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MINIMAX CONFIDENCE BOUND OF THE NORMAL MEAN UNDER AN ASYMMETRIC LOSS FUNCTION

YUSHAN XIAO[1], YOSHIKAZU TAKADA[2] AND NINGZHONG SHI[3]

[1] *College of Applied Science, Changchun University, Changchun 130022, China and Institute of Mathematical Science, Jilin University, Changchun 130012, China*
[2] *Department of Computer Science, Kumamoto University, Kumamoto 860-8555, Japan*
[3] *Department of Mathematics, Northeast Normal University, Changchun 130024, China*

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Abstract. This paper considers a minimax confidence bound of the normal mean under an asymmetric loss function. A minimax confidence bound is obtained for the case that the variance is known or unknown. The admissibility of the minimax confidence bound is also considered for the case of known variance.

Key words and phrases: Confidence bound, LINEX loss function, normal mean, Bayes risk, minimaxity, admissibility.

1. Introduction

Let $X_1, \ldots, X_n$ be *i.i.d.* normal random variables with mean μ and variance σ^2 ($N(\mu, \sigma^2)$). A problem of estimating μ is considered for the case that σ is known or unknown. An estimator $\hat{\mu}_L$ is called a lower confidence bound at confidence level $1 - \alpha$ ($0 < \alpha < \frac{1}{2}$) if

$$P_\theta(\hat{\mu}_L(X) < \mu) \geq 1 - \alpha \quad \text{for all} \quad \theta,$$

where $X = (X_1, \ldots, X_n)$, and $\theta = \mu$ with σ known, $\theta = (\mu, \sigma)$ with σ unknown. An upper confidence bound $\hat{\mu}_U$ of μ is similarly defined. An interval $(\hat{\mu}_L, \hat{\mu}_U)$ is called a confidence interval at confidence level $1 - \alpha$ ($0 < \alpha < \frac{1}{2}$) if

$$P_\theta(\hat{\mu}_L(X) < \mu < \hat{\mu}_U(X)) \geq 1 - \alpha \quad \text{for all} \quad \theta.$$

In order to compare confidence bounds it seems appropriate to use an asymmetric loss function since the loss resulting from overestimating μ is more serious than that from underestimating for the lower confidence bound, and vice versa for the upper confidence bound. For such a case, Zellner (1986) considered a useful asymmetric loss function, which was called LINEX (Linear-Exponential), to estimate the normal mean. See Shafie and Noorbaloochi (1995) for further developments. Xiao (2000) discussed some applications to a prediction problem.

In this paper, we adopt the LINEX loss function to compare confidence bounds. For the lower confidence bound $\hat{\mu}_L$ the loss function is given by

$$L_1(\theta, \hat{\mu}_L) = b_1\{\exp(a_1(\hat{\mu}_L - \mu)/\sigma) - a_1(\hat{\mu}_L - \mu)/\sigma - 1\} \tag{1.1}$$

and for the upper confidence bound $\hat{\mu}_U$

$$L_2(\theta, \hat{\mu}_U) = b_2\{\exp(-a_2(\hat{\mu}_U - \mu)/\sigma) + a_2(\hat{\mu}_U - \mu)/\sigma - 1\},$$

where a_i and b_i $(i = 1, 2)$ are known positive constants. Further, in order to compare confidence intervals $(\hat{\mu}_L, \hat{\mu}_U)$, we adopt the following loss function

$$L(\theta, \hat{\mu}_L, \hat{\mu}_U) = L_1(\theta, \hat{\mu}_L) + L_2(\theta, \hat{\mu}_U).$$

The accuracy of lower and upper confidence bounds $\hat{\mu}_L$ and $\hat{\mu}_U$ are then measured by the risk functions

$$R_1(\theta, \hat{\mu}_L) = E_\theta\{L_1(\theta, \hat{\mu}_L(X))\}$$
$$R_2(\theta, \hat{\mu}_U) = E_\theta\{L_2(\theta, \hat{\mu}_U(X))\}$$

and that of a confidence interval $(\hat{\mu}_L, \hat{\mu}_U)$ is

$$R(\theta, \hat{\mu}_L, \hat{\mu}_U) = R_1(\theta, \hat{\mu}_L) + R_2(\theta, \hat{\mu}_U). \tag{1.2}$$

A $1 - \alpha$ lower confidence bound $\hat{\mu}_L^\star$ is called minimax if

$$\sup_\theta R_1(\theta, \hat{\mu}_L^\star) \leq \sup_\theta R_1(\theta, \hat{\mu}_L)$$

for any other $1 - \alpha$ lower confidence bound $\hat{\mu}_L$. A $1 - \alpha$ lower confidence bound $\hat{\mu}_L^\star$ is called admissible if there exists no other $1 - \alpha$ lower confidence bound $\hat{\mu}_L$ such that

$$R_1(\theta, \hat{\mu}_L) \leq R_1(\theta, \hat{\mu}_L^\star) \qquad \text{for all} \qquad \theta$$

with strict inequality for some θ. The concepts are also adopted to the upper confidence bound and the confidence interval.

In the subsequent section we shall mainly treat the lower confidence bound. However, the method employed can be easily applied to the upper confidence bound and the confidence interval.

In order to get a minimax lower confidence bound, we shall adopt the Bayes approach (e.g. Berger (1985), p. 350). Let

$$\tilde{L}_1(\theta, \hat{\mu}_L) = \lambda I_{(\mu,\infty)}(\hat{\mu}_L) + L_1(\theta, \hat{\mu}_L)$$

be a loss function and let $\tilde{R}_1(\theta, \hat{\mu}_L)$ be the risk function, where λ is a known positive constant which is later determined in relation to the confidence coefficient, and $I_A(x)$ is an indicator function of the set A. Then

$$\tilde{R}_1(\theta, \hat{\mu}_L) = \lambda P_\theta(\hat{\mu}_L > \mu) + R_1(\theta, \hat{\mu}_L). \tag{1.3}$$

From (1.1)

$$\begin{aligned} \tilde{R}_1(\theta, \hat{\mu}_L) &= \lambda P_\theta(\hat{\mu}_L > \mu) \\ &\quad + b_1 E_\theta\{\exp(a_1(\hat{\mu}_L - \mu)/\sigma) - a_1(\hat{\mu}_L - \mu)/\sigma - 1\}. \end{aligned} \tag{1.4}$$

Let $\{\pi_k\}$ be a sequence of prior distributions on θ. Then we shall seek the Bayes estimator $\hat{\mu}_{L_k}$ which minimizes

$$r_1(\pi_k, \hat{\mu}_L) = \int \tilde{R}_1(\theta, \hat{\mu}_L) d\pi_k(\theta), \tag{1.5}$$

and evaluate the asymptotic Bayes risk as $k \to \infty$, from which a minimax lower confidence bound shall be determined.

In Section 2, we seek a minimax lower confidence bound for the case of known variance, and show its admissibility. The case that the variance is unknown is treated in Section 3.

2. Minimax confidence bound when σ is known

In this section, we suppose that σ is known, so that $\theta = \mu$. We take the normal prior distribution $N(0, k^2)$ as π_k with positive constant k. Then the posterior distribution of μ, given X, is $N(\frac{nk^2}{nk^2+\sigma^2}\bar{X}, \frac{\sigma^2 k^2}{nk^2+\sigma^2})$ with $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Write the lower confidence bound $\hat{\mu}_L$ as

$$\hat{\mu}_L = \frac{nk^2}{nk^2+\sigma^2}\bar{X} - \frac{\sigma}{\sqrt{n}}u(X) \tag{2.1}$$

with some function $u(x)$. Substituting (2.1) into (1.4) and after some calculations, we have that the posterior Bayes risk of $\hat{\mu}_L$ with respect to $\tilde{L}_1$, given X, is

$$\Psi^{(k)}_{a_1,b_1}(u(X)), \tag{2.2}$$

where

$$\Psi^{(k)}_{a,b}(x) = \lambda\left(1 - \Phi\left(\frac{x}{\sqrt{c_k}}\right)\right) + b\left\{\exp\left(\frac{a^2 c_k}{2n} - \frac{ax}{\sqrt{n}}\right) + \frac{ax}{\sqrt{n}} - 1\right\}, \tag{2.3}$$

Φ is the distribution function of $N(0,1)$ and $c_k = nk^2/(nk^2+\sigma^2)$.

Let

$$\Psi_{a,b}(x) = \lambda(1 - \Phi(x)) + b\left\{\exp\left(\frac{a^2}{2n} - \frac{ax}{\sqrt{n}}\right) + \frac{ax}{\sqrt{n}} - 1\right\} \tag{2.4}$$

and

$$\psi^{(k)}_{a,b}(x) = \frac{d\Psi^{(k)}_{a,b}(x)}{dx}, \qquad \psi_{a,b}(x) = \frac{d\Psi_{a,b}(x)}{dx}.$$

Then

$$\psi^{(k)}_{a,b}(x) = -\frac{\lambda}{\sqrt{c_k}}\phi\left(\frac{x}{\sqrt{c_k}}\right) - \frac{ab}{\sqrt{n}}\left\{\exp\left(\frac{a^2 c_k}{2n} - \frac{ax}{\sqrt{n}}\right) - 1\right\}$$

and

$$\psi_{a,b}(x) = -\lambda\phi(x) - \frac{ab}{\sqrt{n}}\left\{\exp\left(\frac{a^2}{2n} - \frac{ax}{\sqrt{n}}\right) - 1\right\}, \tag{2.5}$$

where ϕ is the probability density of $N(0,1)$.

We now give a helpful lemma to establish minimaxity, the proof of which is provided in the Appendix.

LEMMA 2.1. *There exist unique values x_k and x_0 which minimize* (2.3) *and* (2.4), *and are obtained by solving*

$$\psi^{(k)}_{a,b}(x) = 0, \qquad \psi_{a,b}(x) = 0.$$

Furthermore, the sequence $\{x_k\}$ converges to x_0 as $k \to \infty$.

Let u_k be the solution of $\psi^{(k)}_{a_1,b_1}(x) = 0$. Then it follows from (2.2) and Lemma 2.1 that the confidence bound

$$\hat{\mu}_{L_k} = \frac{nk^2}{nk^2+\sigma^2}\bar{X} - \frac{u_k\sigma}{\sqrt{n}}$$

is the Bayes estimator of μ with respect to $\tilde{L}_1$ and the Bayes risk is

$$r_1(\pi_k, \hat{\mu}_{L_k}) = \Psi^{(k)}_{a_1,b_1}(u_k). \tag{2.6}$$

THEOREM 2.1. *If $a_1 < 2u_0\sqrt{n}$ with $\Phi(u_0) = 1-\alpha$, then the lower confidence bound*

$$\hat{\mu}^\star_L = \bar{X} - u_0\sigma/\sqrt{n}$$

is minimax among all $1-\alpha$ lower confidence bounds.

PROOF. From the condition,

$$\lambda = \frac{a_1 b_1}{\sqrt{n}}\left\{1 - \exp\left(\frac{a_1^2}{2n} - \frac{a_1 u_0}{\sqrt{n}}\right)\right\} / \phi(u_0)$$

is positive, and from (2.5) $\psi_{a_1,b_1}(u_0) = 0$. It is seen that

$$\tilde{R}_1(\mu, \hat{\mu}^\star_L) = \Psi_{a_1,b_1}(u_0) \quad \text{for all} \quad \mu,$$

which yields

$$\lim_{k\to\infty} r_1(\pi_k, \hat{\mu}_{L_k}) = \sup_\mu \tilde{R}_1(\mu, \hat{\mu}^\star_L)$$

from (2.6) and Lemma 2.1, and hence for any other confidence bound $\hat{\mu}_L$

$$\sup_\mu \tilde{R}_1(\mu, \hat{\mu}_L) \geq \sup_\mu \tilde{R}_1(\mu, \hat{\mu}^\star_L). \tag{2.7}$$

Suppose that $\hat{\mu}_L$ is any other $1-\alpha$ lower confidence bound. Then from (1.3) and (2.7)

$$\begin{aligned}\sup_\mu R_1(\mu, \hat{\mu}_L) &\geq \sup_\mu R_1(\mu, \hat{\mu}^\star_L) + \lambda(P_\mu(\mu < \hat{\mu}^\star_L) - \sup_\mu P_\mu(\mu < \hat{\mu}_L)) \\ &\geq \sup_\mu R_1(\mu, \hat{\mu}^\star_L)\end{aligned}$$

since $P_\mu(\mu < \hat{\mu}_L) \leq \alpha$ and $P_\mu(\mu < \hat{\mu}^\star_L) = \alpha$. Hence the proof is completed.

Remark 2.1. If the condition that $a_1 < 2u_0\sqrt{n}$ is violated, then the present method does not work well. We conjecture that the condition is also necessary for $\hat{\mu}^\star_L$ to be minimax.

Next we shall show that the minimax confidence bound $\hat{\mu}^\star_L$ is admissible. In order to prove it, we need the following lemma, the proof of which is given in the Appendix.

LEMMA 2.2. *Let x_k and x_0 be the values in Lemma 2.1. Then*

$$\lim_{k\to\infty} k(\Psi^{(k)}_{a,b}(x_k) - \Psi_{a,b}(x_0)) = 0.$$

THEOREM 2.2. *If $a_1 < 2u_0\sqrt{n}$, then the lower confidence bound $\hat{\mu}^\star_L$ is admissible among all $1-\alpha$ lower confidence bounds.*

PROOF. Suppose that there exists a $1-\alpha$ lower confidence bound $\hat{\mu}_L$ such that

$$R_1(\mu, \hat{\mu}_L) \leq R_1(\mu, \hat{\mu}_L^\star) \quad \text{for all} \quad \mu$$

with strict inequality for some μ. Since $P_\mu(\mu < \hat{\mu}_L) \leq \alpha$ and $P_\mu(\mu < \hat{\mu}_L^\star) = \alpha$, we have

$$\tilde{R}_1(\mu, \hat{\mu}_L) \leq \tilde{R}_1(\mu, \hat{\mu}_L^\star) \quad \text{for all} \quad \mu$$

with strict inequality for some μ. By using the Limiting Bayes Method (Lehmann (1983), p. 265), we shall lead to a contradiction. Since $\tilde{R}_1(\mu, \hat{\mu}_L)$ is continuous and $\tilde{R}_1(\mu, \hat{\mu}_L^\star)$ is free from μ, there exist $\epsilon > 0$ and $\mu_0 < \mu_1$ such that

$$\tilde{R}_1(\mu, \hat{\mu}_L) < \tilde{R}_1(\mu, \hat{\mu}_L^\star) - \epsilon$$

for all μ with $\mu_0 < \mu < \mu_1$. Then we have

$$\frac{r_1(\pi_k, \hat{\mu}_L^\star) - r_1(\pi_k, \hat{\mu}_L)}{r_1(\pi_k, \hat{\mu}_L^\star) - r_1(\pi_k, \hat{\mu}_{L_k})} \geq \frac{\epsilon \int_{\mu_0}^{\mu_1} e^{-\mu^2/2k^2} d\mu}{\sqrt{2\pi} k (\Psi_{a_1,b_1}(u_0) - \Psi_{a_1,b_1}^{(k)}(u_k))}. \tag{2.8}$$

Since

$$\int_{\mu_0}^{\mu_1} e^{-\mu^2/2k^2} d\mu \to \mu_1 - \mu_0 \quad \text{as} \quad k \to \infty,$$

it follows from Lemma 2.2 that the right side of (2.8) goes to infinity as $k \to \infty$. Hence there exist k_0 such that $r_1(\pi_{k_0}, \hat{\mu}_L) < r_1(\pi_{k_0}, \hat{\mu}_{L_{k_0}})$, which contradicts the fact that $\hat{\mu}_{L_{k_0}}$ is the Bayes estimator.

An argument paralleling those of Theorems 2.1 and 2.2 yields the following theorem.

THEOREM 2.3. *If $a_2 < 2v_0\sqrt{n}$ with $\Phi(v_0) = 1-\alpha$, then the upper confidence bound*

$$\hat{\mu}_U^\star = \bar{X} + v_0\sigma/\sqrt{n}$$

is minimax and admissible among all $1-\alpha$ upper confidence bounds.

Let us now turn to the problem of the confidence interval $(\hat{\mu}_L, \hat{\mu}_U)$. Suppose that

$$\Phi\left(\frac{a_1}{2\sqrt{n}}\right) - \Phi\left(-\frac{a_2}{2\sqrt{n}}\right) < 1 - \alpha, \tag{2.9}$$

which guarantees the existence of the solution (u_0, v_0, λ) with $\lambda > 0$ for the following equations

$$\psi_{a_1,b_1}(u) = 0, \qquad \psi_{a_2,b_2}(v) = 0, \qquad \Phi(u) - \Phi(-v) = 1 - \alpha.$$

Then we have the following theorem by similar argument in Theorems 2.1 and 2.2.

THEOREM 2.4. *If the condition (2.9) holds, then the confidence interval $(\hat{\mu}_L^\star, \hat{\mu}_U^\star)$ with*

$$\hat{\mu}_L^\star = \bar{X} - u_0\sigma/\sqrt{n}, \qquad \hat{\mu}_U^\star = \bar{X} + v_0\sigma/\sqrt{n}$$

is minimax and admissible among all $1-\alpha$ confidence interval.

Remark 2.2. Joshi (1966) discussed an admissibility of the confidence interval of the location parameter when the loss is the length of the interval. Cohen and Strawderman (1973) considered a very wide class of loss functions to evaluate the confidence interval and gave sufficient conditions for admissibility. The result may be applicable to the present problem, but we preferred the direct proof.

3. Minimax confidence bound when σ is unknown

In this section, we suppose that σ is unknown and hence $\theta = (\mu, \sigma)$. We consider a prior distribution π_k on θ which assigns to μ the uniform distribution on $(-k, k)$ and takes σ to be independent of μ with $\tau = 1/2\sigma^2$ the Γ distribution with density $\tau^{c/2-1}e^{-\tau}/\Gamma(\frac{c}{2})$ where $c = (\log\log k)^{-1}$. This prior distribution was first used by Chen (1966) to directly prove that the usual t- interval is minimax among all $1-\alpha$ confidence intervals when the loss function is the length of a confidence interval.

Let

$$
\begin{aligned}
(3.1)\qquad f_k(x) = {} & a_1 b_1 \Gamma\left(\frac{n+c}{2}\right) - a_1 b_1 e^{a_1^2/2n} C_k \int_{\hat{k}}^{\infty} \exp(-a_1\sqrt{2xy}) y^{(n+c)/2-1} e^{-y} dy \\
& - \lambda (1+nx)^{-(n+c)/2} \sqrt{\frac{n}{2\pi}} \Gamma\left(\frac{n+c}{2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.2)\qquad g_k(x) = {} & a_1 b_1 D_k E_k - a_1 b_1 e^{a_1^2/2n} \int_0^{\infty} \exp(-a_1\sqrt{2xy}) y^{(n+c)/2-1} e^{-y} dy \\
& - \lambda (1+nx)^{-(n+c)/2} \sqrt{\frac{n}{2\pi}} \Gamma\left(\frac{n+c}{2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
&\hat{k} = (k^{-5/6} + (n-1)k^{-1/6})/2, \\
&C_k = \Phi(\sqrt{n}(k^{1/12} + a_1/n)) - \Phi(-\sqrt{n}(k^{1/12} - a_1/n)), \\
&D_k = 2\Phi(\sqrt{n}k^{1/12}) - 1, \qquad E_k = \int_{\hat{k}}^{\infty} y^{(n+c)/2-1} e^{-y} dy.
\end{aligned}
$$

Then for large k there exist unique positive values v_k and ω_k such that $f_k(v_k) = g_k(\omega_k) = 0$.

Let

$$
\begin{aligned}
(3.3)\qquad f_{a_1,b_1}(x) = {} & a_1 b_1 \Gamma(n/2) - a_1 b_1 e^{a_1^2/2n} \int_0^{\infty} \exp(-a_1\sqrt{2xy}) y^{n/2-1} e^{-y} dy \\
& - \lambda(1+nx)^{-n/2} \sqrt{\frac{n}{2\pi}} \Gamma(n/2).
\end{aligned}
$$

There exists a unique positive value u_0 such that $f_{a_1,b_1}(u_0) = 0$. It is not difficult to show that

$$
(3.4)\qquad \lim_{k\to\infty} v_k = \lim_{k\to\infty} \omega_k = u_0.
$$

Write the Bayes estimator $\hat{\mu}_{L_k}$ which minimizes the posterior risk given X as

$$
\hat{\mu}_{L_k} = \bar{X} - u_k(X) S,
$$

where $(n-1)S^2 = \sum_{i=1}^n (X_i - \bar{X})^2$. Then we have the following lemma, the proof of which is given in the Appendix.

LEMMA 3.1. *There exist k' and positive constant M such that if $k > k'$, then for any X with $|\bar{X}| < k - \sqrt{k}$ and $S < k^{1/3}$*

$$
(3.5)\qquad \sqrt{(n-1)v_k} \le u_k(X) \le \sqrt{(n-1)\omega_k} + M/S.
$$

Let

$$h_{a_1}(x) = \Gamma(n/2) - e^{a_1^2/2n} \int_0^\infty \exp(-a_1\sqrt{2xy}) y^{n/2-1} e^{-y} dy. \tag{3.6}$$

THEOREM 3.1. *Suppose $h_{a_1}(u^{\star 2}/(n(n-1))) > 0$ with $T_{n-1}(u^\star) = 1 - \alpha$, where T_{n-1} is the distribution function of t distribution with $n-1$ degrees of freedom. Then the lower confidence bound*

$$\hat{\mu}_L^\star = \bar{X} - u^\star S/\sqrt{n}$$

is minimax among all $1-\alpha$ lower confidence bounds.

PROOF. Let $u_0 = u^{\star 2}/(n(n-1))$ and define λ by

$$\lambda = a_1 b_1 h_{a_1}(u_0)(1 + nu_0)^{n/2}/(\Gamma(n/2)\sqrt{n/2\pi}).$$

Then it is seen from (3.3) and (3.6) that $f_{a_1,b_1}(u_0) = 0$. In the appendix, it is shown that there exists an increasing sequence $\{k_v\}$ such that $v < k_v^{1/4}$ and

$$\lim_{v\to\infty} \int_0^v dG_{k_v}(\sigma) = 0 \tag{3.7}$$

where $G_{k_v}(\sigma)$ is the prior distribution of σ. In the subsequent argument, we take $k = k_v$ and $v \to \infty$. We shall show

$$\liminf_{v\to\infty} r_1(\pi_k, \hat{\mu}_{L_k}) \geq \tilde{R}_1(\theta, \hat{\mu}_L^\star) \quad \text{for all} \quad \theta.$$

The result can then be proved by the same argument as in Theorem 2.1.

Let

$$Q_1 = \int R_1(\theta, \hat{\mu}_{L_k}) d\pi_k(\theta)$$

and

$$Q_2 = \int P_\theta(\mu < \hat{\mu}_{L_k}) d\pi_k(\theta).$$

Then from (1.3) and (1.5) it suffices to show

$$\liminf_{v\to\infty} Q_1 \geq R_1(\theta, \hat{\mu}_L^\star) \quad \text{for all} \quad \theta \tag{3.8}$$

and

$$\liminf_{v\to\infty} Q_2 \geq P_\theta(\mu < \hat{\mu}_L^\star) \quad \text{for all} \quad \theta. \tag{3.9}$$

We shall show (3.8). The proof of (3.9) is provided in the Appendix.

Using (3.5),

$$\begin{aligned} E_\theta\{\exp(a_1(\hat{\mu}_{L_k} - \mu)/\sigma)\} &\geq E_\theta\{\exp(a_1(\bar{X} - u_k(X)S - \mu)/\sigma) I_{A_1}(\bar{X}) I_{A_2}(S)\} \\ &\geq E_\theta\{\exp(a_1(\bar{X} - \mu)/\sigma) I_{A_1}(\bar{X})\} \\ &\quad \times E_\theta\{\exp(-a_1\sqrt{(n-1)\omega_k} S/\sigma - a_1 M/\sigma) I_{A_2}(S)\} \end{aligned} \tag{3.10}$$

where A_1 and A_2 are the sets of $|\bar{X}| < k - \sqrt{k}$ and $S < k^{1/3}$, and I_{A_1} and I_{A_2} are the indicator functions of A_1 and A_2. It is seen that when $|\mu| < k - 2\sqrt{k}$ and $\sigma < k^{1/4}$,

$$E_\theta\{\exp(a_1(\bar{X} - \mu)/\sigma)I_{A_1}(\bar{X})\} \geq \int_{-\sqrt{n}k^{1/4}}^{\sqrt{n}k^{1/4}} \exp(a_1 y/\sqrt{n})\phi(y)dy,$$

and when $v < \sigma < k^{1/4}$,

$$\begin{aligned} &E_\theta\{\exp(-a_1\sqrt{(n-1)\omega_k}S/\sigma - a_1 M/\sigma)I_{A_2}(S)\} \\ &\quad \geq e^{-a_1 M/v} \int_0^{k^{1/12}} \exp(-a_1\sqrt{(n-1)\omega_k}S)dF^\star(S), \end{aligned}$$

where $F^\star(S)$ is the distribution function of S with $\sigma = 1$. Using these inequalities, the right side of (3.10) is bounded below by

$$\begin{aligned} &\int_v^{k^{1/4}} dG_k(\sigma) \int_{-k+2\sqrt{k}}^{k-2\sqrt{k}} dF_k(\mu) e^{-a_1 M/v} \\ &\quad \times \int_{-\sqrt{n}k^{1/4}}^{\sqrt{n}k^{1/4}} \exp(a_1 y/\sqrt{n})\phi(y)dy \int_0^{k^{1/12}} \exp(-a_1\sqrt{(n-1)\omega_k}S)dF^\star(S), \end{aligned}$$

where $F_k(\mu)$ is the distribution function of μ. Hence from (3.4) and (3.7)

$$\begin{aligned} &\liminf_{v\to\infty} E\{\exp(a_1(\hat{\mu}_{L_k} - \mu)/\sigma)\} \\ &\quad \geq e^{a_1^2/2n} \int_0^\infty \exp(-a_1\sqrt{(n-1)u_0}S)dF^\star(S) \\ &\quad = E_\theta\{\exp(a_1(\hat{\mu}_L^\star - \mu)/\sigma)\}, \end{aligned} \tag{3.11}$$

since $\sqrt{(n-1)u_0} = u^\star/\sqrt{n}$. It is seen that when $|\mu| < k - 2\sqrt{k}$ and $\sigma < k^{1/4}$,

$$P_\theta(|\bar{X}| < k - \sqrt{k}) \geq 2\Phi(\sqrt{n}k^{1/4}) - 1. \tag{3.12}$$

Using (3.5) and (3.12), when $|\mu| < k - 2\sqrt{k}$ and $\sigma < k^{1/4}$,

$$\begin{aligned} E_\theta\{u_k(X)S/\sigma\} &\geq E_\theta\{u_k(X)S/\sigma I_{A_1}(\bar{X})I_{A_2}(S)\} \\ &\geq (2\Phi(\sqrt{n}k^{1/4}) - 1)\sqrt{(n-1)v_k} \int_0^{k^{1/12}} SdF^\star(S), \end{aligned}$$

which yields

$$\begin{aligned} &E\{u_k(X)S/\sigma\} \\ &\quad \geq (2\Phi(\sqrt{n}k^{1/4}) - 1)\sqrt{(n-1)v_k} \int_0^{k^{1/12}} SdF^\star(S) \int_0^{k^{1/4}} dG_k(\sigma) \int_{-k+2\sqrt{k}}^{k-2\sqrt{k}} dF_k(\mu). \end{aligned}$$

Then it follows from (3.4) that

$$\begin{aligned} \liminf_{v\to\infty} E\{u_k(X)S/\sigma\} &\geq \sqrt{(n-1)u_0} \int_0^\infty SdF^\star(S) \\ &= E_\theta(u^\star S/\sqrt{n}\sigma). \end{aligned} \tag{3.13}$$

By (3.11) and (3.13), (3.8) follows.

Remark 3.1. It may be possible to use the Hunt-Stein theorem to derive the minimax lower confidence bound. See Kiefer (1957). However, we preferred the direct proof by using the Bayes approach. When the variance is unknown, one may consider a prior distribution on θ which assigns to $\tau = 1/2\sigma^2$ the Γ distribution with density $\lambda^\alpha \tau^{\alpha-1} e^{-\lambda\tau}/\Gamma(\alpha)$ and takes the conditional distribution of μ, given σ, as $N(0, r^2\sigma^2)$, where α, λ, and r^2 are known, and let $r \to \infty$ and $\alpha \to 0$ in order to use the Bayes approach. But such a sequence of prior distributions does not seem to work well to seek a minimax solution. See Ferguson ((1967), p. 183). Contrary to this, the sequence of prior distributions considered by Chen (1966) seems useful to directly seek a minimax solution by using the Bayes approach when the variance is unknown.

Now we proceed to the problem of the upper confidence bound and the confidence interval. Let $f_{a_2,b_2}(x)$ be (3.3) with $a_1 = a_2$ and $b_1 = b_2$, and let $h_{a_2}(x)$ be (3.6) with $a_1 = a_2$. A similar argument in Theorem 3.1 gives the following theorem.

THEOREM 3.2. *Suppose $h_{a_2}(v^{\star 2}/(n(n-1))) > 0$ with $T_{n-1}(v^\star) = 1 - \alpha$. Then the upper confidence bound*

$$\hat{\mu}_U^\star = \bar{X} + v^\star S/\sqrt{n}$$

is minimax among all $1 - \alpha$ upper confidence bounds.

Let c_1 and c_2 be the solutions of the equations $h_{a_1}(x^2/n(n-1)) = 0$ and $h_{a_2}(x^2/n(n-1)) = 0$, respectively. Suppose

$$T_{n-1}(c_1) - T_{n-1}(-c_2) < 1 - \alpha. \tag{3.14}$$

It is seen that (3.14) guarantees the existence of the solution $(u^\star, v^\star, \lambda)$ with $\lambda > 0$ for the following equations

$$f_{a_1,b_1}\left(\frac{u^2}{n(n-1)}\right) = 0, \qquad f_{a_2,b_2}\left(\frac{v^2}{n(n-1)}\right) = 0, \qquad T_{n-1}(u) - T_{n-1}(-v) = 1 - \alpha.$$

Then we have the following theorem by the similar argument in Theorem 3.1.

THEOREM 3.3. *If the condition* (3.14) *holds, then the confidence interval $(\hat{\mu}_L^\star, \hat{\mu}_U^\star)$ with*

$$\hat{\mu}_L^\star = \bar{X} - u^\star S/\sqrt{n}, \qquad \hat{\mu}_U^\star = \bar{X} + v^\star S/\sqrt{n},$$

is minimax among all $1 - \alpha$ confidence intervals.

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Appendix

PROOF OF LEMMA 2.1. It is easy to see that $\psi_{a,b}^{(k)}(x) < 0$ and $\psi_{a,b}(x) < 0$ for $x \le 0$, $\psi_{a,b}^{(k)}(x)$ and $\psi_{a,b}(x)$ are increasing in $x > 0$, and $\lim_{x\to\infty} \psi_{a,b}^{(k)}(x) = ab/\sqrt{n}$, $\lim_{x\to\infty} \psi_{a,b}(x) = ab/\sqrt{n}$. Hence we can find unique values $x_k > 0$ and $x_0 > 0$ which minimize (2.3) and (2.4) by solving $\psi_{a,b}^{(k)}(x) = 0$ and $\psi_{a,b}(x) = 0$. Next we shall show that the sequence $\{x_k\}$ converges to x_0 as $k \to \infty$. Note that

$$\lim_{k\to\infty} \Psi_{a,b}^{(k)}(x) = \Psi_{a,b}(x) \quad \text{for any} \quad x > 0,$$

so that for any $\epsilon > 0$

$$\lim_{k\to\infty} \frac{\Psi_{a,b}^{(k)}(x_0)}{\Psi_{a,b}^{(k)}(x_0 \pm \epsilon)} = \frac{\Psi_{a,b}(x_0)}{\Psi_{a,b}(x_0 \pm \epsilon)} < 1.$$

Hence for large k

$$\Psi_{a,b}^{(k)}(x_0) < \Psi_{a,b}^{(k)}(x_0 - \epsilon), \qquad \Psi_{a,b}^{(k)}(x_0) < \Psi_{a,b}^{(k)}(x_0 + \epsilon),$$

which implies that there exists a value $x_0 - \epsilon < \hat{x}_k < x_0 + \epsilon$ at which $\Psi_{a,b}^{(k)}(x)$ has a local minimum, so that $\psi_{a,b}^{(k)}(\hat{x}_k) = 0$. But the solution of $\psi_{a,b}^{(k)}(x) = 0$ is unique, and hence $\hat{x}_k = x_k$, which implies $|x_k - x_0| < \epsilon$. So the proof is completed.

PROOF OF LEMMA 2.2. Let $x_{l_k} = x_0 - 1/k$ and $x_{u_k} = x_0 + 1/k$. Then

$$\begin{aligned}
\psi_{a,b}^{(k)}(x_{l_k}) &= -\frac{\lambda}{\sqrt{c_k}}\phi\left(\frac{x_0 - 1/k}{\sqrt{c_k}}\right) - \frac{ab}{\sqrt{n}}\left\{\exp\left(\frac{a^2 c_k}{2n} - \frac{a}{\sqrt{n}}(x_0 - 1/k)\right) - 1\right\} \\
&= \lambda\left\{\phi(x_0) - \frac{1}{\sqrt{c_k}}\phi\left(\frac{x_0 - 1/k}{\sqrt{c_k}}\right)\right\} + \frac{ab}{\sqrt{n}}\left\{\exp\left(\frac{a^2}{2n} - \frac{a x_0}{\sqrt{n}}\right)\right. \\
&\qquad \left. - \exp\left(\frac{a^2 c_k}{2n} - \frac{a}{\sqrt{n}}(x_0 - 1/k)\right)\right\}
\end{aligned}$$

since from (2.5)

$$\frac{ab}{\sqrt{n}} = \lambda\phi(x_0) + \frac{ab}{\sqrt{n}}\exp\left(\frac{a^2}{2n} - \frac{a x_0}{\sqrt{n}}\right).$$

It is easily verified that

$$\psi_{a,b}^{(k)}(x_{l_k}) = \frac{\lambda\phi'(x_0)}{k} - \frac{a^2 b}{nk}\exp\left(\frac{a^2}{2n} - \frac{a x_0}{\sqrt{n}}\right) + O(1/k^2)$$

as $k \to \infty$. Likewise, we have

$$\psi_{a,b}^{(k)}(x_{u_k}) = -\frac{\lambda\phi'(x_0)}{k} + \frac{a^2 b}{nk}\exp\left(\frac{a^2}{2n} - \frac{a x_0}{\sqrt{n}}\right) + O(1/k^2)$$

as $k \to \infty$. Since $\phi'(x_0) < 0$, $\psi_{a,b}^{(k)}(x_{l_k}) < 0 < \psi_{a,b}^{(k)}(x_{u_k})$ for large k, which implies that $x_{l_k} < x_k < x_{u_k}$. Hence we have

$$x_k - x_0 = O(1/k) \quad \text{as} \quad k \to \infty. \tag{A.1}$$

Substituting x_k and x_0 into (2.3) and (2.4), and using (A.1) and $\psi_{a,b}(x_0) = 0$,

$$\begin{aligned}
\Psi_{a,b}^{(k)}(x_k) - \Psi_{a,b}(x_0) &= \lambda\left\{\Phi(x_0) - \Phi\left(\frac{x_k}{\sqrt{c_k}}\right)\right\} \\
&\quad + b\left\{\exp\left(\frac{a^2 c_k}{2n} - \frac{a}{\sqrt{n}}x_k\right) - \exp\left(\frac{a^2}{2n} - \frac{a}{\sqrt{n}}x_0\right)\right\} \\
&\quad + \frac{ab}{\sqrt{n}}(x_k - x_0) \\
&= \lambda(x_0 - x_k)(\phi(x_0) + o(1)) + O(1/k^2) \\
&\quad + b\exp\left(\frac{a^2}{2n} - \frac{a}{\sqrt{n}}x_0\right)\left(-\frac{a}{\sqrt{n}}(x_k - x_0) + O(1/k^2)\right) \\
&\quad + \frac{ab}{\sqrt{n}}(x_k - x_0) \\
&= (x_k - x_0)\psi_{a,b}(x_0) + o(1/k) \\
&= o(1/k)
\end{aligned}$$

as $k \to \infty$, which completes the proof.

PROOF OF LEMMA 3.1. First we shall show that there exist k_0 and positive constant A such that if $k > k_0$, then for any X with $|\bar{X}| < k - \sqrt{k}$ and $S < k^{1/3}$

$$0 \le u_k(X)S \le Ak^{1/3}. \tag{A.2}$$

Write $\hat{\mu}_L = \bar{X} - u(X)S$ with some function $u(X)$. In order to show that $u_k = u_k(X) \ge 0$, it suffices to show that $E\{L_1(\theta, \hat{\mu}_L) \mid X\}$ is decreasing in $u = u(X) < 0$ since $P(\mu < \hat{\mu}_L \mid X)$ is decreasing in u. Note that the posterior density of θ, given X, is expressed as

$$\frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{n(\bar{X}-\mu)^2}{2\sigma^2}\right)\frac{1}{\sigma^{n+c}}\exp\left(-\frac{1+(n-1)S^2}{2\sigma^2}\right)I_{(-k,k)}(\mu)/B_k(X) \tag{A.3}$$

where

$$B_k(X) = \int_0^\infty \left\{\int_{-k}^k \frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{n(\bar{X}-\mu)^2}{2\sigma^2}\right)d\mu\right\}\frac{1}{\sigma^{n+c}}\exp\left(-\frac{1+(n-1)S^2}{2\sigma^2}\right)d\sigma.$$

It follows from (1.1) that

$$\begin{aligned}
\frac{\partial}{\partial u}E\{L_1(\theta, \hat{\mu}_L) \mid X\} &= -a_1 b_1 S E\left\{\frac{1}{\sigma}\exp(a_1(\bar{X} - uS - \mu)/\sigma) \mid X\right\} \\
&\quad + a_1 b_1 S E\left\{\frac{1}{\sigma} \mid X\right\}.
\end{aligned} \tag{A.4}$$

When $|\bar{X}| < k - \sqrt{k}$ and $\sigma < k^{5/12}$, it is seen that

$$\int_{-k}^k \exp(a_1(\bar{X}-\mu)/\sigma)\frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{n(\bar{X}-\mu)^2}{2\sigma^2}\right)d\mu \ge e^{a_1^2/2n}C_k. \tag{A.5}$$

When $u < 0$ and $S < k^{1/3}$,

$$\int_0^{k^{5/12}} \exp(-a_1 uS/\sigma)\sigma^{-1-n-c} \exp\left(-\frac{1+(n-1)S^2}{2\sigma^2}\right) d\sigma$$
$$\geq \frac{1}{2}\left(\frac{2}{1+(n-1)S^2}\right)^{(n+c)/2} E_k.$$

Then it follows from (A.3) that for any X with $|\bar{X}| < k - \sqrt{k}$ and $S < k^{1/3}$

$$E\left\{\frac{1}{\sigma}\exp(a_1(\bar{X} - uS - \mu)/\sigma) \mid X\right\} \tag{A.6}$$
$$\geq e^{a_1^2/2n}\frac{1}{2}\left(\frac{2}{1+(n-1)S^2}\right)^{(n+c)/2} C_k E_k / B_k(X)$$

when $u < 0$, and

$$E\left\{\frac{1}{\sigma} \mid X\right\} \leq \int_0^{\infty} \sigma^{-1-n-c}\exp\left(-\frac{1+(n-1)S^2}{2\sigma^2}\right) d\sigma / B_k(X) \tag{A.7}$$
$$= \frac{1}{2}\left(\frac{2}{1+(n-1)S^2}\right)^{(n+c)/2} \Gamma\left(\frac{n+c}{2}\right) / B_k(X).$$

Substituting (A.6) and (A.7) into (A.4) yields that the right side of (A.4) is bounded above by

$$a_1 b_1 S \frac{1}{2}\left(\frac{2}{1+(n-1)S^2}\right)^{(n+c)/2}\left(\Gamma\left(\frac{n+c}{2}\right) - e^{a_1^2/2n} C_k E_k\right) / B_k(X),$$

when $u < 0$, which is negative for large k. Hence $E\{L_1(\theta, \hat{\mu}_L) \mid X\}$ is decreasing in $u < 0$, which shows $u_k \geq 0$ for any X with $|\bar{X}| < k - \sqrt{k}$ and $S < k^{1/3}$.

Since the posterior risk of $\hat{\mu}_{L_k}$ is not larger than that of $\bar{X}$,

$$\lambda P(\mu < \hat{\mu}_{L_k} \mid X) + E\{L_1(\theta, \hat{\mu}_{L_k}) \mid X\} \leq \lambda P(\mu < \bar{X} \mid X) + E\{L_1(\theta, \bar{X}) \mid X\},$$

which yields

$$a_1 b_1 u_k S E\left\{\frac{1}{\sigma} \mid X\right\} \leq \lambda + b_1 E\{\exp(a_1(\bar{X} - \mu)/\sigma) \mid X\}. \tag{A.8}$$

It follows from (A.3) that

$$E\{\exp(a_1(\bar{X} - \mu)/\sigma) \mid X\}$$
$$\leq e^{a_1^2/2n}\frac{1}{2}\left(\frac{2}{1+(n-1)S^2}\right)^{(n+c-1)/2}\Gamma\left(\frac{n+c-1}{2}\right) / B_k(X),$$

$$E\left\{\frac{1}{\sigma} \mid X\right\} \geq \int_0^{k^{5/12}}\left(\int_{-k}^{k}\frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{n(\bar{X}-\mu)^2}{2\sigma^2}\right)d\mu\right)\sigma^{-1-n-c}$$
$$\times \exp\left(-\frac{1+(n-1)S^2}{2\sigma^2}\right) d\sigma / B_k(X)$$
$$\geq D_k E_k \frac{1}{2}\left(\frac{2}{1+(n-1)S^2}\right)^{(n+c)/2} / B_k(X),$$

and

$$B_k(X) \le \frac{1}{2}\left(\frac{2}{1+(n-1)S^2}\right)^{(n+c-1)/2} \Gamma\left(\frac{n+c-1}{2}\right).$$

Using these inequalities in (A.8), we have

$$a_1 b_1 u_k S \le \left(\frac{1+(n-1)S^2}{2}\right)^{1/2} \Gamma\left(\frac{n+c-1}{2}\right)(\lambda + b_1 e^{a_1^2/2n})/D_k E_k$$

for any X with $|\bar{X}| < k - \sqrt{k}$ and $S < k^{1/3}$, which yields (A.2) for large k.

It follows from (A.2) that for large k

$$|\bar{X} - u_k S| < k$$

for any X with $|\bar{X}| < k - \sqrt{k}$ and $S < k^{1/3}$. When $|\bar{X} - uS| < k$,

$$\begin{aligned} P(\mu < \bar{X} - uS \mid X) = &\int_0^\infty \left(\int_{-k}^{\bar{X}-uS} \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{n(\bar{X}-\mu)^2}{2\sigma^2}\right) d\mu\right) \\ &\times \frac{1}{\sigma^{n+c}} \exp\left(-\frac{1+(n-1)S^2}{2\sigma^2}\right) d\sigma / B_k(X), \end{aligned}$$

which follows

$$\begin{aligned} &\frac{\partial}{\partial u} P(\mu < \bar{X} - uS \mid X) \\ &\quad = -S \int_0^\infty \frac{\sqrt{n}}{\sqrt{2\pi}} \frac{1}{\sigma^{n+c+1}} \exp\left(-\frac{1+(n-1)S^2 + nu^2S^2}{2\sigma^2}\right) d\sigma / B_k(X). \end{aligned}$$

Combining this with (A.4) gives

$$\begin{aligned} &\frac{\partial}{\partial u}[\lambda P(\mu < \hat{\mu}_L \mid X) + E\{L_1(\theta, \hat{\mu}_L) \mid X\}] \\ &\quad = S(-\lambda T_1 - a_1 b_1 T_2 + a_1 b_1 T_3)/B_k(X), \end{aligned} \tag{A.9}$$

where

$$T_1 = \int_0^\infty \frac{\sqrt{n}}{\sqrt{2\pi}} \frac{1}{\sigma^{n+c+1}} \exp\left(-\frac{1+(n-1)S^2 + nu^2S^2}{2\sigma^2}\right) d\sigma,$$

$$\begin{aligned} T_2 = &\int_0^\infty \left(\int_{-k}^{k} \exp\left(a_1(\bar{X} - uS - \mu)/\sigma\right) \frac{\sqrt{n}}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left(-\frac{n(\bar{X}-\mu)^2}{2\sigma^2}\right) d\mu\right) \\ &\times \frac{1}{\sigma^{n+c+1}} \exp\left(-\frac{1+(n-1)S^2}{2\sigma^2}\right) d\sigma \end{aligned}$$

and

$$T_3 = \int_0^\infty \left(\int_{-k}^{k} \frac{\sqrt{n}}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left(-\frac{n(\bar{X}-\mu)^2}{2\sigma^2}\right) d\mu\right) \frac{1}{\sigma^{n+c+1}} \exp\left(-\frac{1+(n-1)S^2}{2\sigma^2}\right) d\sigma.$$

Note that

$$T_1 = \frac{\sqrt{n}}{\sqrt{2\pi}} \frac{1}{2}\left(\frac{2}{1+(n-1)S^2 + nu^2S^2}\right)^{(n+c)/2} \Gamma\left(\frac{n+c}{2}\right). \tag{A.10}$$

Using (A.5),

$$T_2 \geq e^{a_1^2/2n} C_k \int_0^{k^{5/12}} \exp(-a_1 uS/\sigma) \frac{1}{\sigma^{1+n+c}} \exp\left(-\frac{1+(n-1)S^2}{2\sigma^2}\right) d\sigma \tag{A.11}$$
$$\geq e^{a_1^2/2n} C_k \frac{1}{2}\left(\frac{2}{1+(n-1)S^2}\right)^{(n+c)/2}$$
$$\times \int_{\hat{k}}^{\infty} \exp\left(-\sqrt{2}a_1\left(\frac{u^2S^2y}{1+(n-1)S^2}\right)^{1/2}\right) y^{(n+c)/2-1} e^{-y} dy.$$

It is seen that

$$T_2 \leq e^{a_1^2/2n} \int_0^{\infty} \exp(-a_1 uS/\sigma) \frac{1}{\sigma^{1+n+c}} \exp\left(-\frac{1+(n-1)S^2}{2\sigma^2}\right) d\sigma \tag{A.12}$$
$$= e^{a_1^2/2n} \frac{1}{2}\left(\frac{2}{1+(n-1)S^2}\right)^{(n+c)/2}$$
$$\times \int_0^{\infty} \exp\left(-\sqrt{2}a_1\left(\frac{u^2S^2y}{1+(n-1)S^2}\right)^{1/2}\right) y^{(n+c)/2-1} e^{-y} dy.$$

Note that

$$T_3 \geq D_k \int_0^{k^{5/12}} \frac{1}{\sigma^{n+c+1}} \exp\left(-\frac{1+(n-1)S^2}{2\sigma^2}\right) d\sigma \tag{A.13}$$
$$\geq D_k E_k \frac{1}{2}\left(\frac{2}{1+(n-1)S^2}\right)^{(n+c)/2}$$

and

$$T_3 \leq \frac{1}{2}\left(\frac{2}{1+(n-1)S^2}\right)^{(n+c)/2} \Gamma\left(\frac{n+c}{2}\right). \tag{A.14}$$

Let

$$\ell(u) = -\lambda T_1 - a_1 b_1 T_2 + a_1 b_1 T_3.$$

Using (A.10) to (A.14), we have

$$\frac{1}{2}\left(\frac{2}{1+(n-1)S^2}\right)^{(n+c)/2} g_k\left(\frac{u^2S^2}{1+(n-1)S^2}\right) \leq \ell(u) \tag{A.15}$$
$$\leq \frac{1}{2}\left(\frac{2}{1+(n-1)S^2}\right)^{(n+c)/2} f_k\left(\frac{u^2S^2}{1+(n-1)S^2}\right),$$

where f_k and g_k are (3.1) and (3.2), respectively. From (A.9) $\ell(u_k) = 0$, so that from (A.15) for large k

$$\sqrt{(n-1+1/S^2)v_k} \leq u_k \leq \sqrt{(n-1+1/S^2)\omega_k}$$

for any X with $|\bar{X}| < k - \sqrt{k}$ and $S < k^{1/3}$. The result then follows from (3.4).

PROOF OF (3.7). Note that

$$\int_0^v dG_k(\sigma) = \frac{1}{\Gamma(c/2)} \int_{1/2v^2}^{\infty} y^{c/2-1} e^{-y} dy$$
$$= \frac{1}{\Gamma(c/2+1)} \frac{c}{2} \int_{1/2v^2}^{\infty} y^{c/2-1} e^{-y} dy.$$

Since $c = (\log \log k)^{-1}$, choosing $k_v = \exp(e^v)$ yields

$$\lim_{v \to \infty} c \log v = 0.$$

Hence

$$\frac{c}{2} \int_{1/2v^2}^{1} y^{c/2-1} e^{-y} dy \leq \frac{c}{2} \int_{1/2v^2}^{1} y^{c/2-1} dy = 1 - (2v^2)^{-c/2},$$

which goes to zero as $v \to \infty$, and this completes the proof.

PROOF OF (3.9). Though the proof of (3.9) is almost the same as that given by Chen (1966), we shall provide the proof for the sake of completeness. Let $T = \sqrt{n}(\bar{X} - \mu)/S$. Then it follows from (3.5) that

$$\begin{aligned}
\text{(A.16)} \quad & P_\theta(\mu < \hat{\mu}_{L_k}) \\
&= P_\theta(T > \sqrt{n} u_k) \\
&\geq P_\theta(T > \sqrt{n} u_k, |\bar{X}| < k - \sqrt{k}, S < k^{1/3}) \\
&\geq P_\theta(T > \sqrt{n}(\sqrt{(n-1)\omega_k} + M/S), |\bar{X}| < k - \sqrt{k}, S < k^{1/3}) \\
&\geq P_\theta(T > \sqrt{n}(\sqrt{(n-1)\omega_k} + \epsilon), |\bar{X}| < k - \sqrt{k}, S < k^{1/3}, S > M/\epsilon) \\
&\geq P_\theta(T > \sqrt{n}(\sqrt{(n-1)\omega_k} + \epsilon)) + P_\theta(|\bar{X}| < k - \sqrt{k}) \\
&\quad + P_\theta(S < k^{1/3}) + P_\theta(S > M/\epsilon) - 3,
\end{aligned}$$

where ϵ is an arbitrary constant. When $v < \sigma < k^{1/4}$,

$$P_\theta(S < k^{1/3}) \geq P_{\theta_0}(S < k^{1/12})$$

and

$$P_\theta(S > M/\epsilon) \geq P_{\theta_0}(S > M/v\epsilon),$$

where $\theta_0 = (0, 1)$. Using these inequalities and (3.12), the right side of (A.16) is bounded below by

$$\begin{aligned}
\Delta_k(\theta_0) = {} & P_{\theta_0}(T > \sqrt{n}(\sqrt{(n-1)\omega_k} + \epsilon)) + 2\Phi(\sqrt{n} k^{1/4}) - 1 + P_{\theta_0}(S < k^{1/12}) \\
& + P_{\theta_0}(S > M/v\epsilon) - 3,
\end{aligned}$$

when $|\mu| < k - 2\sqrt{k}$ and $v < \sigma < k^{1/4}$. Hence

$$Q_2 \geq \Delta_k(\theta_0) \int_v^{k^{1/4}} dG_k(\sigma) \int_{-k+2\sqrt{k}}^{k-2\sqrt{k}} dF_k(\mu),$$

so that from (3.4) and (3.7)

$$\liminf_{v \to \infty} Q_2 \geq P_{\theta_0}(T > \sqrt{n}(\sqrt{(n-1)u_0} + \epsilon)).$$

Since $u^\star = \sqrt{n(n-1)u_0}$ and ϵ is arbitary,

$$\begin{aligned}\liminf_{v\to\infty} Q_2 &\geq P_{\theta_0}(T > u^\star)\\ &= P_{\theta_0}(\mu < \hat{\mu}_L^\star),\end{aligned}$$

which proves (3.9).

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