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ASYMPTOTIC PROPERTIES OF THE LEAST SQUARES ESTIMATORS OF THE PARAMETERS OF THE CHIRP SIGNALS

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Abstract. Chirp signals are quite common in different areas of science and engineering. In this paper we consider the asymptotic properties of the least squares estimators of the parameters of the chirp signals. We obtain the consistency property of the least squares estimators and also obtain the asymptotic distribution under the assumptions that the errors are independent and identically distributed. We also consider the generalized chirp signals and obtain the asymptotic properties of the least squares estimators of the unknown parameters. Finally we perform some simulations experiments to see how the asymptotic results behave for small sample and the performances are quite satisfactory.

Key words and phrases: Chirp signal, least squares estimators, asymptotic distribution, consistent estimators.

1. Introduction

In this paper we consider the following signal processing model;

$$y(n) = A^0 e^{j(\alpha^0 n + \beta^0 n^2)} + e(n); \qquad n = 1, \dots, N. \tag{1.1}$$

Here $y(n)$ is the complex valued signal observed at $n = 1, \dots, N$, A^0 is the amplitude and it can be complex valued, $j = \sqrt{-1}$, α^0 is the initial frequency and β^0 is the frequency rate. The additive errors $e(n)$'s are complex valued independent and identically distributed (i.i.d.) random variables with mean zero and finite variance σ^2.

The model (1.1) is known as the chirp signals in statistical signal processing literature. Chirp signals are quite useful in various areas of science and engineering particularly in physics, sonar, radar and communications. For example, chirp signals are used to estimate trajectories of moving objects with respect to fixed receivers. In addition, in situations where interference rejection is important chirp signals provide a successful digital modulation scheme. For instance, consider a radar illuminating a target. Then, the transmitted signal will be affected by a phase shift induced by the distance and relative motion between the target and the receiver. Assuming this motion is continuous and differentiable, the phase shift can be adequately modeled as $\phi(t) = c + \alpha_0 t + \beta_0 t^2$, where α_0 and β_0 are related to speed and acceleration or range and speed depending on what the radar is intended for and on the kind of waveforms transmitted, see for example Rihaczek ((1969), 56–65). It leads to the chirp signal model (1.1).

The problem of estimation of the parameters of chirp signals is quite important. Several methods are available in the literature, see for example the work of Abatzoglou (1986), Kumaresan and Verma (1987), Djuric and Kay (1990), Gini *et al.* (2000), Huang *et al.* (1999), Besson *et al.* (1999) and Saha and Kay (2002). Most of the methods that have been suggested in the literature yield maximum likelihood estimators. Although several methods are available in the literature, but the theoretical properties of the least squares estimators (LSE's) have not been discussed anywhere. The main aim of this paper is to obtain the theoretical properties of the LSE's under the appropriate model assumptions.

Note that the model (1.1) is a non-linear model. Therefore, it is not possible to obtain any finite sample property of the LSE's (Jennrich (1969)). All the results have to be asymptotic in nature. Several sufficient conditions (see for example Jennrich (1969), Wu (1981) or Kundu (1991)) are available in the literature which guarantee the consistency and the asymptotic normality of the LSE's. It is shown in Kundu and Mitra (1996) that even a particular case of this model (1.1), namely when $\beta^0 = 0$, does not satisfy the sufficient conditions of Jennrich (1969) or Wu (1981). Therefore, it is clear that the present model also does not satisfy those sufficient conditions. Because of the complex structure of the model it is not immediate how the LSE's will behave in this situation. For the non-linear models as considered by Jennrich (1969), Wu (1981) or Kundu (1991), it is observed that the rate of convergence of the LSE's is usually $O_p(N^{1/2})$ (here $O_p(N^a) = Z$ means $\frac{Z}{N^a}$ is bounded in probability), whereas here the rate of convergence of the LSE of α^0 is $O_p(N^{3/2})$ and for β^0 it is $O_p(N^{5/2})$.

In this paper we use the following notations: almost sure convergence will be denoted by '$\xrightarrow{a.s.}$', the convergence in distribution will be denoted by '$\xrightarrow{d}$' and C will denote arbitrary constant and it may be different at different places. The rest of the paper is organized as follows. In Section 2, we define the LSE's of the model parameters of (1.1) and obtain their consistency properties. The asymptotic distributions of the LSE's are obtained in Section 3. The consistency and the asymptotic normality properties of an estimator of σ^2 are discussed in Section 4. Generalized chirp signals are considered in Section 5. Some simulation results are presented in Section 6 and finally we draw conclusions in Section 7.

2. Consistency property of the least squares estimators

In this section first we define the LSE's of the parameters of the model (1.1) and then we obtain the consistency property of the LSE's. We use the following notations; $\boldsymbol{\theta} = (A_R, A_I, \alpha, \beta)$. Here A_R and A_I denote the real and imaginary parts of the complex amplitude A and $\boldsymbol{\theta}^0 = (A_R^0, A_I^0, \alpha^0, \beta^0)$ denotes the true parameter value of $\boldsymbol{\theta}$. The LSE's of $\boldsymbol{\theta}^0$, say $\hat{\boldsymbol{\theta}} = (\hat{A}_R, \hat{A}_I, \hat{\alpha}, \hat{\beta})$ can be obtained by minimizing

$$Q(A_R, A_I, \alpha, \beta) = \sum_{n=1}^{N} |y(n) - Ae^{j(\alpha n + \beta n^2)}|^2$$

with respect to A_R, A_I, α and β. We make the following assumptions of the model (1.1).

ASSUMPTION 1. $A^0 = A_R^0 + jA_I^0$ is an arbitrary complex number, $\alpha^0, \beta^0 \in (0, \pi)$ and $e(n)$'s are i.i.d. complex valued random variables. Let us write $e(n) = e_R(n) + je_I(n)$, where $e_R(n)$ and $e_I(n)$ are the real and imaginary parts of $e(n)$. It is assumed that $E(e_R(n)) = E(e_I(n)) = 0$, $V(e_R(n)) = V(e_I(n)) = \frac{\sigma^2}{2}$ and $e_R(n)$ and $e_I(n)$ are independently distributed.

Now we can state the following result:

THEOREM 2.1. *If $\boldsymbol{\theta}^0 = (A_R^0, A_I^0, \alpha^0, \beta^0)$ is an interior point of the parameter space $\Theta = \Re \times \Re \times [0,\pi] \times [0,\pi]$ and if the error random variables $e(n)$'s satisfy Assumption 1 and $|A^0| > 0$, then $\hat{\boldsymbol{\theta}}$, the LSE of $\boldsymbol{\theta}^0$, is a strongly consistent estimator of $\boldsymbol{\theta}^0$.*

To prove Theorem 2.1, we need the following lemmas.

LEMMA 2.1. *Let us denote*

$$S_{C,M} = \{\boldsymbol{\theta} : \boldsymbol{\theta} = (A_R, A_I, \alpha, \beta), |\boldsymbol{\theta} - \boldsymbol{\theta}^0| \geq C, |A_R| \leq M, |A_I| \leq M\}.$$

Suppose $e(n)$'s are i.i.d. random variables satisfying Assumption 1. If for any $C > 0$ and for some $M < \infty$

$$\underline{\lim} \inf_{\boldsymbol{\theta} \in S_{C,M}} \frac{1}{N}[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)] > 0 \qquad a.s.$$

then $\hat{\boldsymbol{\theta}}$, the LSE of $\boldsymbol{\theta}^0$, is a strongly consistent estimator of $\boldsymbol{\theta}^0$.

PROOF OF LEMMA 2.1. The proof is simple and can be obtained along the same line as the proof of Lemma 1 of Wu (1981), therefore it is omitted.

LEMMA 2.2. *Let $\{X(n)\}$ be a sequence of i.i.d. real valued random variables with mean zero and finite variance σ^2, then as $N \to \infty$*

$$\sup_{a,b} \left| \frac{1}{N} \sum_{t=1}^{N} X(t) \cos(at) \cos(bt^2) \right| \xrightarrow{a.s.} 0.$$

PROOF OF LEMMA 2.2. Consider the following random variables:

$$Z(t) = \begin{cases} X(t) & \text{if } |X(t)| \leq t^{3/4}; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \sum_{t=1}^{\infty} P[X(t) \neq Z(t)] &= \sum_{t=1}^{\infty} P[|X(t)| > t^{3/4}] = \sum_{t=1}^{\infty} \sum_{2^{t-1} \leq s < 2^t} P[|X(s)| > s^{3/4}] \\ &\leq \sum_{t=1}^{\infty} \sum_{2^{t-1} \leq s < 2^t} P[|X(1)| > 2^{(t-1)3/4}] \leq \sum_{t=1}^{\infty} 2^t P[|X(1)| > 2^{(t-1)3/4}] \\ &\leq \sum_{t=1}^{\infty} 2^t \frac{E|X(1)|^2}{2^{(t-1)3/2}} \leq C \sum_{t=1}^{\infty} 2^{-t/2} < \infty. \end{aligned}$$

(Note that the third inequality follows from the Markov inequality.) Therefore, $Z(t)$ and $X(t)$ are equivalent sequences (Chung (1974)). So

$$\sup_{a,b}\left|\frac{1}{N}\sum_{t=1}^{N}X(t)\cos(at)\cos(bt^2)\right|\xrightarrow{a.s.}0 \tag{2.1}$$
$$\Leftrightarrow \sup_{a,b}\left|\frac{1}{N}\sum_{t=1}^{N}Z(t)\cos(at)\cos(bt^2)\right|\xrightarrow{a.s.}0.$$

Let $U(t)=Z(t)-E(Z(t))$. Now observe that

$$\sup_{a,b}\left|\frac{1}{N}\sum_{t=1}^{N}E(Z(t))\cos(at)\cos(bt^2)\right|\le\frac{1}{N}\sum_{t=1}^{N}|E(Z(t))|=\frac{1}{N}\sum_{t=1}^{N}\left|\int_{|x|<t^{3/4}}xdF(x)\right|\to 0$$

as $N\to\infty$, where $F(\cdot)$ is the distribution function of $X(n)$. Therefore, proving (2.1) is equivalent to prove

$$\sup_{a,b}\left|\frac{1}{N}\sum_{t=1}^{N}U(t)\cos(at)\cos(bt^2)\right|\xrightarrow{a.s.}0. \tag{2.2}$$

Now we prove (2.2). For any a,b and $\epsilon>0$ and $0\le h\le\frac{1}{4N^{3/4}}$, we have

$$P\left[\left|\frac{1}{N}\sum_{t=1}^{N}U(t)\cos(at)\cos(bt^2)\right|\ge\epsilon\right]\le 2e^{-hN\epsilon}\prod_{t=1}^{N}Ee^{hU(t)\cos(at)\cos(bt^2)}$$
$$\le 2e^{-hN\epsilon}\prod_{t=1}^{N}(1+h^2\sigma^2)\le 2e^{-hN\epsilon+Nh^2\sigma^2}.$$

The first inequality follows from the Markov inequality (see also Kundu and Mitra (1996)). Note that $V(U(t))=V(Z(t))\le V(X(t))=\sigma^2$. It simply follows from the definition of $Z(t)$. Since $|hU(t)\cos(at)\cos(bt^2)|\le\frac{1}{2}$ and for $|x|\le\frac{1}{2}$, $e^x\le 1+x+x^2$, the second inequality holds true.

Choose $h=\frac{1}{4N^{3/4}}$, therefore for large N,

$$P\left[\left|\frac{1}{N}\sum_{t=1}^{N}U(t)\cos(at)\cos(bt^2)\right|\ge\epsilon\right]\le 2e^{-N^{1/4}\epsilon/4+\sigma^2/(16N^{1/2})}\le 2e^{-N^{1/4}\epsilon/4}. \tag{2.3}$$

Let $J=N^6$. Choose J points $(a_1,b_1),\ldots,(a_J,b_J)$ such that for any point (a,b) in $[0,\pi]\times[0,\pi]$, we have a point (a_k,b_k) satisfying

$$|a_k-a|\le\frac{\pi}{N^3}\quad\text{and}\quad|b_k-b|\le\frac{\pi}{N^3}.$$

Now Taylor series expansion can be used to estimate $|\cos(bt^2)-\cos(b_kt^2)|\le t^2|b-b_k|$ and $|\cos(at)-\cos(a_kt)|\le|t||a-a_k|$. Therefore,

$$\left|\frac{1}{N}\sum_{t=1}^{N}U(t)\{\cos(at)\cos(bt^2)-\cos(a_kt)\cos(b_kt^2)\}\right|$$

$$\leq \left| \frac{1}{N} \sum_{t=1}^{N} U(t) \cos(at) \{\cos(bt^2) - \cos(b_k t^2)\} \right|$$
$$+ \left| \frac{1}{N} \sum_{t=1}^{N} U(t) \cos(b_k t^2) \{\cos(at) - \cos(a_k t)\} \right|$$
$$\leq C \left[\frac{1}{N} \sum_{t=1}^{N} t^{3/4} t^2 \frac{\pi}{N^3} + \frac{1}{N} \sum_{t=1}^{N} t^{3/4} t \frac{\pi}{N^3} \right]$$
$$\leq C \left[\frac{\pi}{N^{1/4}} + \frac{\pi}{N^{5/4}} \right] \to 0 \quad \text{as} \quad N \to \infty.$$

Therefore for large N, we have

$$P \left[\sup_{a,b} \left| \frac{1}{N} \sum_{t=1}^{N} U(t) \cos(at) \cos(bt^2) \right| \geq 2\epsilon \right]$$
$$\leq P \left[\max_{k \leq N^6} \left| \frac{1}{N} \sum_{t=1}^{N} U(t) \cos(a_k t) \cos(b_k t^2) \right| \geq \epsilon \right]$$
$$\leq 2N^6 e^{-N^{1/4}\epsilon/4}.$$

Since $\sum_{N=1}^{\infty} 2N^6 e^{-N^{1/4}\epsilon/4} < \infty$, therefore because of Borel-Cantelli lemma (2.2) holds true and that proves Lemma 2.2.

COROLLARY OF LEMMA 2.2. *As $N \to \infty$, the following results are also true.*

(a) $\displaystyle \sup_{a,b} \left| \frac{1}{N} \sum_{t=1}^{N} X(t) \sin(at) \sin(bt^2) \right| \xrightarrow{a.s.} 0,$

(b) $\displaystyle \sup_{a,b} \left| \frac{1}{N} \sum_{t=1}^{N} X(t) \sin(at) \cos(bt^2) \right| \xrightarrow{a.s.} 0,$

(c) $\displaystyle \sup_{a,b} \left| \frac{1}{N} \sum_{t=1}^{N} X(t) \cos(at) \sin(bt^2) \right| \xrightarrow{a.s.} 0,$

(d) $\displaystyle \sup_{a,b} \left| \frac{1}{N^k} \sum_{t=1}^{N} t^k X(t) \cos(at) \cos(bt^2) \right| \xrightarrow{a.s.} 0, \qquad k = 1, 2, \ldots$

(d) *is true for other combinations of sine and cosine functions like* (a), (b) *and* (c).

PROOF OF THEOREM 2.1. In this proof we denote $\hat{\boldsymbol{\theta}}$ by $\hat{\boldsymbol{\theta}}_N = (\hat{A}_{R_N}, \hat{A}_{I_N}, \hat{\alpha}_N, \hat{\beta}_N)$ to emphasize that $\hat{\boldsymbol{\theta}}$ depends on the sample size. If $\hat{\boldsymbol{\theta}}_N$ is not consistent for $\boldsymbol{\theta}^0$, then either

Case I. For all subsequences $\{N_k\}$ of $\{N\}$, $|A_{R_{N_k}}| + |A_{I_{N_k}}| \to \infty$. Then

$$\frac{1}{N_k}(Q(\hat{\boldsymbol{\theta}}_{N_k}) - Q(\boldsymbol{\theta}^0)) \to \infty.$$

But as $\hat{\boldsymbol{\theta}}_{N_k}$ is the LSE of $\boldsymbol{\theta}^0$,

$$Q(\hat{\boldsymbol{\theta}}_{N_k}) - Q(\boldsymbol{\theta}^0) < 0,$$

which leads to a contradiction. So $\hat{\boldsymbol{\theta}}_N$ is consistent for $\boldsymbol{\theta}^0$.

Case II. For at least one subsequence $\{N_k\}$ of $\{N\}$, $\hat{\boldsymbol{\theta}}_{N_k} \in S_{C,M}$ for some $C > 0$, and an $0 < M < \infty$. Now consider

$$\begin{aligned}\frac{1}{N}[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)] &= \frac{1}{N}\left[\sum_{n=1}^{N} |y(n) - Ae^{j(\alpha n+\beta n^2)}|^2 - \sum_{n=1}^{N} |e(n)|^2\right] \\ &= f_1(\boldsymbol{\theta}) + f_2(\boldsymbol{\theta}) \qquad \text{(say)}\end{aligned}$$

where

$$\begin{aligned}f_1(\boldsymbol{\theta}) &= \frac{1}{N}\sum_{n=1}^{N}[(A_R^0\cos(\alpha^0 n+\beta^0 n^2) - A_R\cos(\alpha n+\beta n^2) \\ &\qquad - A_I^0\sin(\alpha^0 n+\beta^0 n^2) + A_I\sin(\alpha n+\beta n^2))^2] \\ &\quad + \frac{1}{N}\sum_{n=1}^{N}[(A_R^0\sin(\alpha^0 n+\beta^0 n^2) - A_R\sin(\alpha n+\beta n^2) \\ &\qquad + A_I^0\cos(\alpha^0 n+\beta^0 n^2) - A_I\cos(\alpha n+\beta n^2))^2] \\ f_2(\boldsymbol{\theta}) &= \frac{2}{N}\sum_{n=1}^{N} e_R(n)(A_R^0\cos(\alpha^0 n+\beta^0 n^2) - A_R\cos(\alpha n+\beta n^2) \\ &\qquad - A_I^0\sin(\alpha^0 n+\beta^0 n^2) + A_I\sin(\alpha n+\beta n^2)) \\ &\quad + \frac{2}{N}\sum_{n=1}^{N} e_I(n)(A_R^0\sin(\alpha^0 n+\beta^0 n^2) - A_R\sin(\alpha n+\beta n^2) \\ &\qquad + A_I^0\cos(\alpha^0 n+\beta^0 n^2) - A_I\cos(\alpha n+\beta n^2)).\end{aligned}$$

Using Lemma 2.2 and its corollary it follows that

$$\lim_{N\to\infty}\ \sup_{\boldsymbol{\theta}\in S_{C,M}} f_2(\boldsymbol{\theta}) = 0 \quad \text{a.s.} \tag{2.4}$$

Now consider the following sets:

$$\begin{aligned}S_{C,M,1} &= \{\boldsymbol{\theta} : \boldsymbol{\theta} = (A_R, A_I, \alpha, \beta), |A_R - A_R^0| \ge C, |A_R| \le M, |A_I| \le M\}, \\ S_{C,M,2} &= \{\boldsymbol{\theta} : \boldsymbol{\theta} = (A_R, A_I, \alpha, \beta), |A_I - A_I^0| \ge C, |A_R| \le M, |A_I| \le M\}, \\ S_{C,M,3} &= \{\boldsymbol{\theta} : \boldsymbol{\theta} = (A_R, A_I, \alpha, \beta), |\alpha - \alpha^0| \ge C, |A_R| \le M, |A_I| \le M\}, \\ S_{C,M,4} &= \{\boldsymbol{\theta} : \boldsymbol{\theta} = (A_R, A_I, \alpha, \beta), |\beta - \beta^0| \ge C, |A_R| \le M, |A_I| \le M\}.\end{aligned}$$

Note that $S_{C,M} \subset S_{C,M,1} \cup S_{C,M,2} \cup S_{C,M,3} \cup S_{C,M,4} = S$ (say). Therefore,

$$\underline{\lim}\ \inf_{\boldsymbol{\theta}\in S_{C,M}} \frac{1}{N}[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)] \ge \underline{\lim}\ \inf_{\boldsymbol{\theta}\in S} \frac{1}{N}[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)]. \tag{2.5}$$

First we show that

$$\underline{\lim} \inf_{\boldsymbol{\theta} \in S_{C,M,j}} \frac{1}{N}[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)] > 0 \quad \text{a.s.} \tag{2.6}$$

for all $j = 1, \ldots, 4$ and because of (2.5) that would imply

$$\underline{\lim} \inf_{\boldsymbol{\theta} \in S_{C,M}} \frac{1}{N}[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)] > 0 \quad \text{a.s.} \tag{2.7}$$

Because of (2.5), using Lemma 2.1, Theorem 2.1 is proved provided we can show (2.6). First consider $j = 1$ to prove (2.6). So using (2.4), it follows that

$$\begin{aligned}
&\underline{\lim} \inf_{\boldsymbol{\theta} \in S_{C,M,1}} \frac{1}{N}[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)] \\
&\quad = \underline{\lim} \inf_{\boldsymbol{\theta} \in S_{C,M,1}} f_1(\boldsymbol{\theta}) \\
&\quad = \underline{\lim} \inf_{|A_R - A_R^0| \geq C} \frac{1}{N} \sum_{n=1}^{N} [(A_R^0 \cos(\alpha^0 n + \beta^0 n^2) - A_R \cos(\alpha^0 n + \beta^0 n^2))^2 \\
&\qquad\qquad + (A_R^0 \sin(\alpha^0 n + \beta^0 n^2) - A_R \sin(\alpha^0 n + \beta^0 n^2))^2] \\
&\quad = \inf_{|A_R - A_R^0| \geq C} (A_R^0 - A_R)^2 \geq C^2 > 0.
\end{aligned}$$

For other j, it can be shown along the same line and that proves (2.7). Therefore, Theorem 2.1 is proved.

3. Asymptotic distribution of the LSE's

In this section we provide the asymptotic distribution of the least squares estimators, obtained in the previous section. The asymptotic distribution of the least squares estimators of the parameters of the model (1.1) can be stated as follows:

THEOREM 3.1. *Under Assumption* 1, *as* $N \to \infty$

$$[N^{1/2}(\hat{A}_R - A_R^0), N^{1/2}(\hat{A}_I - A_I^0), N^{3/2}(\hat{\alpha} - \alpha^0), N^{5/2}(\hat{\beta} - \beta^0)] \xrightarrow{d} N_4(\mathbf{0}, 2\sigma^2 \boldsymbol{\Sigma}),$$

here $N_4(\mathbf{0}, 2\sigma^2\boldsymbol{\Sigma})$ *denotes a* 4*-variate normal distribution with mean vector* $\mathbf{0}$ *and the dispersion matrix* $2\sigma^2\boldsymbol{\Sigma}$*. The matrix* $\boldsymbol{\Sigma}$ *has the following structure*:

$$\boldsymbol{\Sigma} = \frac{1}{|A^0|^2} \begin{bmatrix} \frac{A_R^{0\,2} + 9A_I^{0\,2}}{2} & -4A_R^0 A_I^0 & 18A_I^0 & -15A_I^0 \\ -4A_R^0 A_I^0 & \frac{9A_R^{0\,2} + A_I^{0\,2}}{2} & -18A_R^0 & 15A_R^0 \\ 18A_I^0 & -18A_R^0 & 96 & -90 \\ -15A_I^0 & 15A_R^0 & -90 & 90 \end{bmatrix}. \tag{3.1}$$

It is interesting to note that the rate of convergence of the estimators of the linear parameters is $N^{1/2}$, whereas the rates of convergence of the estimators of the non-linear parameters are $N^{3/2}$ and $N^{5/2}$ respectively. It indicates that the accuracy of $\hat{\beta}$ is maximum followed by $\hat{\alpha}$ and then the linear parameters.

PROOF OF THEOREM 3.1. We use the following notation:

$$\boldsymbol{Q}'(\boldsymbol{\theta}) = \left[\frac{\partial Q(\boldsymbol{\theta})}{\partial A_R}, \frac{\partial Q(\boldsymbol{\theta})}{\partial A_I}, \frac{\partial Q(\boldsymbol{\theta})}{\partial \alpha}, \frac{\partial Q(\boldsymbol{\theta})}{\partial \beta}\right]$$

and $\boldsymbol{Q}''(\boldsymbol{\theta})$ is a 4×4 matrix of the second derivatives of $Q(\boldsymbol{\theta})$. Now expanding $\boldsymbol{Q}'(\hat{\boldsymbol{\theta}})$ around $\boldsymbol{\theta}^0$, by Taylor series, we obtain

$$\boldsymbol{Q}'(\hat{\boldsymbol{\theta}}) - \boldsymbol{Q}'(\boldsymbol{\theta}^0) = (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\boldsymbol{Q}''(\bar{\boldsymbol{\theta}}), \tag{3.2}$$

here $\bar{\boldsymbol{\theta}}$ is a point on the line joining $\bar{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^0$. Suppose $\boldsymbol{D}$ is a 4×4 diagonal matrix as follows:

$$\boldsymbol{D} = \operatorname{diag}\{N^{-1/2}, N^{-1/2}, N^{-3/2}, N^{-5/2}\}.$$

Since $\boldsymbol{Q}'(\hat{\boldsymbol{\theta}}) = \mathbf{0}$, therefore (3.2) can be written as

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\boldsymbol{D}^{-1} = -[\boldsymbol{Q}'(\boldsymbol{\theta}^0)\boldsymbol{D}][\boldsymbol{D}\boldsymbol{Q}''(\bar{\boldsymbol{\theta}})\boldsymbol{D}]^{-1}, \tag{3.3}$$

as $[\boldsymbol{D}\boldsymbol{Q}''(\bar{\boldsymbol{\theta}})\boldsymbol{D}]$ is an invertible matrix almost surely for large N. From Theorem 2.1, it follows that $\bar{\boldsymbol{\theta}}$ converges to $\boldsymbol{\theta}^0$ almost surely as $N \to \infty$. Since $Q(\boldsymbol{\theta})$ is a continuous function of $\boldsymbol{\theta}$,

$$\lim_{N\to\infty} [\boldsymbol{D}\boldsymbol{Q}''(\bar{\boldsymbol{\theta}})\boldsymbol{D}] = \lim_{N\to\infty} [\boldsymbol{D}\boldsymbol{Q}''(\boldsymbol{\theta}^0)\boldsymbol{D}].$$

Now using the facts

$$\lim_{N\to\infty} \frac{1}{N^2}\sum_{n=1}^{N} n = \frac{1}{2}, \qquad \lim_{N\to\infty} \frac{1}{N^3}\sum_{n=1}^{N} n^2 = \frac{1}{3},$$
$$\lim_{N\to\infty} \frac{1}{N^4}\sum_{n=1}^{N} n^3 = \frac{1}{4}, \qquad \lim_{N\to\infty} \frac{1}{N^5}\sum_{n=1}^{N} n^4 = \frac{1}{5},$$

it can be seen that

$$\lim_{N\to\infty} [\boldsymbol{D}\boldsymbol{Q}''(\boldsymbol{\theta}^0)\boldsymbol{D}] = \begin{bmatrix} 2 & 0 & -A_I^0 & -\frac{2}{3}A_I^0 \\ 0 & 2 & A_R^0 & \frac{2}{3}A_R^0 \\ -A_I^0 & A_R^0 & \frac{2}{3}|A^0|^2 & \frac{1}{2}|A^0|^2 \\ -\frac{2}{3}A_I^0 & \frac{2}{3}A_R^0 & \frac{1}{2}|A^0|^2 & \frac{2}{5}|A^0|^2 \end{bmatrix} = \boldsymbol{\Sigma}^{-1}. \tag{3.4}$$

Therefore, using Lemma 2.1

$$\lim_{N\to\infty} [\boldsymbol{D}\boldsymbol{Q}''(\bar{\boldsymbol{\theta}})\boldsymbol{D}]^{-1} = \lim_{N\to\infty} [\boldsymbol{D}\boldsymbol{Q}''(\boldsymbol{\theta}^0)\boldsymbol{D}]^{-1} = \boldsymbol{\Sigma},$$

where $\boldsymbol{\Sigma}$ is the same as defined in (3.1). Now consider the 1×4 vector $[\boldsymbol{Q}'(\boldsymbol{\theta}^0)\boldsymbol{D}]$, where

$$[\boldsymbol{Q}'(\boldsymbol{\theta}^0)\boldsymbol{D}]^T \tag{3.5}$$
$$= \begin{bmatrix} -\frac{2}{\sqrt{N}}[\sum_{n=1}^N e_R(n)\cos(\alpha^0 n + \beta^0 n^2) + \sum_{n=1}^N e_I(n)\sin(\alpha^0 n + \beta^0 n^2)] \\ \frac{2}{\sqrt{N}}[\sum_{n=1}^N e_R(n)\sin(\alpha^0 n + \beta^0 n^2) - \sum_{n=1}^N e_I(n)\cos(\alpha^0 n + \beta^0 n^2)] \\ \frac{2}{N^{3/2}}[A_R^0\sum_{n=1}^N e_R(n)n\sin(\alpha^0 n + \beta^0 n^2) + A_I^0\sum_{n=1}^N e_R(n)n\cos(\alpha^0 n + \beta^0 n^2)] \\ +\frac{2}{N^{3/2}}[A_I^0\sum_{n=1}^N e_I(n)n\sin(\alpha^0 n + \beta^0 n^2) - A_R^0\sum_{n=1}^N e_I(n)n\cos(\alpha^0 n + \beta^0 n^2)] \\ \frac{2}{N^{5/2}}[A_R^0\sum_{n=1}^N e_R(n)n^2\sin(\alpha^0 n + \beta^0 n^2) + A_I^0\sum_{n=1}^N e_R(n)n^2\cos(\alpha^0 n + \beta^0 n^2)] \\ +\frac{2}{N^{5/2}}[A_I^0\sum_{n=1}^N e_I(n)n^2\sin(\alpha^0 n + \beta^0 n^2) - A_R^0\sum_{n=1}^N e_I(n)n^2\cos(\alpha^0 n + \beta^0 n^2)] \end{bmatrix}.$$

Since all the elements of $[\boldsymbol{Q}'(\boldsymbol{\theta}^0)\boldsymbol{D}]$ satisfy the Lindeberg-Feller's condition (Chung (1974)), therefore as $N \to \infty$,

$$[\boldsymbol{Q}'(\boldsymbol{\theta}^0)\boldsymbol{D}] \xrightarrow{d} N_4(\boldsymbol{0}, 2\sigma^2\boldsymbol{\Sigma}^{-1}), \tag{3.6}$$

where $\boldsymbol{\Sigma}^{-1}$ is the same as defined in (3.4). Therefore from (3.2) and (3.6), Theorem 3.1 follows immediately.

4. Consistency and asymptotic normality of $\hat{\sigma}^2$

In this section first we provide the consistency of $\hat{\sigma}^2$, an estimator of σ^2, which is given by

$$\hat{\sigma}^2 = \frac{1}{N}Q(\hat{A}_R, \hat{A}_I, \hat{\alpha}, \hat{\beta}) = \frac{1}{N}Q(\hat{\boldsymbol{\theta}}).$$

We have the following consistency result of $\hat{\sigma}^2$.

THEOREM 4.1. *Under Assumption 1 as given in Theorem 2.1, if $N \to \infty$, then $\hat{\sigma}^2$ is a strongly consistent estimator of σ^2.*

To prove Theorem 4.1, we need the following lemma.

LEMMA 4.1. *If $\hat{\alpha}$ and $\hat{\beta}$ are the LSE's of α^0 and β^0 respectively, then as $N \to \infty$*

$$N(\hat{\alpha} - \alpha^0) \xrightarrow{a.s.} 0 \quad \text{and} \quad N^2(\hat{\beta} - \beta^0) \xrightarrow{a.s.} 0.$$

PROOF OF LEMMA 4.1. Multiply both sides of the equation (3.3) by $\frac{1}{\sqrt{N}}\boldsymbol{I}_4$, where $\boldsymbol{I}_4$ is the 4×4 identity matrix, we obtain

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)(\sqrt{N}\boldsymbol{D})^{-1} = -\left[\frac{\boldsymbol{Q}'(\boldsymbol{\theta}^0)\boldsymbol{D}}{\sqrt{N}}\right]\left[\boldsymbol{D}\boldsymbol{Q}''(\bar{\boldsymbol{\theta}})\boldsymbol{D}\right]^{-1}.$$

It is clear from (3.5) that $\frac{\boldsymbol{Q}'(\boldsymbol{\theta}^0)\boldsymbol{D}}{\sqrt{N}} \xrightarrow{a.s.} 0$. Therefore, $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)(\sqrt{N}\boldsymbol{D})^{-1} \xrightarrow{a.s.} 0$. It implies Lemma 4.1.

PROOF OF THEOREM 4.1. Let us write $\frac{1}{N}Q(\hat{\boldsymbol{\theta}})$ as follows:

$$\frac{1}{N}Q(\hat{\boldsymbol{\theta}}) = T_1 + T_2 + T_3 + T_4 + T_5,$$

where

$$\begin{aligned}
T_1 &= \frac{1}{N}\left[\sum_{n=1}^{N} e_R(n)^2 + \sum_{n=1}^{N} e_I(n)^2\right] \\
T_2 &= \frac{2}{N}\left[\sum_{n=1}^{N} e_R(n)(A_R^0 \cos(\alpha^0 n + \beta^0 n^2) - \hat{A}_R \cos(\hat{\alpha} n + \hat{\beta} n^2))\right] \\
&\quad - \frac{2}{N}\left[\sum_{n=1}^{N} e_R(n)(A_I^0 \sin(\alpha^0 n + \beta^0 n^2) - \hat{A}_I \sin(\hat{\alpha} n + \hat{\beta} n^2))\right]
\end{aligned}$$

$$\begin{aligned}
T_3 &= \frac{2}{N}\left[\sum_{n=1}^{N} e_I(n)(A_I^0\cos(\alpha^0 n+\beta^0 n^2) - \hat{A}_I\cos(\hat{\alpha}n+\hat{\beta}n^2))\right] \\
&\quad + \frac{2}{N}\left[\sum_{n=1}^{N} e_I(n)(A_R^0\sin(\alpha^0 n+\beta^0 n^2) - \hat{A}_R\sin(\hat{\alpha}n+\hat{\beta}n^2))\right] \\
T_4 &= \frac{1}{N}\sum_{n=1}^{N}[A_R^0\cos(\alpha^0 n+\beta^0 n^2) - \hat{A}_R\cos(\hat{\alpha}n+\hat{\beta}n^2) \\
&\qquad - A_I^0\sin(\alpha^0 n+\beta^0 n^2) + \hat{A}_I\sin(\hat{\alpha}n+\hat{\beta}n^2)]^2 \\
T_5 &= \frac{1}{N}\sum_{n=1}^{N}[A_I^0\cos(\alpha^0 n+\beta^0 n^2) - \hat{A}_I\cos(\hat{\alpha}n+\hat{\beta}n^2) \\
&\qquad + A_R^0\sin(\alpha^0 n+\beta^0 n^2) - \hat{A}_R\sin(\hat{\alpha}n+\hat{\beta}n^2)]^2.
\end{aligned}$$

Note that because of strong law of large number $T_1 \xrightarrow{a.s.} \sigma^2$. Because of Lemma 2.2 and its corollary, $T_2 \xrightarrow{a.s.} 0$ and $T_3 \xrightarrow{a.s.} 0$. Also note that

$$\begin{aligned}
0 \le T_4 \le& \frac{1}{N}\sum_{n=1}^{N}[A_R^0\cos(\alpha^0 n+\beta^0 n^2) - \hat{A}_R\cos(\hat{\alpha}n+\hat{\beta}n^2)]^2 \\
&+ \frac{1}{N}\sum_{n=1}^{N}[A_I^0\sin(\alpha^0 n+\beta^0 n^2) - \hat{A}_I\sin(\hat{\alpha}n+\hat{\beta}n^2)]^2.
\end{aligned}$$

Consider

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N}[A_R^0\cos(\alpha^0 n+\beta^0 n^2) - \hat{A}_R\cos(\hat{\alpha}n+\hat{\beta}n^2)]^2 \\
&\quad \le \frac{2}{N}\sum_{n=1}^{N}(A_R^0-\hat{A}_R)^2\cos^2(\alpha^0 n+\beta^0 n^2) \\
&\qquad + \frac{2}{N}\sum_{n=1}^{N}\hat{A}_R^2(\cos(\alpha^0 n+\beta^0 n^2) - \cos(\hat{\alpha}n+\hat{\beta}n^2))^2 \\
&\qquad\qquad\qquad (\text{as } 2x^2+2y^2-(x+y)^2 \ge 0) \\
&\quad \le 2(A_R^0-\hat{A}_R)^2 + \frac{2}{N}\sum_{n=1}^{N}\hat{A}_R^2[(\hat{\alpha}-\alpha^0)n + (\hat{\beta}-\beta^0)n^2]^2 \\
&\qquad\qquad\qquad (\text{using Taylor series expansion}) \\
&\quad \xrightarrow{a.s.} 0,
\end{aligned}$$

because $\hat{A}_R \xrightarrow{a.s.} A_R^0$, $N(\hat{\alpha}-\alpha^0) \xrightarrow{a.s.} 0$ and $N^2(\hat{\beta}-\beta^0) \xrightarrow{a.s.} 0$. Along the same line it follows that

$$\frac{1}{N}\sum_{n=1}^{N}[A_I^0\sin(\alpha^0 n+\beta^0 n^2) - \hat{A}_I\sin(\hat{\alpha}n+\hat{\beta}n^2)]^2 \xrightarrow{a.s.} 0,$$

therefore, it follows that $T_4 \xrightarrow{a.s.} 0$. Similarly $T_5 \xrightarrow{a.s.} 0$. It proves Theorem 4.1.

Now we make the following assumption to prove the asymptotic normality results of $\hat{\sigma}^2$.

ASSUMPTION 2. Other than Assumption 1, $e_R(n)$ and $e_I(n)$ also satisfy the following conditions:

$$E(e_R(n))^4 < \infty \quad \text{and} \quad E(e_I(n))^4 < \infty.$$

Now we state the asymptotic normality result of $\hat{\sigma}^2$.

THEOREM 4.2. *Under Assumption* 2, *as* $N \to \infty$

$$\sqrt{N}(\hat{\sigma}^2 - \sigma^2) \xrightarrow{d} N(0, \sigma^*),$$

where $\sigma^* = E(e_R(n))^4 + E(e_I(n))^4 - \frac{\sigma^4}{2}$.

PROOF OF THEOREM 4.2. First we show that as $N \to \infty$,

$$\left\{\sqrt{N}\hat{\sigma}^2 - \frac{1}{\sqrt{N}}\left[\sum_{n=1}^{N}(e_R(n)^2 + e_I(n)^2)\right]\right\} \xrightarrow{d} 0. \tag{4.1}$$

Note that

$$\left\{\sqrt{N}\hat{\sigma}^2 - \frac{1}{\sqrt{N}}\left[\sum_{n=1}^{N}(e_R(n)^2 + e_I(n)^2)\right]\right\} = \sqrt{N}(T_2 + T_3 + T_4 + T_5).$$

Expanding $\cos(\hat{\alpha}n + \hat{\beta}n^2)$ and $\sin(\hat{\alpha}n + \hat{\beta}n^2)$ around $(\alpha^0 n + \beta^0 n^2)$ and applying Lemma 4.1, it follows by some calculations that

$$\sqrt{N}T_2 \xrightarrow{d} 0, \qquad \sqrt{N}T_3 \xrightarrow{d} 0, \qquad \sqrt{N}T_4 \xrightarrow{d} 0, \qquad \sqrt{N}T_5 \xrightarrow{d} 0.$$

Therefore (4.1) follows. Now using Central Limit Theorem and because of Assumption 2, it follows that

$$\frac{1}{\sqrt{N}}\left[\sum_{n=1}^{N}(e_R(n)^2 + e_I(n)^2 - \sigma^2)\right] \xrightarrow{d} N(0, \sigma^*).$$

Therefore, the result follows immediately.

5. Generalized chirp signals

In this section we consider the generalized chirp signals as considered by Djuric and Kay (1990) and discuss the properties of the least squares estimators of the unknown parameters. Djuric and Kay (1990) defined the generalized chirp signal as follows:

$$y(n) = A^0 e^{j(\omega_1^0 n + \omega_2^0 n^2 + \cdots + \omega_p^0 n^p)} + e(n).$$

Here $y(n)$'s are the observed complex valued signals, A^0 and $e(n)$'s are the same as defined in Section 1. The unknown frequencies $\omega_1^0, \ldots, \omega_p^0$ all belong to $(0, \pi)$ and p is a known fixed integer. Here also the problem is to estimate the unknown frequencies

$\omega_1^0, \ldots, \omega_p^0$ and σ^2 from a sample of size N, namely $y(1), \ldots, y(N)$. We mainly consider the least squares estimators of the unknown parameters obtained by minimizing:

$$Q(\boldsymbol{\theta}_p) = \sum_{n=1}^{N} |y(n) - Ae^{j(\omega_1 n + \cdots + \omega_p n^p)}|^2,$$

with respect to the unknown parameters $\boldsymbol{\theta}_p = (A_R, A_I, \omega_1, \ldots, \omega_p)$. Here A_R and A_I are the same as defined before. We denote, $\boldsymbol{\theta}_p^0 = (A_R^0, A_I^0, \omega_1^0, \ldots, \omega_p^0)$ as the true parameter value and $\hat{\boldsymbol{\theta}}_p = (\hat{A}_R, \hat{A}_I, \hat{\omega}_1, \ldots, \hat{\omega}_p)$ as the LSE's of $\boldsymbol{\theta}_p^0$. Now we can state the consistency result of $\hat{\boldsymbol{\theta}}_p$ as follows.

THEOREM 5.1. *If $\boldsymbol{\theta}_p^0 = (A_R^0, A_I^0, \omega_1^0, \ldots, \omega_p^0)$ is an interior point of the parameter space $\Theta_p = \Re \times \Re \times [0, \pi] \times \cdots \times [0, \pi]$, the error random variables $e(n)$ satisfy Assumption 1 and $|A^0| > 0$, then $\hat{\boldsymbol{\theta}}_p$ is a consistent estimator of $\boldsymbol{\theta}_p^0$.*

Note that to prove Theorem 5.1, we need to extend Lemmas 2.1 and 2.2 for general p. Lemma 2.1 can be extended very easily for general p therefore it is not stated here explicitly. The following lemma is an extension of Lemma 2.2 for general p.

LEMMA 5.1. *Let $\{X(n)\}$ be a sequence of i.i.d. random variables with mean zero and finite variance, then*

$$\sup_{a_1, \ldots, a_p} \left| \frac{1}{N} \sum_{n=1}^{N} X(n) \cos(a_1 n) \cdots \cos(a_p n^p) \right| \xrightarrow{a.s.} 0.$$

PROOF OF LEMMA 5.1. The proof goes exactly the same way as the line of proof of Lemma 2.2 up to (2.3), with the obvious modifications that everywhere $\cos(an)\cos(bn^2)$ is replaced by $\cos(a_1 n) \cdots \cos(a_p n^p)$. After (2.3), the following changes have to be made.

Let $J = N^{p(p+1)}$, choose J points $(a_{11}, \ldots, a_{p1}), \ldots, (a_{1J}, \ldots, a_{pJ})$, such that for any $(a_1, \ldots, a_p) \in [0, \pi] \times \cdots \times [0, \pi]$, there exists a point $(a_{1k}, \ldots, a_{pk})$, such that

$$|a_1 - a_{1k}| \leq \frac{\pi}{N^{p+1}}, \ldots, |a_p - a_{pk}| \leq \frac{\pi}{N^{p+1}},$$

for some $1 \leq k \leq J$. Now as $N \to \infty$,

$$\left| \frac{1}{N} \sum_{n=1}^{N} U(n) \{\cos(a_1 n) \cdots \cos(a_p n^p) - \cos(a_{1k} n) \cdots \cos(a_{pk} n^p)\} \right|$$
$$\leq C \left[\frac{1}{N} \sum_{n=1}^{N} n^{3/4} \cdot n \cdot \frac{\pi}{N^{p+1}} + \cdots + \frac{1}{N} \sum_{n=1}^{N} n^{3/4} \cdot n^p \cdot \frac{\pi}{N^{p+1}} \right] \to 0.$$

Therefore, for large N, we have

$$P\left\{ \sup_{a_1, \ldots, a_p} \left| \frac{1}{N} \sum_{n=1}^{N} U(n) \cos(a_1 n) \cdots \cos(a_p n^p) \right| \geq 2\epsilon \right\}$$
$$\leq P\left\{ \max_{k \leq N^{p(p+1)}} \left| \frac{1}{N} \sum_{n=1}^{N} U(n) \cos(a_{1k} n) \cdots \cos(a_{pk} n^p) \right| \geq \epsilon \right\} \leq 2N^{p(p+1)} e^{-N^{1/4}\epsilon/4}.$$

Since $\sum_{N=1}^{\infty} N^{p(p+1)} e^{-N^{1/4}\epsilon/4} < \infty$, therefore by Borel-Cantelli lemma the result follows.

With the help of Lemma 5.1 and following the same line of proof as of Theorem 2.1, Theorem 5.1 can be proved. Now we provide the asymptotic distribution of $\hat{\boldsymbol{\theta}}_p$ in the following theorem.

THEOREM 5.2. *Under Assumption* 1, *as* $N \to \infty$

$$[\sqrt{N}(\hat{A}_R - A_R^0), \sqrt{N}(\hat{A}_I - A_I^0), N^{3/2}(\hat{\omega}_1 - \omega_1^0), \dots, N^{(2p+1)/2}(\hat{\omega}_p - \omega_p^0)] \xrightarrow{d} N_{p+2}(\mathbf{0}, \sigma^2 \boldsymbol{\Sigma}_{p+2}),$$

where $\boldsymbol{\Sigma}_{p+2}$ *is* $(p+2) \times (p+2)$ *positive definite matrix and it is defined through its inverse as follows*:

$$\boldsymbol{\Sigma}_{p+2}^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{2}A_I^0 & -\frac{1}{3}A_I^0 & \cdots & -\frac{1}{p+1}A_I^0 \\ 0 & 1 & \frac{1}{2}A_R^0 & \frac{1}{3}A_R^0 & \cdots & \frac{1}{p+1}A_R^0 \\ -\frac{1}{2}A_I^0 & \frac{1}{2}A_R^0 & \frac{1}{3}|A^0|^2 & \frac{1}{4}|A^0|^2 & \cdots & \frac{1}{p+2}|A^0|^2 \\ -\frac{1}{3}A_I^0 & \frac{1}{3}A_R^0 & \frac{1}{4}|A^0|^2 & \frac{1}{5}|A^0|^2 & \cdots & \frac{1}{p+3}|A^0|^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{p+1}A_I^0 & \frac{1}{p+1}A_R^0 & \frac{1}{p+2}|A^0|^2 & \frac{1}{p+3}|A^0|^2 & \cdots & \frac{1}{2p+1}|A^0|^2 \end{bmatrix}.$$

PROOF OF THEOREM 5.2. The proof can be obtained along the same line of proof of Theorem 3.1 and it is omitted.

It may be mentioned that although we could not provide the explicit expression of $\boldsymbol{\Sigma}_{p+2}$, but it can be obtained recursively using the standard matrix theory results (see Rao (1973)). For example, since we know the explicit expression for $p = 2$, therefore using that we can obtain the result for $p = 3$ and so on. Note that σ^2 can be estimated in this case as $\frac{1}{N}Q(\hat{\boldsymbol{\theta}}_p)$ and Theorems 4.1 and 4.2 are valid in this case without any change.

6. Numerical experiments

In this section, we present some numerical experiments results to see how the proposed LSE's behave for finite samples and whether the asymptotic results can be used for small sample inferences. We consider the following model with $p = 2$:

$$y(n) = Ae^{j(\alpha n + \beta n^2)} + e(n) \tag{6.1}$$

with $A = A_R + jA_I = 4.93 + 1.91j$, $\alpha = 1.84$ and $\beta = 1.99$. We consider that $e_R(n)$ and $e_I(n)$, the real and imaginary parts of $e(n)$ are normally distributed i.i.d. random variables with mean zero and finite variance σ^2 and they are independently distributed. We consider $\sigma^2 = 0.04$, 0.10, 0.50 and 1.00 and the sample size $N = 100$. For each σ^2, we generate a dataset from the model (6.1) and compute the LSE's of A_R, A_I, α and β. We replicate the process five thousand times and compute the average estimates (AE) and mean squared errors (MSE) of all the parameters over five thousand replications. The

Table 1. The average estimates, their mean squared errors and asymptotic variances of parameter α and β for different values of σ^2.

		$\sigma^2 = 0.04$	$\sigma^2 = 0.10$	$\sigma^2 = 0.50$	$\sigma^2 = 1.00$
	AE	1.839920	1.839888	1.840095	1.840286
α	MSE	0.195047e-05	0.432194e-5	0.399056e-04	0.109926e-03
	AVAR	0.274747e-06	0.686867e-6	0.343434e-05	0.686868e-05
	AE	1.989986	1.990078	1.990501	1.991370
β	MSE	0.411410e-09	0.891097e-9	0.249901e-04	0.764466e-04
	AVAR	0.182669e-09	0.456672e-9	0.228336e-08	0.456672e-08

Table 2. The 95% coverage percentages, the average confidence lengths and the expected confidence lengths of the LSE's of the initial frequency α and the frequency rate β for different values of σ^2.

		$\sigma^2 = 0.04$	$\sigma^2 = 0.10$	$\sigma^2 = 0.50$	$\sigma^2 = 1.00$
	COVPER	0.96	0.94	0.86	0.81
α	AVLEN	0.234893e-02	0.379345e-2	0.999770e-02	0.163162e-01
	EXLEN	0.205472e-02	0.324880e-2	0.726453e-02	0.102736e-01
	COVPER	0.93	0.94	0.83	0.73
β	AVLEN	0.227434e-04	0.367300e-4	0.968022e-04	0.157981e-03
	EXLEN	0.198947e-04	0.314563e-4	0.703385e-04	0.994737e-04

asymptotic variances (AVAR) of each case are also reported for comparison purposes. The results are presented in Table 1 for the nonlinear parameters α and β only. In the first column in Table 1 the first row represents the average estimate of α for $\sigma^2 = .04$ and second and third rows represent the MSE and AVAR of α respectively. Similarly fourth, fifth and sixth rows represent the corresponding results for β. The other columns represent results for $\sigma^2 = 0.10$, 0.50 and 1.00. We also compute the 95% confidence intervals for α and β over five thousand replications. The results are reported in Table 2. In Table 2 for different σ^2, the first row represents the coverage percentages (COVPER) of α and the corresponding average confidence lengths (AVLEN) are in the second row. The expected confidence lengths (EXLEN), obtained from theorem 3.1 using the true parameter values are reported in third row. Similarly the results for β are given in fourth, fifth and sixth rows.

The following observations are clear from the entries of Tables 1 and 2. It is observed that as σ^2 increases, the MSE's and biases of the estimators increase. It verifies the consistency property of the LSE's. The biases are quite small and the MSE's are close to the asymptotic variances when the error variances are small but when the error variances are large then they are quite different as expected. Similarly, for different values of σ^2, the average confidence lengths are quite close to the expected confidence lengths of both the parameters when the error variances are small but for large error variances they are quite different. Moreover, for large error variances, the corresponding coverage probabilities are much lower than the nominal level. These findings are not that surprising, apparently for $\sigma^2 = 0.5$ or $\sigma^2 = 1.0$, the sample size needs to be much larger for the asymptotic theories to work.

It should be mentioned here that in simulation study the true parameter values are used as initial estimators to obtain the LSE's. But in practical problems the following function which is analogous to the periodogram function

$$I(\lambda,\mu)=\frac{1}{N}\left|\sum_{n=1}^{N} y(n)e^{-j(\lambda n+\mu n^2)}\right|^2$$

can be used. The initial estimates of α and β can be obtained by maximizing $I(\lambda,\mu)$ by two-dimensional grid search method. It is a very important practical problem. It is well known that even for a much simpler sum of sinusoidal model, it is necessary to start with an extremely precise initial guess of the frequencies, namely $o_p(\frac{1}{N})$, where $Z=o_p(\frac{1}{N^a})$, means $N^aZ \to 0$ in probability. In case of chirp signal, it is expected that we need the initial estimates of α and β as $o_p(\frac{1}{N})$ and $o_p(\frac{1}{N^2})$ respectively. How to obtain those initial estimates is a challenging computational issue and more work is needed in that direction.

7. Conclusions

In this paper we consider the LSE's of the parameters of the chirp signals and discuss their theoretical properties when the additive errors are independent and identically distributed random variables with mean zero and finite variance. It is observed that the LSE's are consistent and asymptotically normally distributed. We also obtain the rates of convergence of the LSE's. These results are not available in the literature. We also consider the generalized chirp signals which was originally discussed by Djuric and Kay (1990) and discuss the asymptotic properties of the LSE's of the unknown parameters of the generalized chirp signals. It is observed that the LSE's are strongly consistent and asymptotically normally distributed. The simulation study indicates that the asymptotic results can be used in small sample inferences.

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References

Abatzoglou, T. (1986). Fast maximum likelihood joint estimation of frequency and frequency rate, *IEEE Transactions on Aerospace and Electronic Systems*, **22**, 708–714.

Besson, O., Ghogho, M. and Swami, A. (1999). Parameter estimation for random amplitude chirp signals, *IEEE Transactions on Signal Processing*, **47**, 3208–3219.

Chung, K. L. (1974). *A First Course in Probability*, Academic Press, New York.

Djuric, P. M. and Kay, S. M. (1990). Parameter estimation of chirp signals, *IEEE Transactions on Acoustics, Speech and Signal Processing*, **38**, 2118–2126.

Gini, F., Montanari, M. and Verrazzani, L. (2000). Estimation of chirp signals in compound Gaussian clutter: A cyclostationary approach, *IEEE Transactions on Acoustics, Speech and Signal Processing*, **48**(4), 1029–1039.

Huang, D., Sando, S. and Wen, L. (1999). Least squares estimation of polynomial phase signals via stochastic tree-search, ICASSP, Phoenix, Arizona, Paper No. 1355.

Jennrich, R. I. (1969). Asymptotic properties of the nonlinear least squares estimators, *Annals of Mathematical Statistics*, **40**, 633–643.

Kumaresan, R. and Verma, S. (1987). On estimating the parameters of chirp signals using rank reduction techniques, *Proceedings of 21st Asilomar Conference*, 555–558, Pacific Grove, California.

Kundu, D. (1991). Asymptotic properties of the complex valued nonlinear regression model, *Communications in Statistics: Theory and Methods*, **20**, 3793–3803.

Kundu, D. and Mitra, A. (1996). A note on the consistency of the undamped exponential signals model, *Statistics*, **28**, 25–33.

Rao, C. R. (1973). *Linear Statistical Inferences and Its Applications*, 2nd ed., Wiley and Sons, New York.

Rihaczek, A. W. (1969). *Principles of High Resolution Radar*, McGraw Hill, New York.

Saha, S. and Kay, S. (2002). Maximum likelihood parameter estimation of superimposed chirps using Monte Carlo importance sampling, *IEEE Transactions on Signal Processing*, **50**, 224–230.

Wu, C. F. J. (1981). Asymptotic theory of the nonlinear least squares estimation, *Annals of Statistics*, **9**, 501–513.