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IMPROVING ON THE MINIMUM RISK EQUIVARIANT ESTIMATOR OF A LOCATION PARAMETER WHICH IS CONSTRAINED TO AN INTERVAL OR A HALF-INTERVAL

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Abstract. For location families with densities $f_0(x-\theta)$, we study the problem of estimating θ for location invariant loss $L(\theta, d) = \rho(d-\theta)$, and under a lower-bound constraint of the form $\theta \geq a$. We show, that for quite general (f_0, ρ), the Bayes estimator δ_U with respect to a uniform prior on (a, ∞) is a minimax estimator which dominates the benchmark minimum risk equivariant (MRE) estimator. In extending some previous dominance results due to Katz and Farrell, we make use of Kubokawa's *IERD* (Integral Expression of Risk Difference) method, and actually obtain classes of dominating estimators which include, and are characterized in terms of δ_U. Implications are also given and, finally, the above dominance phenomenon is studied and extended to an interval constraint of the form $\theta \in [a, b]$.

Key words and phrases: Lower-bounded parameter, location family, constrained parameter space, minimax estimation, minimum risk equivariant estimator, dominating estimators.

1. Introduction

Consider estimating, under loss $L(\theta, d) = \rho(d-\theta)$, based on one observation X, the parameter θ of a location family with probability density functions of the form $f_\theta(x) = f_0(x-\theta)$, with known f_0. For an unconstrained parameter space $(-\infty, \infty)$, the minimum risk equivariant (MRE) estimator is, under mild conditions, uniquely given by $\delta_0(X) = X + c_0$, where c_0 minimizes in c the constant risk $R(\theta, X+c) = E_0[\rho(X+c)]$ (see for instance Lehmann and Casella (1998), Chapter 3, for a presentation). Also, the representation of the MRE as the generalized Bayes estimator with respect to the right invariant Haar measure $\pi(\theta) = 1$; (see Lemma 2.3, part (c)) is well known, and will play a pivotal role in our results, as well as their interpretation.

The first part of this paper (Sections 2, 3 and 4) is concerned with the estimation framework above, but for a restricted parameter space of the form $\Theta = [a, \infty)$ with a known. For estimators written in the form $\delta_h(X) = \delta_0(X) + h(X)$, we produce simple conditions on h for δ_h to dominate δ_0. Namely, these conditions are fundamentally

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related to the generalized Bayes estimator δ_U associated with the uniform prior density $\pi_a(\theta) = [\theta > a]$, which is shown as well to dominate δ_0.

Although δ_0 produces implausible estimates in taking values outside the parameter space $[a, \infty)$, it serves as a useful benchmark for assessing estimators in our truncated parameter space problem. More precisely, we establish in Section 4 that, for quite general location family f_θ and continuous bowl-shaped loss, the MRE estimator's constant risk matches the minimax risk which implies, in such situations, that dominating estimators of δ_0 are necessarily minimax. We initially thought this result may have been known, but to our knowledge the strongest result in this direction is due to Farrell (1964) and applies (only) to cases where ρ is strictly convex.

For the normal case with squared-error loss (i.e., $\rho(y) = y^2$), Katz (1961) showed that the generalized Bayes estimator δ_U dominates $\delta_0(X)$ (which is X here), and is a minimax and admissible estimator of θ. For the general model f_0, Farrell (1964) established: (i) the minimaxity of δ_U (and hence (ii) domination of δ_0) under general strictly convex loss ρ, and (iii) the admissibility of δ_U under squared-error loss ρ. For the particular case of an exponential location model with $f_0(x) = \frac{1}{\sigma}e^{-x/\sigma}[x > 0]$; with known σ, Parsian and Sanjari Farsipour (1997) gave a proof of (i) and (ii) and established (iii) for Linex losses $\rho(y) = e^{dy} - dy - 1$ with $d \neq 0$ and $d < \frac{1}{\sigma}$.

The dominance results given in Theorem 3.1 and Corollary 3.1 not only extend Farrell's result (ii) to strictly bowl-shaped loss for families of densities f_θ with a strict monotone likelihood ratio property, but also provide explicit classes of dominating estimators. We also: (a) obtain an alternative proof of Farrell's result (ii) for strictly convex loss ρ, (b) obtain a version of these dominance results for convex loss and strictly positive densities f_θ, and (c) show how various implications follow from our conditions for dominance (i.e., Remark 3.1 and Theorem 3.2). Finally, as previously mentioned, we extend in Section 4 Farrell's minimax result (i) for general location families to general bowl-shaped loss.

The key technique used in the derivation of our dominance results is Kubokawa's Integral Expression of Risk Difference (*IERD*) method. Introduced by Kubokawa (1994*a*), and further illustrated by Kubokawa (1998, 1999), the method has proven useful in various and diverse settings. In particular, the work of Kubokawa (1994*b*) and Kubokawa and Saleh (1994), involves restricted parameters and an approach similar to Section 3's. We note that our results are not limited to samples of size 1, and hold for a general location parameter family with density $f(x_1 - \theta, x_2 - \theta, \ldots, x_n - \theta)$ provided the conditional distribution of x_n given the maximum invariant $y = (x_1 - x_n, x_2 - x_n, \ldots, x_{n-1} - x_n)$ satisfies the conditions required for f_0 (a.e. y). Also, one of the key features of the results below resides with the generality of the loss function and the family of distributions under study.

The usefullness of Kubokawa's *IERD* method is further illustrated in Section 5 where we consider parameter spaces which are intervals; of the form $[-m, m]$ without loss of generality. Again here, our dominance results involve the Bayes estimator associated with a uniform prior on the parameter space, which is shown to dominate the MRE δ_0 for general strictly convex loss ρ and general location families. A discussion of related historical developments and implications is relegated to Section 5.

We now proceed in Section 2 in specifying some further notations and assumptions, as well collecting some properties for later use.

2. Preliminaries and notations

Throughout, we work with loss functions of the form $L(\theta, d) = \rho(d - \theta)$, with absolutely continuous strictly bowl-shaped ρ, which we characterize (without loss of generality) by the properties: $\rho \geq 0$; $\rho(0) = 0$, $\rho'(u) < 0$ for $u < 0$, $\rho'(u) > 0$ for $u > 0$. We also work with loss functions ρ such that the associated MRE $\delta_0(X) = X + c_0$ has finite risk.

It is convenient to define C as the following class of estimators.

DEFINITION 2.1. Let $C = \{\delta_h : \delta_h(X) = \delta_0(X) + h(X)$ with h absolutely continuous, nonincreasing, and $\lim_{t\to\infty} h(t) = 0\}$.

Except for those estimators considered in Section 5, all the estimators considered will belong (or be shown to belong) to the class C. Moreover, these properties of the class C will imply that we are working with a.e. differentiable functions h, as in part (i) of Theorem 3.1. Finally, to assure the validity of the interchange of limit and integration in part (a) of Lemmas 2.3 and 5.1, we assume there exists a function k such that $|\rho'(y)| \leq k(y)$, and $\int_{-\infty}^{\infty} k(z) f_\theta(z)dz < \infty$.

Now, we pursue with two technical lemmas concerning functions with a sign change. We will say that a function g changes sign from $-$ to $+$, if there exists on its domain a pair of values c_0 and c_0' with $c_0 \leq c_0'$ such that $g(x) < 0$ for $x < c_0$, $g(x) \geq 0$ for $c_0 \leq x \leq c_0'$, and $g(x) > 0$ for $x > c_0'$.

LEMMA 2.1. *Let X be a continuous random variable with density f. Let g be a non-constant real valued function, continuous almost everywhere, that changes sign from $-$ to $+$ on the support of X. Then, for any pair c_1, c_2 such that $c_1 > c_2$, $P(X \leq c_2) > 0$, and $P(c_2 < X \leq c_1) > 0$; it follows that*

$$E[g(X) \mid X \leq c_1] \leq 0 \Rightarrow E[g(X) \mid X \leq c_2] < 0.$$

PROOF. First if $g(c_2) < 0$, the result is immediate. Now suppose $g(c_2) \geq 0$, and define $I(c) = \int_{(-\infty,c]} g(x)f(x)dx$. Since $g(x) \geq 0$ for $x \in (c_2, c_1]$, we obtain $I(c_1) - I(c_2) = \int_{(c_2,c_1]} g(x)f(x)dx \geq 0$. It then follows that $I(c_2) \leq 0$ with $I(c_2) = 0$ only possible if $I(c_1) = 0$ and $g(x) = 0$ for $x \in (c_2, c_1]$. But these conditions would imply $g(c_2) = 0$, and coupled with the condition $I(c_2) = 0$ would lead: $P(g(X) = 0 \mid X \leq c_2) = 1$, thus contradicting the sign change assumption on g. Hence $I(c_2) < 0$ and the stated result follows.

LEMMA 2.2. *Consider a family of continuous distributions with strict monotone likelihood ratio (MLR) in X with θ being the parameter; (and hence strictly positive densities f_θ on $\Re$). Suppose g is a function that changes sign once on the real line from $-$ to $+$, and suppose c_1 and θ_1 are such that either*

$$g(c_1) > 0 \quad \textit{and} \quad \int_{-\infty}^{c_1} g(x) f_{\theta_1}(x)dx \leq 0, \qquad \textit{or} \qquad \int_{-\infty}^{c_1} g(x) f_{\theta_1}(x)dx = 0.$$

Then, for all $\theta_0 < \theta_1$, we must have

$$\int_{-\infty}^{c_1} g(x) f_{\theta_0}(x)dx < 0.$$

PROOF. Set c_0 as a point where g changes sign, and note that we must have $c_1 \geq c_0$ given the characterization of c_1. Now observe that

$$\begin{aligned} 0 \geq \int_{-\infty}^{c_1} g(x) f_{\theta_1}(x)dx &= \int_{-\infty}^{c_0} g(x)\frac{f_{\theta_1}(x)}{f_{\theta_0}(x)} f_{\theta_0}(x)dx + \int_{c_0}^{c_1} g(x)\frac{f_{\theta_1}(x)}{f_{\theta_0}(x)} f_{\theta_0}(x)dx \\ &> \int_{-\infty}^{c_0} g(x)\frac{f_{\theta_1}(c_0)}{f_{\theta_0}(c_0)} f_{\theta_0}(x)dx + \int_{c_0}^{c_1} g(x)\frac{f_{\theta_1}(c_0)}{f_{\theta_0}(c_0)} f_{\theta_0}(x)dx \\ &= \frac{f_{\theta_1}(c_0)}{f_{\theta_0}(c_0)} \int_{-\infty}^{c_1} g(x) f_{\theta_0}(x)dx, \end{aligned}$$

which yields the result. In the above expansion, note that the second inequality is indeed strict given the assumed MLR property and the assumptions on g and f_θ which imply $P_{\theta_0}[g(X) < 0 \mid X < c_0] > 0$ and $P_{\theta_0}[g(X) \leq 0 \mid X < c_0] = P_{\theta_0}[g(X) \geq 0 \mid c_0 < X \leq c_1] = 1$.

LEMMA 2.3. *For general strict bowl-shaped loss ρ and a general location family of densities f_θ having a strict monotone increasing likelihood ratio, the generalized Bayes estimator $\delta_U(X) = \delta_0(X) + h_U(X)$, associated with the prior density $\pi_a(\theta) = 1[\theta > a]$; $a \in \Re$ or $a = -\infty$, satisfies the following properties:*

(a) $E_0[\rho'(\delta_0(X) + h_U(t)) \mid X \leq t - a] = 0$, *for all* $t \in \Re$.

(b) $\delta_U \in C$.

(c) *When* $a = -\infty$, $\delta_U = \delta_0$ *(i.e., here* $\delta_U(X) = \delta_0(X) = X + c_0$*).*

PROOF. (a) By definition, the generalized Bayes estimator δ_U minimizes the posterior risk, so that it also minimizes in $h(t)$, for all $t \in \Re$,

$$\int_a^\infty \rho(t + c_0 + h(t) - \theta) f_\theta(t)d\theta = \int_{-\infty}^{t-a} \rho(u + c_0 + h(t)) f_0(u)du.$$

Hence, by assumptions on ρ, we must have

$$\int_{-\infty}^{t-a} \rho'(u + c_0 + h_U(t)) f_0(u)du = 0,$$

or again $E_0[\rho'(\delta_0(X) + h_U(t)) \mid X \leq t - a] = 0$, as stated.

(b) Without loss of generality, we set $a = 0$. We need to show (1) that h_U is nonincreasing with (2) $\lim_{t\to\infty} h_U(t) = 0$.

(1) To show that h_U is nonincreasing, assume in order to arrive at a contradiction that there exists $t_2 > t_1$ such that $h_U(t_2) > h_U(t_1)$. From part (a),

$$\begin{aligned} 0 = E_0[\rho'(X + c_0 + h_U(t_2)) \mid X \leq t_2] &= E_{h_U(t_2)}[\rho'(X + c_0) \mid X \leq t_2 + h_U(t_2)] \\ &> E_{h_U(t_1)}[\rho'(X + c_0) \mid X \leq t_2 + h_U(t_2)]; \end{aligned}$$

with the inequality being a consequence of Lemma 2.2, and assumptions on ρ and f_θ. Now, if the latter inequality were actually true, we could deduce from Lemma 2.1 the implication:

$$E_{h_U(t_1)}[\rho'(X + c_0) \mid X \leq t_2 + h_U(t_2)] < 0 \Rightarrow E_{h_U(t_1)}[\rho'(X + c_0) \mid X \leq t_1 + h_U(t_1)] < 0,$$

which would contradict the property in part (a) of this lemma, i.e., since $E_{h_U(t_1)}[\rho'(X + c_0) \mid X \le t_1 + h_U(t_1)] = E_0[\rho'(X + c_0 + h_U(t_1)) \mid X \le t_1]$. Therefore, $h_U(t_2) \le h_U(t_1)$ as claimed.

(2) To show that $\lim_{t\to\infty} h_U(t) = 0$, we begin by showing that we must have $h_U \ge 0$. Coupled with the nonincreasing property of h_U, this will permit us to set as a contradiction:

$$\lim_{t\to\infty} h_U(t) = \epsilon, \qquad \text{with} \quad \epsilon > 0. \tag{2.1}$$

So suppose that $h_u(t_0) < 0$ for some t_0. Property (a) of this lemma (with $t = t_0$) would then, in conjunction with Lemma 2.1, imply $E_0[\rho'(\delta_0(X) + h_u(t_0))] > 0$. But $E_0[\rho'(\delta_0(X) + h_u(t_0))] = E_{h_u(t_0)}[\rho'(\delta_0(X))]$, and the inequality $E_{h_u(t_0)}[\rho'(\delta_0(X))] > 0$, in conjunction with Lemma 2.2, would imply $E_0[\rho'(\delta_0(X))] > 0$, which would contradict properties (a) and (c) of this lemma.

Hence, $h_U \ge 0$, and suppose now that (2.1) holds, which would imply $h_U(t) \ge \epsilon$ for all $t \in \Re$. Note that the quantity $E_0[\rho'(\delta_0(X)+\epsilon)] = E_\epsilon[\rho'(\delta_0(X))]$ must be positive since, otherwise, Lemma 2.2 would tell us (with $c_1 = \infty$, $\theta_1 = \epsilon$, and $\theta_0 = 0$): $E_0[\rho'(\delta_0(X))] < 0$, which is false given parts (a) and (c) of this lemma. But if $E_\epsilon[\rho'(\delta_0(X))]$ is positive, so must it be the case that $\int_{-\infty}^{t'} \rho'(\delta_0(X)) f_\epsilon(x)dx$ is positive for a large enough t', and so must $\int_{-\infty}^{t'+h_U(t')} \rho'(\delta_0(X)) f_{h_U(t')}(x)dx$ be positive, given Lemma 2.2 and the assumption $h_U(t') \ge \epsilon$. Finally, this contradicts part (a) of this lemma. Hence, expression (2.1) cannot be true and we must have $\lim_{t\to\infty} h_U(t) = 0$ as claimed.

(c) This is well known. See for instance Berger ((1985), p. 410).

To conclude this section, we should point out and prove that, under the conditions of strictly bowl-shaped loss and of monotone likelihood ratio for the family f_θ of densities, the Bayes estimate $\delta_U(t_0)$ is unique for all $t_0 \in \Re$. This can be shown by contradiction with the help of Lemmas 2.1 and 2.2. Suppose indeed, as required by part (a) of Lemma 2.3, that

$$\int_{-\infty}^{t_0 - a} \rho'(\delta_0(x) + h_0(t_0)) f_0(x)dx = 0 \tag{2.2}$$

where $h_0(t_0) - h_U(t_0) = \epsilon > 0$ (the case $\epsilon < 0$ can be handled with a similar development). Now, Lemma 2.2 can be applied to yield

$$\int_{-\infty}^{t_0 - a} \rho'(\delta_0(x) + h_0(t_0)) f_\epsilon(x)dx > 0.$$

Then, $0 < \int_{-\infty}^{t_0-a} \rho'(\delta_0(x) + h_0(t_0)) f_\epsilon(x)dx = \int_{-\infty}^{t_0-a-\epsilon} \rho'(\delta_0(x) + h_U(t_0)) f_0(x)dx$, and an application of Lemma 2.1 with $c_2 = t_0 - a - \epsilon$ and $c_1 = t_0 - a$ tells us that, if (2.2) were indeed true, then $\int_{-\infty}^{t_0-a} \rho'(\delta_0(x) + h_U(t_0)) f_0(x)dx > 0$, which contradicts the fact that $\delta_U(t_0)$ is the Bayes estimate for $x = t_0$. Therefore, the uniqueness of δ_U follows.

3. Main dominance results for the parameter space $\Theta = [a, \infty)$

THEOREM 3.1. *For general strict bowl-shaped loss ρ and a general location family of densities f_θ having a strict monotone increasing likelihood ratio, either one of the following two conditions are sufficient for estimators $\delta_h \in C$, $\delta_h \ne \delta_0$, to dominate δ_0:*

(i) $E_0[\rho'(\delta_0(X)+h(t)) \mid X \le t-a] \le 0$; *for all* $t \in \Re$;
(ii) $h \le h_U$.

PROOF. (i) Following Kubokawa (1994*a*, 1998, 1999), we may write

$$\begin{aligned}\rho(\delta_0(x)-\theta)-\rho(\delta_0(x)+h(x)-\theta) &= \rho(\delta_0(x)+h(t)-\theta)\,\Big|_{t=x}^{t=\infty}\\ &= \int_x^\infty \rho'(\delta_0(x)+h(t)-\theta)h'(t)dt,\end{aligned}$$

so that

$$\begin{aligned}\Delta_h(\theta) &= R(\theta,\delta_0)-R(\theta,\delta_h)\\ &= \int_{-\infty}^{\infty}\int_x^\infty \rho'(\delta_0(x)+h(t)-\theta)h'(t)f_\theta(x)dtdx\\ &= \int_{-\infty}^{\infty} h'(t)\left\{\int_{-\infty}^{t}\rho'(\delta_0(x)+h(t)-\theta)f_\theta(x)dx\right\}dt.\end{aligned}$$

Given that $h' \le 0$, the difference in risks $\Delta_h(\theta)$ will be nonnegative whenever, for all $t \in \Re$,

$$\begin{aligned}&E_\theta[\rho'(\delta_0(X)+h(t)-\theta) \mid X \le t] \le 0\\ &\Leftrightarrow E_\theta[\rho'(X-\theta+c_0+h(t)) \mid X-\theta \le t-\theta] \le 0\\ &\Leftrightarrow E_0[\rho'(\delta_0(X)+h(t)) \mid X \le t-\theta] \le 0. \qquad (3.1)\end{aligned}$$

Now since $\rho'(\delta_0(x)+h(t))$ changes sign at most once as a function of x from $-$ to $+$ on $(-\infty,\infty)$, Lemma 2.1 tells us that $\Delta_h(\theta)$ will be nonnegative for all $\theta \ge a$, whenever for all $t \in \Re$, $E_0[\rho'(\delta_0(X)+h(t)) \mid X \le t-a] \le 0$. Now, observe that for $\delta_h \in C$ with $\delta_h \ne \delta_0$, we must have $h' < 0$ on a set of positive measure; so that, if condition (i) holds, $\Delta_h(\theta) > 0$ for $\theta > a$ by virtue of Lemma 2.1.

(ii) Here, we show that (ii) implies (i). Defining $I(u,v) = \int_{-\infty}^{u-a}\rho'(\delta_0(x)+v)f_0(x)dx$, it will suffice to establish that $I(t,h(t)) \le 0$ whenever $h(t) \le h_u(t)$. By setting $\tau = h(t)-h_U(t) \le 0$, we use part (a) of Lemma 2.3 to obtain

$$0 = I(t,h_U(t)) = I(t,h(t)-\tau) = \int_{-\infty}^{t-\tau-a}\rho'(\delta_0(x)+h(t))f_{-\tau}(x)dx,$$

and deduce, with the aid of Lemma 2.2, that $I(t-\tau,h(t)) \le 0$. Finally, use Lemma 2.1 to obtain the implication $I(t-\tau,h(t)) \le 0 \Rightarrow I(t,h(t)) \le 0$, and the desired result.

Part (a) of the following is now immediate from Theorem 3.1 and Lemma 2.3, while part (b) follows from the development that led to expression (3.1) and the uniqueness of the estimates $\delta_U(t)$.

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1,

(a) *The generalized Bayes estimator* δ_U *with respect to the uniform prior density* $\pi_a(\theta) = 1[\theta > a]$ *dominates* δ_0 *on the parameter space* $[a,\infty)$.

(b) *For estimators* $\delta_h \in C$ *satisfying either sufficient condition* (i) *or* (ii) *of Theorem* 3.1, *we have* $R(\theta,\delta_h) \le R(\theta,\delta_0)$ *for* $\theta \ge a$, *with equality iff* $\delta_h = \delta_U$ *and* $\theta = a$.

Before moving on to further remarks and implications, we wish to point out that Lemma 2.3 (and the uniqueness of δ_U), Theorem 3.1 and Corollary 3.1, which were established under the assumptions:

(A) of a strictly bowl-shaped loss and a strict MLR family of densities f_θ;

also hold under the assumptions:

(B) of a strictly convex loss ρ and a general family f_θ;

or alternatively under the assumptions:

(C) of a convex loss ρ and of strictly positive densities f_θ.

The key observation is that we may still make use of Lemma 2.1 and we can invoke, given that ρ' is increasing the stochastically increasing property in ℓ of the family of distributions $X \mid X \leq \ell$ (under f_θ). In the case of assumptions (B), Corollary 3.1 is due to Farrell (1964), but our proof differs and, as seen with its validity under assumption (A) or (C), permits to extend the result to more general losses for some families of densities f_θ. Furthermore, for all three sets of assumptions, Theorem 3.1 gives a novel and simple characterization of a class of dominating estimators of the MRE δ_0.

Remark 3.1. Further interesting implications follow from the benchmark estimator δ_U, which, ubiquitously, plays the same role in our dominance conditions irrespective of loss ρ and location family f_0 being considered. For instance, the dominance result of part (ii) of Theorem 3.1 is simply stated as: if $\delta \in C$ ($\delta \neq \delta_0$) and $\delta \leq \delta_U$, then δ dominates δ_0. Examples include the truncation of δ_0 onto the parameter space $[a, \infty)$, the generalized Bayes estimators (say $\delta_{U,b}$) with respect to uniform priors on $[b, \infty)$, with $b < a$; as well as their truncations $\max(\delta_{U,b}, a)$. On the other hand, we may infer from part (b) of Corollary 3.1 the following condition for non-dominance: if $\delta' \geq \delta_U$ (with strict inequality on some set of positive probability when $\theta = 0$), then $R(a, \delta') > R(a, \delta_0)$, and such an estimator δ' neither dominates δ_0, nor is minimax. This latter situation arises for instance when we consider $\delta' = \delta_{U,b}$ with $b > a$.

Finally, observe from part (b) of Corollary 3.1, that δ_U fails to dominate any other of the dominating estimators of Theorem 3.1. In particular, δ_U does not dominate the truncation of δ_0 on the parameter space $[a, \infty)$, which corresponds to the maximum likelihood estimator for symmetric and unimodal densities f_θ. This latter observation has previously been observed in the normal case with squared-error loss (see Rukhin (1990)).

Example 3.1. For squared-error loss, the generalized Bayes estimator is given by $\delta_U(x) = x - E_0[X \mid X \leq x - a]$ while the MRE is given by $\delta_0(x) = x - E_0[X]$. For absolute value loss, the estimators become $\delta_U(x) = x - \text{Median}_0[X \mid X \leq x - a]$ and $\delta_0(x) = x - \text{Median}_0[X]$. These expressions can be derived from part (a) of Lemma 2.3; and Corollary 3.1 tells us that δ_U dominates δ_0 in both situations for strict MLR families of densities f_θ. The dominance of δ_U over δ_0 for squared-error loss also holds for a general location family (subject to risk-finiteness of δ_0), while the dominance in the absolute value loss case also holds for a general location family with strictly positive densities (see points (B) and (C) raised following Corollary 3.1).

Now, let us write δ_{U,ρ_0} and δ_{0,ρ_0} to emphasize the association between our generalized Bayes estimators and our MRE's with the loss ρ_0. Observe that δ_{U,ρ_0} belongs to the class C for any (other) ρ, and therefore will also dominate δ_{0,ρ_0} for other loss functions ρ as long as condition (i) of Theorem 3.1 is satisfied. Motivated by this idea, we state and prove the following result which may be viewed as either providing dominating estimators of δ_{0,ρ_0} under convex loss function ρ_1, or as providing a class $\mathcal{L}_1$ of convex losses such that a given estimator $\delta_h \in C$ dominates δ_{0,ρ_0} simultaneously for all losses in $\mathcal{L}_1$.

THEOREM 3.2. *An estimator $\delta_h \in C$ which satisfies sufficient condition* (i) *of Theorem* 3.1 *in dominating the MRE δ_{0,ρ_0} under convex loss $L_0(d,\theta) = \rho_0(d-\theta)$, also dominates δ_{0,ρ_0} under all losses $L_1(d,\theta) = \rho_1(d-\theta)$ for which*

$$\frac{\rho_1'(y)}{\rho_0'(y)} \text{ is nonincreasing in } y, \qquad y \in \Re. \tag{3.2}$$

PROOF. We show that, with condition (3.2), $W(\rho_1) \leq 0$, where $W(\rho) = E_0[\rho'(\delta_0(X)+h(t)) \mid X \leq t-a]$. Indeed, for such losses ρ_1,

$$\begin{aligned} W(\rho_1) &= E_0\left[\frac{\rho_1'(\delta_0(X)+h(t))}{\rho_0'(\delta_0(X)+h(t))}\rho_0'(\delta_0(X)+h(t)) \mid X \leq t-a\right] \\ &\leq W(\rho_0) E_0\left[\frac{\rho_1'(\delta_0(X)+h(t))}{\rho_0'(\delta_0(X)+h(t))} \mid X \leq t-a\right] \\ &\leq 0, \end{aligned}$$

given: (i) condition (3.2) and the convexity of ρ_0; (ii) that $\rho_1(y)$ and $\rho_0(y)$ are both strictly bowl-shaped implying that $\rho_1'(y)$ and $\rho_0'(y)$ have matching signs (for all $y \in \Re$); and (iii) $W(\rho_0) \leq 0$ by assumption on δ_h under loss ρ_0.

Example 3.2. Note that if both ρ_0 and ρ_1 are symmetric, then so is the ratio $\frac{\rho_1'(y)}{\rho_0'(y)}$; whence the impossibility of (3.2) whenever $\rho_1 \neq \rho_0$. However if either ρ_0 or ρ_1 is not symmetric, then we can entertain the possibility of (3.2) being satisfied. Suppose for instance that ρ_0 is L^p loss with $p > 1$ (i.e., $\rho_0(y) = |y|^p$). Then (3.2) is satisfied for asymmetric losses of the form $\rho_1(y) = |y|^{p_1}[y \leq 0] + |y|^{p_2}[y > 0]$ with $p_2 \leq p \leq p_1$. For the particular case where ρ_0 is squared-error loss, the class of losses ρ_1 which satisfy (3.2) include losses for which ρ_1' is concave, as well as losses for which ρ_1' is concave on $(-\infty, 0)$ and ρ_1 is concave on $(0,\infty)$. An interesting subclass of losses for which ρ_1' is concave consist of Linex type losses of the form $\rho_1(y) = e^{dy} - dy - 1$ with $d < 0$. More generally, it is possible to show that, for strictly convex ρ_0, the concavity of $\rho_1'(\rho_0'^{-1})$ is sufficient for (3.2) to hold. Finally, observe again that the above inferences of this example are applicable for general strictly positive f_0 (see point (C) above).

We conclude this section by pointing that our work does not address the interesting issue of the admissibility of the estimator δ_U (but recall Farrell's general admissibility result for squared-error loss). We now turn to the issue of minimaxity.

4. Minimaxity of δ_U and of other dominating estimators

In this section, we show whenever an unique MRE estimator exists for the unconstrained parameter space $\theta \in (-\infty,\infty)$, that the minimax risk for the constrained parameter space $\theta \in [a,\infty)$ matches the minimax risk for the unconstrained problem for general (f_0,ρ). This result may be viewed as an extension of Farrell's (1964) result established for general (f_0, ρ) with strictly convex ρ. As well, Theorem 4.1 follows for squared-error loss ρ from more general results given by Blumenthal and Cohen (1968), and Kumar and Sharma (1988). We show the result by following a well-known minimax search strategy (for instance see Lehmann and Casella (1998), p. 316) which consists in identifying a sequence of priors whose Bayes risks converge to $R_0 = \sup_{\theta \in [a,\infty)}\{R(\theta, \delta_0)\}$. Below, we denote r_π as the Bayes risk associated with prior π.

THEOREM 4.1. *Whenever an unique MRE estimator δ_0 exists for the unconstrained problem $\theta \in \Re$ with constant risk R_0, then R_0 is also the minimax risk for any constrained problem $\theta \geq a$.*

PROOF. Without loss of generality, we set $a = 0$. The proof is subdivided into two parts: (A) first we prove the result for a bounded loss $\rho \leq M$; (B) secondly, we use the result (A) to obtain the result for unbounded losses ρ.

(A) Here we assume $\rho \leq M$, for some constant $M < \infty$.

Let $\{\pi_n\}_{n=1}^{\infty}$ be a sequence of proper prior densities such that

$$\lim_{n\to\infty} r_{\pi_n} = R_0. \tag{4.1}$$

Such a sequence exists (e.g., Wald (1950), p. 90) since $R_0 = \sup_\pi\{r_\pi\}$. Define, for a fixed element π_n of this sequence, the sequence of densities $\{\pi_{n,m}\}_{m=1}^{\infty}$ as the truncations of π_n onto $[-m, m]$, i.e.,

$$\pi_{n,m}(\theta) = \frac{\pi_n(\theta)[-m \leq \theta \leq m]}{H_{n,m}};$$

where $H_{n,m} = \int_{-m}^{m} d\pi_n(\theta)$. We next show that for an element π_n; $n \geq 1$;

$$\lim_{m\to\infty} r_{\pi_{n,m}} = r_{\pi_n}. \tag{4.2}$$

To establish (4.2), first observe that with the boundedness of ρ,

$$\begin{aligned} r_{\pi_n} &\leq \int_{-\infty}^{\infty} R(\theta, \delta_{\pi_{n,m}}) d\pi_n(\theta) \\ &= \left\{\int_{-\infty}^{\infty} R(\theta, \delta_{\pi_{n,m}}) d\pi_{n,m}(\theta)\right\} H_{n,m} + \int_{|\theta|>m} R(\theta, \delta_{\pi_{n,m}}) d\pi_n(\theta) \\ &\leq (r_{\pi_{n,m}})(H_{n,m}) + M(1 - H_{n,m}), \end{aligned}$$

which implies

$$r_{\pi_n} \leq \liminf_{m\to\infty} r_{\pi_{n,m}}. \tag{4.3}$$

Secondly,

$$r_{\pi_n} \geq \int_{-m}^{m} R(\theta, \delta_n) d\pi_n(\theta) = \left\{\int_{-m}^{m} R(\theta, \delta_n) d\pi_{n,m}(\theta)\right\} H_{n,m} \geq (r_{\pi_{n,m}})(H_{n,m}),$$

so that

$$r_{\pi_n} \geq \limsup_{m\to\infty} r_{\pi_{n,m}}. \tag{4.4}$$

From (4.3) and (4.4), we obtain (4.2).

Now, it follows from (4.1) that: for all $\epsilon > 0$, there exists n_0 such that $r_{\pi_n} \geq R_0 - \epsilon$ for all $n \geq n_0$. Also, from (4.2) it is true that: for all $\epsilon > 0$ and $n \geq n_0$, there exists $m(n)$ such that $r_{\pi_{n,m}} \geq r_{\pi_n} - \epsilon$ for all $m \geq m(n)$. Hence, for all $\epsilon > 0$, there exists n_0 such that $r_{\pi_{n,m(n)}} \geq r_{\pi_n} - \epsilon \geq R_0 - 2\epsilon$ for all $n \geq n_0$. But, since R_0 represents the integrated risk under $\pi_{n,m(n)}$ of δ_0, we also have $r_{\pi_{n,m(n)}} \leq R_0$. Therefore, we have a sequence of priors $\pi_n^* = \pi_{n,m(n)}$, such that

$$\lim_{n\to\infty} r_{\pi_n^*} = R_0. \tag{4.5}$$

Finally, as the sequence $\{\pi_n^*\}$ is not supported on the parameter space $[0, \infty)$, define the sequence of prior densities $\{\pi_n^{**}\}$ such that

$$\pi_n^{**}(\theta) = \pi_n^*(\theta - m(n)).$$

By location invariance and (4.5), $\lim_{n\to\infty}\{r_{\pi_n^{**}}\} = \lim_{n\to\infty}\{r_{\pi_n^*}\} = R_0$. Hence, we have a sequence of prior densities $\{\pi_n^{**}\}$ on compact support such that the associated Bayes risks converge to $R_0 = \sup_{\theta\geq 0}\{R(\theta, \delta_0)\}$. This completes the proof that the minimax risk is indeed given by R_0 for bounded losses ρ (note that for this part, it is not necessary that the MRE estimator be unique).

(B) Here, we define truncated loss $\rho^M(y) = (M \wedge \rho(y))$, and we approach an unbounded loss ρ by a sequence of bounded losses ρ^{M_i}, with $M_i \to \infty$, and take advantage of the result in (A). To do so, begin by defining $\delta_0^M(X) = X + c_{0,M}$ and R_0^M as the MRE estimator and risk for loss $L(\theta, d) = \rho^M(d - \theta)$. Note that, from (A), R_0^M is the minimax risk under loss ρ^M, not only for the parameter space $(-\infty, \infty)$, but also for $[0, \infty)$.

Suppose now that:

$$\lim_{M\to\infty} R_0^M = R_0. \tag{4.6}$$

If property (4.6) holds, there exists an increasing sequence $\{M_i\}_{i=1}^\infty$ such that $\lim_{M_i\to\infty} R_0^{M_i} = R_0$, whence the existence, for all $\epsilon > 0$, of a value M^* such that $R_0^{M_i} \geq R_0 - \epsilon$ for all $M_i \geq M^*$. Also we can infer from (4.5) that for all M_i, there exists a sequence of prior densities $\{\pi_{n,M_i}^{**}\}_{n=1}^\infty$ such that $\lim_{n\to\infty} r_{\pi_{n,M_i}^{**}} \geq R_0^{M_i}$, whence the existence, for all $\epsilon > 0$ and M_i, of a value $n_1(M_i)$ such that $r_{\pi_{n,M_i}^{**}} \geq R_0^{M_i} - \epsilon$, for all $n \geq n_1(M_i)$. From the above, it follows that, for all $\epsilon > 0$, there exists a sequence of priors $\pi_{n_1(M_i),M_i}^{**}$ such that, for all $M_i \geq M^*$, $r_{\pi_{n_1(M_i),M_i}^{**}} \geq R_0 - 2\epsilon = \sup_{\theta\geq 0} R(\theta, \delta_0)$, which proves the theorem as long as (4.6) is valid.

Now to prove (4.6), first observe that $\rho^{M'} \geq \rho^M$ for $M' > M$ which implies that the (risk and) maximum risk of any estimator $\delta(X)$ under loss ρ^M is a non-decreasing function of M. Hence, the minimax risk R_0^M is also a non-decreasing function of M, so that $\limsup_{M\to\infty} R_0^M = R^* \leq R_0$. There remains to prove that $\liminf_{M\to\infty} R_0^M \geq R_0$, and we handle separately the two cases:

(i) there exists a subsequence $\{M_j\}_{j=1}^\infty$ with $\lim_{j\to\infty} M_j = \infty$, such that $\lim_{j\to\infty} c_{0,M_j} = c_0^*$ and $|c_0^*| < \infty$;

(ii) there exists a subsequence $\{M_j\}_{j=1}^\infty$ with $\lim_{j\to\infty} M_j = \infty$, such that $\lim_{j\to\infty} c_{0,M_j} = \pm\infty$.

Now observe that we may represent R_0^M as the expected posterior loss ρ^M (for any x) corresponding to the uniform prior on $(-\infty, \infty)$. Hence in case (i), by Fatou's lemma and since ρ^M is continuous in M,

$$\begin{aligned}\liminf_{j\to\infty} R_0^{M_j} &= \liminf_{j\to\infty} \int_{-\infty}^{\infty} \rho^{M_j}(\delta_0^{M_j}(x)-\theta) f_0(x-\theta)d\theta \\ &\geq \int_{-\infty}^{\infty} \liminf_{j\to\infty}\{\rho^{M_j}(\delta_0^{M_j}(x)-\theta)\} f_0(x-\theta)d\theta \\ &= \int_{-\infty}^{\infty} \rho(x+c_0^*-\theta) f_0(x-\theta)d\theta \geq R_0.\end{aligned}$$

Similarly, in case (ii), we have

$$\begin{aligned}\liminf_{j\to\infty} R_0^{M_j} &= \liminf_{j\to\infty} \int_{-\infty}^{\infty} \rho^{M_j}(\delta_0^{M_j}(x)-\theta) f_0(x-\theta)d\theta \\ &\geq \int_{-\infty}^{\infty} \liminf_{j\to\infty}\{\rho^{M_j}(\delta_0^{M_j}(x)-\theta)\} f_0(x-\theta)d\theta \\ &= \int_{-\infty}^{\infty} \sup_{u\in\Re} \rho(u) f_0(x-\theta)d\theta \geq R_0.\end{aligned}$$

yielding (4.6) and completing the proof.

Remark 4.1. As mentioned above, the fact that the minimax risks coincide for the unconstrained and constrained problems implies that dominating estimators of δ_0 are necessarily minimax for the constrained parameter space $[a,\infty)$. These dominating estimators include δ_U, its truncated version $\max(\delta_U, a)$, and all the dominating estimators which can be generated by Theorem 3.1.

We conclude this section with a result which offers (we believe) additional insight into the minimax phenomenon behind Theorem 4.1. Indeed, the following is a short version of Theorem 4.1, which applies to the subclass C^* of estimators $\delta_h(x) = \delta_0(x) + h(x) \in C$ such that $\lim_{x\to\infty} h(x)$ exists.

THEOREM 4.2. *Whenever δ_0 is unique minimax for the unconstrained problem with minimax risk R_0, the minimax risk among estimators in C^* for the constrained problem $[a,\infty)$ is (also) given by R_0, for any pair (f_0, ρ) (with ρ continuous).*

PROOF. It will suffice to show that $\liminf_{\theta\to\infty} R(\theta,\delta_h) \geq R_0$ for any $\delta_h \in C^*$. Let $\lim_{x\to\infty} h(x) = c$. We have by Fatou's lemma

$$\begin{aligned}\liminf_{\theta\to\infty} R(\theta,\delta_h) &= \liminf_{\theta\to\infty} \int_{-\infty}^{\infty} \rho(\delta_0(x)+h(x)-\theta) f_\theta(x)dx \\ &= \liminf_{\theta\to\infty} \int_{-\infty}^{\infty} \rho(\delta_0(x)+h(x+\theta)) f_0(x)dx \\ &\geq \int_{-\infty}^{\infty} \liminf_{\theta\to\infty}\{\rho(\delta_0(x)+h(x+\theta))\} f_0(x)dx \\ &= \int_{-\infty}^{\infty} \rho(\delta_0(x)+c) f_0(x)dx \geq R_0;\end{aligned}$$

establishing the result.

5. The case of the parameter space $\Theta = [-m, m]$

In this section, we present a similar development for the restricted parameter space $\Theta = [-m, m]$, where m is known. In particular, we show that the (proper) Bayes estimator δ_U^* with respect to the uniform prior density $\pi_U^*(\theta) = (2m)^{-1}[-m < \theta < m]$ dominates the MRE δ_0, for strictly convex losses $\rho(d-\theta)$ and general location families (but also see Remark 5.1). This generalizes the result in the normal case under squared-error loss, which was obtained by Gatsonis *et al.* (1987), and offers a different method of proof (in contrast to their sign change arguments). Marchand and Perron (2001) showed that, in a multivariate generalization of the normal case with $X \overset{d}{=} N_p(\theta, I_p)$, $\|\theta\| \le m$ and $L(\theta, d) = \|d-\theta\|^2$, the Bayes estimator δ_U^* with respect to the uniform prior on the ball $\{\theta : \|\theta\| \le m\}$ dominates the maximum likelihood estimator δ_{mle} for sufficiently small m, i.e., $m \le m_0(p)$. Namely, for $p = 1$, they obtain that δ_U^* dominates δ_{mle} for the sufficient condition $m \le m_0(1) \approx 0.5230$. Recently, Hartigan (2004) proved for $X \sim N_p(\theta, I_p)$ with $\theta \in A$, where A ($A \neq \Re^p$) is a convex set with a non-empty interior but otherwise arbitrary, that the Bayes estimator δ_U with respect to the uniform prior on A dominates X under loss $\|d-\theta\|^2$. For $p = 1$, this result applies here for a normal model f_0 and squared-error loss ρ, and also applies to the case $\Theta = [a, \infty)$ studied in Section 3.

Although implications following from such dominance results are more limited for restricted parameter spaces of the form $\Theta = [-m, m]$, it is nevertheless of interest to characterize and obtain estimators that are adapted to the restricted parameter space in the sense of passing the minimum test of improving upon the MRE δ_0. Furthermore, it is particularly interesting to witness (see Corollary 5.1):

(1) that the dominance property of δ_U^* occurs for quite general (f_0, ρ);

(2) the phenomenon for which the truncation π_U^* of the prior $\pi_U(\theta) = 1$ onto the parameter space $[-m, m]$ leads to a dominating estimator δ_U^* of the original Bayes estimator δ_0 associated with the untruncated prior π_U.

Note that the estimators $\delta_h \in C$ of Section 3 which satisfy condition (i) or (ii) of Theorem 3.1 in dominating the MRE δ_0 on the parameter space $[-m, \infty)$ (or $(-\infty, m]$) necessarily dominate δ_0 on the parameter space $[-m, m]$, but that these estimators take some values outside $[-m, m]$ as $\lim_{x\to\infty} h(x) = 0$. Rather, we describe dominating estimators below which take values in $[-m, m]$ only.

The estimators considered in this section will be written in the form $\delta_h(X) = \delta_0(X) + h(X)$, and will belong (or be shown to belong) to the class D of estimators defined as:

$$D = \{\delta_h : \delta_h(X) = \delta_0(X) + h(X) \text{ with } h \text{ absolutely continuous, nonincreasing,} \\ \text{and } \delta_h(x) \in [-m, m] \text{ for all } x \in \Re\}.$$

As before, these properties will imply that we are working with a.e. differentiable function h. Also, given these properties, observe that there will exist, for the estimators $\delta_h \in D$, a value $x_0(h)$, such that $\delta_h(x_0(h)) = \delta_0(x_0(h))$. The dominance conditions of Theorem 5.1 are expressed in terms of $x_0(h)$ and $x_0(h_U^*)$. Note, with the applicability of condition (ii) of Theorem 5.1 in mind, that for symmetric f_0 and ρ, we will have $\delta_0(X) = X$, and $x_0(h) = 0$ for sign invariant δ_h's (i.e., $\delta_h(x) = -\delta_h(-x)$) such as δ_U^*.

The next lemma establishes some useful properties of the estimator δ_U^*, and it is followed by general conditions for an estimator $\delta_h \in D$ to dominate δ_0 on the parameter space $[-m, m]$.

LEMMA 5.1. *The Bayes estimator $\delta_U^*(X) = \delta_0(X) + h_U^*(X)$ associated with the uniform prior on $[-m, m]$ satisfies the following properties:*

(a)

$$E_0[\rho'(\delta_0(X) + h_U^*(t)) \mid t - m \leq X \leq t + m] = 0, \qquad \textit{for all} \quad t \in \Re; \tag{5.1}$$

(b) $\delta_U^* \in D$.

PROOF. We prove (b) only, as part (a) is quite similar to part (a) of Lemma 2.3. We only need to show that h_U^* is nonincreasing. Define $B(v, w) = \int_{v-m}^{v+m} \rho'(u+c_0+w) f_0(u) du$. Suppose, in order to arrive at a contradiction, that there exists values a pair of t_1 and t_2, both on the support of f_0, such that $t_2 > t_1$ and $h_U^*(t_2) > h_U^*(t_1)$. If such were the case, we would have by (5.1), and since ρ' is increasing: $0 = B(t_2, h_U^*(t_2)) > B(t_1, h_U^*(t_2)) \geq B(t_1, h_U^*(t_1))$ which contradicts the property $B(t_1, h_U^*(t_1)) = 0$ given in (5.1).

THEOREM 5.1. *Either one of following two conditions are sufficient for estimators $\delta_h \in D$ to dominate δ_0 on $\Theta = [-m, m]$:*

(i) *$E_0[\rho'(\delta_0(X)+h(t)) \mid X \leq t+m] \leq 0$, for all $t \leq x_0(h)$ and $E_0[\rho'(\delta_0(X)+h(t)) \mid X \geq t - m] \geq 0$, for all $t \geq x_0(h)$.*

(ii) *$x_0(h) = x_0(h_U^*)$ and $|h| \leq |h_U^*|$.*

PROOF. (i) Following Kubokawa (1994*a*, 1998, 1999), we may write

$$\begin{aligned} \rho(\delta_0(x) - \theta) - \rho(\delta_0(x) + h(x) - \theta) &= \rho(\delta_0(x) + h(t) - \theta) \mid_{t=x}^{t=x_0(h)} \\ &= \int_x^{x_0(h)} \rho'(\delta_0(x) + h(t) - \theta) h'(t) dt, \end{aligned}$$

so that

$$\begin{aligned} \Delta_h(\theta) &= R(\theta, \delta_0) - R(\theta, \delta_h) \\ &= \int_{-\infty}^{\infty} \int_x^{x_0(h)} \rho'(\delta_0(x) + h(t) - \theta) h'(t) f_\theta(x) dt dx \\ &= \int_{-\infty}^{x_0(h)} \int_x^{x_0(h)} \rho'(\delta_0(x) + h(t) - \theta) h'(t) f_\theta(x) dt dx \\ &\quad - \int_{x_0(h)}^{\infty} \int_{x_0(h)}^{x} \rho'(\delta_0(x) + h(t) - \theta) h'(t) f_\theta(x) dt dx \\ &= I_{1,h}(\theta) - I_{2,h}(\theta), \end{aligned}$$

where

$$I_{1,h}(\theta) = \int_{-\infty}^{x_0(h)} h'(t) \left\{ \int_{-\infty}^{t} \rho'(\delta_0(x) + h(t) - \theta) f_\theta(x) dx \right\} dt$$

and

$$I_{2,h}(\theta) = \int_{x_0(h)}^{\infty} h'(t) \left\{ \int_t^{\infty} \rho'(\delta_0(x) + h(t) - \theta) f_\theta(x) dx \right\} dt.$$

Now, observe that $I_{1,h}(\theta) \geq 0$ whenever, for all $t \leq x_0(h)$,

$$\begin{aligned} &E_\theta[\rho'(\delta_0(X) + h(t) - \theta) \mid X \leq t] \leq 0 \\ &\quad \Leftrightarrow E_0[\rho'(\delta_0(X) + h(t)) \mid X \leq t - \theta] \leq 0, \end{aligned}$$

as in (3.1). From this, Lemma 5.1 tells us that $I_{1,h}(-m) \geq 0$ and $I_{1,h}(\theta) > 0$ for all $\theta \in (-m, m]$ whenever, for all $t \leq x_0(h)$, $E_0[\rho'(\delta_0(X) + h(t)) \mid X \leq t + m] \leq 0$. Similarly, $I_{2,h}(m) \leq 0$ and $I_{2,h}(\theta) < 0$ for all $\theta \in [-m, m)$ whenever, for all $t \geq x_0(h)$, $E_0[\rho'(\delta_0(X) + h(t)) \mid X \geq t - m] \geq 0$. Hence, the two inequations of condition (i) are jointly sufficient for $\Delta_h(\theta)$ to be positive.

(ii) We show that condition (ii) implies (i). Indeed, if $x_0(h) = x_0(h_U^*)$ and $|h| \leq |h_U^*|$, then

$$E_0[\rho'(\delta_0(X) + h(t)) \mid X \leq t + m] \leq E_0[\rho'(\delta_0(X) + h(t)) \mid t - m \leq X \leq t + m] \leq 0$$

for $t \leq x_0(h)$, and

$$E_0[\rho'(\delta_0(X) + h(t)) \mid X \geq t - m] \geq E_0[\rho'(\delta_0(X) + h(t)) \mid t - m \leq X \leq t + m] \geq 0$$

for $t \geq x_0(h)$, given the convexity of ρ, the nonincreasing property of h for $\delta_h \in D$, and part (a) of Lemma 5.1.

COROLLARY 5.1. *Under general strictly convex loss ρ and for general location families, the Bayes estimator δ_U^* with respect to the uniform prior on $[-m, m]$ dominates the MRE δ_0 on the parameter space $\Theta = [-m, m]$.*

Observe that the dominating estimators $\delta_h \in D$ shrink δ_0 towards $x_0(h)$ (i.e., $|\delta_h - x_0(h)| \leq |\delta_0 - x_0(h)|$), while dominating estimators $\delta_h \in D$ with $x_0(h) = x_0(h_U^*)$ shrink less towards the origin than the benchmark estimator δ_U^*, (i.e., $|\delta_h| \geq |\delta_U^*|$). We also note the interesting case of squared-error loss where the uniform Bayes estimator, as derived from part (a) of Lemma 5.1, is given by $\delta_U^*(x) = x - E_0[X \mid x - m \leq X \leq x + m]$, and dominates the MRE $\delta_0(X) = X - E_0[X]$.

Remark 5.1. Theorem 5.1 and Corollary 5.1 also hold under strictly bowl-shaped losses ρ and for densities f_θ having a strict monotone increasing likelihood ratio. This is so since, in such cases and as in part (b) of Lemma 2.3, part (b) of Lemma 5.1 can be established with the aid of Lemma 2.2. Finally, as in Section 3, the assumption of convex loss and strictly positive densities f_θ also suffice for the validity of Theorem 5.1 and Corollary 5.1.

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