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# SEMIPARAMETRIC SPATIO-TEMPORAL COVARIANCE MODELS WITH THE ARMA TEMPORAL MARGIN

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**Abstract.** Starting from a purely spatial variogram, this paper derives a class of semiparametric spatio-temporal covariance models that are stationary in time but not necessarily stationary in space. In particular, we obtain spatio-temporal covariance models with the continuous-time autoregressive and moving average (ARMA) temporal margin and long-range dependent spatial margin.

*Key words and phrases*: Autoregressive and moving average, covariance, intrinsically stationary, long-range dependence, stationary, variogram.

## 1. Introduction

There is a great demand for statistical modeling of phenomena that evolve in both space and time. Practical examples are those in Haslett and Raftery (1989), Handcock and Wallis (1994), Cressie and Huang (1999), Brix and Diggle (2001), Stroud *et al.* (2001), De Iaco *et al.* (2002), Gneiting (2002), and Hartfield and Gunst (2003), to mention but a few. Two commonly used tools to describe the space-time interaction and dependence are the covariance function and variogram, a wide variety of which are demanded for the practical use. The aim of this paper is to introduce a flexible class of spatio-temporal covariance models.

For a real-valued random field $Z(\boldsymbol{s};t)$ defined over a spatial domain $\mathcal{S}$ and a temporal domain $\mathcal{T}$, where $\mathcal{S} = \mathbb{R}^d$ or $\mathbb{Z}^d$, and $\mathcal{T} = \mathbb{Z}$ or $\mathbb{R}$, we denote its covariance function by

$$C(\boldsymbol{s}_1, \boldsymbol{s}_2; t_1, t_2) = \mathrm{E}[\{Z(\boldsymbol{s}_1;t_1) - \mathrm{E}Z(\boldsymbol{s}_1;t_1)\}\{Z(\boldsymbol{s}_2;t_2) - \mathrm{E}Z(\boldsymbol{s}_2;t_2)\}],$$

and its variogram by

$$\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2; t_1, t_2) = \frac{1}{2}\operatorname{var}\{Z(\boldsymbol{s}_1;t_1) - Z(\boldsymbol{s}_2;t_2)\}, \qquad (\boldsymbol{s}_1;t_1), (\boldsymbol{s}_2;t_2) \in \mathcal{S} \times \mathcal{T}.$$

Under the assumption that $\operatorname{var}(Z(\boldsymbol{s};t)) < \infty$ for all $(\boldsymbol{s};t) \in \mathcal{S} \times \mathcal{T}$, the covariance function and the variogram are well-defined, with a simple identity,

$$\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2; t_1, t_2) = \frac{1}{2}\{C(\boldsymbol{s}_1, \boldsymbol{s}_1; t_1, t_1) + C(\boldsymbol{s}_2, \boldsymbol{s}_2; t_2, t_2)\} - C(\boldsymbol{s}_1, \boldsymbol{s}_2; t_1, t_2).$$

An important relation between covariance and variogram, due mainly to Schoenberg ((1938), p. 828) (cf. Berg *et al.* (1984), p. 74), says that $\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2; t_1, t_2)$ is a variogram on $\mathcal{S} \times \mathcal{T}$ if and only if $\exp\{-\alpha\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2; t_1, t_2)\}$ is a covariance on $\mathcal{S} \times \mathcal{T}$ for all $\alpha \geq 0$.

However, as one of the referees critically points out, the continuity of the variogram is a necessary condition for many authors; see Yaglom ((1957), p. 289), Johansen ((1966), p. 305), Cressie ((1993), p. 87), and Chilès and Delfiner ((1999), pp. 66–67). Since this is an important point and need to be better explained, a proof is provided in Appendix A.1 for the general case, as the referee suggests.

The random field $Z(s;t)$ is said to be (weakly, or second-order) stationary in time if

$$\mathrm{E}Z(s;t_1) = \mathrm{E}Z(s;t_2), \text{ and } C(s_1, s_2; t_1, t_2) \text{ depends only on } t_1 - t_2 \text{ and } s_1, s_2;$$

stationary in space if

$$\mathrm{E}Z(s_1;t) = \mathrm{E}Z(s_2;t), \text{ and } C(s_1, s_2; t_1, t_2) \text{ depends only on } s_1 - s_2 \text{ and } t_1, t_2;$$

and stationary (in space-time) if

$$\mathrm{E}Z(s_1;t_1) = \mathrm{E}Z(s_2;t_2), \text{ and } C(s_1, s_2; t_1, t_2) \text{ depends only on } s_1 - s_2 \text{ and } t_1 - t_2.$$

When the random field is stationary in time (or space), its covariance is denoted by $C(s_1, s_2; t)$ (or $C(s; t_1, t_2)$) for simplicity, and when the random field is stationary in space-time, its covariance is denoted by $C(s;t)$ and called a stationary covariance function.

In the same manner the intrinsic stationarity is defined in terms of the variogram. Equivalently, the random field $Z(s;t)$ is intrinsically stationary in space-time if the increment process $Z(s + s_0; t + t_0) - Z(s;t)$ is stationary in space-time for any fixed $(s_0; t_0) \in \mathcal{S} \times \mathcal{T}$. The intrinsic stationarity is more general than the second-order stationarity, since there are processes for which the variogram is well-defined but the covariance is not. If $Z(s;t)$ is stationary, then it is also intrinsically stationary.

It is not uncommon that a spatio-temporal random field is not stationary in space. Starting from a purely spatial variogram $\gamma(s_1, s_2)$, in this paper we derive a class of spatio-temporal covariance models that are stationary in time but may not stationary in space, by using the cosine transform method (cf. Ma (2003*a*)), which, in the stationary case, closely relates to the (inverse) Fourier transform method of Cressie and Huang (1999) derived from Bochner's theorem. The general form of our model is proposed in Section 2, and special cases with the continuous-time autoregressive and moving average (CARMA) temporal margin are given in Section 3. Section 4 offers a summary and related concluding remarks.

## 2. The general form of the model

In what follows assume that $p$ is a positive integer, $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_p$ are positive constants, and $\gamma(s_1, s_2)$ is a purely spatial variogram on $\mathcal{S}$. Our proposed spatio-temporal covariance function is of the form (2.1).

THEOREM 1. *If $\kappa(\omega)$ is a nonnegative function on $[0, \infty)$ such that the integral in (2.1) is finite for all $s_1, s_2 \in \mathcal{S}, t \in \mathbb{R}$, then*

$$C(s_1, s_2; t) = \int_0^\infty \left[ \prod_{k=1}^p \{\alpha_k^2 + \beta_k^2 \gamma(s_1, s_2) + \omega^2\} \right]^{-1} \cos(t\omega)\kappa(\omega) d\omega, \quad s_1, s_2 \in \mathcal{S}, t \in \mathbb{R} \tag{2.1}$$

*is a spatio-temporal covariance function on $\mathcal{S} \times \mathbb{R}$ stationary in time.*

PROOF. For each $k \in \{1, \ldots, p\}$ and $\omega \geq 0$, $\{\alpha_k^2 + \beta_k^2 \gamma(s_1, s_2) + \omega^2\}^{-1}$ is a purely spatial covariance on $\mathcal{S}$, since it can be expressed as

$$\{\alpha_k^2 + \beta_k^2 \gamma(s_1, s_2) + \omega^2\}^{-1} = \int_0^\infty \exp\{-(\alpha_k^2 + \beta_k^2 \gamma(s_1, s_2) + \omega^2) u\} du, \qquad s_1, s_2 \in \mathcal{S},$$

where the integrand is a purely spatial covariance on $\mathcal{S}$ by Schoenberg's theorem (cf. Appendix A.1). Thus, the product $\prod_{k=1}^p \{\alpha_k^2 + \beta_k^2 \gamma(s_1, s_2) + \omega^2\}^{-1}$ is a purely spatial covariance on $\mathcal{S}$, and the integrand of (2.1) is a separable spatio-temporal covariance function on $\mathcal{S} \times \mathbb{R}$ for every fixed $\omega \geq 0$. As a result, (2.1) is a valid spatio-temporal covariance function on $\mathcal{S} \times \mathbb{R}$. □

An important feature of the model (2.1) is that at fixed locations $s_1, s_2$, (2.1) is a purely temporal covariance on the real line, since the integrand of (2.1) is a purely temporal covariance for fixed $\omega, s_1, s_2$.

The basic feature of the model (2.1) is that it is semi-parametric: it is nonparametric with respect to the flexibility of the purely spatial variogram $\gamma(s_1, s_2)$, but it depends on parameters $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_p$. Sources for purely spatial variograms are Cressie (1993), and Chilès and Delfiner (1999), among others.

The model (2.1) is stationary in time, but not stationary in space unless $\gamma(s_1, s_2)$ is intrinsically stationary on $\mathcal{S}$. In a particular case $\gamma(s_1, s_2) = \|g(s_1) - g(s_2)\|$, where $g(\cdot)$ is a bijective deformation of the geographic coordinate system and $\|s_1 - s_2\|$ denotes the usual Euclidean distance, the space deformation approach of Sampson and Guttorp (1992) can be employed for modeling space-time data.

Various covariance models can be derived from (2.1) by appropriate selection of the function $\kappa(\omega)$ and parameters $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_p$. As an example, let $\kappa(\omega) = \frac{4}{\pi} \exp(-\omega^2)$, $\omega \in \mathbb{R}$, $p = 1$, $\alpha_1^2 = \alpha$ and $\beta_1^2 = \beta$ in (2.1). Using the formula (cf. Bateman (1954), p. 15)

$$\begin{aligned} &\frac{4}{\pi} \int_0^\infty \frac{e^{-\omega^2} \cos(t\omega) d\omega}{\omega^2 + u^2} \\ &\quad = u^{-1} e^{u^2} \left\{ e^{-u|t|} \mathrm{Erfc}\left(u - \frac{|t|}{2}\right) + e^{u|t|} \mathrm{Erfc}\left(u + \frac{|t|}{2}\right) \right\}, \qquad u > 0, t \in \mathbb{R}, \end{aligned}$$

where

$$\mathrm{Erfc}(x) = \begin{cases} \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-\omega^2) d\omega, & \text{if} \quad x \geq 0, \\ 2 - \mathrm{Erfc}(-x), & \text{if} \quad x < 0, \end{cases}$$

we obtain a spatio-temporal covariance function on $\mathcal{S} \times \mathbb{R}$,

$$\begin{aligned} C(s_1, s_2; t) = &\{\alpha + \beta\gamma(s_1, s_2)\}^{-1/2} e^{\alpha + \beta\gamma(s_1, s_2)} \\ &\times \left\{ e^{-(\alpha + \beta\gamma(s_1, s_2))^{1/2}|t|} \mathrm{Erfc}\left( (\alpha + \beta\gamma(s_1, s_2))^{1/2} - \frac{|t|}{2} \right) \right. \\ &\quad \left. + e^{(\alpha + \beta\gamma(s_1, s_2))^{1/2}|t|} \mathrm{Erfc}\left( (\alpha + \beta\gamma(s_1, s_2))^{1/2} + \frac{|t|}{2} \right) \right\}, \\ &\qquad s_1, s_2 \in \mathcal{S}, t \in \mathbb{R}, \end{aligned}$$

which is stationary in time.

## 3. Spatio-temporal models with the CARMA temporal margin

Among the classes of spatio-temporal models generated from (2.1), one in particular stands out, with the temporal margin being the continuous-time autoregressive and moving average model, and the spatial margin having long-range dependence in the stationary case. We derive these models below by appropriately choosing the function $\kappa(\omega)$ and constants in (2.1). When a purely spatial variogram $\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2)$ is intrinsically stationary on $\mathcal{S}$, we write $\gamma(\boldsymbol{s}_1 - \boldsymbol{s}_2)$ instead of $\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2)$ for simplicity.

### 3.1 *Spatio-temporal models with the* CAR(1) *temporal margin*

Let $p = 1$, $\alpha_1 = \beta_1 = \alpha$, and $\kappa(\omega) = \frac{2\alpha}{\pi}$, $\omega \geq 0$. It follows from (2.1) that

$$\begin{aligned} C(\boldsymbol{s}_1, \boldsymbol{s}_2; t) &= (1 + \gamma(\boldsymbol{s}_1, \boldsymbol{s}_2))^{-1/2} \\ &\quad \times \exp\{-\alpha|t|(1 + \gamma(\boldsymbol{s}_1, \boldsymbol{s}_2))^{1/2}\}, \qquad \boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}, t \in \mathbb{R}. \end{aligned} \tag{3.1}$$

At a fixed location $\boldsymbol{s}$, setting $\boldsymbol{s}_1 = \boldsymbol{s}_2 = \boldsymbol{s}$ in (3.1) yields the temporal margin of the model (3.1),

$$C(\boldsymbol{s}, \boldsymbol{s}; t) = \exp(-\alpha|t|), \qquad t \in \mathbb{R},$$

which is the Ornstein-Uhlenbeck or CAR(1) model. The spatial margin of the model (3.1) is obtained by setting $t = 0$,

$$C(\boldsymbol{s}_1, \boldsymbol{s}_2; 0) = (1 + \gamma(\boldsymbol{s}_1, \boldsymbol{s}_2))^{-1/2}, \qquad \boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}.$$

In a specific case $\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2) = \|\boldsymbol{s}_1 - \boldsymbol{s}_2\|^\theta$, which is an intrinsically stationary variogram when $\theta$ is a constant between 0 and 2, $C(\boldsymbol{s}_1, \boldsymbol{s}_2; 0)$ is a power-law covariance. The reader is referred to Whittle (1956, 1962) for earlier systematic studies of power-law covariances, where evidence from agricultural uniformity trials strongly indicates that the covariance function of yield in the plane decays ultimately as the inverse of distance, and to Ma (2003*c*) for recent development of the exact power-law and other long-range dependent models on a planar lattice.

The model (3.1) is the product of $(1 + \gamma(\boldsymbol{s}_1, \boldsymbol{s}_2))^{-1/2}$ and $\exp\{-\alpha|t|(1 + \gamma(\boldsymbol{s}_1, \boldsymbol{s}_2))^{1/2}\}$, where the former is a purely spatial covariance on $\mathcal{S}$. But, the latter is not a valid covariance on $\mathcal{S} \times \mathbb{R}$ unless $\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2) \equiv 0$, as shown in Example 2 of Ma (2003*b*).

Interestingly, at fixed locations $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}$, (3.1) is a purely temporal model on $\mathbb{R}$ with

$$\frac{\partial}{\partial t} C(\boldsymbol{s}_1, \boldsymbol{s}_2; t) + \alpha(1 + \gamma(\boldsymbol{s}_1, \boldsymbol{s}_2))^{1/2} C(\boldsymbol{s}_1, \boldsymbol{s}_2; t) = 0, \qquad t > 0.$$

This may suggest a similar first-order stochastic differential equation for the process itself, in case $\gamma(\boldsymbol{s})$ is an intrinsically stationary variogram on $\mathcal{S}$,

$$\frac{\partial}{\partial t} Z(\boldsymbol{s}; t) + \alpha(1 + \gamma(\boldsymbol{s}))^{1/2} Z(\boldsymbol{s}; t) = 0, \qquad t > 0.$$

In the discrete-time setting, the temporal margin of (3.1) corresponds to a stationary process $\{Z(\boldsymbol{s}; t), t \in \mathbb{Z}\}$ satisfying the following first-order stochastic difference equation for every fixed $\boldsymbol{s} \in \mathcal{S}$,

$$Z(\boldsymbol{s}; t) - \exp\{-\alpha(1 + \gamma(\boldsymbol{s}))^{1/2}\} Z(\boldsymbol{s}; t - 1) = \varepsilon(\boldsymbol{s}; t), \qquad t \in \mathbb{Z},$$

where $\varepsilon(s;t)$, $t \in \mathbb{Z}$, is the white noise. The correlation structure of a class of stationary, unilateral, linear autoregressions defined on a lattice were studied by Whittle (1954) and Besag (1972).

For a fixed temporal lag $t \in \mathbb{R}$, (3.1) is also a purely spatial covariance model on $\mathcal{S}$. To see this, notice that $\gamma(s_1, s_2)$ is a variogram, so that the positive function $1+\gamma(s_1, s_2)$ is negative definite on $\mathcal{S}$. By Corollary 2.10 of Berg *et al.* (1984), $(1+\gamma(s_1, s_2))^{1/2}$ and $\ln(1+\gamma(s_1, s_2))$ are negative definite. It follows from Schoenberg's theorem that

$$(1+\gamma(s_1, s_2))^{-1/2} = \exp\left\{-\frac{1}{2}\ln(1+\gamma(s_1, s_2))\right\}$$

and $\exp\{-\alpha|t|(1+\gamma(s_1, s_2))^{1/2}\}$ are positive definite on $\mathcal{S}$. Their product, $(1+\gamma(s_1, s_2))^{-1/2}\exp\{-\alpha|t|(1+\gamma(s_1, s_2))^{1/2}\}$, is thus a purely spatial covariance on $\mathcal{S}$ when $t$ is fixed.

3.2 *Spatio-temporal models with the* CARMA(2, 1) *temporal margin*

Let $p = 2$. In this subsection we derive the spatio-temporal model whose temporal margin $C(s, s; t)$ satisfying $C(s, s; -t) = C(s, s; t)$, $t \in \mathbb{R}$, and a second-order differential equation

$$\frac{\partial^2}{\partial^2 t}C(s, s; t) + a_1\frac{\partial}{\partial t}C(s, s; t) + a_2 C(s, s; t) = 0, \qquad t > 0, \tag{3.2}$$

where $a_1$ and $a_2$ are nonnegative numbers subject to $a_1^2 - 4a_2 \geq 0$. Three specific cases are considered as follows.

*Case* (i). In (2.1) taking $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \alpha > 0$ and $\kappa(\omega) \equiv \frac{4\alpha^3}{\pi}$, $\omega \geq 0$, and using the formula

$$\int_0^\infty (\omega^2 + u^2)^{-2}\cos(t\omega)d\omega = \frac{\pi}{4u^3}(1+u|t|)\exp(-u|t|), \qquad u > 0, t \in \mathbb{R},$$

we obtain, for $s_1, s_2 \in \mathcal{S}$, $t \in \mathbb{R}$,

$$\begin{aligned} C(s_1, s_2; t) &= (1+\gamma(s_1, s_2))^{-3/2}\{1+\alpha|t|(1+\gamma(s_1, s_2))^{1/2}\} \\ &\quad \times \exp\{-\alpha|t|(1+\gamma(s_1, s_2))^{1/2}\}. \end{aligned} \tag{3.3}$$

Clearly, its temporal margin,

$$C(s, s; t) = (1+\alpha|t|)\exp(-\alpha|t|), \qquad t \in \mathbb{R},$$

is a CAR(2) model satisfying equation (3.2) with $a_1 = 2\alpha$ and $a_2 = \alpha^2$. The spatial margin of (3.3) is

$$C(s_1, s_2; 0) = (1+\gamma(s_1, s_2))^{-3/2}, \qquad s_1, s_2 \in \mathcal{S},$$

which is a power-law covariance in the particular case where $\gamma(s_1, s_2) = \|s_1 - s_2\|^\theta$ and $\theta \in (0, 2]$.

At fixed locations $s_1, s_2$, (3.3) is a purely temporal CAR(2) model and satisfies the second-order differential equation

$$\begin{aligned} &\frac{\partial^2}{\partial^2 t}C(s_1, s_2; t) + 2\alpha(1+\gamma(s_1, s_2))^{1/2}\frac{\partial}{\partial t}C(s_1, s_2; t) \\ &\quad + \alpha^2(1+\gamma(s_1, s_2))C(s_1, s_2; t) = 0, \qquad t > 0. \end{aligned}$$

This property is useful when one makes the statistical inference for the parameter $\alpha$.

*Case* (ii). Suppose now that $\alpha_1 > \alpha_2 > 0$. Letting $\kappa(\omega) \equiv \frac{2}{\pi}\alpha_1\alpha_2(\alpha_1^2 - \alpha_2^2)^{-1}$, $\omega \geq 0$, $\beta_1 = \alpha_1$, and $\beta_2 = \alpha_2$ in (2.1), yields the model

$$(3.4) \quad C(s_1, s_2; t) = (1 + \gamma(s_1, s_2))^{-3/2}[\alpha_1 \exp\{-\alpha_2|t|(1 + \gamma(s_1, s_2))^{1/2}\} - \alpha_2 \exp\{-\alpha_1|t|(1 + \gamma(s_1, s_2))^{1/2}\}], \quad s_1, s_2 \in \mathcal{S}, t \in \mathbb{R}.$$

The spatial margin of the model (3.4) is

$$C(s_1, s_2; 0) = (\alpha_1 - \alpha_2)\{1 + \gamma(s_1, s_2)\}^{-3/2}, \qquad s_1, s_2 \in \mathcal{S},$$

which allows for long-range dependence when $\gamma(s_1, s_2)$ is intrinsically stationary. Its temporal margin,

$$C(s, s; t) = \alpha_1 \exp(-\alpha_2|t|) - \alpha_2 \exp(-\alpha_1|t|), \qquad t \in \mathbb{R},$$

is a CAR(2) model satisfying equation (3.2) everywhere on the real line with $a_1 = \alpha_1 + \alpha_2$ and $a_2 = \alpha_1\alpha_2$.

The spatio-temporal covariance function (3.4) is twice continuously differentiable with respect to $t \in \mathbb{R}$, and satisfies a second-order partial differential equation everywhere on the real line,

$$(3.5) \quad \frac{\partial^2}{\partial^2 t}C(s_1, s_2; t) + (\alpha_1 + \alpha_2)(1 + \gamma(s_1, s_2))^{1/2}\frac{\partial}{\partial t}C(s_1, s_2; t) + \alpha_1\alpha_2(1 + \gamma(s_1, s_2))C(s_1, s_2; t) = 0, \qquad t \in \mathbb{R}.$$

As a result, the partial derivative process $\{\frac{\partial}{\partial t}Z(s; t), s \in \mathcal{S}, t \in \mathbb{R}\}$ exists in the mean squared sense, and possesses the covariance function

$$-\frac{\partial^2}{\partial^2 t}C(s_1, s_2; t) = \alpha_1\alpha_2(1 + \gamma(s_1, s_2))^{-1/2}[\alpha_1 \exp\{-\alpha_1|t|(1 + \gamma(s_1, s_2))^{1/2}\} - \alpha_2 \exp\{-\alpha_2|t|(1 + \gamma(s_1, s_2))^{1/2}\}], \quad s_1, s_2 \in \mathcal{S}, t \in \mathbb{R}.$$

In other words, the spatio-temporal covariance function (3.6) below can be alternatively derived as the negative of the second partial derivative of (3.4) with respect to $t$.

The covariance (3.3) has a close link to (3.4) as well. Indeed, one can obtain (3.3) from (3.4) by letting $\alpha_1 = \alpha$ and taking the left limit of $(\alpha - \alpha_2)^{-1}C(s_1, s_2; t)$ as $\alpha_2 \to \alpha$.

Equation (3.5) holds also for a purely temporal CAR(2) model obtained from (3.4) with fixed locations $s_1, s_2 \in \mathcal{S}$.

*Case* (iii). In case $\beta_1 = \alpha_1, \beta_2 = \alpha_2$, $\alpha_1 > \alpha_2 > 0$, and $\kappa(\omega) = \frac{2\omega^2}{(\alpha_1^2 - \alpha_2^2)\pi}$, $\omega \geq 0$, from (2.1) we obtain the model

$$(3.6) \quad C(s_1, s_2; t) = (1 + \gamma(s_1, s_2))^{-1/2}[\alpha_1 \exp\{-\alpha_1|t|(1 + \gamma(s_1, s_2))^{1/2}\} - \alpha_2 \exp\{-\alpha_2|t|(1 + \gamma(s_1, s_2))^{1/2}\}].$$

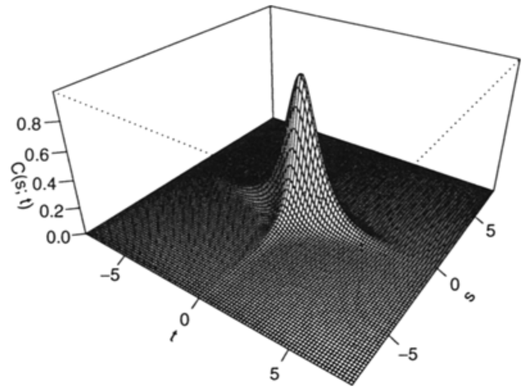


Fig. 1. The plot of the covariance function (3.4) versus $(s;t)$, where $d = 1$, $\gamma(s_1, s_2) = (s_1 - s_2)^2$, $\alpha_1 = 2$, and $\alpha_2 = 1$.

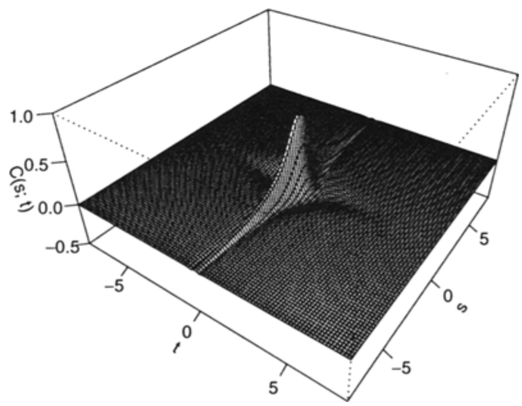


Fig. 2. The plot of the covariance function (3.6) versus $(s;t)$, where $d = 1$, $\gamma(s_1, s_2) = (s_1 - s_2)^2$, $\alpha_1 = 2$, and $\alpha_2 = 1$.

Its temporal margin is a CARMA(2, 1) model satisfying equation (3.2) with $a_1 = \alpha_1 + \alpha_2$ and $a_2 = \alpha_1 \alpha_2$,

$$C(\boldsymbol{s}, \boldsymbol{s}; t) = \alpha_1 \exp(-\alpha_1 |t|) - \alpha_2 \exp(-\alpha_2 |t|), \qquad t \in \mathbb{R},$$

and its spatial margin is

$$C(\boldsymbol{s}_1, \boldsymbol{s}_2; 0) = (\alpha_1 - \alpha_2)\{1 + \gamma(\boldsymbol{s}_1, \boldsymbol{s}_2)\}^{-1/2}, \qquad \boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}.$$

Just like (3.4), the spatio-temporal covariance function (3.6) satisfies the second-order partial differential equation (3.5) on $\mathbb{R}$ except at $t = 0$. Also, at fixed locations $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}$, (3.6) is a temporal CARMA(2, 1) model satisfying (3.5) on $\mathbb{R}$ except at $t = 0$.

A major difference between the model (3.4) and the model (3.6) is that the former is nonnegative for all $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}, t \in \mathbb{R}$, but the latter can assume negative values, as Figs. 1 and 2 illustrate. The following theorem describes a more general case that includes (3.4) and (3.6) as special cases.

THEOREM 2. *Let $\gamma(s_1, s_2)$ be a purely spatial variogram on $\mathcal{S}$, and $\alpha_1 > \alpha_2 > 0$. Then the function*

$$
\begin{aligned}
(3.7) \qquad C(s_1, s_2; t) = {} & (1 + \gamma(s_1, s_2))^{-1/2} \\
& \times [\{\theta\alpha_1 - (1-\theta)\alpha_2(1 + \gamma(s_1, s_2))^{-1}\} \\
& \quad \times \exp\{-\alpha_1|t|(1 + \gamma(s_1, s_2))^{1/2}\} \\
& \quad + \{(1-\theta)\alpha_1(1 + \gamma(s_1, s_2))^{-1} - \theta\alpha_2\} \\
& \quad \times \exp\{-\alpha_2|t|(1 + \gamma(s_1, s_2))^{1/2}\}], \qquad s_1, s_2 \in \mathcal{S}, t \in \mathbb{R},
\end{aligned}
$$

*is a spatio-temporal covariance function on $\mathcal{S} \times \mathbb{R}$ and stationary in time if and only if $\theta$ is a constant between* 0 *and* 1.

PROOF. If $0 \le \theta \le 1$, then as the convex combination of (3.4) and (3.6), (3.7) is a spatio-temporal covariance function on $\mathcal{S} \times \mathbb{R}$ and stationary in time.

On the other hand, let (3.7) be a spatio-temporal covariance function on $\mathcal{S} \times \mathbb{R}$. Then its temporal margin,

$$C(s, s; t) = \{\theta\alpha_1 - (1-\theta)\alpha_2\}\exp(-\alpha_1|t|) + \{(1-\theta)\alpha_1 - \theta\alpha_2\}\exp(-\alpha_2|t|), \qquad t \in \mathbb{R},$$

must be a stationary temporal covariance function on the real line, for which it is necessary that $\theta$ is a constant between 0 and 1, according to Lemma 1 of Ma (2003*d*). □

Obviously, the temporal margin of (3.7) satisfies the second-order differential equation (3.2), where $a_1$ and $a_2$ are nonnegative numbers subject to $a_1^2 - 4a_2 \ge 0$. It would be of interest to derive a spatio-temporal model with the CARMA(2, 1) temporal margin of the form (3.2) but $a_1^2 - 4a_2 < 0$.

3.3 *Spatio-temporal models with the* CARMA(p, q) *temporal margin*

Having discussed the spatio-temporal models with the CAR(1) and CARMA(2, 1) temporal margins in some details, we now extend the idea to obtain spatio-temporal models with the CARMA(p, q) ($0 \le$ q $<$ p) temporal margin that satisfies the $p$-th-order differential equation

$$\frac{\partial^p}{\partial^p t} C(s, s; t) + b_1 \frac{\partial^{p-1}}{\partial t^{p-1}} C(s, s; t) + \cdots + b_p C(s, s; t) = 0, \qquad t > 0,$$

where $b_1, \ldots, b_p$ are real numbers such that the roots of the polynomial $x^p + b_1 x^{p-1} + \cdots + b_p$ are all negative numbers.

As an illustration, suppose that $\alpha_1, \ldots, \alpha_p$ are distinct positive numbers. Decompose $[\prod_{k=1}^p \{\alpha_k^2 + \beta_k^2 \gamma(s_1, s_2) + \omega^2\}]^{-1}$ into partial fractions

$$
\begin{aligned}
& \left[\prod_{k=1}^p \{\alpha_k^2 + \beta_k^2 \gamma(s_1, s_2) + \omega^2\}\right]^{-1} \\
& \quad = \sum_{k=1}^p \left\{\prod_{j \ne k} (\alpha_k^2 - \alpha_j^2 + (\beta_k^2 - \beta_j^2)\gamma(s_1, s_2))\right\}^{-1} \{\alpha_k^2 + \beta_k^2 \gamma(s_1, s_2) + \omega^2\}^{-1}.
\end{aligned}
$$

Taking $\kappa(\omega) \equiv \frac{2}{\pi}$ $(\omega \geq 0)$ in (2.1) yields the model

$$
\begin{aligned}
(3.8)\qquad C(\boldsymbol{s}_1, \boldsymbol{s}_2; t) = \sum_{k=1}^{p} & \left\{ \prod_{j \neq k} (\alpha_k^2 - \alpha_j^2 + (\beta_k^2 - \beta_j^2)\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2)) \right\}^{-1} \\
& \times (\alpha_k^2 + \beta_k^2 \gamma(\boldsymbol{s}_1, \boldsymbol{s}_2))^{-1/2} \exp\{-|t|(\alpha_k^2 + \beta_k^2\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2))^{1/2}\}, \\
& \qquad \boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}, t \in \mathbb{R},
\end{aligned}
$$

whose temporal margin is a CARMA model,

$$
C(\boldsymbol{s}, \boldsymbol{s}; t) = \sum_{k=1}^{p} \left\{ \alpha_k \prod_{j \neq k} (\alpha_k^2 - \alpha_j^2) \right\}^{-1} \exp\{-\alpha_k |t|\}, \qquad t \in \mathbb{R}.
$$

Observe that $C(\boldsymbol{s}_1, \boldsymbol{s}_2; t)$ defined by (3.8) is twice continuously differentiable everywhere on the real line with respect to $t$. We will show that

$$
\begin{aligned}
-\frac{\partial^2}{\partial^2 t} C(\boldsymbol{s}_1, \boldsymbol{s}_2; t) = -\sum_{k=1}^{p} & \left\{ \prod_{j \neq k} (\alpha_k^2 - \alpha_j^2 + (\beta_k^2 - \beta_j^2)\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2)) \right\}^{-1} \\
& \times (\alpha_k^2 + \beta_k^2 \gamma(\boldsymbol{s}_1, \boldsymbol{s}_2))^{1/2} \exp\{-|t|(\alpha_k^2 + \beta_k^2\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2))^{1/2}\}, \\
& \qquad \boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}, t \in \mathbb{R},
\end{aligned}
$$

is also a spatio-temporal covariance function on $\mathcal{S} \times \mathbb{R}$.

Suppose that $\{Z(\boldsymbol{s}; t), (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R}\}$ is a spatio-temporal random field with mean zero and covariance (3.8). For a fixed $h \in \mathbb{R}$, consider the increment process $\{Z(\boldsymbol{s}; t + h) - Z(\boldsymbol{s}; t), (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R}\}$, whose covariance function can be easily verified to be $2C(\boldsymbol{s}_1, \boldsymbol{s}_2; t) - C(\boldsymbol{s}_1, \boldsymbol{s}_2; t + h) - C(\boldsymbol{s}_1, \boldsymbol{s}_2; t - h)$. Thus, for every $h \neq 0$,

$$
\frac{2C(\boldsymbol{s}_1, \boldsymbol{s}_2; t) - C(\boldsymbol{s}_1, \boldsymbol{s}_2; t + h) - C(\boldsymbol{s}_1, \boldsymbol{s}_2; t - h)}{h^2}
$$

is a spatio-temporal covariance function. Finally, by letting $h \to 0$, we obtain that

$$
-\frac{\partial^2}{\partial^2 t} C(\boldsymbol{s}_1, \boldsymbol{s}_2; t) = \lim_{h \to 0} \frac{2C(\boldsymbol{s}_1, \boldsymbol{s}_2; t) - C(\boldsymbol{s}_1, \boldsymbol{s}_2; t + h) - C(\boldsymbol{s}_1, \boldsymbol{s}_2; t - h)}{h^2}, \qquad \boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}, t \in \mathbb{R}
$$

is a spatio-temporal covariance function on $\mathcal{S} \times \mathbb{R}$. This means that the partial derivative process $\{\frac{\partial}{\partial t} Z(\boldsymbol{s}; t), \boldsymbol{s} \in \mathcal{S}, t \in \mathbb{R}\}$ exists in the mean squared sense.

## 4. Conclusion

The models proposed in this paper offer a much greater degree of flexibility for data analysis because of the free choice of the purely spatial variogram $\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2)$, for which stationarity or intrinsic stationarity does not necessarily require. Of course, parametric models can be easily obtained by specifying $\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2)$. This kind of flexibility suggests the following practical strategy for space-time data analysis. First, by fitting the data at a fixed time we could obtain the form of $\gamma(\boldsymbol{s}_1, \boldsymbol{s}_2)$ using purely spatial techniques, such as the space deformation approach or other nonparametric methods. After estimating $\gamma(s_1, s_2)$, we then fit the space-time data through a parametric version of the covariance $C(s_1, s_2; t)$. See De Iaco *et al.* (2002) for some other practical aspects.

The cosine transform method is employed here for our derivation of spatio-temporal covariance models. This simple approach is easy to be used, as examples of Ma (2003*a*) illustrate. As another example, Appendix A.2 gives a relatively simple proof of Theorem 2 of Gneiting (2002). It can be seen that (11) of Gneiting (2002) is essentially a special case of (3.3) of Ma (2003*a*). More examples can be found in Ma (2005).

Another method we use here to derive the spatio-temporal covariance function is to take the second partial derivative, whenever it exists, of a spatio-temporal covariance function with respect to the time lag, and then add a negative sign. Alternatively, one may consider the partial integral of a spatio-temporal random field with respect to the time lag. Aggregation in time for a spatio-temporal stationary random field produces a random field intrinsically stationary in time. We explore this idea in a forthcoming manuscript.

There are two research lines following empirical observations in the literature on long-range dependence, long memory, or persistence. One line is concerned with spatial long-range dependence, and the other is primarily focused on temporal long memory. Historically, the first line started from agricultural uniformity trials made by Fairfield Smith in 1930s that strongly indicated that the covariance function of yield in the plane decays ultimately as the inverse of the Euclidean distance. To explain such asymptotic behavior of variation, some random fields with a power-law covariance at large distances were derived in Whittle (1956, 1962) via stochastic partial differential equations. Exact power-law covariances on a planar lattice are recently obtained by Ma (2003*c*). The second line was initially based on the so-called Hurst phenomenon, which was observed on river flow data by H. E. Hurst in 1950s. A rapidly expanding empirical literature has found evidence of long-range dependence in many time series data; see, for instance, Doukhan *et al.* (2003). By randomizing the time scale of a spatio-temporal random field (cf. Ma (2003*d*)), one could obtain a long-range dependent spatio-temporal random field.

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## Appendix

### A.1 *A proof of Schoenberg's theorem*

THEOREM A.1. (Schoenberg) *Assume that $\gamma(s_1, s_2; t_1, t_2)$ is a nonnegative function on $\mathcal{S} \times \mathcal{T}$ with $\gamma(s, s; t, t) = 0$ for all $(s; t) \in \mathcal{S} \times \mathcal{T}$. Then $\gamma(s_1, s_2; t_1, t_2)$ is a variogram on $\mathcal{S} \times \mathcal{T}$ if and only if* $\exp\{-\alpha\gamma(s_1, s_2; t_1, t_2)\}$ *is a covariance function on $\mathcal{S} \times \mathcal{T}$ for all $\alpha \geq 0$.*

PROOF. Suppose that $\exp\{-\alpha\gamma(s_1, s_2; t_1, t_2)\}$ is the covariance function of a random field $\{Z_\alpha(s; t), (s; t) \in \mathcal{S} \times \mathcal{T}\}$. Then for $\alpha > 0$, the random field $\{\alpha^{-1/2} Z_\alpha(s; t), (s; t) \in \mathcal{S} \times \mathcal{T}\}$ possesses the variogram

$$\gamma_\alpha(s_1, s_2; t_1, t_2) = \frac{1}{2\alpha} \operatorname{var}(Z_\alpha(s_1; t_1) - Z_\alpha(s_2; t_2))$$

$$= \frac{1}{2\alpha}\{\mathrm{var}(Z_\alpha(s_1;t_1)) + \mathrm{var}(Z_\alpha(s_2;t_2)) - 2\,\mathrm{cov}(Z_\alpha(s_1;t_1), Z_\alpha(s_2;t_2))\}$$
$$= \alpha^{-1}[1 - \exp\{-\alpha\gamma(s_1, s_2; t_1, t_2)\}].$$

Letting $\alpha$ tend to zero, we obtain that

$$\lim_{\alpha\to 0_+} \gamma_\alpha(s_1, s_2; t_1, t_2) = \lim_{\alpha\to 0_+} \frac{1 - \exp\{-\alpha\gamma(s_1, s_2; t_1, t_2)\}}{\alpha} = \gamma(s_1, s_2; t_1, t_2)$$

is a variogram on $\mathcal{S} \times \mathcal{T}$.

On the other hand, let $\gamma(s_1, s_2; t_1, t_2)$ be a variogram associated with a (Gaussian) random field $\{Z(s;t), (s;t) \in \mathcal{S}\times\mathcal{T}\}$. For a fixed $(s_0;t_0) \in \mathcal{S}\times\mathcal{T}$, consider an increment process $\{Z(s;t) - Z(s_0;t_0), (s;t) \in \mathcal{S}\times\mathcal{T}\}$, which possesses second-order moments, and in particular, its covariance is

$$\begin{aligned}
&\mathrm{cov}(Z(s_1;t_1) - Z(s_0;t_0), Z(s_2;t_2) - Z(s_0;t_0))\\
&\quad = \mathrm{E}[\{Z(s_1;t_1) - Z(s_0;t_0) - \mathrm{E}(Z(s_1;t_1) - Z(s_0;t_0))\}\\
&\qquad\quad \times \{Z(s_2;t_2) - Z(s_0;t_0) - \mathrm{E}(Z(s_2;t_2) - Z(s_0;t_0))\}]\\
&\quad = \frac{1}{2}\mathrm{E}[\{Z(s_1;t_1) - Z(s_0;t_0) - \mathrm{E}(Z(s_1;t_1) - Z(s_0;t_0))\}^2\\
&\qquad\quad + \{Z(s_2;t_2) - Z(s_0;t_0) - \mathrm{E}(Z(s_2;t_2) - Z(s_0;t_0))\}^2\\
&\qquad\quad - \{Z(s_1;t_1) - Z(s_2;t_2) - \mathrm{E}(Z(s_1;t_1) - Z(s_2;t_2))\}^2]\\
&\quad = \gamma(s_1, s_0; t_1, t_0) + \gamma(s_2, s_0; t_2, t_0) - \gamma(s_1, s_2; t_1, t_2).
\end{aligned}$$

Now for any nonnegative constant $\alpha$ and nonnegative integer $k$, $\frac{\alpha^k}{k!}\{\gamma(s_1, s_0; t_1, t_0) + \gamma(s_2, s_0; t_2, t_0) - \gamma(s_1, s_2; t_1, t_2)\}^k$ is also a covariance function on $\mathcal{S}\times\mathcal{T}$. So is their sum,

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{\alpha^k}{k!}\{\gamma(s_1, s_0; t_1, t_0) + \gamma(s_2, s_0; t_2, t_0) - \gamma(s_1, s_2; t_1, t_2)\}^k\\
&\quad = \exp[\alpha\{\gamma(s_1, s_0; t_1, t_0) + \gamma(s_2, s_0; t_2, t_0) - \gamma(s_1, s_2; t_1, t_2)\}].
\end{aligned}$$

If $\exp[\alpha\{\gamma(s_1, s_0; t_1, t_0) + \gamma(s_2, s_0; t_2, t_0) - \gamma(s_1, s_2; t_1, t_2)\}]$ is the covariance function of a random field $\{Y(s;t), (s;t) \in \mathcal{S}\times\mathcal{T}\}$, then the random field $\{\exp(-\alpha\gamma(s, s_0; t, t_0))Y(s;t), (s;t) \in \mathcal{S}\times\mathcal{T}\}$ has the covariance

$$\begin{aligned}
&\mathrm{cov}\{\exp(-\alpha\gamma(s_1, s_0; t_1, t_0))Y(s_1;t_1), \exp(-\alpha\gamma(s_2, s_0; t_2, t_0))Y(s_2;t_2)\}\\
&\quad = \exp(-\alpha\gamma(s_1, s_0; t_1, t_0) - \alpha\gamma(s_2, s_0; t_2, t_0))\,\mathrm{cov}\{Y(s_1;t_1), Y(s_2;t_2)\}\\
&\quad = \exp(-\alpha\gamma(s_1, s_2; t_1, t_2)). \qquad \square
\end{aligned}$$

A.2 *A simple proof of Theorem* 2 *of Gneiting* (2002)

THEOREM A.2. (Gneiting (2002)) *Let $\varphi(x)$, $x \ge 0$, be a completely monotone function, and let $\psi(x)$, $x \ge 0$, be a positive function with a completely monotone derivative. Then*

$$C(s;t) = \frac{\sigma^2}{\psi(|t|^2)^{d/2}}\varphi\left(\frac{\|s\|^2}{\psi(|t|^2)}\right), \qquad (s;t) \in \mathbb{R}^d \times \mathbb{R},$$

*is a space-time covariance function, where $\sigma^2$ is a positive constant.*

PROOF. By Bernstein's theorem, $\varphi(x)$ is the Laplace transform of a bounded, nondecreasing function $F(u)$, $u \geq 0$,

$$\varphi(x) = \int_0^\infty \exp(-xu)dF(u), \qquad x \geq 0.$$

As a result, $C(\boldsymbol{s};t)$ can be expressed as

$$C(\boldsymbol{s};t) = \int_0^\infty \frac{\sigma^2}{\psi(|t|^2)^{d/2}} \exp\left(-\frac{\|\boldsymbol{s}\|^2 u}{\psi(|t|^2)}\right) dF(u), \qquad (\boldsymbol{s};t) \in \mathbb{R}^d \times \mathbb{R}.$$

Thus, it suffices to show that for each constant $u \geq 0$,

$$C_u(\boldsymbol{s};t) = \frac{\sigma^2}{\psi(|t|^2)^{d/2}} \exp\left(-\frac{\|\boldsymbol{s}\|^2 u}{\psi(|t|^2)}\right), \qquad (\boldsymbol{s};t) \in \mathbb{R}^d \times \mathbb{R},$$

is a space-time covariance function.

To apply Corollary 2.2 of Ma (2003$a$), we rewrite $C_u(\boldsymbol{s};t)$ as

$$\begin{aligned} C_u(\boldsymbol{s};t) &= (2\pi)^{d/2}\sigma^2 \int_{\mathbb{R}^d} \cos(\sqrt{u}\boldsymbol{\omega}'\boldsymbol{s}) \exp\left(-\frac{\|\boldsymbol{\omega}\|^2}{2}\psi(|t|^2)\right) d\boldsymbol{\omega} \\ &= (2\pi)^{d/2}\sigma^2 \int_{\mathbb{R}^d} \cos(\sqrt{u}\boldsymbol{\omega}'\boldsymbol{s}) \exp\left(-\frac{\|\boldsymbol{\omega}\|^2}{2}\int_0^{t^2} \psi'(v)dv\right) \\ &\qquad \times \exp\left(-\frac{\|\boldsymbol{\omega}\|^2}{2}\psi(0)\right) d\boldsymbol{\omega}, \end{aligned}$$

where $\psi(0) > 0$, and the derivative function of $\psi(x)$, $\psi'(x)$, is completely monotone. It remains to show that $\exp(-\frac{\|\boldsymbol{\omega}\|^2}{2}\int_0^{t^2} \psi'(v)dv)$ is a purely temporal covariance on $\mathbb{R}$ for every fixed $\boldsymbol{\omega} \in \mathbb{R}^d$, or by Schoenberg's theorem, to show that $\int_0^{t^2} \psi'(v)dv$ is a purely temporal variogram on $\mathbb{R}$. This is true, since $\psi'(v)$ is completely monotone, so that there exists a bounded, nondecreasing function $G(x)$ on $[0,\infty)$ such that

$$\psi'(v) = \int_0^\infty \exp(-vx)dG(x), \qquad v \geq 0,$$

and

$$\int_0^{t^2} \psi'(v)dv = \int_0^{t^2}\int_0^\infty \exp(-vx)dG(x)dv = \int_0^\infty \frac{1-\exp(-xt^2)}{x}dG(x), \qquad t \in \mathbb{R},$$

which is a purely temporal variogram on $\mathbb{R}$ as the mixture of $1-\exp(-xt^2)$, $t \in \mathbb{R}$. □

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