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JOINT DISTRIBUTIONS OF NUMBERS OF SUCCESS RUNS OF SPECIFIED LENGTHS IN LINEAR AND CIRCULAR SEQUENCES*

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Abstract. In this paper, we study two joint distributions of the numbers of success runs of several lengths in a sequence of n Bernoulli trials arranged on a line (linear sequence) or on a circle (circular sequence) based on four different enumeration schemes. We present formulae for the evaluation of the joint probability functions, the joint probability generating functions and the higher order moments of these distributions. Besides, the present work throws light on the relation between the joint distributions of the numbers of success runs in the circular and linear binomial model. We give further insights into the run-related problems arisen from the circular sequence. Some examples are given in order to illustrate our theoretical results. Our results have potential applications to other problems such as statistical run tests for randomness and reliability theory.

Key words and phrases: Bernoulli trials, circular success runs, enumeration schemes, recursive scheme, circular binomial distribution of order k, probability function, probability generating function, double generating function.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a fixed number of Bernoulli trials with success (S) probability $p = P(X_i = 1)$ and failure (F) probability $q = P(X_i = 0) = 1 - p$, $i = 1, 2, \ldots, n$. The concept of success runs has been used effectively in a wide range of areas such as reliability theory, start-up demonstration tests and statistical quality control (see Chao *et al.* (1995), Balakrishnan *et al.* (1997), Shmueli and Cohen (2000) and references therein). The distribution theory of success runs has been developed by many authors under various enumeration schemes. There are different ways of counting the number of success runs of length k in the literature (see Fu and Koutras (1994) and Balakrishnan and Koutras (2002)). It depends on the practical problem which way of counting should be adopted. The four best-known types of the ways of counting the number of success runs of length k are as follows.

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(i) Type I enumeration scheme: the way of counting the number of non-overlapping and recurrent success runs of length k, in the sense of Feller's (1968) counting,

(ii) Type II enumeration scheme: the way of counting the number of success runs of length at least k, in the sense of Goldstein's (1990) counting (see Gibbons (1971)),

(iii) Type III enumeration scheme: the way of counting the number of overlapping success runs of length k, in the sense of Ling's (1988) counting,

(iv) Type IV enumeration scheme: the way of counting the number of success runs of size exactly k, in the sense of Mood's (1940) counting.

In the case where the n Bernoulli trials are arranged on a line (linear sequence), the distribution theory of success runs has been developed very actively (see Aki and Hirano (1988), Koutras and Alexandrou (1995), Antzoulakos and Chadjiconstantinidis (2001), Godbole *et al.* (1997) and Han and Aki (1998)). On the other hand, in the case where the n Bernoulli trials are arranged on a circle (circular sequence), although the need to study the circular success runs was recognized very early (see Derman *et al.* (1982) and Asano (1965)), the development of the relevant distribution theory was very slow and insufficient. Here, we assume that the outcomes of the n Bernoulli trials are bent into a circle so that additional success runs may be formed by combining successes at the beginning and end of the sequence. Several authors have made effort to establish formulae for the evaluation of the probability function and the probability generating function (p.g.f.) of the distribution of the number of success runs in the circular binomial model (see Charalambides (1994), Makri and Philippou (1994), Koutras and Papastavridis (1993) and Koutras *et al.* (1995)). However, the formulae obtained were usually complicated, and the higher order moments of these distributions have never been examined.

Our purpose of the present paper is to develop the formulae for the derivation of the joint probability function, the joint p.g.f. and the higher order moments of the joint distribution of the numbers of success runs of several lengths in the circular binomial model. We provide the perspectives on the run-related problems arisen from the circular sequence. We elucidate the relation between the joint distributions of the numbers of success runs in the circular and linear binomial model.

For $\boldsymbol{k} = (k_1, k_2, \dots, k_r)$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$, let $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ and $\boldsymbol{N}^c(n, \boldsymbol{k}; \boldsymbol{\alpha})$ be the r-dimensional random variables $(N(n, k_1; \alpha_1), \dots, N(n, k_r; \alpha_r))$ and $(N^c(n, k_1; \alpha_1), \dots, N^c(n, k_r; \alpha_r))$, where $N(n, k_i; \alpha_i)$ $(N^c(n, k_i; \alpha_i))$ represents the number of linear (circular) success runs of length k_i $(i = 1, 2, \dots, r)$ in the linear (circular) sequence by engaging Type $\alpha_i (= I, II, III, IV)$ enumeration scheme. In Section 2, we discuss the joint distribution of the numbers of success runs in the linear binomial model. We present recursive schemes for the evaluation of the joint probability function, the joint p.g.f. and the mixed $\boldsymbol{m} = (m_1, \dots, m_r)$-th moment about zero of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$. The expression for the double generating function of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ is given. Section 3 studies the joint distribution of $\boldsymbol{N}^c(n, \boldsymbol{k}; \boldsymbol{\alpha})$ from the viewpoint of the joint distribution of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$. We focus on the relation between the joint distributions of the number of success runs in the circular and linear binomial model. In Section 4, we discuss the joint distribution of the number of success runs in the circular binomial model. Recursive schemes for the derivation of the joint probability function, the joint p.g.f. and the mixed $\boldsymbol{m} = (m_1, \dots, m_r)$-th moment about zero of $\boldsymbol{N}^c(n, \boldsymbol{k}; \boldsymbol{\alpha})$ are given. The expression for the double generating function of $\boldsymbol{N}^c(n, \boldsymbol{k}; \boldsymbol{\alpha})$ is given. We derived the exact formula for the joint probability function and the joint p.g.f. of $\boldsymbol{N}^c(n, \boldsymbol{k}; \boldsymbol{\alpha})$. We can obtain useful information from the joint distribution of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ $(\boldsymbol{N}^c(n, \boldsymbol{k}; \boldsymbol{\alpha}))$. For example, in the linear(circular)-m-

consecutive-k-out-of-n:F system, the number of failure runs of length k and the number of failures provide useful clues to the maintenance. Here, we regard a "success" as a failed component. In Section 5, some examples are given in order to illustrate our theoretical results.

2. Linear binomial model

In this section, we establish recursive formulae for the evaluation of the joint probability function, the joint p.g.f. and the mixed $\boldsymbol{m} = (m_1, \dots, m_r)$-th moment about zero of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$. The double generating function of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ is given.

We define

$$\boldsymbol{\mu}(i; \boldsymbol{\alpha}) = (\mu(i; \alpha_1), \dots, \mu(i; \alpha_r)), \tag{2.1}$$

where

$$\mu(i; \alpha_j) = \begin{cases} \left[\frac{i}{k_j}\right] & \alpha_j = I, \\ I(i \geq k_j) & \alpha_j = II, \\ (i - (k_j - 1))^+ & \alpha_j = III, \\ I(i = k_j) & \alpha_j = IV, \end{cases}$$

$(i - (k_j - 1))^+ = \max\{0, i - (k_j - 1)\}$ and

$$I(u) = \begin{cases} 1 & u \text{ is true}, \\ 0 & \text{otherwise}. \end{cases}$$

2.1 *The joint probability functions and generating functions*

For $n \geq 0$ and $\boldsymbol{x} = (x_1, x_2, \dots, x_r)$, let

$$B_n(\boldsymbol{x}; \boldsymbol{\alpha}) = P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x}) \qquad x_i \geq 0 \qquad i = 1, 2, \dots, r$$

be the joint probability function of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$, with convention

$$\begin{cases} B_0(\boldsymbol{x}; \boldsymbol{\alpha}) = \delta_{\boldsymbol{x}, \boldsymbol{0}}, \\ B_n(\boldsymbol{x}; \boldsymbol{\alpha}) = 0 \qquad n \geq 0 \quad \text{and if } x_i < 0 \quad \text{for some } i, \end{cases} \tag{2.2}$$

where Kronecker delta $\delta_{\boldsymbol{x}, \boldsymbol{y}}$ equals one if $\boldsymbol{x} = \boldsymbol{y}$ and zero otherwise.

PROPOSITION 2.1. *Under the condition* (2.2), *the joint probability function* $B_n(\boldsymbol{x}; \boldsymbol{\alpha})$ *of* $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ *satisfies the following recursive relation*:

$$B_n(\boldsymbol{x}; \boldsymbol{\alpha}) = q \sum_{i=0}^{n-1} p^i B_{n-i-1}(\boldsymbol{x} - \boldsymbol{\mu}(i; \boldsymbol{\alpha}); \boldsymbol{\alpha}) + p^n B_0(\boldsymbol{x} - \boldsymbol{\mu}(n; \boldsymbol{\alpha}); \boldsymbol{\alpha}) \qquad n \geq 1,$$
$$B_0(\boldsymbol{x}; \boldsymbol{\alpha}) = \delta_{\boldsymbol{x}, \boldsymbol{0}},$$

$\boldsymbol{\mu}(i; \alpha)$ *is as in* (2.1).

PROOF. For $i = 0, 1, \dots, n-1$, let A_i be the event that the first failure occurs at the $(i+1)$-th trial and let A_n be the event that the first failure does not occur in

$X_1, X_2, \ldots, X_n$, that is,

$$A_0 = \{X_1 = 0\}, \tag{2.3}$$

$$A_i = \{X_1 = X_2 = \cdots = X_i = 1, X_{i+1} = 0\} \quad i = 1, 2, \ldots, n-1, \tag{2.4}$$

$$A_n = \{X_1 = X_2 = \cdots = X_{n-1} = X_n = 1\}. \tag{2.5}$$

Then, we have

$$\begin{aligned} P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x}) &= \sum_{i=0}^{n-1} P(A_i) P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} \mid A_i) \\ &\quad + P(A_n) P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} \mid A_n) \\ &= \sum_{i=0}^{n-1} q p^i P(\boldsymbol{N}(n-i-1, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} - \boldsymbol{\mu}(i; \boldsymbol{\alpha})) \\ &\quad + p^n P(\boldsymbol{N}(0, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} - \boldsymbol{\mu}(n; \boldsymbol{\alpha})). \end{aligned}$$

The proof is completed. □

The joint p.g.f. and the double generating function of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ will be denoted by $\phi_n(\boldsymbol{t}; \boldsymbol{\alpha})$ and $\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha})$, respectively, that is,

$$\phi_n(\boldsymbol{t}; \boldsymbol{\alpha}) = E[t_1^{N(n,k_1;\alpha_1)} t_2^{N(n,k_2;\alpha_2)} \cdots t_r^{N(n,k_r;\alpha_r)}] = \sum_{x_1, x_2, \ldots, x_r} B_n(\boldsymbol{x}; \boldsymbol{\alpha}) t_1^{x_1} t_2^{x_2} \cdots t_r^{x_r},$$

$$\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha}) = \sum_{n=0}^{\infty} \phi_n(\boldsymbol{t}; \boldsymbol{\alpha}) z^n = \sum_{n=0}^{\infty} \sum_{x_1, x_2, \ldots, x_r} B_n(\boldsymbol{x}; \boldsymbol{\alpha}) t_1^{x_1} t_2^{x_2} \cdots t_r^{x_r} z^n,$$

where $\boldsymbol{t} = (t_1, t_2, \ldots, t_r)$. Using Proposition 2.1, we can obtain the following proposition.

PROPOSITION 2.2. *The joint p.g.f.* $\phi_n(\boldsymbol{t}; \boldsymbol{\alpha})$ *of* $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ *satisfies the following recursive relation*:

$$\begin{aligned} &\phi_n(\boldsymbol{t}; \boldsymbol{\alpha}) = q \sum_{i=0}^{n-1} p^i \boldsymbol{t}^{\boldsymbol{\mu}(i;\boldsymbol{\alpha})} \phi_{n-i-1}(\boldsymbol{t}; \boldsymbol{\alpha}) + p^n \boldsymbol{t}^{\boldsymbol{\mu}(n;\boldsymbol{\alpha})} \qquad n \geq 1, \\ &\phi_0(\boldsymbol{t}; \boldsymbol{\alpha}) = 1, \end{aligned}$$

where

$$\boldsymbol{t}^{\boldsymbol{\mu}(i;\boldsymbol{\alpha})} = t_1^{\mu(i;\alpha_1)} t_2^{\mu(i;\alpha_2)} \cdots t_r^{\mu(i;\alpha_r)}. \tag{2.6}$$

Using Proposition 2.2, we have

$$\begin{aligned} \Phi(\boldsymbol{t}, z; \boldsymbol{\alpha}) &= 1 + \sum_{n=1}^{\infty} \phi_n(\boldsymbol{t}; \boldsymbol{\alpha}) z^n \\ &= qz \sum_{i=0}^{\infty} (pz)^i \boldsymbol{t}^{\boldsymbol{\mu}(i;\boldsymbol{\alpha})} \sum_{n=i+1}^{\infty} \phi_{n-i-1}(\boldsymbol{t}; \boldsymbol{\alpha}) z^{n-i-1} + \sum_{n=0}^{\infty} (pz)^n \boldsymbol{t}^{\boldsymbol{\mu}(n;\boldsymbol{\alpha})} \\ &= qz P(\boldsymbol{t}, pz; \boldsymbol{\alpha}) \Phi(\boldsymbol{t}, z; \boldsymbol{\alpha}) + P(\boldsymbol{t}, pz; \boldsymbol{\alpha}), \end{aligned}$$

where

$$P(\boldsymbol{t}, pz; \boldsymbol{\alpha}) = \sum_{i=0}^{\infty} (pz)^i \boldsymbol{t}^{\boldsymbol{\mu}(i;\boldsymbol{\alpha})}. \tag{2.7}$$

Therefore, we obtain the following proposition.

PROPOSITION 2.3. *The double generating function $\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha})$ of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ is given by*

$$\Phi(\boldsymbol{t}, z; \boldsymbol{\alpha}) = \frac{P(\boldsymbol{t}, pz; \boldsymbol{\alpha})}{1 - qzP(\boldsymbol{t}, pz; \boldsymbol{\alpha})}, \tag{2.8}$$

where $P(\boldsymbol{t}, pz; \boldsymbol{\alpha})$ is as in (2.7).

2.2 *Evaluation of moments*

Let $\eta_{n,m_1,\dots,m_r}(\boldsymbol{\alpha}) = E[\prod_{i=1}^{r}(N(n, k_i; \alpha_i))^{m_i}]$ denote the mixed $\boldsymbol{m} = (m_1, \dots, m_r)$-th moment about zero. We can establish a recursive formula for the evaluation of $\eta_{n,m_1,\dots,m_r}(\boldsymbol{\alpha})$. Replacing t_i by e^{t_i} $(i = 1, 2, \dots, r)$ in $\phi_n(\boldsymbol{t}; \boldsymbol{\alpha})$, the moment generating function (m.g.f.) $M_n(\boldsymbol{t}; \boldsymbol{\alpha})$ of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ is obtained. Since $M_n(\boldsymbol{t}; \boldsymbol{\alpha}) = \phi_n(e^{t_1}, \dots, e^{t_r}; \boldsymbol{\alpha})$, it follows from Proposition 2.2 that $\boldsymbol{m} = (m_1, \dots, m_r)$-th order derivative of $M_n(\boldsymbol{t}; \boldsymbol{\alpha})$ satisfies the recursive relation

$$\begin{aligned}
M_{n,m_1,\dots,m_r}(\boldsymbol{t}; \boldsymbol{\alpha}) &= \sum_{i=0}^{n-1} \sum_{\boldsymbol{b}=0}^{\boldsymbol{m}} qp^i e^{\Sigma_{j=1}^{r} t_j \mu(i;\alpha_j)} \binom{\boldsymbol{m}}{\boldsymbol{b}} (\mu(i; \alpha_j))^{m_j - b_j} M_{n-i-1,b_1,\dots,b_r}(\boldsymbol{t}; \boldsymbol{\alpha}) \\
&\quad + p^n \prod_{j=1}^{r} (\mu(n; \alpha_j))^{m_j} e^{\Sigma_{j=1}^{r} t_j \mu(n;\alpha_j)} \qquad n \geq 1, \\
M_{0,m_1,\dots,m_r}(\boldsymbol{t}; \boldsymbol{\alpha}) &= 0 \qquad (m_1, \dots, m_r) \neq (0, \dots, 0), \\
M_{0,m_1,\dots,m_r}(\boldsymbol{t}; \boldsymbol{\alpha}) &= 1 \qquad (m_1, \dots, m_r) = (0, \dots, 0),
\end{aligned}$$

where

$$M_{n,m_1,\dots,m_r}(\boldsymbol{t}; \boldsymbol{\alpha}) = \frac{\partial^{m_1+\cdots+m_r}}{\partial t_1^{m_1} \cdots \partial t_r^{m_r}} M_n(\boldsymbol{t}; \boldsymbol{\alpha}),$$

$$\sum_{\boldsymbol{b}=0}^{\boldsymbol{m}} = \sum_{b_1=0}^{m_1} \cdots \sum_{b_r=0}^{m_r} \quad \text{and} \quad \binom{\boldsymbol{m}}{\boldsymbol{b}} = \prod_{i=1}^{r} \binom{m_i}{b_i}.$$

Then, we obtain the following proposition.

PROPOSITION 2.4. *The mixed $\boldsymbol{m} = (m_1, \dots, m_r)$-th moment about zero $\eta_{n,m_1,\dots,m_r}(\boldsymbol{\alpha})$ of $\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha})$ satisfies the following recursive relation:*

$$\begin{aligned}
\eta_{n,m_1,\dots,m_r}(\boldsymbol{\alpha}) &= \sum_{i=0}^{n-1} \sum_{\boldsymbol{b}=0}^{\boldsymbol{m}} qp^i \binom{\boldsymbol{m}}{\boldsymbol{b}} (\mu(i; \alpha_j))^{m_j - b_j} \eta_{n-i-1,b_1,\dots,b_r}(\boldsymbol{\alpha}) \\
&\quad + p^n \prod_{j=1}^{r} (\mu(n; \alpha_j))^{m_j} \qquad n \geq 1, \\
\eta_{0,m_1,\dots,m_r}(\boldsymbol{\alpha}) &= 0 \qquad (m_1, \dots, m_r) \neq (0, \dots, 0), \\
\eta_{0,m_1,\dots,m_r}(\boldsymbol{\alpha}) &= 1 \qquad (m_1, \dots, m_r) = (0, \dots, 0).
\end{aligned}$$

Remark 2.1. Recently, for the linear sequence, Aki and Hirano (2000) introduced a generalized enumeration scheme which is called ℓ-overlapping counting (see Antzoulakos (2003) and Inoue and Aki (2003)). Setting

$$\mu(j;\alpha_i) = \max\left\{0, \left[\frac{j-\ell_i}{k_i-\ell_i}\right]\right\},$$

where $0 \le \ell_i \le k_i - 1$, the results presented in Propositions 2.1–2.4 can be extended to cover this case easily.

3. Relation between the circular and linear binomial models

In this section, we study the joint distribution of $\boldsymbol{N}^c(n,\boldsymbol{k};\boldsymbol{\alpha})$ through the joint distribution of $\boldsymbol{N}(n,\boldsymbol{k};\boldsymbol{\alpha})$. We elucidate the relation between the joint distributions of the number of success runs in the circular and linear binomial models.

We define

$$\boldsymbol{\nu}(i;\boldsymbol{\alpha}) = (\nu(i;\alpha_1),\dots,\nu(i;\alpha_r)), \tag{3.1}$$

where

$$\nu(i;\alpha_j) = \begin{cases} \left[\frac{i}{k_j}\right] & \alpha_j = I, \\ I(i \ge k_j) & \alpha_j = II, \\ iI(i \ge k_j) & \alpha_j = III, \\ I(i = k_j) & \alpha_j = IV. \end{cases}$$

3.1 *The joint probability functions and generating functions*

For $n \ge 0$ and $\boldsymbol{x} = (x_1, x_2, \dots, x_r)$, let

$$B_n^c(\boldsymbol{x};\boldsymbol{\alpha}) = P(\boldsymbol{N}^c(n,\boldsymbol{k};\boldsymbol{\alpha}) = \boldsymbol{x}) \qquad x_i \ge 0 \quad i = 1,2,\dots,r$$

be the joint probability function of $\boldsymbol{N}^c(n,\boldsymbol{k};\boldsymbol{\alpha})$, with convention

$$\begin{aligned} &B_0^c(\boldsymbol{x};\boldsymbol{\alpha}) = \delta_{\boldsymbol{x},\boldsymbol{0}}, \\ &B_n^c(\boldsymbol{x};\boldsymbol{\alpha}) = 0 \quad n \ge 0 \quad \text{and if} \quad x_i < 0 \quad \text{for some } i. \end{aligned} \tag{3.2}$$

THEOREM 3.1. *Under the conditions* (2.2) *and* (3.2), *the joint probability functions* $B_n^c(\boldsymbol{x};\boldsymbol{\alpha})$ *and* $B_n(\boldsymbol{x};\boldsymbol{\alpha})$ *satisfy the following recursive relation:*

$$\begin{aligned} B_n^c(\boldsymbol{x};\boldsymbol{\alpha}) &= q^2 \sum_{i=0}^{n-2} (i+1)p^i B_{n-i-2}(\boldsymbol{x} - \boldsymbol{\mu}(i;\alpha);\boldsymbol{\alpha}) \\ &\quad + nqp^{n-1} B_0(\boldsymbol{x} - \boldsymbol{\mu}(n-1;\boldsymbol{\alpha});\boldsymbol{\alpha}) + p^n B_0(\boldsymbol{x} - \boldsymbol{\nu}(n;\boldsymbol{\alpha});\boldsymbol{\alpha}) \qquad n \ge 1, \\ B_0^c(\boldsymbol{x};\boldsymbol{\alpha}) &= \delta_{\boldsymbol{x},\boldsymbol{0}}, \end{aligned}$$

$\boldsymbol{\mu}(i;\alpha)$, $\boldsymbol{\nu}(i;\alpha)$ *are as in* (2.1), (3.1). *The summation* $\sum_{i=b}^{a}$ *is ignored for* $b > a$. *Such a convention is frequently used in the following.*

PROOF. For $j = 0, 1, \dots, n-2$, let C_j be the event that the last failure occurs at the $(n-j)$-th trial and for $i = 1, 2, \dots, n$, let D_i be the event that only one failure occurs at the i-th trial, that is,

$$\begin{aligned}
&C_0 = \{X_n = 0\}, \\
&C_j = \{X_{n-j} = 0, X_{n-j+1} = X_{n-j+2} = \cdots = X_n = 1\} \quad j = 1, 2, \dots, n-1, \\
&D_i = \{X_i = 0, X_{i'} = 1, 1 \le i' \ne i \le n\} \quad i = 1, 2, \dots, n.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&P(\boldsymbol{N}^c(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x}) \\
&\quad = \sum_{0 \le i+j \le n-2} P(A_i)P(C_j)P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} \mid A_i \cap C_j) \\
&\qquad + \sum_{i=1}^{n} P(D_i)P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} \mid D_i) \\
&\qquad + P(A_n)P(\boldsymbol{N}(n, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} \mid A_n) \\
&\quad = \sum_{i+j=0}^{n-2} (i+j+1)q^2 p^{i+j} P(\boldsymbol{N}(n-i-j-2, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} - \boldsymbol{\mu}(i+j; \boldsymbol{\alpha})) \\
&\qquad + nqp^{n-1} P(\boldsymbol{N}(0, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} - \boldsymbol{\mu}(n-1; \boldsymbol{\alpha})) \\
&\qquad + p^n P(\boldsymbol{N}(0, \boldsymbol{k}; \boldsymbol{\alpha}) = \boldsymbol{x} - \boldsymbol{\nu}(n; \boldsymbol{\alpha}))
\end{aligned}$$

A_i $(i = 0, 1, \dots, n)$ are as in (2.3), (2.4), (2.5). The proof is completed. □

The methodology employed for establishing the recursive relation in Theorem 3.1 has been introduced by Makri and Philippou (1994). They tackled the univariate case of $\alpha = I, III$ (see also Makri and Philippou (2003)).

The joint p.g.f. and the double generating function of $\boldsymbol{N}^c(n, \boldsymbol{k}; \boldsymbol{\alpha})$ will be denoted by $\phi_n^c(\boldsymbol{t}; \boldsymbol{\alpha})$ and $\Phi^c(\boldsymbol{t}, z; \boldsymbol{\alpha})$, respectively, that is,

$$\phi_n^c(\boldsymbol{t}; \boldsymbol{\alpha}) = E[t_1^{N^c(n,k_1;\alpha_1)} t_2^{N^c(n,k_2;\alpha_2)} \cdots t_r^{N^c(n,k_r;\alpha_r)}] = \sum_{x_1, x_2, \dots, x_r} B_n^c(\boldsymbol{x}; \boldsymbol{\alpha}) t_1^{x_1} t_2^{x_2} \cdots t_r^{x_r},$$

$$\Phi^c(\boldsymbol{t}, z; \boldsymbol{\alpha}) = \sum_{n=0}^{\infty} \phi_n^c(\boldsymbol{t}; \boldsymbol{\alpha}) z^n = \sum_{n=0}^{\infty} \sum_{x_1, x_2, \dots, x_r} B_n^c(\boldsymbol{x}; \boldsymbol{\alpha}) t_1^{x_1} t_2^{x_2} \cdots t_r^{x_r} z^n.$$

Using Theorem 3.1, we can obtain the following theorem.

THEOREM 3.2. *The joint p.g.f.'s $\phi_n^c(\boldsymbol{t}; \boldsymbol{\alpha})$ and $\phi_n(\boldsymbol{t}; \boldsymbol{\alpha})$ satisfy the following recursive relation*:

$$\begin{aligned}
\phi_n^c(\boldsymbol{t}; \boldsymbol{\alpha}) &= q^2 \sum_{i=0}^{n-2} (i+1) p^i \boldsymbol{t}^{\boldsymbol{\mu}(i; \boldsymbol{\alpha})} \phi_{n-i-2}(\boldsymbol{t}; \boldsymbol{\alpha}) \\
&\qquad + nqp^{n-1} \boldsymbol{t}^{\boldsymbol{\mu}(n-1; \boldsymbol{\alpha})} + p^n \boldsymbol{t}^{\boldsymbol{\nu}(n; \boldsymbol{\alpha})} \qquad n \ge 1, \\
\phi_0^c(\boldsymbol{t}; \boldsymbol{\alpha}) &= 1,
\end{aligned}$$

where

$$t^{\nu(i;\alpha)} = t_1^{\nu(i;\alpha_1)} t_2^{\nu(i;\alpha_2)} \cdots t_r^{\nu(i;\alpha_r)}, \tag{3.3}$$

and $t^{\mu(i;\alpha)}$ *is as in* (2.6).

It is interesting to note that the double generating function $\Phi^c(t, z; \alpha)$ can be easily captured through $\Phi(t, z; \alpha)$. The next theorem provides the details.

THEOREM 3.3. *The double generating function* $\Phi^c(t, z; \alpha)$ *of* $N^c(n, k; \alpha)$ *is given by*

$$\Phi^c(t, z; \alpha) = (qz)^2 Q(t, pz; \alpha)\Phi(t, z; \alpha) + qzQ(t, pz; \alpha) + R(t, pz; \alpha), \tag{3.4}$$

where

$$Q(t, pz; \alpha) = \sum_{i=0}^{\infty} (i+1)(pz)^i t^{\mu(i;\alpha)}, \tag{3.5}$$

$$R(t, pz; \alpha) = \sum_{i=0}^{\infty} (pz)^i t^{\nu(i;\alpha)}, \tag{3.6}$$

and $\Phi(t, z; \alpha)$ *are as in* (2.8).

PROOF. Using Theorem 3.2, we have

$$\begin{aligned} \Phi^c(t, z; \alpha) &= 1 + \sum_{n=1}^{\infty} \phi_n^c(t; \alpha) z^n \\ &= (qz)^2 \sum_{i=0}^{\infty} (i+1)(pz)^i t^{\mu(i;\alpha)} \sum_{n=i+2}^{\infty} \phi_{n-i-2}(t; \alpha) z^{n-i-2} \\ &\quad + \sum_{n=1}^{\infty} nqp^{n-1} t^{\mu(n-1;\alpha)} + \sum_{n=0}^{\infty} p^n t^{\nu(n;\alpha)}. \end{aligned}$$

Therefore, the proof of (3.4) is completed. □

3.2 *Evaluation of moments*

Let $\eta^c_{n,m_1,\dots,m_r}(\alpha) = E[\prod_{i=1}^r (N^c(n, k_i; \alpha_i))^{m_i}]$ denote the mixed $m = (m_1, \dots, m_r)$-th moment about zero. We can evaluate $\eta^c_{n,m_1,\dots,m_r}(\alpha)$ through the $\eta_{n,m_1,\dots,m_r}(\alpha)$. Replacing t_i by e^{t_i} $(i = 1, 2, \dots, r)$ in the recursive formula provided by Theorem 3.2 and considering the $m = (m_1, \dots, m_r)$-th order derivative of the m.g.f. $\phi_n^c(e^{t_1}, \dots, e^{t_r}; \alpha)$, we may easily obtain the following theorem.

THEOREM 3.4. *The mixed* $m = (m_1, \dots, m_r)$*-th moments about zero* $\eta^c_{n,m_1,\dots,m_r}(\alpha)$ *and* $\eta_{n,m_1,\dots,m_r}(\alpha)$ *satisfy the following recursive relation:*

$$\eta^c_{n,m_1,\dots,m_r}(\alpha) = \sum_{i=0}^{n-1} \sum_{b=0}^{m} (i+1)qp^i \binom{m}{b} (\mu(i; \alpha_j))^{m_j - b_j} \eta_{n-i-2,b_1,\dots,b_r}(\alpha)$$

$$+ nqp^{n-1} \prod_{j=1}^{r} (\mu(n-1;\alpha_j))^{m_j} + p^n \prod_{j=1}^{r} (\nu(n;\alpha_j))^{m_j} \qquad n \geq 1,$$

$$\eta^c_{0,m_1,\ldots,m_r}(\boldsymbol{\alpha}) = 0 \qquad (m_1,\ldots,m_r) \neq (0,\ldots,0),$$
$$\eta^c_{0,m_1,\ldots,m_r}(\boldsymbol{\alpha}) = 1 \qquad (m_1,\ldots,m_r) = (0,\ldots,0).$$

4. Circular binomial model

In Section 3, we consider the joint distribution of $\boldsymbol{N}^c(n,\boldsymbol{k};\boldsymbol{\alpha})$ from the viewpoint of the joint distribution of $\boldsymbol{N}(n,\boldsymbol{k};\boldsymbol{\alpha})$. In this section, we examine the joint distribution of $\boldsymbol{N}^c(n,\boldsymbol{k};\boldsymbol{\alpha})$ directly.

4.1 *The joint probability functions and generating functions*

To begin with, we consider the double generating function $\Phi^c(\boldsymbol{t},z;\boldsymbol{\alpha})$ of $\boldsymbol{N}^c(n,\boldsymbol{k};\boldsymbol{\alpha})$. By making use of Theorem 3.3, we can establish compact formulae for the double generating function $\Phi^c(\boldsymbol{t},z;\boldsymbol{\alpha})$.

THEOREM 4.1. *The double generating function $\Phi^c(\boldsymbol{t},z;\boldsymbol{\alpha})$ of $\boldsymbol{N}^c(n,\boldsymbol{k};\boldsymbol{\alpha})$ is given by*

$$\Phi^c(\boldsymbol{t},z;\boldsymbol{\alpha}) = \frac{qzQ(\boldsymbol{t},pz;\boldsymbol{\alpha})}{1-qzP(\boldsymbol{t},pz;\boldsymbol{\alpha})} + R(\boldsymbol{t},pz;\boldsymbol{\alpha}), \tag{4.1}$$

or, equivalently,

$$\Phi^c(\boldsymbol{t},z;\boldsymbol{\alpha}) = -z\frac{\partial}{\partial z}\log[1-qzP(\boldsymbol{t},pz;\boldsymbol{\alpha})] + R(\boldsymbol{t},pz;\boldsymbol{\alpha}), \tag{4.2}$$

where $P(\boldsymbol{t},pz;\boldsymbol{\alpha})$, $Q(\boldsymbol{t},pz;\boldsymbol{\alpha})$, $R(\boldsymbol{t},pz;\boldsymbol{\alpha})$ are as in (2.7), (3.5), (3.6).

PROOF. From the equations (2.8) and (3.4), the proof of (4.1) is easily completed. The proof of (4.2) is completed if we take into account that

$$\frac{\partial}{\partial z}(zP(\boldsymbol{t},pz;\boldsymbol{\alpha})) = Q(\boldsymbol{t},pz;\boldsymbol{\alpha}). \qquad \square$$

We will establish recursive schemes for the evaluation of the joint p.g.f. $\phi^c_n(\boldsymbol{t};\boldsymbol{\alpha})$ and probability function $B^c_n(\boldsymbol{x};\boldsymbol{\alpha})$ of $\boldsymbol{N}^c(n,\boldsymbol{k};\boldsymbol{\alpha})$. From the equation (4.1), we have

$$(1-qzP(\boldsymbol{t},pz;\boldsymbol{\alpha}))\Phi^c(\boldsymbol{t},z;\boldsymbol{\alpha}) = qzQ(\boldsymbol{t},pz;\boldsymbol{\alpha}) + (1-qzP(\boldsymbol{t},pz;\boldsymbol{\alpha}))R(\boldsymbol{t},pz;\boldsymbol{\alpha}).$$

Equating the coefficients of z^n on the both sides of the above equation, we obtain the following theorem.

THEOREM 4.2. *The joint p.g.f. $\phi^c_n(\boldsymbol{t};\boldsymbol{\alpha})$ of $\boldsymbol{N}^c(n,\boldsymbol{k};\boldsymbol{\alpha})$ satisfies the following recursive relation:*

$$\phi^c_n(\boldsymbol{t};\boldsymbol{\alpha}) = q\sum_{i=0}^{n-1} p^i \boldsymbol{t}^{\boldsymbol{\mu}(i;\boldsymbol{\alpha})}\phi^c_{n-i-1}(\boldsymbol{t};\boldsymbol{\alpha}) + nqp^{n-1}\boldsymbol{t}^{\boldsymbol{\mu}(n-1;\boldsymbol{\alpha})} + p^n\boldsymbol{t}^{\boldsymbol{\nu}(n;\boldsymbol{\alpha})} \tag{4.3}$$

$$- qp^{n-1} \sum_{i=0}^{n-1} \boldsymbol{t}^{\boldsymbol{\mu}(n-i-1;\boldsymbol{\alpha})+\boldsymbol{\nu}(i;\boldsymbol{\alpha})} \qquad n \geq 1,$$

(4.4) $$\phi_0^c(\boldsymbol{t};\boldsymbol{\alpha}) = 1,$$

where $\boldsymbol{t}^{\boldsymbol{\mu}(i;\boldsymbol{\alpha})}$, $\boldsymbol{t}^{\boldsymbol{\nu}(i;\boldsymbol{\alpha})}$ *are as in* (2.6), (3.3).

Equating the coefficients of $t_1^{x_1} t_2^{x_2} \cdots t_r^{x_r}$ on the both sides of the equations (4.3) and (4.4) in Theorem 4.2, we obtain the following theorem.

THEOREM 4.3. *Under the condition* (3.2), *the joint probability function* $B_n^c(\boldsymbol{x};\boldsymbol{\alpha})$ *of* $\boldsymbol{N}^c(n,\boldsymbol{k};\boldsymbol{\alpha})$ *satisfies the following recursive relation:*

$$B_n^c(\boldsymbol{x};\boldsymbol{\alpha}) = q \sum_{i=0}^{n-1} p^i B_{n-i-1}^c(\boldsymbol{x} - \boldsymbol{\mu}(i;\boldsymbol{\alpha});\boldsymbol{\alpha}) + nqp^{n-1}\delta_{\boldsymbol{x},\boldsymbol{\mu}(n-1;\boldsymbol{\alpha})} + p^n \delta_{\boldsymbol{x},\boldsymbol{\nu}(n;\boldsymbol{\alpha})}$$

$$- qp^{n-1} \sum_{i=0}^{n-1} \delta_{\boldsymbol{x},\boldsymbol{\mu}(n-i-1;\boldsymbol{\alpha})+\boldsymbol{\nu}(i;\boldsymbol{\alpha})} \qquad n \geq 1,$$

$$B_0^c(\boldsymbol{x};\boldsymbol{\alpha}) = \delta_{\boldsymbol{x},\boldsymbol{0}},$$

where $\boldsymbol{\mu}(i;\boldsymbol{\alpha})$, $\boldsymbol{\nu}(i;\boldsymbol{\alpha})$ *are as in* (2.1), (3.1).

4.2 *Evaluation of moments*

We will establish a recursive formula for the evaluation of the mixed $\boldsymbol{m} = (m_1, \ldots, m_r)$-th moment about zero $\eta_{n,m_1,\ldots,m_r}^c(\boldsymbol{\alpha})$. Replacing t_i by e^{t_i} $(i = 1, 2, \ldots, r)$ in the recursive formula provided by Theorem 4.2 and considering the $\boldsymbol{m} = (m_1, \ldots, m_r)$-th order derivative of the m.g.f. $\phi_n^c(e^{t_1}, \ldots, e^{t_r};\boldsymbol{\alpha})$, we may easily derive a recursive scheme for the mixed $\boldsymbol{m} = (m_1, \ldots, m_r)$-th moment about zero $\eta_{n,m_1,\ldots,m_r}^c(\boldsymbol{\alpha})$.

THEOREM 4.4. *The mixed* $\boldsymbol{m} = (m_1, \ldots, m_r)$*-th moment about zero* $\eta_{n,m_1,\ldots,m_r}^c(\boldsymbol{\alpha})$ *of* $\boldsymbol{N}^c(n,\boldsymbol{k};\boldsymbol{\alpha})$ *satisfies the following recursive relation:*

$$\eta_{n,m_1,\ldots,m_r}^c(\boldsymbol{\alpha}) = \sum_{i=0}^{n-1} \sum_{\boldsymbol{b}=\boldsymbol{0}}^{\boldsymbol{m}} qp^i \binom{\boldsymbol{m}}{\boldsymbol{b}} (\mu(i;\alpha_j))^{m_j-b_j} \eta_{n-i-1,b_1,\ldots,b_r}^c(\boldsymbol{\alpha})$$

$$+ nqp^{n-1} \prod_{j=1}^{r} (\mu(n-1;\alpha_j))^{m_j} + p^n \prod_{j=1}^{r} (\nu(n;\alpha_j))^{m_j}$$

$$- qp^{n-1} \sum_{i=0}^{n-1} \prod_{j=1}^{r} (\mu(n-i-1;\alpha_j) + \nu(i;\alpha_j))^{m_j} \qquad n \geq 1,$$

$$\eta_{0,m_1,\ldots,m_r}^c(\boldsymbol{\alpha}) = 0 \qquad (m_1, \ldots, m_r) \neq (0, \ldots, 0),$$

$$\eta_{0,m_1,\ldots,m_r}^c(\boldsymbol{\alpha}) = 1 \qquad (m_1, \ldots, m_r) = (0, \ldots, 0).$$

4.3 *Closed form expressions*

Expanding the equation (4.1) in a power series of z and picking out the coefficient of z^n, we may easily obtain the expression for the joint p.g.f. $\phi_n^c(\boldsymbol{t};\boldsymbol{\alpha})$.

THEOREM 4.5. *The joint p.g.f.* $\phi_n^c(\boldsymbol{t};\boldsymbol{\alpha})$ *of* $\boldsymbol{N}^c(n,\boldsymbol{k};\boldsymbol{\alpha})$ *is given by*

$$\phi_n^c(\boldsymbol{t};\boldsymbol{\alpha}) = p^n \sum_{m=1}^{n} m \sum_{n_1+2n_2+\cdots+nn_n=n-m} \binom{n_1+n_2+\cdots+n_n}{n_1,n_2,\dots,n_n} \left(\frac{q}{p}\right)^{n_1+\cdots+n_n+1}$$
$$\times\, \boldsymbol{t}^{\sum_{i=2}^{n} n_i \boldsymbol{\mu}(i-1;\boldsymbol{\alpha}) + \boldsymbol{\mu}(m-1;\boldsymbol{\alpha})} + p^n \boldsymbol{t}^{\boldsymbol{\nu}(n;\boldsymbol{\alpha})} \qquad n \geq 0,$$

where $\boldsymbol{\mu}(i;\boldsymbol{\alpha})$, $\boldsymbol{\nu}(i;\boldsymbol{\alpha})$ *are as in* (2.1), (3.1).

Picking out the coefficient of $t_1^{x_1} t_2^{x_2} \cdots t_r^{x_r}$ in $\phi_n^c(\boldsymbol{t};\boldsymbol{\alpha})$ provided in Theorem 4.5, we may obtain an explicit form of the joint probability function $B_n^c(\boldsymbol{x};\boldsymbol{\alpha})$.

THEOREM 4.6. *The joint probability function* $B_n^c(\boldsymbol{x};\boldsymbol{\alpha})$ *of* $\boldsymbol{N}^c(n,\boldsymbol{k};\boldsymbol{\alpha})$ *is given by*

$$B_n^c(\boldsymbol{x};\boldsymbol{\alpha}) = p^n \sum_{m=1}^{n} m \sum_{1} \binom{n_1+n_2+\cdots+n_n}{n_1,n_2,\dots,n_n} \left(\frac{q}{p}\right)^{n_1+\cdots+n_n+1} + p^n \delta_{\boldsymbol{x},\boldsymbol{\nu}(n;\boldsymbol{\alpha})}$$
$$n \geq 0,$$

where the inner summation $\sum_1$ *is over all non-negative integers* $\{n_i\}_{i=1}^n$ *satisfying the conditions*

$$\sum_{i=1}^{n} i n_i = n - m,$$
$$\sum_{i=2}^{n} n_i \boldsymbol{\mu}(i-1;\boldsymbol{\alpha}) + \boldsymbol{\mu}(m-1;\boldsymbol{\alpha}) = \boldsymbol{x},$$

and $\boldsymbol{\mu}(i;\boldsymbol{\alpha})$, $\boldsymbol{\nu}(i;\boldsymbol{\alpha})$ *are as in* (2.1), (3.1).

The expressions given in Theorems 4.5 and 4.6 may be unsuitable for the calculations, since the exact formulae involve the multinomial coefficients and the inner summations on index sets determined by the solutions of the conditions. However, we think that they are very helpful for explaining the combinatorial meanings.

Remark 4.1. We mention the circular-m-consecutive-k-out-of-n:F system where the circular binomial model is appropriate (see Chang *et al.* (2000)). This system fails whenever at least m non-overlapping failure runs of length k occur (suppose we define a "success" as a failed component). Then the system's reliability is given by $\sum_{x=0}^{m-1} P(N^c(n,k;I) = x)$ and can be easily calculated by making use of Theorem 4.3 for $r = 1$, $k_1 = k$, $\alpha_1 = I$. Furthermore we can obtain useful information for the more efficient study of this model. In addition to the reliability, the numbers of failure runs of length s ($s = 1,2,\dots,k-1$) provide useful clues to the maintenance.

Remark 4.2. Our results presented in Section 4 are useful for circular statistical problems in many fields. Recently, Agin and Godbole (1992) proposed a non-parametric test for randomness by making use of the number of non-overlapping linear success runs of length k. Koutras and Alexandrou (1997) suggested a test for randomness based on the number of linear success runs of length k by engaging Type I, II, III enumeration schemes. An analogous run test for a circular distribution can be established based on the number of circular success runs of length k by engaging Type I, II, III, IV enumeration schemes. More details on this topic will be presented in a subsequent paper.

5. Examples

In this section, we present computational results for the distributions of run statistics. Theorems and propositions obtained in Sections 2–4 are useful for the numerical and symbolic calculations. To illustrate our theoretical results for deriving the joint distribution of the numbers of success runs, several examples are given below.

5.1 *Multivariate distributions as special cases*

For $r = 4$, $k_1 = k_2 = k_3 = k_4 = k$ and $\boldsymbol{\alpha} = (I, II, III, IV)$, we consider the joint distributions of $(N(n,k;I), N(n,k;II), N(n,k;III), N(n,k;IV))$, $(N^c(n,k;I), N^c(n,k;II), N^c(n,k;III), N^c(n,k;IV))$. The double generating functions are given by the expressions (2.8) and (4.2) with

$$(5.1)\qquad P(\boldsymbol{t}, pz; \boldsymbol{\alpha}) = \frac{1-(pz)^k}{1-pz} + (pz)^k t_1 t_2 t_3 (t_4 - 1) + \frac{1-(pzt_3)^k}{1-pzt_3} \cdot \frac{(pz)^k t_1 t_2 t_3}{1-(pzt_3)^k t_1},$$

$$(5.2)\qquad R(\boldsymbol{t}, pz; \boldsymbol{\alpha}) = \frac{1-(pz)^k}{1-pz} + (pzt_3)^k t_1 t_2 (t_4 - 1) + \frac{1-(pzt_3)^k}{1-pzt_3} \cdot \frac{(pzt_3)^k t_1 t_2}{1-(pzt_3)^k t_1},$$

where $\boldsymbol{t} = (t_1, t_2, t_3, t_4)$.

The marginal distribution of $N^c(n,k;\alpha)$ ($\alpha = I, II, III, IV$) is called circular binomial distribution of order k. For $\alpha = I$, the corresponding distribution has studied by several authors (see Charalambides (1994), Makri and Philippou (1994) and Koutras *et al.* (1995)). For $\alpha = III$, Koutras *et al.* (1994) have given formulae for the probability function.

5.2 *Correlation coefficients*

The correlation coefficients between $N(n,k_1;\alpha_1)$ and $N(n,k_2;\alpha_2)$, $N^c(n,k_1;\alpha_1)$ and $N^c(n,k_2;\alpha_2)$ will be denoted by $\rho_{n,k_1,k_2}(\alpha_1,\alpha_2)$ and $\rho^c_{n,k_1,k_2}(\alpha_1,\alpha_2)$, respectively, for $\alpha_1, \alpha_2 = I, II, III, IV$.

For the special case $k_1 = k_2 = 3$, the correlation coefficients are plotted in Figs. 1–12.

5.3 *Numerical examples*

We consider the joint distribution of $(N^c(n,k_1;I), N^c(n,k_2;II))$. For $n = 10$, $k_1 = 2$, $k_2 = 3$ and $p = 0.7$, the joint p.g.f. is

$$\begin{aligned}\phi^c_{10}(\boldsymbol{t};\boldsymbol{\alpha}) = {} & 0.0102122451 + 0.038662029t_1 + 0.043535961t_1t_2 + 0.043174782t_1^2 \\ & + 0.175097727t_1^2t_2 + 0.0365300145t_1^2t_2^2 + 0.009529569t_1^3 \\ & + 0.267886773t_1^3t_2 + 0.096354531t_1^3t_2^2 + 0.0259416045t_1^4t_2^2 \\ & + 0.2248272390t_1^4t_2 + 0.02824752490t_1^5t_2.\end{aligned}$$

For $n = 100$, we give Fig. 13, which is the three-dimensional plot of the exact joint probability function of $(N^c(100,2;I), N^c(100,3;II))$.

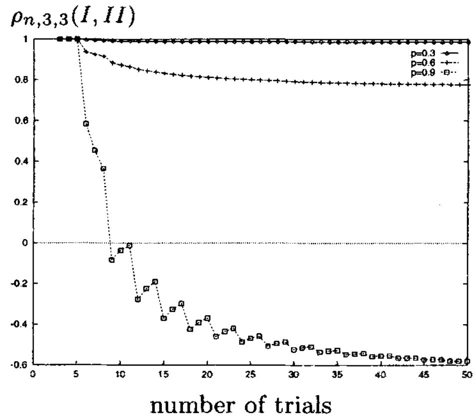

Fig. 1. The correlation coefficient between $N(n, 3; I)$ and $N(n, 3; II)$.

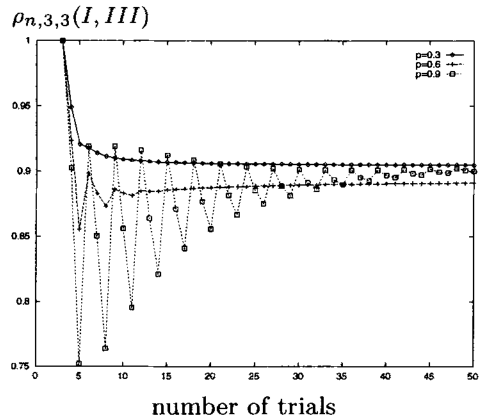

Fig. 2. The correlation coefficient between $N(n, 3; I)$ and $N(n, 3; III)$.

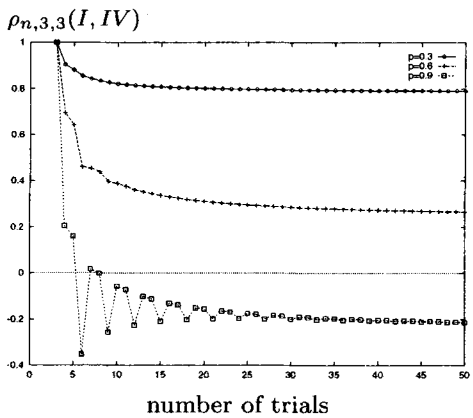

Fig. 3. The correlation coefficient between $N(n, 3; I)$ and $N(n, 3; IV)$.

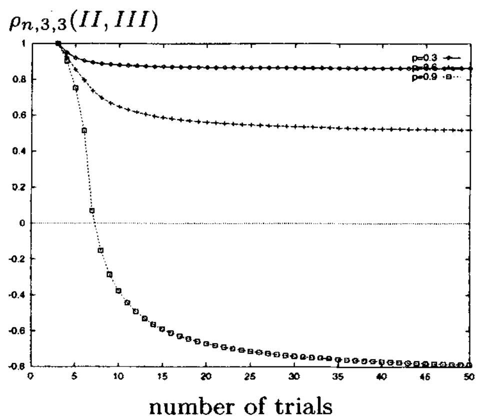

Fig. 4. The correlation coefficient between $N(n, 3; II)$ and $N(n, 3; III)$.

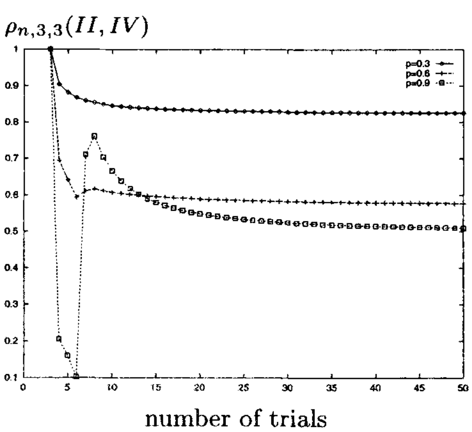

Fig. 5. The correlation coefficient between $N(n, 3; II)$ and $N(n, 3; IV)$.

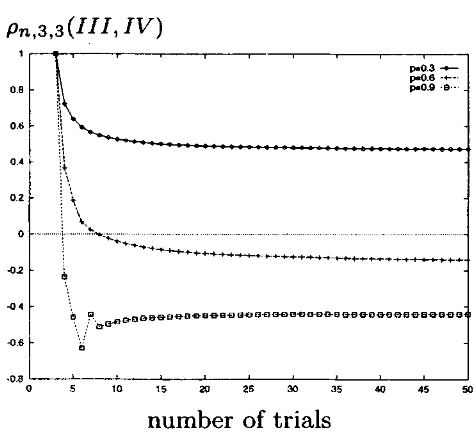

Fig. 6. The correlation coefficient between $N(n, 3; III)$ and $N(n, 3; IV)$.

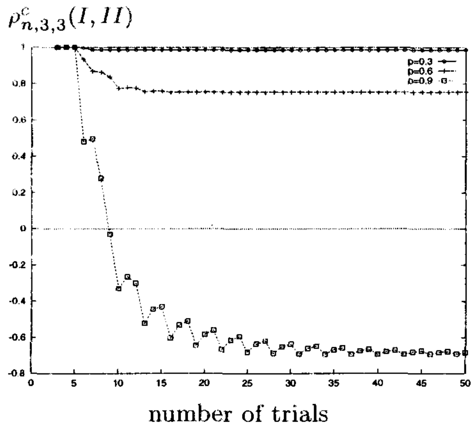

Fig. 7. The correlation coefficient between $N^c(n, 3; I)$ and $N^c(n, 3; II)$.

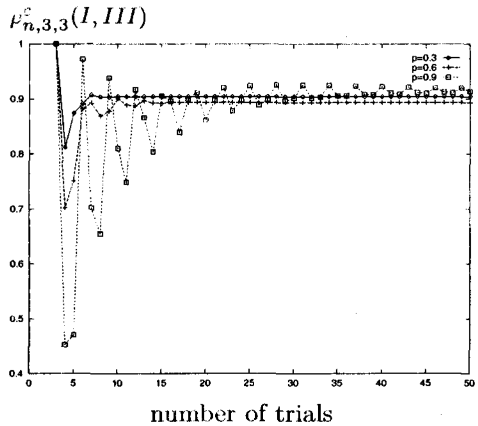

Fig. 8. The correlation coefficient between $N^c(n, 3; I)$ and $N^c(n, 3; III)$.

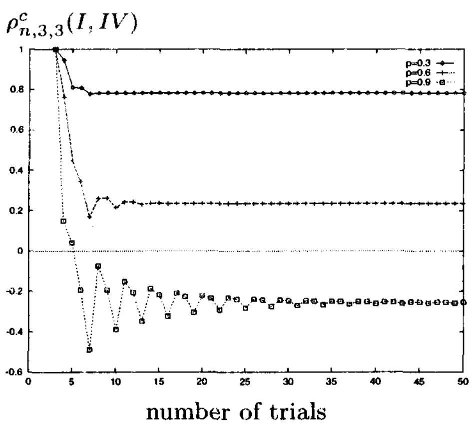

Fig. 9. The correlation coefficient between $N^c(n, 3; I)$ and $N^c(n, 3; IV)$.

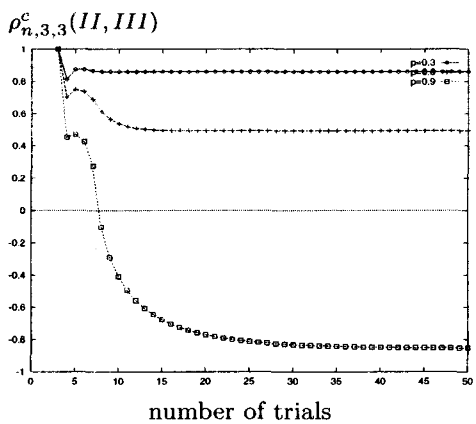

Fig. 10. The correlation coefficient between $N^c(n, 3; II)$ and $N^c(n, 3; III)$.

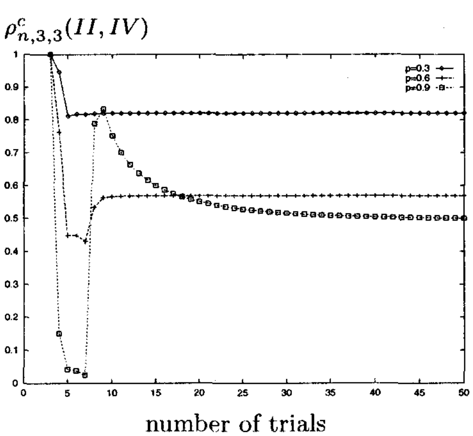

Fig. 11. The correlation coefficient between $N^c(n, 3; II)$ and $N^c(n, 3; IV)$.

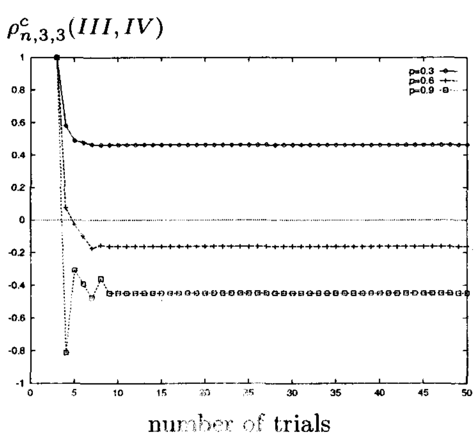

Fig. 12. The correlation coefficient between $N^c(n, 3; III)$ and $N^c(n, 3; IV)$.

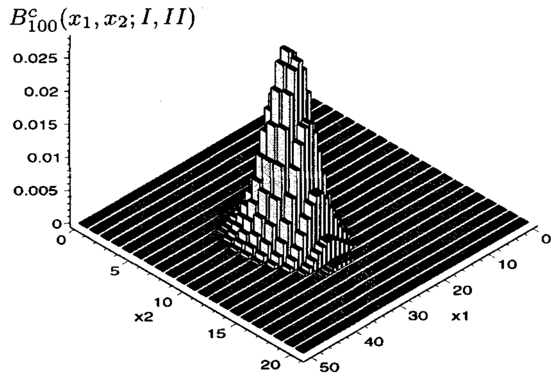

Fig. 13. The joint probability function of $(N^c(100, 2; I), N^c(100, 3; II))$, given $p = 0.7$.

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