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NECESSARY CONDITIONS FOR DOMINATING THE JAMES-STEIN ESTIMATOR

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Abstract. This paper develops necessary conditions for an estimator to dominate the James-Stein estimator and hence the James-Stein positive-part estimator. The ultimate goal is to find classes of such dominating estimators which are admissible. While there are a number of results giving classes of estimators dominating the James-Stein estimator, the only admissible estimator known to dominate the James-Stein estimator is the generalized Bayes estimator relative to the fundamental harmonic function in three and higher dimension. The prior was suggested by Stein and the domination result is due to Kubokawa. Shao and Strawderman gave a class of estimators dominating the James-Stein positive-part estimator but were unable to demonstrate admissibility of any in their class. Maruyama, following a suggestion of Stein, has studied generalized Bayes estimators which are members of a point mass at zero and a prior similar to the harmonic prior. He finds a subclass which is minimax and admissible but is unable to show that any in his class with positive point mass at zero dominate the James-Stein estimator. The results in this paper show that a subclass of Maruyama's procedures including the class that Stein conjectured might contain members dominating the James-Stein estimator cannot dominate the James-Stein estimator. We also show that under reasonable conditions, the "constant" in shrinkage factor must approach $p-2$ for domination to hold.

Key words and phrases: The James-Stein estimator, unbiased estimator of risk, admissibility, generalized Bayes.

1. Introduction

Let X be a random variable having p-variate normal distribution $N_p(\theta, I_p)$. Then we consider the problem of estimating the mean vector θ by $\delta(X)$ relative to quadratic loss. Therefore every estimator is evaluated based on the risk function

$$R(\theta,\delta) = E_\theta[\|\delta(X)-\theta\|^2] = \int_{R^p} \frac{\|\delta(x)-\theta\|^2}{(2\pi)^{p/2}} \exp\left(-\frac{\|x-\theta\|^2}{2}\right) dx.$$

The usual estimator X with the constant risk p is minimax. Stein (1956a) showed that equivariant estimators relative to the orthogonal transformation group are of the form $\delta_\phi(X) = (1-\phi(\|X\|^2)/\|X\|^2)X$ and that there exists an estimator dominating X among these when $p \geq 3$. James and Stein (1961) succeeded in giving an explicit form of an estimator improving on X as

$$\delta_{JS}(X) = (1 - (p-2)/\|X\|^2)X,$$

which is called the James-Stein estimator. It is however noted that when $\|x\|^2 < p-2$, the James-Stein estimator yields an over-shrinkage and changes the sign of each component of X. The James-Stein positive-part estimator

$$\delta^{+}_{JS}(X) = \max(0, 1 - (p-2)/\|X\|^2)X,$$

eliminates this drawback and dominates the James-Stein estimator as shown in Baranchik (1964). Furthermore, a complete class result of Brown (1971), implies that the James-Stein positive-part estimator is not analytic and is thus inadmissible.

Estimators which dominate the James-Stein estimator have been given by several authors. Li and Kuo (1982) and Guo and Pal (1992) considered the class of estimators of forms $\delta_{LK} = (1 - \phi_{LK}(\|X\|^2)/\|X\|^2)X$ where

$$\phi_{LK}(w) = p - 2 - \sum_{i=1}^{n} a_i w^{-b_i},$$

where $a_i \geq 0$ for any i and $0 < b_1 < b_2 < \cdots < b_n$. For example when $n = 1$, they both showed that, $\delta_{LK}(X)$ for $0 < b_1 < 4^{-1}(p-2)$ and $a_1 = 2b_1 2^{b_1}\Gamma(p/2 - b_1 - 1)/\Gamma(p/2 - 2b_1 - 1)$ is superior to the James-Stein estimator. Kuriki and Takemura (2000) gave two estimators which shrink toward the ball with center 0, $\delta^{i}_{KT}(X) = (1 - \phi^{i}_{KT}(\|X\|^2)/\|X\|^2)X$ for $i = 1, 2$ where

$$\phi^1_{KT}(w) = \begin{cases} 0 & w \leq r^2 \\ p - 2 - \sum_{i=1}^{p-2}(r/w^{1/2})^i & w > r^2, \end{cases}$$

$$\phi^2_{KT}(w) = \begin{cases} 0 & w \leq \{(p-1)/(p-2)\}^2 r^2 \\ p - 2 - r/(w^{1/2} - r) & w > \{(p-1)/(p-2)\}^2 r^2. \end{cases}$$

They showed that when r is sufficiently small, these two estimators dominate the James-Stein estimator. However since the shrinkage factor $(1 - \phi_{LK}(w)/w)$ becomes negative for some w and ϕ^{i}_{KT} for $i = 1, 2$ fail to be analytic, δ_{LK} and δ^{i}_{KT} are inadmissible. To the best of our knowledge, the sole admissible estimator known to dominate the James-Stein estimator is Kubokawa's (1991) estimator $\delta_K = (1 - \phi_K(\|X\|^2)/\|X\|^2)X$ where

$$\phi_K(w) = p - 2 - 2\frac{\exp(-w/2)}{\int_0^1 \lambda^{p/2-2}\exp(-w\lambda/2)d\lambda}.$$

The estimator is a generalized Bayes with respect to the density $\|\theta\|^{2-p}$ which was suggested by Stein (1973).

A more challenging and long standing open problem is to find admissible estimators dominating the James-Stein positive-part estimator. Kubokawa's estimator δ_K fails to dominate the James-Stein positive-part estimator because the risks of δ_K and δ_{JS} are same at $\|\theta\| = 0$ (see Pal and Chang (1996)). Shao and Strawderman (1994) gave a class of estimators which dominate the James-Stein positive-part estimator. They were unable to show that any estimators in their class were admissible. Our goal in this paper is to give necessary conditions for domination of the James-Stein (and hence James-Stein positive-part) estimator.

In particular, most statisticians working in the area believe that $\lim_{w\to\infty} \phi(w) = p-2$ should be a necessary condition for δ_ϕ to improve on the James-Stein estimator. However we know of no such result in the literature. We give such a result under natural conditions. Additionally we show that no estimator for which $\phi(w)$ approaches $p-2$ from above can dominate the James-Stein estimator. It follows from this result that a conjecture of Stein (1973) to the effect that a mixture of a point mass at 0 and the harmonic prior $\|\theta\|^{2-p}$ may give a generalized Bayes estimator that dominates the James-Stein estimator is false if the mass at 0 is strictly positive. An interesting feature of our main result is that it combines two of Stein's most important techniques—namely the unbiased estimator of risk and the admissibility of convex acceptance regions for tests for exponential families.

Section 2 is devoted to developing the necessary conditions. Proofs are given in the Appendix.

2. Necessary conditions

In this section, we develop some necessary conditions for dominance over the James-Stein estimator. We consider only orthogonally invariant estimators.

The risk of the estimator $\delta_\phi(X) = (1 - \phi(\|X\|^2)/\|X\|^2)X$ with any absolutely continuous ϕ is given by

$$\begin{aligned} R(\theta, \delta_\phi) &= E\left[\left\|\left(1 - \frac{\phi(\|X\|^2)}{\|X\|^2}\right)X - \theta\right\|^2\right] \\ &= p + E\left[\frac{\phi(\|X\|^2)\{\phi(\|X\|^2) - 2(p-2)\}}{\|X\|^2} - 4\phi'(\|X\|^2)\right] \end{aligned} \tag{2.1}$$

which was derived by Stein (1973) and Efron and Morris (1976). The absolute continuity of ϕ in the above is a sufficient condition for the indefinite integral of ϕ' to be equal to ϕ. See Hudson (1978) for the detail. The James-Stein estimator δ_{JS} corresponds to $\phi_{JS}(w) = p-2$ and hence the difference in risks between δ_{JS} and δ_ϕ is

$$\Delta = R(\theta, \delta_{JS}) - R(\theta, \delta_\phi) = E\left[-\frac{\{\phi(\|X\|^2) - (p-2)\}^2}{\|X\|^2} + 4\phi'(\|X\|^2)\right].$$

Now we will propose some necessary conditions for domination of δ_ϕ over δ_{JS}. Let

$$g_\phi(w) = -\frac{\{\phi(w) - (p-2)\}^2}{w} + 4\phi'(w). \tag{2.2}$$

Our basic result is following:

THEOREM 2.1. *Assume that $g_\phi(w)$ is bounded above. If δ_ϕ dominates the James-Stein estimator, then, for every w, there exists w_0 ($> w$) such that $g_\phi(w_0) \geq 0$.*

PROOF. See the Appendix.

It follows that under the above boundedness assumption that δ_ϕ cannot dominate the James-Stein estimator if $g_\phi(w) < 0$ for all w greater than some arbitrary fixed w_0. It is interesting to note that the Proof of Theorem 2.1 is closely related to Stein's (1956*b*) proof of the admissibility of convex acceptance regions for testing a simple hypothesis about the natural parameters of an exponential family. Hence our results depend on two of the basic tools introduced by Stein.

COROLLARY 2.1. *Assume that $\phi(w)$ is bounded and absolutely continuous. Necessary conditions for an estimator $\delta_\phi(X)$ to dominate the James-Stein estimator are that*

(i) *for every w, there exists w_0 $(> w)$ such that $\phi'(w_0) \geq 0$,*
(ii) *if $w\phi'(w)$ has a limiting value as w approaches infinity, it must be 0,*
(iii) *if $\phi(w)$ has a limiting value as w approaches infinity and $w\phi'(w)$ converges to 0 as w approaches infinity, the limit value for $\phi(w)$ must be $p-2$.*

PROOF. See the Appendix.

It follows from (i) that if $\phi(w)$ approaches $p-2$ from above (i.e. $\phi'(w) \leq 0$ for any $w > w_0$), then δ_ϕ cannot dominate either the James-Stein or positive-part James-Stein estimator. Theorem 2.1 however cannot rule out the possibility that $\phi(w)$ for a dominating estimator does not have a limiting value. The following result rules out this possibility under natural conditions as also assumed in Berger (1976) and Casella (1980) in the context of tail minimaxity of estimators related to δ_ϕ.

THEOREM 2.2. *Assume that both $\phi(w)$ and $\phi(w)/w$ are bounded, that $\phi'(w) = o(w^{-1})$ and that $\phi''(w) = o(w^{-3/2})$. If $\delta_\phi(X)$ dominates the James-Stein estimator, $\lim_{w\to\infty}\phi(w) = p-2$.*

PROOF. See the Appendix.

3. Discussion

Stein (1973) suggested that a generalized Bayes estimator with respect to a prior which is mixture of a point mass at 0 and the generalized (harmonic) density $\|\theta\|^{2-p}$ may dominate the James-Stein estimator or the James-Stein positive-part estimator. Kubokawa (1991) showed that the generalized Bayes estimator with respect to $\|\theta\|^{2-p}$ is admissible and in fact dominates the James-Stein estimator. This suggests that the addition of a small point mass at 0 might also produce dominating estimators. Efron and Morris (1976) also studied numerically some properties of Stein's (1973) estimator.

Maruyama (2004) investigates various properties of the behavior of generalized Bayes estimators corresponding to priors which are a mixture of a point mass with weight $\beta/(1+\beta)$ at the origin 0 and a scale mixture of normal distributions given by

$$(2\pi)^{-p/2}\int_0^1 \left(\frac{\lambda}{1-\lambda}\right)^{p/2} \exp\left(-\frac{\lambda\|\theta\|^2}{2(1-\lambda)}\right)\lambda^{-a}h(\lambda)d\lambda, \tag{3.1}$$

with weight $1/(1+\beta)$ for $\beta \geq 0$. Let denote the generalized Bayes estimator by $\delta_\beta(X) = (1-\phi_\beta(\|X\|^2)/\|X\|^2)X$. It is assumed that $h(\lambda)$ is a measurable nonnegative function on $(0,1)$ and that $\lim_{\lambda\to 0}h(\lambda)=1$. This prior distribution is a generalization of Stein's (1973). Maruyama (2004) showed that $\phi_\beta(w)$ for $a=2$ approaches $p-2$ as $w\to\infty$

regardless of $h(\lambda)$ and β and that $\lim_{w\to\infty} w \cdot \frac{d}{dw}\phi_\beta(w) = 0$. In particular, Maruyama (2004) showed that ϕ_β with $a = 2$, $h(\lambda) \equiv 1$ and $\beta > 0$, which is just Stein (1973)'s case, approaches $p-2$ from above. This, together with (i) of Corollary 2.1, implies that Stein's (1973) generalized Bayes estimator fails to dominate the James-Stein estimator. Maruyama (2004) also showed that, for $h(\lambda)$ which is bounded, not a constant function and for which $\lambda h'(\lambda)/h(\lambda)$ is monotone nondecreasing and also for $h(\lambda) = (1-\lambda)^b$ for $-1 < b < 0$, $\phi(w)$ with $a = 2$ and $\beta \geq 0$ approaches $p-2$ from above and hence fails to dominate the James-Stein estimator.

Maruyama (2004) showed that under mild conditions on $\phi(w)$, admissible domination of the James-Stein positive-part estimator requires that there exists $w_0 (> p-2)$ so that $\phi(w_0)$ exceeds $p-2$. Thus most simple behavior of $\phi(w)$ desirable for admissible domination over the James-Stein positive-part estimator is that the function $\phi(w)$ first increases above $p-2$ then decreases back below $p-2$ and finally, increases to $p-2$ from below. Maruyama (2004) showed that for $h(\lambda)$ which is bounded, not a constant function and $\lambda h'(\lambda)/h(\lambda)$ is monotone nonincreasing, $\phi_\beta(w)$ with $a = 2$ and suitable β behaves in this way. It is hoped that generalized Bayes estimator with respect to a subclass of such priors will give classes of admissible minimax estimators which dominate both the James-Stein and James-Stein positive-part estimator.

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Appendix

PROOF OF THEOREM 2.1. Suppose to the contrary that there exists w_0 such that $g_\phi(w) < 0$ for any $w \geq w_0$. Under the assumption on boundedness of g_ϕ, there exists an $M (> 0)$ such that $g_\phi(w) \leq M$ for any w. Under the assumption of absolute continuity of g_ϕ there exist two points $(w_0 <) w_1 < w_2$ and $\epsilon (> 0)$ such that $g_\phi(w) < -\epsilon$ on $w \in [w_1, w_2]$. Using M and ϵ, we define $g_{M,\epsilon}(w)$ as

$$\text{(A.1)} \qquad g_{M,\epsilon}(w) = \begin{cases} M & w \leq w_0 \\ 0 & w_0 < w < w_1 \\ -\epsilon & w_1 \leq w \leq w_2 \\ 0 & w > w_2. \end{cases}$$

The inequality $g_{M,\epsilon}(w) \geq g_\phi(w)$ for any w implies that

$$\text{(A.2)} \qquad \Delta = E[g_\phi(W)] \leq M\mathrm{P}_\theta[W \leq w_0] - \epsilon \mathrm{P}_\theta[w_1 \leq W \leq w_2],$$

where $W = \|X\|^2$.

Now let a be a fixed p-dimensional unit vector (see Fig. 1). Then the half plane $\{x : a'x \leq \sqrt{w_0}\}$ includes the p-dimensional sphere $\{x : \|x\|^2 \leq w_0\}$. For $\theta = (\sqrt{w_0}+\lambda)a$, we have

$$\text{(A.3)} \qquad \mathrm{P}_\theta[W \leq w_0] < \int_{a'x \leq \sqrt{w_0}} \frac{1}{(2\pi)^{p/2}} \exp\left(-\frac{\|x-\theta\|^2}{2}\right) dx$$

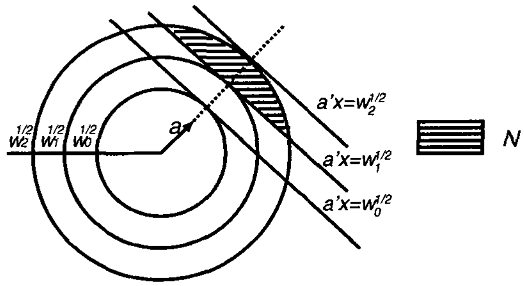

Fig. 1.

$$
\begin{aligned}
&= \int_{a'x \le \sqrt{w_0}} \frac{1}{(2\pi)^{p/2}} \exp\left(-\frac{\|x\|^2}{2} + (\sqrt{w_0}a + \lambda a)'x - \frac{\|\theta\|^2}{2}\right) dx \\
&\le \exp(\lambda\sqrt{w_0}) \exp\left(-\frac{\|\theta\|^2}{2}\right) \\
&\quad \times \int_{a'x \le \sqrt{w_0}} \frac{1}{(2\pi)^{p/2}} \exp\left(-\frac{\|x\|^2}{2} + \sqrt{w_0}a'x\right) dx \\
&\le \exp(\lambda\sqrt{w_0}) \exp\left(-\frac{\|\theta\|^2}{2} + \frac{w_0}{2}\right).
\end{aligned}
$$

For $N = \{x : w_1 \le \|x\|^2 \le w_2, \sqrt{w_1} \le a'x \le \sqrt{w_2}\}$ and $\theta = (\sqrt{w_0} + \lambda)a$, we have

$$
\begin{aligned}
\text{(A.4)} \quad P_\theta[w_1 \le W \le w_2] &> \int_N \frac{1}{(2\pi)^{p/2}} \exp\left(-\frac{\|x-\theta\|^2}{2}\right) dx \\
&= \int_N \frac{1}{(2\pi)^{p/2}} \exp\left(-\frac{\|x\|^2}{2} + (\sqrt{w_0}a + \lambda a)'x - \frac{\|\theta\|^2}{2}\right) dx \\
&\ge \exp(\lambda\sqrt{w_1}) \exp\left(-\frac{\|\theta\|^2}{2}\right) \\
&\quad \times \int_N \frac{1}{(2\pi)^{p/2}} \exp\left(-\frac{\|x\|^2}{2} + \sqrt{w_0}a'x\right) dx.
\end{aligned}
$$

Combining (A.3) and (A.4), we have

$$
\text{(A.5)} \qquad \Delta \le c_1 \exp\left(\sqrt{w_0}\lambda - \frac{\|\theta\|^2}{2}\right)(1 - c_2 \exp\{(\sqrt{w_1} - \sqrt{w_0})\lambda\}),
$$

where $c_1 = M \exp(w_0/2)$ and

$$
c_2 = \frac{\epsilon}{M} \exp\left(-\frac{w_0}{2}\right) \int_N \frac{1}{(2\pi)^{p/2}} \exp\left(-\frac{\|x\|^2}{2} + \sqrt{w_0}a'x\right) dx.
$$

Since c_1 and c_2 do not depend on λ, Δ is negative for sufficiently large λ. This completes the proof. Note that this proof is closely related to Stein's (1956*b*) proof of admissibility of convex acceptance regions for testing simple null hypotheses on the natural parameters of an exponential family.

PROOF OF COROLLARY 2.1. (i) If there exists w_0 such that $\phi'(w) < 0$ for every $w \geq w_0$, $g_\phi(w) < 0$ for $w \geq w_0$.

(ii) Let $\lim_{w\to\infty} w\phi'(w) = a(\geq 0)$. Note that we have only to consider the case $a \geq 0$ by (i). Suppose that $a > 0$. For any $\epsilon > 0$, there exists w_0 such that $a-\epsilon < w\phi'(w) < a+\epsilon$ for any $w \geq w_0$. Thus

$$(a-\epsilon)\{\log w - \log w_0\} + \phi(w_0) \leq \phi(w) \leq (a+\epsilon)\{\log w - \log w_0\} + \phi(w_0),$$

and

$$g_\phi(w) < -(a-\epsilon)^2 \frac{\{\log w\}^2}{w} + o\left(\frac{\{\log w\}^2}{w}\right).$$

Hence $g_\phi(w)$ for $a > 0$ is always negative for sufficiently large w and the result follows from Theorem 2.1.

(iii) Let $\lim_{w\to\infty} \phi(w) = b$. Because for every $\epsilon > 0$, there exists w_0 such that $b - \epsilon \leq \phi(w) \leq b + \epsilon$ for any $w > w_0$, we have

$$g_\phi(w) < -\frac{\min\{(b-\epsilon-p+2)^2, (b+\epsilon-p+2)^2\}}{w} + o(w^{-1}).$$

Hence when $b \neq p-2$, $g_\phi(w)$ is always negative for sufficiently large w and it follows from Theorem 2.1.

PROOF OF THEOREM 2.2. Note that the risk of δ_ϕ is given by (2.1). By using Lemma A.1 below, the asymptotic behavior of risk of δ_ϕ for sufficiently large $\lambda = \|\theta\|^2$ is

$$p + \frac{\phi(\lambda)\{\phi(\lambda) - 2(p-2)\}}{\lambda} + o(\lambda^{-1})$$

when both ϕ and $\phi(w)/w$ are bounded, that $\phi'(w) = o(w^{-1})$ and that $\phi''(w) = o(w^{-3/2})$. On the other hand, the risk of the James-Stein estimator is given by $p-(p-2)^2 E[\|X\|^{-2}]$. Note that

$$\begin{aligned} E[\|X\|^{-2}] &= \sum_{i=0}^{\infty} \frac{(\lambda/2)^i}{i!} \exp(-\lambda/2) \int_0^\infty y^{-1} f_{p+2i}(y)dy \\ &= \sum_{i=0}^{\infty} \frac{(\lambda/2)^i}{i!} \exp(-\lambda/2) \frac{\Gamma(\lambda/2 - 1 + i)}{2\Gamma(\lambda/2+i)} \\ &= \exp(-\lambda/2)(p-2)^{-1} M(p/2-1, p/2, \lambda/2) \end{aligned}$$

where $f_{p+2j}(y)$ is a density function of χ^2_{p+2j} and $M(\cdot,\cdot,\cdot)$ is a confluent hypergeometric function given by

$$M(a,b,z) = 1 + \frac{a}{b}z + \cdots + \frac{a\cdot(a+1)\cdots(a+n-1)}{b\cdot(b+1)\cdots(b+n-1)}\frac{z^n}{n!} + \cdots.$$

A confluent hypergeometric function has an asymptotic form

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)} \exp(z) z^{a-b} \left(\sum_{n=0}^{l} \frac{(b-a)_n (1-a)_n}{n!} z^{-n} + O(z^{-l}) \right)$$

for sufficiently large z (see Abramowitz and Stegun (1964)), which implies that

$$E[\|X\|^{-2}] = \lambda^{-1} + (2 - p/2)\lambda^{-2} + O(\lambda^{-3}).$$

Hence the asymptotic behavior of risk of the James-Stein estimator is given by

$$R(\theta, \delta_{JS}) = p - (p-2)^2\lambda^{-1} - (p-2)^2(4-p)\lambda^{-2} + O(\lambda^{-3}).$$

The difference in risks between δ_{JS} and δ_ϕ is

$$-\frac{\{\phi(\lambda) - (p-2)\}^2}{\lambda} + o(\lambda^{-1})$$

which takes negative values for some sufficiently large λ unless $\lim_{\lambda\to\infty} \phi(\lambda) = p - 2$.

LEMMA A.1. *Assume that $g : R^p \to R$ is a bounded function with first partial derivatives $g^{(i)}(x) = (\partial/\partial x_i)g(x)$ for $i = 1, \ldots, p$ and that $g(x) = g_*(\|x\|^2) + o(\|x\|^{-2})$ and $g^{(i)}(x) = o(\|x\|^{-2})$ for sufficiently large $\|x\|^2$. Then for $X \sim N_p(\mu, I_p)$,*

$$E[g(X)] = g_*(\|\mu\|^2) + o(\|\mu\|^{-2})$$

for sufficiently large $\|\mu\|^2$.

PROOF. This proof is essentially the same as Lemma 4.1 in Maruyama and Strawderman (2005). We omit the details.

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