1}(n)=\sum_{i=n-k+1}^{n}(r_i-R)X_i$. Let $Y'_k(n)=\sum_{i=k}^{n}|(r_i-R)X_i|$. By (1) we can take the number k_0 such that $\sum_{i=k_0+1}^{m}(r_i-R)^2<\epsilon^2$ for all $m>k_0$ and therefore

$$\begin{aligned}
&P(Y'_{n-k+1}(n)<c\epsilon, 1\le k\le n-k_0, k_0<n\le m)\\
&=P(Y'_{k_0+1}(m)<c\epsilon)>1-\frac{\sum_{i=k_0+1}^{m}(r_i-R)^2}{(c\epsilon)^2}\\
&\sim\left(1-o\left(\frac{1}{c}\right)\right)
\end{aligned}$$

for large c. Let

$$H_{m,k_0}(\epsilon)=\left\{\sum_{i=n-k+1}^{n}X_i/\sqrt{k}<c(1-2\epsilon/R), 1\le k\le n-k_0, k_0<n\le m\right\}.$$

Then

$$P(H_{m,k_0})\ge P(H_{m,k_0}(\epsilon))P(Y_{k_0+1}(m)<c\epsilon)\sim P(H_{m,k_0}(\epsilon))\left(1-o\left(\frac{1}{c}\right)\right).$$

Taking $m-k_0=t\sqrt{2\pi}e^{[c(1-2\epsilon/R)]^2/2}/(c(1-2\epsilon/R)K)$, it follows from Theorem 1 in Siegmund and Venkatraman (1995) that $P(H_{m,k_0}(\epsilon))\ge e^{-t}$ as $c\to\infty$. Note that $P(G_{k_0})\to 1$ and

$$P(I_{m,k_0})\ge[\Phi(c)]^{k_0(m-k_0)}\to 1$$

as $c\to\infty$. Hence,

$$\begin{aligned}
P(T_{GL}>m)&\ge P(G_{k_0})P(H_{m,k_0})P(I_{m,k_0})\\
&\ge\left(1-o\left(\frac{1}{c}\right)\right)P(G_{k_0})P(H_{m,k_0}(\epsilon))[\Phi(c)]^{k_0(m-k_0)}\\
&\ge\left(1-o\left(\frac{1}{c}\right)\right)e^{-t}
\end{aligned}$$

as $c\to\infty$. By using the properties of exponential distribution, we have

$$ARL_0(T_{GL})\ge\sqrt{2\pi}e^{[c(1-2\epsilon/R)]^2/2}/(c(1-2\epsilon/R)K)$$

as $c\to\infty$. This completes the proof of Theorem 4.1.

PROOF OF THEOREM 4.2. Let $\mu<R\delta/2$. For any small fixed $\epsilon>0$ satisfying $R'_1=(R-2\mu/\delta-\epsilon)>0$, we can take a natural number $k_0\ge\tau$ such that

$$R'_1<\frac{\sum_{i=n-k+1}^{n}(r_i^2-2r_i\mu_i/\delta)}{[k\sum_{i=n-k+1}^{n}r_i^2]^{1/2}}<(R-2\mu/\delta+\epsilon)=R'_2$$

holds for $n\ge k_0$ and $1\le k\le n-k_0$. Let

$$\begin{aligned}
B_{m,k_0}(\epsilon,\mu)=\Bigg\{&\frac{\sum_{i=n-k+1}^{n}r_i(X_i-\mu_i)}{[\sum_{i=n-k+1}^{n}r_i^2]^{1/2}}<\frac{c}{R_1\sqrt{k}}+\frac{\delta R'_2\sqrt{k}}{2},\\
&1\le k\le n-k_0, k_0<n\le m\Bigg\}.
\end{aligned}$$

Then $B_{m,k_0} \subset B_{m,k_0}(\epsilon,\mu)$. As (6.1) we can obtain

$$\begin{aligned} ARL_{\tau\mu_i}(T_{SO}) &= \sum_{m=1}^{k_0} P_{\tau\mu_i}(A_m) + \sum_{m=k_0+1}^{\infty} P_{\tau\mu_i}(A_{k_0}B_{m,k_0}C_{m,k_0}) \\ &\le k_0 + \sum_{m=k_0+1}^{\infty} P(B_{m,k_0}(\epsilon,\mu)) \\ &\le k_0 + dc + 2dc\sqrt{\pi\delta c R_2'/R_1}\, e^{\delta c R_2'/R_1} \end{aligned}$$

for large c. Thus, there exits a constant M_2' such that $M_2' > 2d\sqrt{\pi\delta R_2'/R_1}$ and

$$ARL_{\tau\mu_i}(T_{SO}) \le M_2'(c)^{3/2} e^{\delta c R_2'/R_1}$$

for large c. This is the upward inequality of (4.4). Similarly, we can prove the downward inequality of (4.4) as (6.2). Moreover, taking $R_1 = R_2 = 1$ and $R = 1$ in (4.4) we can obtain (4.5).

Suppose that $\mu > R\delta/2$. Taking $k_0 \ge \tau$ such that

$$R_1'' = (2\mu/\delta - R - \epsilon) < \frac{\sum_{i=n-k+1}^{n}(2r_i\mu_i/\delta - r_i^2)}{[k\sum_{i=n-k+1}^{n} r_i^2]^{1/2}} < (2\mu/\delta - R + \epsilon) = R_2''$$

holds for $n \ge k_0$ and $1 \le k \le n - k_0$, where $R_1'' = (2\mu/\delta - R - \epsilon) > 0$.

Let $N = k_0 + 2c/(\delta R_1'' R_1) + 4d\sqrt{2c\ln c})$ and $n = N - k_0 + k$, where $d = (\delta R_1''\sqrt{\delta R_1'' R_1})^{-1}$. It follows that

$$\begin{aligned} \frac{c}{R_1\sqrt{n}} - \frac{\delta R_1''}{2}\sqrt{n} &= -\left(\frac{\delta R_1''}{2}\sqrt{N - k_0 + k} - \frac{c}{R_1\sqrt{n}}\right) \\ &= -\frac{\delta R_1''}{2}\sqrt{N - k_0 + k}\left\{1 - \frac{1}{1 + 2d\delta R_1'' R_1\sqrt{\ln c/c} + \delta R_1'' R_1 k/(2c)}\right\} \\ &\le -\delta R_1'' A_N \sqrt{N - k_0 + k} \sim -2\sqrt{\ln c} \to -\infty \end{aligned}$$

as $c \to \infty$ since $(\delta R_1'' A_N)^2(N - k_0) \to 4\ln c$ as $c \to \infty$, where $A_N = [1 - (1 + 2d\delta R_1'' R_1\sqrt{\ln c/c})^{-1}]/2$.

As $c \to \infty$ and $\varepsilon \to 0$, we have

$$\begin{aligned} &\sum_{n=N-k_0+1}^{\infty} P_{\tau\mu_i}(T_{SO} > n) \\ &\quad \le \sum_{n=N-k_0+1}^{\infty} P\left(\frac{\sum_{i=n-k+1}^{n} r_i(X_i - \mu_i)}{[\sum_{i=n-k+1}^{n} r_i^2]^{1/2}} < \frac{c}{R_1\sqrt{k}} - \frac{\delta R_1''\sqrt{k}}{2},\right. \\ &\qquad\qquad \left. 1 \le k \le n - k_0, k_0 < n \le m\right) \\ &\quad \le \sum_{n=N-k_0+1}^{\infty} P\left(\frac{\sum_{i=n-k+1}^{n} r_i(X_i - \mu_i)}{[\sum_{i=n-k+1}^{n} r_i^2]^{1/2}} < \frac{c}{R_1\sqrt{n}} - \frac{\delta R_1''\sqrt{n}}{2}\right) \end{aligned}$$

$$\leq \sum_{n=N-k_0+1}^{\infty} \int_{-\infty}^{\delta R_1'' A_N \sqrt{n}} \phi(x)dx \leq \sum_{n=N-k_0+1}^{\infty} \int_{\delta R_1'' A_N \sqrt{n}}^{+\infty} \phi(x)dx$$
$$\leq \sum_{k=1}^{\infty} \frac{\exp\left\{-\frac{1}{2}(\delta R_1'' A_N)^2 (N-k_0+k)\right\}}{\delta R_1'' \sqrt{2\pi} A_N \sqrt{N-k_0}}$$
$$\leq \frac{\exp\left\{-\frac{1}{2}(\delta R_1'' A_N)^2 (N-k_0)\right\}}{\delta R_1'' \sqrt{2\pi} A_N \sqrt{N-k_0}} (1-e^{-1/2(\delta R_1'' A_N)^2})$$
$$\leq \frac{1}{\sqrt{2\pi}(\delta R_1'')^2 (\ln c)^{3/2}}.$$

Thus

$$ARL_{\tau\mu_i}(T_{SO}) \leq \sum_{n=1}^{N-k_0} P_{\tau\mu_i}(T_{SO} > n) + \frac{1}{\sqrt{2\pi}(\delta R_1'')^2 (\ln c)^{3/2}} \tag{6.3}$$
$$\leq N - k_0 + \frac{1}{\sqrt{2\pi}(\delta R_1'')^2 (\ln c)^{3/2}}$$
$$\leq \frac{2c}{\delta R_1'' R_1} + 4d\sqrt{c \ln c} + o\left(\frac{1}{\ln c}\right)$$
$$\sim \frac{2c}{(2\mu - \delta R)R}(1 + o(1))$$

as $c \to \infty$.

On the other hand, let $M = k_0 + 2c/(\delta R_2'' R_2) - 4d'\sqrt{2c \ln c}$ and $M' = M - k_0$, where $d' = (\delta R_2'' \sqrt{\delta R_2'' R_2})^{-1}$. Since

$$\frac{c}{R_2\sqrt{M'}} - \frac{\delta R_2''}{2}\sqrt{M'} \sim 2\sqrt{\ln c}$$

as $c \to \infty$ and $\varepsilon \to 0$, it follows that $\Phi(\frac{c}{R_2\sqrt{M'}} - \frac{\delta R_2''}{2}\sqrt{M'}) \sim 1 - (2\sqrt{2\pi \ln c}c^2)^{-1}$ as $c \to \infty$ and $\varepsilon \to 0$. Let $m' = m - k_0$ and $n' = n - k_0$. Then, by using Theorem 5.1 in Esary *et al.* (1967) we have

$$\sum_{n=k_0+1}^{M} P_{\tau\mu_i}(B_{m,k_0}) \tag{6.4}$$
$$\geq \sum_{n=k_0+1}^{M} P\left\{ \frac{\sum_{i=m-k+1}^{m} r_i(X_i - \mu_i)}{[\sum_{i=n-k+1}^{n} r_i^2]^{1/2}} < \frac{c}{R_2\sqrt{k}} - \frac{\delta R_2''\sqrt{k}}{2}, \right.$$
$$\left. 1 \leq k \leq m - k_0, k_0 < m \leq n \right\}$$
$$\geq \sum_{n'=1}^{M'} \prod_{m'=1}^{n'} \prod_{k=1}^{m'} P\left\{ \frac{\sum_{i=m'+k_0-k+1}^{m'+k_0} r_i(X_i - \mu_i)}{[\sum_{i=m'+k_0-k+1}^{m'+k_0} r_i^2]^{1/2}} < \frac{c}{R_2\sqrt{k}} - \frac{\delta R_2''\sqrt{k}}{2} \right\}$$
$$= \sum_{n'=1}^{M'} \prod_{m'=1}^{n'} \prod_{k=1}^{m'} \Phi\left(\frac{c}{R_2\sqrt{M'}} - \frac{\delta R_2''}{2}\sqrt{M'}\right)$$

$$
\begin{aligned}
&\geq \sum_{n'=1}^{M'} \left(\left[\Phi\left(\frac{c}{R_2\sqrt{M'}} - \frac{\delta R_2''}{2}\sqrt{M'} \right) \right]^{(M'+1)/2} \right)^{n'} \\
&= \frac{\left[\Phi\left(\frac{c}{R_2\sqrt{M'}} - \frac{\delta R_2''}{2}\sqrt{M'} \right) \right]^{(M'+1)/2} \left(1 - \left[\Phi\left(\frac{c}{R_2\sqrt{M'}} - \frac{\delta R_2''}{2}\sqrt{M'} \right) \right]^{(M'+1)M'/2} \right)}{1 - \left[\Phi\left(\frac{c}{R_2\sqrt{M'}} - \frac{\delta R_2''}{2}\sqrt{M'} \right) \right]^{(M'+1)/2}} \\
&\sim M' \left(1 - o\left(\frac{1}{\sqrt{\ln c}} \right) \right)
\end{aligned}
$$

as $c \to \infty$ and $\varepsilon \to 0$. Note that $P_{\tau\mu_i}(A_{k_0}) \to 1$ and

$$
\begin{aligned}
P(C_{M,k_0}) &\geq \left[\Phi\left(\frac{c}{R_2\sqrt{M'}} - \frac{\delta R_2''}{2}\sqrt{M'} \right) \right]^{k_0 M'} \\
&\sim \left[1 - \frac{1}{2\sqrt{2\pi \ln c}c^2} \right]^{k_0 M'} \sim \left(1 - o\left(\frac{1}{c} \right) \right)
\end{aligned}
\tag{6.5}
$$

as $c \to \infty$. Thus, we have

$$
\begin{aligned}
&ARL_{\tau\mu_i}(T_{SO}) \\
&\geq \sum_{m=1}^{k_0} P_{\tau\mu_i}(A_m) + \sum_{m=k_0+1}^{M} P_{\tau\mu_i}(A_{k_0}) P_{\tau\mu_i}(B_{m,k_0}) P_{\tau\mu_i}(C_{m,k_0}) \\
&\geq \sum_{m=1}^{k_0} P(A_m) + P_{\tau\mu_i}(A_{k_0}) P_{\tau\mu_i}(C_{M,k_0}) \sum_{m=k_0+1}^{M} P_{\tau\mu_i}(B_{m,k_0}) \\
&\geq k_0 + M' \left(1 - o\left(\frac{1}{\sqrt{\ln c}} \right) \right) \left(1 - o\left(\frac{1}{c} \right) \right) \sim \frac{2c}{(2\mu - \delta R)R}(1 - o(1)),
\end{aligned}
\tag{6.6}
$$

as $c \to \infty$. By (6.3) and (6.6), we obtain (4.6). Taking $R = 1$ in (4.6) we can get (4.7).

To prove (4.8) let $\epsilon > 0$ such that $\mu - \epsilon > 0$. Obviously, we can take a natural number $k_0 \geq \tau$ such that

$$
\mu *_1 = \mu - \epsilon < \frac{\sum_{i=n-k+1}^{n} r_i \mu_i}{[k \sum_{i=n-k+1}^{n} r_i^2]^{1/2}} < \mu + \epsilon = \mu *_2
$$

holds for $n \geq k_0$ and $1 \leq k \leq n - k_0$. Let $N = k_0 + (c^2 + 4c\sqrt{\ln c})/(\mu *_1)^2$ and $n = N + k$. It follows that

$$
\begin{aligned}
c - \mu *_1 \sqrt{n - k_0} &= -(\mu \sqrt{N - k_0 + k} - c) \\
&= -\mu *_1 \sqrt{N - k_0 + k} \left\{ 1 - \frac{1}{\sqrt{1 + 4\sqrt{\ln c}/c + (\mu *_1)^2 k/c}} \right\} \\
&\leq -\mu *_1 A_N \sqrt{N - k_0 + k} \sim -2\sqrt{\ln c} \to -\infty
\end{aligned}
$$

and $(\mu *_1)^2 A_N^2 (N - k_0) \sim 4 \ln c$ as $c \to \infty$, where $A_N = [1 - (1 + 4\sqrt{\ln c}/c)^{-1/2}]$. Let

$$
A'_m = \left\{ \left(\sum_{i=n-k+1}^{n} r_i^2 \right)^{-1/2} \sum_{i=n-k+1}^{n} r_i X_i < c, 1 \leq k \leq n, 1 \leq n \leq m \right\}
$$

for $m \le k_0$,

$$B'_{m-k_0} = \left\{ \left(\sum_{i=n-k+1}^{n} r_i^2 \right)^{-1/2} \sum_{i=n-k+1}^{n} r_i X_i < c, 1 \le k \le n-k_0, k_0 < n \le m \right\}$$

and

$$C'_{m-k_0} = \left\{ \left(\sum_{i=n-k+1}^{n} r_i^2 \right)^{-1/2} \sum_{i=n-k+1}^{n} r_i X_i < c, n-k_0+1 \le k \le n, k_0 < n \le m \right\}$$

for $m > k_0$. Note that

$$\begin{aligned} B'_{m,k_0} &= \left\{ \left(\sum_{i=n-k+1}^{n} r_i^2 \right)^{-1/2} \sum_{i=n-k+1}^{n} r_i X_i < c, 1 \le k \le n-k_0, k_0 < n \le m \right\} \\ &\subset \left\{ \left(\sum_{i=n-k+1}^{n} r_i^2 \right)^{-1/2} \sum_{i=n-k+1}^{n} r_i (X_i - \mu_i) < c - \mu *_1 \sqrt{k}, \right. \\ &\qquad\qquad \left. 1 \le k \le n-k_0, k_0 < n \le m \right\} \end{aligned}$$

for $m > k_0$. As (6.3) we can prove that

$$\begin{aligned} ARL_{\tau\mu_i}(T_{GLP}) &\le \sum_{n=1}^{N} P_{\tau\mu_i}(T_{GLP} > n) + \frac{1}{4\sqrt{2\pi}(\mu *_1)^2 (\ln c)^{3/2}} \qquad (6.7) \\ &\le N + \frac{1}{4\sqrt{2\pi}(\mu *_1)^2 (\ln c)^{3/2}} \\ &\le k_0 + \frac{1}{(\mu *_1)^2}(c^2 + 4c\sqrt{\ln c}) + o\left(\frac{1}{\ln c}\right) \\ &\sim \frac{c^2}{\mu^2} \end{aligned}$$

$c \to \infty$ and $\varepsilon \to 0$.

On the other hand, let $M = k_0 + \frac{1}{(\mu *_2)^2}(c^2 - 4c\sqrt{2\ln c})$ and $M' = M - k_0$. Note that

$$\begin{aligned} B'_{m,k_0} \supset &\left\{ \left(\sum_{i=n-k+1}^{n} r_i^2 \right)^{-1/2} \sum_{i=n-k+1}^{n} r_i (X_i - \mu_i) < c - \mu *_2 \sqrt{k}, \right. \\ &\qquad\qquad \left. 1 \le k \le n-k_0, k_0 < n \le m \right\} \end{aligned}$$

for $m > k_0$ and $\Phi(c - \mu *_2 \sqrt{M'}) \sim 1 - (4\sqrt{\pi \ln c} c^4)^{-1}$ as $c \to \infty$ and $\varepsilon \to 0$. As (6.4) and (6.5) we have

$$\sum_{n=k_0+1}^{M} P_{\tau\mu_i}(B'_{m,k_0}) \ge M' \left(1 - o\left(\frac{1}{\sqrt{\ln c}} \right) \right)$$

and

$$P(C'_{M,k_0}) \geq \left(1 - o\left(\frac{1}{c}\right)\right)$$

as $c \to \infty$ and $\varepsilon \to 0$. Thus

$$\begin{aligned} (6.8) \qquad & ARL_{\tau\mu_i}(T_{GLP}) \\ & \geq \sum_{m=1}^{k_0} P_{\tau\mu_i}(A'_m) + \sum_{m=k_0+1}^{M} P_{\tau\mu_i}(A'_{k_0}) P_{\tau\mu_i}(B'_{m,k_0}) P_{\tau\mu_i}(C'_{m,k_0}) \\ & \geq \sum_{m=1}^{k_0} P_{\tau\mu_i}(A'_m) + P_{\tau\mu_i}(A'_{k_0}) P_{\tau\mu_i}(C'_{M,k_0}) \sum_{m=k_0+1}^{M} P_{\tau\mu_i}(B'_{m,k_0}) \\ & \geq k_0 + M'\left(1 - o\left(\frac{1}{\sqrt{\ln c}}\right)\right) \sim \frac{1}{(\mu*_2)^2}(c^2 - 4c\sqrt{2\ln c}) \sim \frac{c^2}{\mu^2}, \end{aligned}$$

since $P_{\tau\mu_i}(A'_{k_0}) P_{\tau\mu_i}(C'_{M,k_0}) \to 1$ as $c \to \infty$. Obviously, (4.8) follows from (6.7) and (6.8). This completes the proof of Theorem 4.2.

PROOF OF THEOREM 4.3. Let $0 < R < 1$. Note that $ARL_0 \to \infty$ means that the control limit $c \to \infty$. It follows from (4.1) and (4.2) that the Cuscore and CUSUM charts have the same order of infinite large control limits, c and c', that is, $c'/c \to 1$ when $ARL_0(T_{SO}(c)) = ARL_0(T_C(c')) \to \infty$. By (4.4) and (4.5) we have

$$(6.9) \qquad M_1 \frac{e^{\delta c(1-2\mu/(\delta R)-o(1))}}{c} \leq ARL_{\tau\mu_i}(T_{SO}(c)) \leq M'_2(c)^{3/2} e^{\delta c(1-2\mu/(\delta R)+o(1))}$$

for $0 < \mu < \delta R/2$ and

$$(6.10) \qquad M_1 \frac{e^{\delta c'(1-2\mu/\delta-o(1))}}{c'} \leq ARL_{\tau\mu_i}(T_C(c')) \leq M'_2(c')^{3/2} e^{\delta c'(1-2\mu/\delta+o(1))}$$

for $0 < \mu < \delta/2$ as $c, c' \to \infty$. It follows from (6.9) and (6.10) that $ARL_{\tau\mu_i}(T_{SO}(c)) < ARL_{\tau\mu_i}(T_C(c'))$ when $ARL_0(T_{SO}(c)) = ARL_0(T_C(c')) \to \infty$ for $\mu < \delta R/2$ since $(\delta R - 2\mu)/R < (\delta - 2\mu)$ and $c'/c \to 1$. By (4.6) and (6.10) we see that $ARL_{\tau\mu_i}(T_{SO}(c)) < ARL_{\tau\mu_i}(T_C(c'))$ for $\delta R/2 < \mu < \delta/2$. Let $\delta/2 < \mu$. It follows from (4.6) and (4.7) that $ARL_{\tau\mu_i}(T_{SO}(c)) < ARL_{\tau\mu_i}(T_C(c'))$ if and only if

$$\frac{1}{(\mu - \delta R/2)R} < \frac{1}{(\mu - \delta/2)}$$

as $ARL_0(T_{SO}(c)) = ARL_0(T_C(c')) \to \infty$. It can be checked that $\frac{1}{(\mu-\delta R/2)R} < \frac{1}{(\mu-\delta/2)}$ for $\delta/2 < \mu < \delta(R+1)/2$, $\frac{1}{(\mu-\delta R/2)R} = \frac{1}{(\mu-\delta/2)}$ for $\delta(R+1)/2 = \mu$ and $\frac{1}{(\mu-\delta R/2)R} > \frac{1}{(\mu-\delta/2)}$ for $\mu > \delta(R+1)/2$. This completes the proof of Theorem 4.3.

PROOF OF THEOREM 4.4. Theorem 4.4 can be proved similarly as Theorem 4.3.

PROOF OF THEOREM 4.5. From (4.1) and (4.3) it follows that there exists a positive increasing function $l(c)$ such that $l(c) \sim \sqrt{2\delta c}$ and $ARL_0(T_{SO}(c)) = ARL_0(T_{GL}(l(c))) \to \infty$ as $c \to \infty$. Thus, by (4.4) and (4.8) we see that $ARL_{\tau\mu_i}(T_{SO}(c)) > ARL_{\tau\mu_i}(T_{GL}(l(c))) \sim \frac{2\delta c}{\mu^2}$ for $\delta R/2 > \mu$ as $c \to \infty$. Let $\mu > \delta R/2$.

It follows from (4.6) and (4.8) that $ARL_{\tau\mu_i}(T_{SO}(c)) > ARL_{\tau\mu_i}(T_{GL}(l(c)))$ for large c if and only if

$$\frac{1}{(\mu - \delta R/2)R} > \frac{2\delta}{\mu^2},$$

since

$$\frac{1}{(\mu - \delta R/2)R} - \frac{2\delta}{\mu^2} = \frac{(\mu - \delta R)^2}{(\mu - \delta R/2)R\mu^2} \geq 0.$$

Thus, $ARL_{\tau\mu_i}(T_{SO}(c)) > ARL_{\tau\mu_i}(T_{GL}(l(c)))$ for $\mu \neq \delta R$ and $ARL_{\tau\mu}(T_{SO}(c)) \sim ARL_{\tau\mu}(T_{GL}(l(c)))$ for $\mu = \delta R$ as $c \to \infty$. Similarly, $ARL_{\tau\mu_i}(T_C(c)) > ARL_{\tau\mu_i}(T_{GL}(l(c)))$ for $\mu \neq \delta$ and $ARL_{\tau\mu_i}(T_C(c)) \sim ARL_{\tau\mu_i}(T_{GL}(l(c)))$ for $\mu = \delta$ as $c \to \infty$.

PROOF OF COROLLARY 4.1. Taking $\mu = \delta R$ in Theorems 4.3 and 4.4 we have $ARL_{\tau\mu_i}(T_{SO}) < ARL_{\tau\mu_i}(T_C)$ for $0 < R \neq 1$. Similarly, $ARL_{\tau\mu_i}(T_{SO}) > ARL_{\tau\mu_i}(T_C)$ for $\mu = \delta$ and $R \neq 1$. It follows from (c_3) and (d_3) in Theorem 4.5 that

$$ARL_{\tau\mu_i}(T_{SO}) \sim ARL_{\tau\mu_i}(T_C)$$

for $\mu = \delta$ and $R = 1$ as $ARL_0(T_{SO}) = ARL_0(T_C) \to \infty$.

7. Conclusions

It is known that the ARL is a very important quantity to measure the detecting capability of the control charts. Therefore, how to calculate or estimate the ARL becomes one of the major tasks in the study of SPC. As can be seen in the literature that most work on a study of the performance of various control charts is mainly based on the numerical simulation of the ARL. Though the theoretical approximations of ARLs of the LRT and CUSUM chart in detecting a constant mean shift have been done by Siegmund and Venkatraman (1995) and Wu (1994), their methods are not efficient in estimating the ARLs of the Cuscore, GLGT and CUSUM tests in detecting a dynamic mean change $\{\mu_i\}$. Thus, we present a different approach to estimate the ARLs of the three tests and compare their performance both in theoretical approximation and numerical simulation in detecting a dynamic mean change which finally goes to a steady-state value.

From the theoretical study we find that, when the steady-state value is greater or less than a critical value, $R\delta/2 + \delta/2$, the Cuscore and CUSUM charts have a contrary performance in detecting the mean change. Moreover, the detecting performance of the GLRT is more robust than that of the Cuscore and CUSUM charts in the sense that the ARLs of the GLRT do not depend on the steady-stable value of the reference pattern or reference value but the Cuscore and CUSUM charts do. The simulation results in Tables 1 and 2 show that the above conclusions are still true even if the in-control ARL_0 (≈ 870) is not large. We prove also that the GLRT has the best performance in detecting any mean change except that the steady-state value of the mean change is equal to δ or δR, and the CUSUM and Cuscore charts have the best performance if the steady-stable value is equal to δ and δR, respectively, when the in-control ARL is large enough. Since the ARL_0 is not large enough in the numerical simulation, the detecting performance of the GLRT is not always best as shown in the theoretical comparison. This shows that the condition, $ARL_0 \to \infty$, is necessary for the results of Theorem 4.5.

It is known that the efficiency of Cuscore, GLRT and CUSUM charts depends on the knowledge of the change point τ, the reference value δ, and particularly the reference pattern $\{r_k\}$. It is worthwhile to investigate robust charting techniques that are not sensitive to these parameters. In addition, how to detect a dynamic mean shift without a steady state value when ARL_0 is not large enough also warrants future research.

Acknowledgements

We thank the editor and the referees for valuable comments and suggestions that have improved both the content and presentation of this work. The authors also thank Dr. Xinjian Hu for his help in the numerical simulation and Dr. Lianjie Shu for discussions on the Cuscore chart. This research was supported by RGC Competitive Earmarked Research Grants HKUST6011/01E and HKUST6183/03E.

References

Alwan, L. C. and Roberts, H. V. (1988). Time-series modeling for statistical process control, *Journal of Business Economic Statistics*, **6**, 87–95.

Apley, D. W. and Shi, J. (1994). A statistical process control method for autocorrelated data using a GLRT, *Proceeding of the International Symposiumon Manufacturing Science and Technology for 21st Century*, 165–170, Tsinghua Press, Beijing.

Apley, D. W. and Shi, J. (1999). The GLRT for statistical process control of autocorrelated processes, *IIE Transactions*, **31**, 1123–1134.

Bagshaw, M. and Johnson, R. A. (1977). Sequential procedures for detecting parameter changes in a time-series model, *Journal of the American Statistical Association*, **72**, 593–597.

Box, G. E. P. and Luceño, A. (1997). *Statistical Control by Monitoring and Feedback Adjustment*, Wiley, New York.

Box, G. E. P. and Ramírez, J. G. (1992). Cumulative score charts, *Quality Reliability Engineering International*, **8**, 17–27.

Esary, J. D., Proschan, F. and Walkup, D. W. (1967). Association of random variables with applications, *The Annals of Mathematical Statistics*, **38**, 1466–1474.

Fisher, R. A. (1925). Theory of statistical estimation, *Proceeding of the Cambridge Philosophical Society*, **22**, 700–725.

Hu, S. J. and Roan, C. (1996). Change patterns of time series-based control charts, *Journal of Quality Technology*, **28**, 302–312.

Keats, J. B., Montgomery, D. C., Runger, G. C. and Messina, W. S. (1996). Feedback control and statistical process monitoring, *International Journal of Reliability, Quality, and Safety Engineering*, **3**(2), 231–241.

Lai, T. L. (1974). Control charts based on weighted sums, *The Annals of Statistics*, **2**, 134–147.

Luceño, A. (1999). Average run lengths and run length probability distributions for cuscore charts to control normal mean, *Computational Statistics & Data Analysis*, **32**, 177–195.

Montgomery, D. C. and Mastrangelo, C. M. (1991). Some statistical process control charts methods for autocorrelated data, *Journal of Quality Technology*, **23**, 179–193.

Moustakides, G. V. (1986). Optimal stopping times for detecting changes in distribution, *The Annals of Statistics*, **14**, 1379–1387.

Ramírez, G. J. (1998). Monitoring clean room air using Cuscore charts, *Quality Reliability Engineering International*, **14**, 281–289.

Ritov, Y. (1990). Decision theoretic optimality of the Cusum procedure, *The Annals of Statistics*, **18**, 1464–1469.

Shu, L. J., Apley, D. W. and Tsung, F. (2002). Autocorrelated process monitoring using triggered Cuscore charts, *Quality and Reliability Engineering International*, **18**, 411–421.

Siegmund, D. and Venkatraman, E. S. (1995). Using the generalized likelihood ratio statistic for sequential detection of a change-point, *The Annals of Statistics*, **23**, 255–271.

Tsung, F. and Tsui, K.-L. (2003). A mean shift pattern study on integration of SPC and APC for process monitoring, *IIE Transactions*, **35**, 231–242.

Tsung, F., Shi, J. and Wu, C. F. J. (1999). Joint monitoring of PID controlled processes, *Journal of Quality Technology*, **31**, 275–285.

Vasilopoulos, A. V. and Stamboulis, A. P. (1978). Modification of control chart limits in the presence of data correlation, *Journal of Quality Technology*, **10**, 20–30.

Wardell, D. G., Moskowitz, H. and Plante, R. D. (1992). Control charts in the presence of data correlation, *Management Science*, **38**, 1084–1105.

Wu, Y. H. (1994). Design of control charts for detecting the change, *Change—Point Problems* (eds. E. Carlstein, H. Muller and D. Siegmund), **23**, 330–345, Institute of Mathematical Statistics, Hayward, California.

Yashchin, E. (1993). Performance of CUSUM control schemes for serially correlated observations, *Technometrics*, **35**(1), 37–52.