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SENSITIVITY OF MINIMAXITY AND ADMISSIBILITY IN THE ESTIMATION OF A POSITIVE NORMAL MEAN

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Abstract. On the problem of estimating a positive normal mean with known variance, it is well known that one minimax admissible estimator is the generalized Bayes one with respect to the non-informative prior measure, the Lebesgue measure, restricted on the positive half-line. When the true variance is misspecified, however, it is shown that this estimator does not always retain minimaxity and admissibility. In particular, it is almost surely inadmissible in the misspecification case.

Key words and phrases: Admissibility, minimaxity, complete class theorem, positive normal mean, misspecification.

1. Introduction

Let X be a normal random variable with unknown mean θ and known variance 1. Under the restriction $\theta \geq 0$, we consider the problem of estimating the mean θ by $\delta(X)$ under the quadratic loss function. Therefore, any estimator $\delta(X)$ is evaluated on the risk function $R(\theta, \delta) = E_\theta[(\delta(X) - \theta)^2]$.

The usual estimator of θ in the unrestricted problem is clearly X. This estimator is minimax with constant risk 1, generalized Bayes with respect to the non-informative prior distribution, the Lebesgue measure, and admissible. Under the restriction above, X is still minimax, but inadmissible because X takes negative values with positive probability and the improved estimator $\delta_+(X) = \max(0, X)$ is easily found. Furthermore, $\delta_+(X)$ is also inadmissible because any admissible estimator should be generalized Bayes, as was shown by Sacks (1963). To our knowledge, the sole admissible minimax estimator previously derived is $\delta_{\phi_1}(X) = X + \phi_1(X)$ where

$$\phi_1(w) = -\frac{\int_{-\infty}^{w} t\exp(-t^2/2)dt}{\int_{-\infty}^{w}\exp(-t^2/2)dt}, \tag{1.1}$$

which was derived by Katz (1961) and Sacks (1963) independently. This estimator is generalized Bayes with respect to the non-informative prior distribution, that is, the Lebesgue measure restricted on the positive half-line. See Lehman and Casella (1998) for details.

In this paper, we consider the situation where the true variance is not 1, that is, we misspecify $X \sim N(\theta, 1)$ while $X \sim N(\theta, \sigma^2)$ with $\sigma^2 \neq 1$ truly. In the unrestricted case, the natural estimator X clearly retains both minimaxity and admissibility for any

σ^2. In the restricted case, however, the decision-theoretic properties of $\delta_{\phi_1}(X)$ are not clear. Hence we are interested in determining when $\delta_{\phi_1}(X)$ retains minimaxity and admissibility under $\sigma^2 \neq 1$.

In Section 2, we show that $\delta_{\phi_1}(X)$ retains minimaxity if and only if $\sigma^2 > 1$. In Section 3 we consider the problem of determining whether $\delta_{\phi_1}(X)$ under $\sigma^2 \neq 1$ is admissible or not. We show that it reduces to the problem of whether or not the function $f(z)^\alpha$ for $\alpha = \sigma^{-2}$ with

$$f(z) = \int_0^\infty \exp(-zt - t^2/2)dt \tag{1.2}$$

can be expressed as the Laplace transform of some nonnegative measure. When α is a positive integer, we see that such a positive measure exists and hence $\delta_{\phi_1}(X)$ is admissible. However when α is not a positive integer, the admissibility is not apparent. Note that $f(z)^\alpha$ is a multi-valued function in this case. To solve it we consider the inverse Laplace transform of $f(z)^\alpha$, that is,

$$g(t) = \frac{1}{2\pi i} \int_\Gamma f(z)^\alpha e^{zt} dz, \qquad t \geq 0, \tag{1.3}$$

where Γ is a suitable contour in the complex z-plane (see Fig. 1 in Section 5). In Section 4, we investigate some properties of the entire function $f(z)$, i.e., its zeros and asymptotic behavior as $z \to \infty$, and we show in Section 5 that, when α is not a positive integer, the function $g(t)$ for sufficiently large $t > 0$ oscillates between positive and negative values (see Theorem 5.1). Thus, $\delta_{\phi_1}(X)$ is not generalized Bayes by the unicity of the inverse Laplace transform and hence is inadmissible when σ^{-2} is not a positive integer for the true variance σ^2. These results suggest that the decision-theoretic properties of the estimator $\delta_{\phi_1}(X)$ are quite sensitive to the misspecification of the variance. In particular, it is inadmissible for any $\sigma^2 > 1$ and for almost every $\sigma^2 < 1$. Such sensitivity of $\delta_{\phi_1}(X)$ in the restricted problem exhibits a striking contrast to the robustness of X in the unrestricted problem.

The additional contributions of our article are the following.

(i) As far as we know, the Sacks-Brown complete class theorem has been applied only to show the inadmissibility of non-differentiable estimators, for example, the James-Stein positive-part estimator on the estimation of a multivariate normal mean. Our work is the first attempt at determining whether an estimator having infinite differentiability is admissible.

(ii) We demonstrate that classical complex and asymptotic analysis can be a very powerful tool in discussing statistical estimation problems.

2. Minimaxity

In this section, we consider the minimaxity of $\delta_{\phi_1}(X)$ given by (1.1) when the true variance is not 1. First we derive a sufficient condition for minimaxity when the true variance is σ^2. For an estimator of the form $\delta_\phi(X) = X + \phi(X)$ with $\lim_{x\to\infty} \phi(x) = 0$, the difference in risks between X, which has constant minimax risk, and $\delta_\phi(X)$ is

$$
\begin{aligned}
(2.1)\qquad \Delta &= R(\theta, X) - R(\theta, \delta_\phi(X)) \\
&= c\int_{-\infty}^{\infty} \{(x-\theta)^2 - (x+\phi(x)-\theta)^2\} \exp\{-(x-\theta)^2/(2\sigma^2)\} dx \\
&= c\int_{-\infty}^{\infty} \left\{\int_0^{\infty} \frac{d}{dt}\{(x+\phi(x+t)-\theta)^2\} dt\right\} \exp\{-(x-\theta)^2/(2\sigma^2)\} dx \\
&= 2c\int_{-\infty}^{\infty}\int_0^{\infty} \phi'(x+t)(x+\phi(x+t)-\theta) \exp\{-(x-\theta)^2/(2\sigma^2)\} dxdt \\
&= 2c\int_{-\infty}^{\infty}\int_0^{\infty} \phi'(w)(w-t+\phi(w)-\theta) \exp\{-(w-t-\theta)^2/(2\sigma^2)\} dwdt \\
&= 2c\int_{-\infty}^{\infty} \phi'(w)\{\phi(w) - \psi_{\sigma^2}(w,\theta)\} \left\{\int_0^{\infty} \exp\left\{-\frac{(w-t-\theta)^2}{2\sigma^2}\right\} dt\right\} dw,
\end{aligned}
$$

where $c = (2\pi\sigma^2)^{-1/2}$ and

$$
(2.2)\qquad \psi_{\sigma^2}(w,\theta) = \frac{\int_0^{\infty}(t-w+\theta)\exp\{-(t-w+\theta)^2/(2\sigma^2)\}dt}{\int_0^{\infty}\exp\{-(t-w+\theta)^2/(2\sigma^2)\}dt}
$$

$$
(2.3)\qquad = \sigma^2\left(\int_0^{\infty}\exp\left(-\frac{t^2}{2\sigma^2} + \frac{t(w-\theta)}{\sigma^2}\right)dt\right)^{-1}.
$$

By (2.3), $\psi_{\sigma^2}(w,\theta)$ is increasing in θ and hence $\psi_{\sigma^2}(w,\theta) \geq \psi_{\sigma^2}(w,0)$ for $\theta \geq 0$. Let $\phi_{\sigma^2}(w) = \psi_{\sigma^2}(w,0)$. Hence Δ is nonnegative for any $\theta \geq 0$ when $\phi(w)$ is nonincreasing and $\phi(w) \leq \phi_{\sigma^2}(w)$. Noting that $\lim_{w\to\infty}\phi_{\sigma^2}(w) = 0$, we have the following result, which is almost the same as one in Kubokawa (1999).

THEOREM 2.1. *For $X \sim N(\theta, \sigma^2)$ with $\theta \geq 0$, $\delta_\phi(X) = X + \phi(X)$ is minimax if $\phi(w)$ is monotone decreasing (or nonincreasing) and $0 \leq \phi(w) \leq \phi_{\sigma^2}(w)$.*

Using Theorem 2.1, we have the following result on the minimaxity of $\delta_{\phi_1}(X)$ given by (1.1). Note that $\phi_1(w)$ corresponds to $\phi_{\sigma^2}(w)$ with $\sigma^2 = 1$ because, by (2.2), $\phi_{\sigma^2}(w)$ is also written as

$$
(2.4)\qquad \phi_{\sigma^2}(w) = -\frac{\int_{-\infty}^{w} t\exp\{-t^2/(2\sigma^2)\}dt}{\int_{-\infty}^{w}\exp\{-t^2/(2\sigma^2)\}dt}.
$$

THEOREM 2.2. *The estimator $\delta_{\phi_1}(X)$ is minimax if and only if the true variance σ^2 is greater than or equal to one.*

PROOF. The function $\phi_1(w)$ is monotone decreasing because

$$
\frac{d}{dw}\phi_1(w) = -\frac{\exp\{-w^2/2\}\int_{-\infty}^{w}(w-t)\exp\{-t^2/2\}dt}{\{\int_{-\infty}^{w}\exp\{-t^2/2\}dt\}^2} \leq 0.
$$

The function $\phi_{\sigma^2}(w)$ is increasing in σ^2, as shown in the lemma below. When $\sigma^2 > 1$, we have $\phi_1(w) < \phi_{\sigma^2}(w)$ for all w and hence $\delta_{\phi_1}(X)$ is minimax. When $\sigma^2 < 1$, we have

$\phi_1(w) > \phi_{\sigma^2}(w)$ and from the right-hand side of (2.1), $R(0, \delta_{\phi_1}(X)) > R(0, X)$, which implies that $\delta_{\phi_1}(X)$ is not minimax.

The following lemma is used in the above proof.

LEMMA 2.1. *The function $\phi_{\sigma^2}(w)$ is monotone increasing in σ^2.*

PROOF. Let $\eta = \sigma^2$. The derivative of $\phi_\eta(w)$ with respect to η is calculated as

$$\frac{\partial \phi_\eta}{\partial \eta}(w) = \frac{\int_{-\infty}^{w} t^2 e^{-t^2/2\eta} dt \int_{-\infty}^{w} t e^{-t^2/2\eta} dt - \int_{-\infty}^{w} t^3 e^{-t^2/2\eta} dt \int_{-\infty}^{w} e^{-t^2/2\eta} dt}{2\eta^2 \{\int_{-\infty}^{w} e^{-t^2/2\eta} dt\}^2}.$$

The correlation inequality implies that the derivative $(\partial \phi_\eta / \partial \eta)(w)$ for $w \le 0$ is nonnegative because the function t^2 is monotone decreasing for $t \le 0$. Next we consider the case $w > 0$. Letting $Z(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ and $P(x) = \int_{-\infty}^{x} Z(t) dt$, we have

$$\phi_\eta(w) = -\sqrt{\eta} \frac{\int_{-\infty}^{\eta^{-1/2} w} t \exp(-t^2/2) dt}{\int_{-\infty}^{\eta^{-1/2} w} \exp(-t^2/2) dt} = \sqrt{\eta} \frac{Z(\eta^{-1/2} w)}{P(\eta^{-1/2} w)}. \tag{2.5}$$

Because $P(x)$ is expressed as the power series

$$P(x) = \frac{1}{2} + Z(x) \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!!},$$

(see e.g., Abramowitz and Stegun (1964)), $Z(x)/P(x)$ is monotone decreasing for $x > 0$. Hence $\phi_\eta(w)$ is increasing in η.

3. Admissibility

As was explained in Section 1, Sacks (1963) showed that any admissible estimator should be generalized Bayes in this estimation problem. Under the quadratic loss function, the generalized Bayes estimator with respect to the prior measure $\tau(\theta)$ is written as

$$\frac{\int_0^\infty \theta \exp\{-(X-\theta)^2/(2\sigma^2)\} d\tau(\theta)}{\int_0^\infty \exp\{-(X-\theta)^2/(2\sigma^2)\} d\tau(\theta)} = X + \sigma^2 \frac{d}{dX} \log m_\tau^B(X),$$

where

$$m_\tau^B(x) = \int_0^\infty \exp\{-(x-\theta)^2/(2\sigma^2)\} d\tau(\theta).$$

Brown (1971) extended the complete class theorem to the multiparameter case and presented a powerful sufficient condition for a generalized Bayes estimator to be admissible. The sufficient condition in our setting is stated as follows.

THEOREM 3.1. *The generalized Bayes estimator with respect to $\tau(\theta)$ is admissible if $\sigma^2 (d/dx) \log m_\tau^B(x)$ is uniformly bounded for $x \ge 0$ and if the two integrals $\int_1^\infty m_\tau^B(x)^{-1} dx$ and $\int_{-\infty}^{-1} m_\tau^B(x)^{-1} dx$ diverge.*

The estimator $\delta_{\phi_1}(X)$ is rewritten as $X + \sigma^2(d/dX)\log m_1(X)$, where

$$m_1(x) = \left(\int_0^\infty \exp(-(x-\theta)^2/2)d\theta\right)^{\sigma^{-2}}.$$

As we have asymptotic representations, for $\alpha = \sigma^{-2}$,

$$m_1(x)^{1/\alpha} = \begin{cases} \sqrt{2\pi} + o(1) & (x \to \infty) \\ \exp(-x^2/2)(-x)^{-1}\{1 + o(1/x)\} & (x \to -\infty), \end{cases}$$

which are special cases of (4.4) and (4.3) in Section 4, respectively, both $\int_1^\infty m_1(x)^{-1}dx$ and $\int_{-\infty}^{-1} m_1(x)^{-1}dx$ diverge. Moreover, $\sigma^2(d/dw)\log m_1(w)$, which is equal to $\phi_1(w)$, is uniformly bounded for $w \geq 0$ because $\lim_{w\to\infty}\phi_1(w) = 0$. Therefore, by Theorem 3.1, if $\delta_{\phi_1}(X)$ is generalized Bayes, that is, if there is a nonnegative measure τ such that

$$m_1(x) = \int_0^\infty \exp\{-(x-\theta)^2/(2\sigma^2)\}d\tau(\theta),$$

then it is admissible. This is equivalent to the condition that the function $f(x)^\alpha$ for $\alpha = \sigma^{-2}$, where $f(x)$ is given by (1.2), can be expressed as the Laplace transform of a nonnegative measure, say, $G(t)$:

$$f(x)^\alpha = \int_0^\infty e^{-xt}dG(t). \tag{3.1}$$

Indeed, if (3.1) holds then we have

$$\begin{aligned} m_1(x) &= \exp\{-x^2/(2\sigma^2)\}\left(\int_0^\infty \exp(x\theta - \theta^2/2)d\theta\right)^\alpha \\ &= \exp\{-x^2/(2\sigma^2)\}\int_0^\infty \exp(xt)dG(t) \\ &= \exp\{-x^2/(2\sigma^2)\}\int_0^\infty \exp(x\alpha\theta)dG(\alpha\theta) \\ &= \int_0^\infty \exp\{-(x-\theta)^2/(2\sigma^2)\}d\tau(\theta) \end{aligned}$$

where $d\tau(\theta) = \exp(\alpha\theta^2/2)dG(\alpha\theta)$.

If α is a positive integer, $\{(2/\pi)^{1/2}f(x)\}^\alpha$ is a moment generating function of the distribution of $Z = \sum_{i=1}^\alpha Y_i$ where $Y_1, \dots, Y_\alpha$ are independent random variables having a probability density $(2/\pi)^{1/2}\exp(-y^2/2)I_{(0,\infty)}(y)$. Therefore $\delta_{\phi_1}(X)$ is admissible when σ^{-2} is a positive integer.

Next we consider the case where α is not a positive integer. Taking the inverse Laplace transform of (3.1) yields

$$dG(t) = g(t)dt,$$

where $g(t)$ is given by (1.3). Hence, if such a nonnegative measure $G(t)$ as in (3.1) exists, then the function $g(t)$ must be nonnegative. In Section 5, however, we will show that the function $g(t)$ for non-integer α oscillates between positive and negative values for sufficiently large $t > 0$ and therefore cannot be nonnegative for all $t \geq 0$ (see Theorem 5.1). This means that when σ^{-2} is not a positive integer, $\delta_{\phi_1}(X)$ is not generalized Bayes and hence is inadmissible.

THEOREM 3.2. *The estimator $\delta_{\phi_1}(X)$ is admissible if and only if σ^{-2} is a positive integer for the true variance σ^2.*

4. Laplace transform

We discuss some function-theoretic properties of the function $f(z)$ in (1.2), which is the Laplace transform of the Gaussian function $e^{-x^2/2}$. It is easy to see that $f(z)$ is an entire function satisfying the functional equation

$$f(z) + f(-z) = \sqrt{2\pi}e^{z^2/2}, \tag{4.1}$$

as well as the first-order inhomogeneous linear differential equation

$$f'(z) = zf(z) - 1. \tag{4.2}$$

As for the asymptotic behavior of the function $f(z)$, Watson's lemma (see e.g., §3.3 of Olver (1997)) yields a uniform asymptotic representation

$$f(z) \sim \sum_{n=0}^{\infty}(-1)^n(2n-1)!!z^{-2n-1} \tag{4.3}$$

as $z \to \infty$ in any proper subsector of the sector $|\arg z| < 3\pi/4$, where we set $(-1)!! = 1$ by convention. Because the change of variable $z \mapsto -z$ takes the sector $|\arg z| < 3\pi/4$ onto the sector $\pi/4 < \arg z < 7\pi/4$, the asymptotic formula (4.3), together with the functional equation (4.1), leads to another uniform asymptotic representation

$$f(z) - \sqrt{2\pi}e^{z^2/2} \sim \sum_{n=0}^{\infty}(-1)^n(2n-1)!!z^{-2n-1} \tag{4.4}$$

as $z \to \infty$ in any proper subsector of the sector $\pi/4 < \arg z < 7\pi/4$. Note that formulas (4.3) and (4.4) cover asymptotic behaviors as $z \to \infty$ in every directions. They imply that $f(z)$ is an entire function of order two.

The following information about the zeros of $f(z)$ will be important in the sequel.

LEMMA 4.1. (i) *The function $f(z)$ has infinitely many zeros. Each zero, say, w, has the derivative $f'(w) = -1$ and hence is simple.*

(ii) *The function $f(z)$ has no zeros on the closed half-plane* $\mathrm{Re}\, z \geq 0$ *or on the closed sector $3\pi/4 \leq \arg z \leq 5\pi/4$.*

(iii) *The function $f(z)$ has at most finitely many zeros outside the sectors $3\pi/4 - \delta < |\arg z| < 3\pi/4$ for any $0 < \delta < 3\pi/4$.*

PROOF. Assertion (i): Assume the contrary, that $f(z)$ has at most finitely many zeros. Because $f(z)$ is an entire function of order two, Hadamard's theorem (see e.g., Chapter 5, §3.2 of Ahlfors (1979)) implies that $f(z)$ can be represented as $f(z) = e^{uz^2+vz}p(z)$

with some complex numbers u, v and a polynomial $p(z)$. On the other hand, the asymptotic formula (4.3) implies that $f(z) = O(1/z)$ as $z \to \infty$ in any proper subsector of the sector $|\arg z| < 3\pi/4$ and hence one must have $u = v = 0$ and $p(z) \equiv 0$, which is a contradiction. Hence $f(z)$ has infinitely many zeros. It follows from the differential equation (4.2) that $f'(w) = -1$ at each zero $z = w$. In particular the zero w is simple.

Assertion (ii): If z is real, the integrand of (1.2) is positive and so is $f(z)$. Put $z = \xi + i\eta$. It follows from (1.2) that

$$\operatorname{Im} f(z) = -e^{\xi^2/2} \int_0^\infty e^{-(x+\xi)^2/2} \sin(\eta x) dx.$$

If $\xi = \operatorname{Re} z$ is nonnegative then the function $e^{-(x+\xi)^2/2}$ is strictly decreasing in $x \geq 0$. Hence the alternating series test implies that if $\eta = \operatorname{Im} z$ is positive (negative) then so is the negative of $\operatorname{Im} f(z)$. Therefore $f(z)$ has no zeros on the closed half-space $\operatorname{Re} z \geq 0$. To show that $f(z)$ has no zeros on the sector $3\pi/4 \leq \arg z \leq 5\pi/4$, we notice that (1.2) yields

$$|f(-z)| \leq \int_0^\infty e^{-x^2/2+(\operatorname{Re} z)x} dx \leq \int_0^\infty e^{-x^2/2} dx = \sqrt{\pi/2} \quad \text{for} \quad \operatorname{Re} z \leq 0.$$

This estimate and the functional equation (4.1) lead to an estimate

$$|f(z)| \geq \sqrt{2\pi}|e^{z^2/2}| - |f(-z)| \geq \sqrt{2\pi} - \sqrt{\pi/2} = \sqrt{\pi/2} > 0$$

on the sector $3\pi/4 \leq \arg z \leq 5\pi/4$. Hence Assertion (ii) is proved.

Assertion (iii): The asymptotic formula (4.3) implies that $f(z) = 1/z + O(1/z^3)$ as $z \to \infty$ in the sector $|\arg z| \leq 3\pi/4 - \delta$. Hence $f(z)$ has at most finitely many zeros there. Then, in view of Assertion (ii), we have Assertion (iii).

Let $\{z_n\}_{n=1}^\infty$ be the zeros of $f(z)$ in the upper half-plane $\operatorname{Im} z > 0$, totally ordered so that $m < n$ implies either (i) $\operatorname{Re} z_m > \operatorname{Re} z_n$, or (ii) $\operatorname{Re} z_m = \operatorname{Re} z_n$ and $\operatorname{Im} z_m < \operatorname{Im} z_n$. Because $f(z)$ is a real entire function, the zeros of $f(z)$ consist of $\{z_n\}_{n=1}^\infty$ and their complex conjugates $\{\bar{z}_n\}_{n=1}^\infty$. We put

$$z_1 = a + ib,$$

where $a < 0$ and $b > 0$ (see Fig. 1 in Section 5).

Remark 1. Let $m \geq 1$ be the unique integer such that

$$\operatorname{Re} z_1 = \cdots = \operatorname{Re} z_m = a > \operatorname{Re} z_{m+1}. \tag{4.5}$$

Numerical computations strongly suggest that $m = 1$, although it has not been logically established. Hereafter we shall assume that $m = 1$ is the case. This assumption is not essential for our discussion, but is made only for simplicity of presentation. At the end of the next section, we shall indicate how to modify the argument if m happens to be greater than one (see Remark 2 in Section 5).

5. Inverse Laplace transform

We consider the inverse Laplace transform $g(t)$ in (1.3) of the function $f(z)^\alpha$ with $\alpha > 0$, where $f(z)$ is defined by (1.2). We are only interested in the case where α is not a positive integer as we have already treated the positive integer case in Section 3. Take a real number c such that $a < c < 0$, as indicated in Fig. 1. Note that $f(c) > 0$ by (1.2). Then the branch of the multi-valued function $f(z)^\alpha$ is specified by $f(c)^\alpha > 0$ at the point $z = c$.

We specify the contour of integration Γ in (1.3). Usually, the contour of integration of an inverse Laplace transform is chosen to be a suitable line parallel to the imaginary axis. In the current situation, such a choice is feasible when $\alpha > 1/2$. One can take Γ to be the vertical line passing through the point c, as indicated in Fig. 1 (left). Indeed, the asymptotic formula (4.3) implies that $f(z)^\alpha = O(z^{-\alpha})$ as $z \to \infty$ along Γ and hence, if $\alpha > 1/2$, then $f(z)^\alpha$ is square integrable on Γ and the integral (1.3) converges in the mean. With this choice of Γ, however, the integral is divergent when $0 < \alpha \leq 1/2$. To cover this case also, we should replace the line Γ by a contour as in Fig. 1 (right), which is the union of two rays meeting at c, slightly inclined toward the negative real axis, and lying to the right of all the zeros of $f(z)$. Because the integrand $f(z)^\alpha e^{zt}$ of (1.3) is exponentially decreasing along the new contour, this replacement makes the integral (1.3) absolutely convergent for every $\alpha > 0$, without changing the values of (1.3) for $\alpha > 1/2$ by Cauchy's integral theorem. Thus, from the beginning, we may and shall assume that Γ is the contour as in Fig. 1 (right).

To investigate the asymptotic behavior of the function $g(t)$ for sufficiently large $t > 0$, we modify the contour of integration Γ, via the one in Fig. 2 (left), to $\gamma + (-\bar{\gamma}) + \Gamma'$ in Fig. 2 (right). Here γ is a loop that starts and ends at $z_1 + e^{i\theta}\infty$ with an angle θ such that $\pi/2 < \theta < 3\pi/4$ and encircles the first zero z_1 in the positive direction, and $-\bar{\gamma}$ is the complex conjugate of γ with orientation reversed. The contour Γ' consists of a line

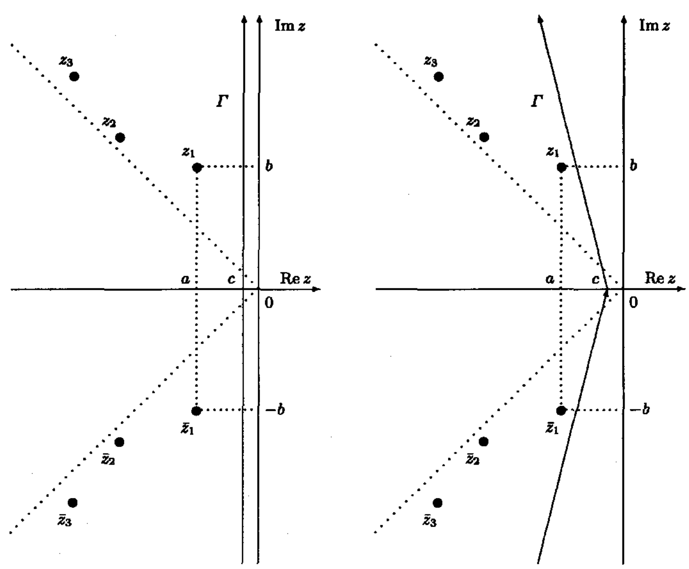

Fig. 1. Contour of integration Γ.

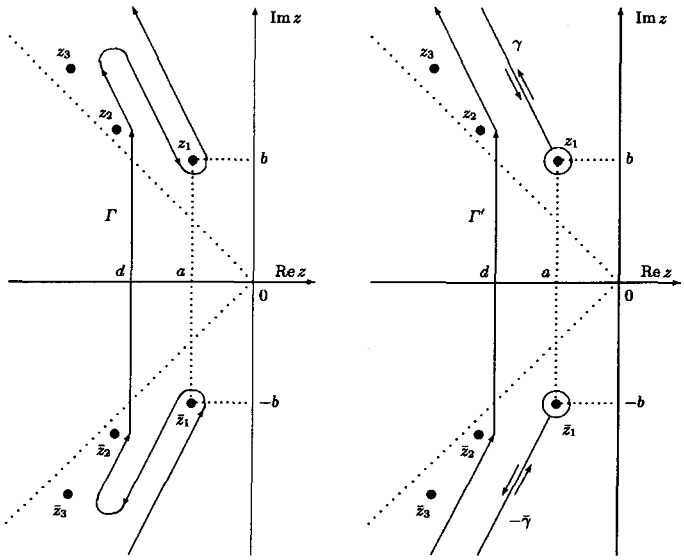

Fig. 2. Modification of the contour Γ.

segment on the vertical line $\operatorname{Re} z = d$ with $d < a$ and two rays parallel to γ and $-\bar{\gamma}$; it should be located to the right of the remaining zeros $\{z_n, \bar{z}_n\}_{n=2}^{\infty}$. This modification of Γ leads to a decomposition

$$g(t) = h(t) + \bar{h}(t) + H(t), \tag{5.1}$$

where the functions $h(t)$ and $H(t)$ are defined by

$$h(t) = \frac{1}{2\pi i}\int_{\gamma} f(z)^{\alpha} e^{zt} dz, \qquad H(t) = \frac{1}{2\pi i}\int_{\Gamma'} f(z)^{\alpha} e^{zt} dz, \tag{5.2}$$

and $\bar{h}(t)$ is the complex conjugate of $h(t)$. We shall investigate $h(t)$ and $H(t)$ separately and see that the asymptotic behavior of $g(t)$ for sufficiently large $t > 0$ is controlled by $h(t) + \bar{h}(t)$. Let us first treat the minor term $H(t)$.

LEMMA 5.1. *There is a constant M, independent of t, such that*

$$|H(t)| \le M e^{dt} \qquad (t \ge 1). \tag{5.3}$$

PROOF. If we put $H(t) = K(t)e^{dt}$, then

$$K(t) = \frac{1}{2\pi i}\int_{\Gamma'} f(z)^{\alpha} e^{(z-d)t} dz.$$

The asymptotic formula (4.3) implies that $f(z)$ is bounded on Γ' so that there is a constant N such that $|f(z)|^{\alpha} \le N$ on Γ'. Hence for any $t \ge 1$,

$$|K(t)| \le \frac{N}{2\pi}\int_{\Gamma'} e^{\operatorname{Re}(z-d)t}|dz| \le \frac{N}{2\pi}\int_{\Gamma'} e^{\operatorname{Re}(z-d)}|dz| =: M < +\infty,$$

where we used the fact that $\mathrm{Re}(z-d) \leq 0$ on Γ'. This proves the lemma.

We proceed to the investigation of the principal term $h(t)$. Recall that it is only necessary to consider the case where α is not a positive integer.

LEMMA 5.2. *There exists an asymptotic representation as $t \to +\infty$,*

$$h(t) = -\frac{\sin(\pi\alpha)}{\pi}\Gamma(\alpha+1)\exp(z_1 t - i\pi\alpha)t^{-\alpha-1}\{1+O(1/t)\}. \tag{5.4}$$

PROOF. The function $f(z)$ has a simple zero at $z = z_1$ with derivative $f'(z_1) = -1$, so the function $f(z)^\alpha$ is expressed as $f(z)^\alpha = (z-z_1)^\alpha\{e^{i\pi\alpha}+O(z-z_1)\}$ in a neighborhood of $z = z_1$, and hence is multi-valued across the branch cut $L = \{z = z_1 + xe^{i\theta} : 0 \leq x < \infty\}$. By the change of variable $z = z_1 + we^{i\theta}$, the first integral in (5.2) is converted to

$$h(t) = \frac{\exp(z_1 t + i\theta)}{2i\pi}\int_{\gamma'}\phi(w)\exp(we^{i\theta}t)dw, \tag{5.5}$$

where $\phi(w) = f(z_1 + we^{i\theta})^\alpha$ and $\gamma' = \ell_\varepsilon^- + C_\varepsilon + \ell_\varepsilon^+$ is the loop indicated in Fig. 3. Here $\ell_\varepsilon^\pm = \{x \pm i0 : \varepsilon \leq x < \infty\}$ are half-lines directed as in Fig. 3 and C_ε is the circle of radius ε, centered at the origin, directed anti-clockwise. The function $\phi(w)$ is expressed as $\phi(w) = w^\alpha\psi(w)$, where $\psi(w)$ is a single-valued holomorphic function having a Taylor expansion $\psi(w) = c_0 + c_1 w + c_2 w^2 + \cdots$ with $c_0 = e^{i\alpha(\theta+\pi)}$. The branch of $\phi(w)$ is chosen so that w^α is positive on the half-line ℓ_ε^-. The multi-valuedness of $\phi(w)$ across the branch cut $0 \leq x < \infty$ is given by $\phi(x+i0) = e^{-2i\pi\alpha}\phi(x-i0) = e^{-2i\pi\alpha}\phi(x)$, where $x - i0$ is identified with x. This yields

$$\begin{aligned}\int_{\gamma'}\phi(w)\exp(we^{i\theta}t)dw &= \int_{C_\varepsilon}\phi(w)\exp(we^{i\theta}t)dw \\ &\quad + (1-e^{-2i\pi\alpha})\int_\varepsilon^\infty \phi(x)\exp(xe^{i\theta}t)dx.\end{aligned}$$

Because $\alpha > 0$, the first integral in the right-hand side tends to zero as $\varepsilon \to 0$. Hence, by letting $\varepsilon \to 0$, (5.5) becomes

$$\begin{aligned}h(t) &= \frac{\exp(z_1 t + i\theta)}{2i\pi}(1-e^{-2i\pi\alpha})\int_0^\infty \phi(x)\exp(xe^{i\theta}t)dx \\ &= \frac{\sin\pi\alpha}{\pi}\exp(z_1 t + i\theta - i\pi\alpha)\int_0^\infty \phi(x)\exp(xe^{i\theta}t)dx.\end{aligned} \tag{5.6}$$

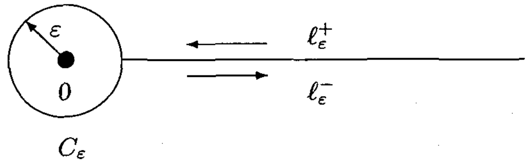

Fig. 3. The loop γ'.

On the other hand, Watson's lemma in asymptotic analysis (see e.g., §3.3 of Olver (1997)) yields

$$\int_0^\infty \phi(x)e^{xs}dx = -c_0 e^{-i\pi\alpha}\Gamma(\alpha+1)s^{-\alpha-1}\{1+O(1/s)\}$$

as $s \to \infty$ in any proper subsector of the sector $|\arg s - \pi| < \pi/2$. Note that the ray $s = e^{i\theta}t$, $t > 0$, lies in this sector, because $\pi/2 < \theta < 3\pi/4$. Substituting the above formula into (5.6) and using $c_0 = e^{i\alpha(\theta+\pi)}$, we obtain (5.4).

Putting Lemmas 5.1 and 5.2 together, we have the following:

THEOREM 5.1. *There exists an asymptotic representation as $t \to +\infty$,*

$$g(t) = M_\alpha e^{at}t^{-\alpha-1}\{\cos(bt-\pi\alpha)+O(1/t)\}, \tag{5.7}$$

where $M_\alpha = -2\sin(\pi\alpha)\Gamma(\alpha+1)/\pi$. In particular, for sufficiently large $t > 0$, the real function $g(t)$ oscillates between positive and negative values.

PROOF. Substituting (5.3) and (5.4) into (5.1) and using $z_1 = a + ib$, we obtain

$$\begin{aligned} g(t) &= (M_\alpha/2)\exp(z_1 t - i\pi\alpha)t^{-\alpha-1}\{1+O(1/t)\} \\ &\quad + (M_\alpha/2)\exp(\bar{z}_1 t + i\pi\alpha)t^{-\alpha-1}\{1+O(1/t)\} + O(e^{dt}) \\ &= M_\alpha e^{at}t^{-\alpha-1}\cos(bt-\pi\alpha) + O(e^{at}t^{-\alpha-2}) + O(e^{dt}) \\ &= M_\alpha e^{at}t^{-\alpha-1}\{\cos(bt-\pi\alpha)+O(1/t)\}. \end{aligned}$$

Here the last equality follows from $O(e^{at}t^{-\alpha-2})+O(e^{dt}) = O(e^{at}t^{-\alpha-2})$, which is because of the inequality $d < a$. Hence formula (5.7) is proved. The oscillation property of $g(t)$ readily follows from (5.7).

Remark 2. Theorem 5.1 was established under the assumption that $m = 1$ in (4.5) (see Remark 1 in Section 4). If $m \geq 2$, formula (5.7) should be replaced by

$$g(t) = M_\alpha e^{at}t^{-\alpha-1}\left\{\sum_{j=1}^{m}\cos(b_j t-\pi\alpha)+O(1/t)\right\} \tag{5.8}$$

where $z_j = a + ib_j$ for $j = 1, \ldots, m$ with $b_1 < \cdots < b_m$. In this case, the oscillation property in Theorem 5.1 also remains true for the function $g(t)$.

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