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TENSORS AND LIKELIHOOD EXPANSIONS IN THE PRESENCE OF NUISANCE PARAMETERS

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Abstract. Stochastic expansions of likelihood quantities are usually derived through ordinary Taylor expansions, rearranging terms according to their asymptotic order. The most convenient form for such expansions involves the score function, the expected information, higher order log-likelihood derivatives and their expectations. Expansions of this form are called expected/observed. If the quantity expanded is invariant or, more generally, a tensor under reparameterisations, the entire contribution of a given asymptotic order to the expected/observed expansion will follow the same transformation law. When there are no nuisance parameters, explicit representations through appropriate tensors are available. In this paper, we analyse the geometric structure of expected/observed likelihood expansions when nuisance parameters are present. We outline the derivation of likelihood quantities which behave as tensors under interest-respecting reparameterisations. This allows us to write the usual stochastic expansions of profile likelihood quantities in an explicitly tensorial form.

Key words and phrases: Asymptotic expansion, higher order asymptotics, interest respecting reparameterisation, nuisance parameter, profile likelihood, tensor.

1. Introduction

Inference problems, such as estimation and testing, are unaffected by reparametrisations of the model. Accordingly, inference procedures are required to follow a coherent behaviour under reparameterisations. This means that inferential conclusions should not depend on the choice of parameterisation. Many likelihood based procedures meet the requirement of parameterisation invariance. Notable instances are the maximum likelihood estimator and the likelihood ratio test statistic. On the other hand, the Wald test statistic is a well known example of a likelihood procedure which is affected by the parameterisation. For a discussion, see Barndorff-Nielsen and Cox ((1994), Section 1.5) and Pace and Salvan ((1997), Section 2.11).

Asymptotic expansions are widely used in likelihood theory. They provide valuable insight into inference procedures and are a basic tool for studying first and higher order properties. It is of course desirable that, when taking an asymptotic expansion of an invariant likelihood quantity, parameterisation invariance is maintained throughout the expansion, both in the leading and in the higher order terms.

The main approaches for obtaining invariant or "geometric" asymptotic expansions are reviewed in Barndorff-Nielsen and Cox ((1994), Chapter 5). An intrinsically geometric technique for invariant Taylor expansions is introduced in Barndorff-Nielsen (1987). However, the most useful form of a likelihood expansion is the so-called expected/observed expansion, whose derivation is not directly based on geometric arguments. Geometrical aspects can nonetheless be brought in, as is illustrated in Barndorff-Nielsen and Cox ((1994), Section 5.3) and further elucidated in Pace and Salvan (1994). These geometric results are based on the assumption that the whole parameter is of interest.

Most of modern likelihood theory deals with inference in the presence of nuisance parameters and is based on the profile likelihood and related statistics. In this setting, invariance under interest respecting reparameterisations is a key requirement (cf. Barndorff-Nielsen and Cox (1994), Section 1.5). No systematic study concerning invariance of likelihood expansions in the presence of nuisance parameters seems to be available so far.

The aim of the present paper is to provide a framework that allows us to write expected/observed likelihood expansions in a geometric form when nuisance parameters are present. To this end, we first define interest respecting tensors, which are quantities that behave tensorially under interest respecting reparameterisations. We show how to build recursively interest respecting tensors. The construction is inspired by the manipulation needed to get interest respecting tensors from the more familiar tensors under global reparameterisations. Terms of a given order in the expected/observed expansion of an invariant profile likelihood quantity may then be represented through contractions of interest respecting tensors, so that invariance under interest respecting reparameterisations is apparent. Throughout the paper, the expected/observed expansion of the profile log-likelihood ratio statistic will be used as a key example.

Here we take a coordinate-bound approach with an explicit representation for the nuisance parameter. The choice of a coordinate-bound approach is close to the usual algorithmic way of doing likelihood expansions, see e.g. Barndorff-Nielsen and Cox ((1994), Chapter 5), DiCiccio and Stern (1994), Li (2001). Of course, the requirement of an explicit representation for the nuisance parameter leaves out invariant likelihood expansions in the more general and natural setting. For some hints in this direction, see Severini ((2000), Section 7.4.2). In addition, no attempt will be made to provide an interpretation of terms of expected/observed expansions according to the geometrical theory of statistical manifolds. Hence, interest respecting tensors will be used as a mere device to bring out parameterisation invariance.

The layout of the paper is as follows. Section 2 introduces some notation and preliminary material. Geometric aspects of expected/observed likelihood expansions are reviewed in Section 3. Section 4 deals with interest respecting tensors. Some notable instances of these new tensorial quantities derived from the log-likelihood function are illustrated in Section 5. Section 6 describes how interest respecting tensors may be used to write interest respecting expected/observed expansions. Technical details related to the material in Section 4 are collected in the Appendix.

2. Notation and preliminaries

Let $\mathcal{F} = \{P_\theta : \theta \in \Theta \subseteq \mathbb{R}^p\}$ be a parametric family of probability distributions defined on a sample space $\mathcal{Y}$ and dominated by a σ-finite measure μ. The parameter space Θ is assumed to be an open non-empty subset of $\mathbb{R}^p$. Let us denote by $p(y;\theta)$, $y \in \mathcal{Y}$, the density of P_θ with respect to μ and by $l(\theta) = l(\theta; y) = \log p(y;\theta)$ the log-likelihood function based on the sample data y. We assume, for each $\theta \in \Theta$, that $p(y;\theta) > 0$ for every $y \in \mathcal{Y}$. We assume in addition that $l(\theta)$ is a smooth function of θ and that the maximum likelihood estimator $\hat{\theta}$ exists and is a solution of the likelihood equation $(\partial l/\partial\theta)(\theta) = 0$. Moreover, we assume that the usual additional regularity conditions hold ensuring validity of the Bartlett identities (cf. Barndorff-Nielsen and Cox (1994), Section 5.2).

Throughout the paper we use index notation and the Einstein summation convention. We denote generic components of θ by $\theta^r, \theta^s, \ldots$, with $r, s, \ldots = 1, \ldots, p$. The elements of the score vector are $l_r = (\partial l/\partial\theta^r)(\theta)$. Higher order log-likelihood derivatives are denoted by

$$l_{R_m} = l_{r_1 \cdots r_m} = \frac{\partial^m l}{\partial\theta^{r_1} \cdots \partial\theta^{r_m}}(\theta).$$

The expected information matrix $i = i(\theta)$ has generic element $i_{rs} = E_\theta(-l_{rs})$. We denote by i^{rs} an element of the matrix inverse of i. Further likelihood quantities to be considered are $l^r = i^{rs}l_s$, $\nu_{R_m} = E_\theta(l_{R_m})$, $H_{R_m} = l_{R_m} - \nu_{R_m}$, $\nu_{R_m,S_n,\ldots,U_q} = E_\theta(l_{R_m}l_{S_n}\cdots l_{U_q})$.

Let $\omega = \omega(\theta)$ be an alternative parameterisation of $\mathcal{F}$, i.e. a smooth one-to-one transformation from θ to ω. We denote components of ω by $\omega^{\bar{r}}, \omega^{\bar{s}}, \ldots$, with $\bar{r}, \bar{s}, \ldots = 1, \ldots, p$. Let $\theta(\omega)$ be the inverse function of $\omega(\theta)$ and let $\theta^r_{\bar{r}} = (\partial\theta^r/\partial\omega^{\bar{r}})(\omega)$, $\theta^r_{\bar{r}\bar{s}} = (\partial^2\theta^r/\partial\omega^{\bar{r}}\partial\omega^{\bar{s}})(\omega)$, and so on, denote partial derivatives of components of $\theta(\omega)$ with respect to components of ω. Conversely, let $\omega^{\bar{r}}_r = (\partial\omega^{\bar{r}}/\partial\theta^r)(\theta)$, $\omega^{\bar{r}}_{rs} = (\partial^2\omega^{\bar{r}}/\partial\theta^r\partial\theta^s)(\theta)$, and so on, denote partial derivatives of $\omega(\theta)$ with respect to components of $\boldsymbol{\theta}$. Notice that $\omega^{\bar{r}}_r\theta^s_{\bar{r}} = \delta^s_r$, where δ^s_r is the Kronecker delta ($\delta^s_r = 1$ if $s = r$ and $\delta^s_r = 0$ if $s \neq r$). A likelihood quantity with indices $\bar{r}, \bar{s}, \ldots$ is understood as referred to the ω parameterisation.

Reparameterisation does not alter the log-likelihood function itself, whereas it affects log-likelihood derivatives and their moments. For instance,

$$\begin{aligned}
l_{\bar{r}} &= l_r\theta^r_{\bar{r}} \\
l_{\bar{r}\bar{s}} &= l_{rs}\theta^r_{\bar{r}}\theta^s_{\bar{s}} + l_r\theta^r_{\bar{r}\bar{s}} \\
i_{\bar{r}\bar{s}} &= i_{rs}\theta^r_{\bar{r}}\theta^s_{\bar{s}} \\
i^{\bar{r}\bar{s}} &= i^{rs}\omega^{\bar{r}}_r\omega^{\bar{s}}_s \\
l^{\bar{r}} &= l^r\omega^{\bar{r}}_r.
\end{aligned}$$

A collection of smooth real functions $T^{R_m}_{S_n} = T^{R_m}_{S_n}(\theta) = T^{r_1\cdots r_m}_{s_1\cdots s_n}(\theta)$ is called an (m,n) tensor on $\mathcal{F}$, or, equivalently, a tensor of contravariant rank m and covariant rank n, if under reparameterisation it obeys the transformation rule

$$T^{\bar{r}_1\cdots\bar{r}_m}_{\bar{s}_1\cdots\bar{s}_n} = T^{r_1\cdots r_m}_{s_1\cdots s_n}\omega^{\bar{r}_1}_{r_1}\cdots\omega^{\bar{r}_m}_{r_m}\theta^{s_1}_{\bar{s}_1}\cdots\theta^{s_n}_{\bar{s}_n}. \tag{2.1}$$

For instance, l_r is a $(0,1)$ tensor, i_{rs} is a $(0,2)$ tensor, l^r is a $(1,0)$ tensor. A $(0,0)$ tensor is a parameterisation-invariant quantity. If T^{R_m} is an $(m,0)$ tensor and U_{S_m} is a $(0,m)$ tensor, their contraction $T^{R_m}U_{R_m}$ is invariant. Tensors are therefore instrumental in writing likelihood expansions in a geometric form.

3. Invariant expected/observed expansions: a review

3.1 *Nuisance parameters absent*

Let $f(\hat{\theta}) = f(\hat{\theta}; y)$ be a parameterisation-invariant statistic, also called a scalar function, defined on a copy of Θ that represents the range space of the maximum likelihood estimator. Let us assume that $f(\hat{\theta})$ is of order $O_p(n^\beta)$ under repeated sampling of size n. Stochastic expansions for $f(\hat{\theta})$ are often obtained from the ordinary Taylor formula, which is not parameterisation-invariant, depending on the coordinate system adopted for the statistical manifold $\mathcal{F}$. An important aim of a geometric stochastic calculus is to obtain an asymptotic expansion for $f(\hat{\theta})$ of the form

$$f(\hat{\theta}) = f(\theta) + b_1 + b_2 + b_3 + b_4 + O_p(n^{\beta-5/2}) \tag{3.1}$$

where each term b_m is a scalar function of order $O_p(n^{\beta-m/2})$, $m = 1, 2, 3, 4$. More generally, if $f(\theta)$ is a geometric object, such as a tensor, it is desirable that each term b_m should follow the same transformation law as $f(\theta)$.

One possibility for obtaining the geometric expansion (3.1) is through the ordinary Taylor formula followed by *a posteriori* elicitation of geometric ingredients. Taylor's formula gives

$$\begin{aligned} f(\hat{\theta}) &= f(\theta) + f_r(\hat{\theta} - \theta)^r + \frac{1}{2} f_{rs}(\hat{\theta} - \theta)^{rs} \\ &\quad + \frac{1}{6} f_{rst}(\hat{\theta} - \theta)^{rst} + \frac{1}{24} f_{rstu}(\hat{\theta} - \theta)^{rstu} + \cdots, \end{aligned} \tag{3.2}$$

where $f_r = (\partial f/\partial\theta^r)(\theta)$, $f_{rs} = (\partial^2 f/\partial\theta^r\partial\theta^s)(\theta)$, and so on, $(\hat{\theta} - \theta)^r = (\hat{\theta}^r - \theta^r)$, $(\hat{\theta} - \theta)^{rs} = (\hat{\theta} - \theta)^r(\hat{\theta} - \theta)^s$ and so on. In order to preserve invariance a suitable expansion for $\hat{\theta} - \theta$ has to be inserted into (3.2), see Barndorff-Nielsen and Cox ((1994), Section 5.3) and Pace and Salvan ((1994), Section 3). The resulting expansions are called expected/observed. They are expressed in terms of the score function, the expected information, higher order log-likelihood derivatives and their expectations. They also depend on derivatives of $f(\theta)$ and on their expectations.

Example 1. *Expected/observed expansion of the log-likelihood ratio.* The expected/observed expansion of $W(\theta) = 2(l(\hat{\theta}) - l(\theta))$ is

$$W(\theta) = B_1 + B_2 + B_3 + O_p(n^{-3/2}), \tag{3.3}$$

with

$$\begin{aligned} B_1 &= i_{rs} l^r l^s \\ B_2 &= \frac{1}{3} \nu_{rst} l^r l^s l^t + H_{rs} l^r l^s \\ B_3 &= \frac{1}{12} (\nu_{rstu} + 3 i^{vw} \nu_{rsv} \nu_{tuw}) l^r l^s l^t l^u + \frac{1}{3} H_{rst} l^r l^s l^t \\ &\quad + i^{vw} \nu_{rsv} H_{tw} l^r l^s l^t + i^{vw} H_{rv} H_{sw} l^r l^s. \end{aligned}$$

Unlike the leading term B_1, neither B_2 nor B_3 is written as a contraction of tensors.

Expected/observed expansions such as (3.3) are not explicitly written in terms of tensors and further manipulation is required for a geometric structure to become visible. In McCullagh and Cox (1986) a technique is suggested for recovering invariant terms in the expansion of the log-likelihood ratio statistic. Their contribution inspired much of subsequent work on invariant Taylor series in statistics (see Barndorff-Nielsen and Cox

(1994), Section 5.6). In Pace and Salvan (1994) a geometric formulation is produced for the expected/observed expansion of a generic smooth function $f(\hat{\theta})$. In particular, it is shown that expected/observed expansions may be derived also from invariant Taylor expansions. As a result, expected/observed expansions may be written using ingredients clearly recognizable as tensors.

Example 2. Geometric expected/observed expansion of the log-likelihood ratio. In Pace and Salvan (1994) the terms B_2 and B_3 of (3.3) are re-written as

$$B_2 = \frac{1}{3}\tau_{rst}l^r l^s l^t + T^-_{rs}l^r l^s \tag{3.4}$$

$$B_3 = \frac{1}{12}\tau_{rstu}l^r l^s l^t l^u + \frac{1}{3}T^-_{rst}l^r l^s l^t + i^{vw}T^+_{rv}T^+_{sw}l^r l^s. \tag{3.5}$$

Above, τ_{R_m}, $T^-_{R_m}$, $T^+_{R_m}$ are $(0,m)$ tensors obtained recursively from the collections of log-likelihood derivatives and their expectations, following a procedure introduced by McCullagh and Cox (1986) and further developed by Barndorff-Nielsen (1986) and Barndorff-Nielsen and Blæsild (1987), see also McCullagh ((1987), Section 7.2.3). Their particular instances appearing in (3.4) and (3.5) are given by

$$\tau_{rst} = \nu_{rst} + \nu_{r;st}[3] = 2\nu_{r,s,t} \tag{3.6}$$

$$\tau_{rstu} = \nu_{rstu} + \nu_{r;stu}[4] + i^{vw}\nu_{v;rs}\nu_{w;tu}[3] - i^{vw}\tau_{rsv}\nu_{w;tu}[6]$$

$$T^+_{rs} = H_{rs} - \nu_{t,rs}l^t \tag{3.7}$$

$$T^-_{rs} = H_{rs} - \nu_{t;rs}l^t \tag{3.8}$$

$$T^-_{rst} = H_{rst} - i^{vw}\nu_{v;rs}H_{wt}[3] - (\nu_{u;rst} - i^{vw}\nu_{v;rs}\nu_{u;tw}[3])l^u.$$

The symbol $[k]$ indicates the sum of k similar terms obtained by all suitable permutations of the indices, except for the obvious symmetry relations. Moreover, the quantities $\nu_{R_m;S_n}$ above are defined as

$$\nu_{R_m;S_n} = \sum_{h=1}^{n}\sum_{S_n/h}\nu_{R_m,S_{n_1},\ldots,S_{n_h}},$$

where, for $h \leq n$, the symbol $\sum_{S_n/h}$ indicates summation over all the partitions of S_n into h non-empty subsets $S_{n_1},\ldots,S_{n_h}$. For instance $\nu_{r;st} = \nu_{r,st} + \nu_{r,s,t}$ and $\nu_{r;stu} = \nu_{r,stu} + \nu_{r,s,tu}[3] + \nu_{r,s,t,u}$.

3.2 *Nuisance parameters present*

Consider now inference about a function of the parameter, $\psi = \psi(\theta)$, with $\psi(\cdot)$ a smooth function with range $\Psi \subseteq \mathbb{R}^k$, $k < p$. Often ψ is a component of a given partition $\theta = (\psi, \chi)$ of θ into subparameters ψ and χ, with χ considered as a $(p-k)$-dimensional nuisance parameter. Inference problems about ψ are unaffected by one-to-one reparameterisations of ψ. Moreover, the choice of the nuisance parameterisation is immaterial. For simplicity of discussion, we assume hereafter that $\theta = (\psi, \chi)$. An interest respecting reparameterisation of $\mathcal{F}$ is then a reparameterisation $\omega = (\varphi, \xi)$, where $\varphi = \varphi(\psi)$ and $\xi = \xi(\psi, \chi)$, with $\varphi(\psi)$ a one-to-one function of ψ. Conversely, $\psi = \psi(\varphi)$ and $\chi = \chi(\varphi, \xi)$.

Denote by l_ψ and l_χ blocks of the score vector corresponding to ψ and to χ, respectively. Let, in addition, $i_{\psi\psi}$, $i_{\psi\chi}$, $i_{\chi\chi}$ denote blocks of the expected information i and $i^{\psi\psi}$, $i^{\psi\chi}$, $i^{\chi\chi}$ denote blocks of the matrix inverse of i.

We will use indices $a, b, c, \ldots$, with range $\{1, \ldots, k\}$, when referring to components of ψ, and Greek letters $\alpha, \beta, \gamma, \ldots$, with range $\{1, \ldots, p-k\}$, when referring to components of χ. Hence, for instance, l_ψ has generic component $l_a = (\partial l/\partial\psi^a)(\psi, \chi)$, $a = 1, \ldots, k$, while l_χ has generic component $l_\alpha = (\partial l/\partial\chi^\alpha)(\psi, \chi)$, $\alpha = 1, \ldots, p-k$. Moreover, $i_{\chi\chi}^{\alpha\beta}$ denotes a generic element of the matrix inverse of $i_{\chi\chi}$.

Likelihood inference about ψ is usually based on the profile log-likelihood $l_P(\psi) = l(\psi, \hat{\chi}_\psi)$, where $\hat{\chi}_\psi$ is the maximum likelihood estimator of χ for a given ψ (see Barndorff-Nielsen and Cox (1994), Section 3.4). We assume that $\hat{\chi}_\psi$ is a solution with respect to χ of the partial likelihood equation $l_\chi(\psi, \chi) = 0$. Hypothesis testing and construction of confidence regions about ψ are based on the profile log-likelihood ratio statistic

$$W_P(\psi) = 2(l_P(\hat{\psi}) - l_P(\psi)) = 2(l(\hat{\psi}, \hat{\chi}) - l(\psi, \hat{\chi}_\psi)),$$

whose asymptotic null distribution is χ^2_k under regularity conditions.

Under an interest respecting reparameterisation (φ, ξ), indices $\bar{a}, \bar{b}, \ldots$ are used for the components of φ, while indices $\bar{\alpha}, \bar{\beta}, \ldots$ refer to the components of ξ. A likelihood quantity denoted by indices $\bar{a}, \bar{b}, \ldots, \bar{\alpha}, \bar{\beta}, \ldots$ is understood as referred to the (φ, ξ) parameterisation. Notice that $\psi^a_{\bar{\alpha}} = 0$ and $\varphi^{\bar{a}}_\alpha = 0$. Moreover, from $\omega^{\bar{r}}_r \theta^s_{\bar{r}} = \delta^s_r$, we get

$$\begin{aligned}
\psi^a_{\bar{a}} \varphi^{\bar{a}}_b &= \delta^a_b \\
\chi^\alpha_{\bar{a}} \varphi^{\bar{a}}_a &= -\chi^\alpha_{\bar{\alpha}} \xi^{\bar{\alpha}}_a \\
\chi^\alpha_{\bar{\alpha}} \xi^{\bar{\alpha}}_\beta &= \delta^\alpha_\beta .
\end{aligned}$$

The profile log-likelihood and the profile log-likelihood ratio statistic are invariant under interest respecting reparameterisations. An interest respecting reparameterisation, however, affects log-likelihood derivatives and their moments. For instance, from $l(\varphi, \xi) = l(\psi(\varphi), \chi(\varphi, \xi))$ we have

$$\begin{aligned}
l_{\bar{a}} &= l_a \psi^a_{\bar{a}} + l_\alpha \chi^\alpha_{\bar{a}} \\
l_{\bar{\alpha}} &= l_\alpha \chi^\alpha_{\bar{\alpha}} \\
i_{\bar{a}\bar{b}} &= i_{ab} \psi^a_{\bar{a}} \psi^b_{\bar{b}} + i_{\alpha b} \chi^\alpha_{\bar{a}} \psi^b_{\bar{b}}[2] + i_{\alpha\beta} \chi^\alpha_{\bar{a}} \chi^\beta_{\bar{b}} \\
i_{\bar{\alpha}\bar{\beta}} &= i_{\alpha\beta} \chi^\alpha_{\bar{\alpha}} \chi^\beta_{\bar{\beta}} .
\end{aligned} \tag{3.9}$$

The profile score $(\partial l_P/\partial\psi)(\psi)$ with generic component $l_a(\psi, \hat{\chi}_\psi)$ transforms under interest respecting reparameterisations as

$$\frac{\partial l_P}{\partial \varphi^{\bar{a}}}(\varphi) = l_a(\psi, \hat{\chi}_\psi) \psi^a_{\bar{a}},$$

see e.g. formulae (4.36) and (9.99) in Pace and Salvan (1997).

Expected/observed expansions are used also in the study of asymptotic properties of profile likelihood quantities. The most notable instance is analysed in the following example.

Example 3. Expected/observed expansion of the profile log-likelihood ratio: invariance of the leading term. The expected/observed expansion of $W_P(\psi)$ has the form

$$W_P(\psi) = \overset{P}{B_1} + \overset{P}{B_2} + \overset{P}{B_3} + O_p(n^{-3/2}).$$

It is obtained by writing $W_P(\psi)$ as the difference between $W(\psi,\chi)=2(l(\hat{\psi},\hat{\chi})-l(\psi,\chi))$ and $W^{\psi}(\chi)=2(l(\psi,\hat{\chi}_\psi)-l(\psi,\chi))$ and applying (3.3) to both $W(\psi,\chi)$ and $W^{\psi}(\chi)$. In particular,

$$W^{\psi}(\chi)=B_1^{\psi}+B_2^{\psi}+B_3^{\psi}+O_p(n^{-3/2}),$$

where $B_1^{\psi}=i_{\alpha\beta}\bar{l}^{\alpha}\bar{l}^{\beta}$, with $\bar{l}^{\alpha}=i_{\chi\chi}^{\alpha\beta}l_{\beta}$. Hence, the leading term of $W_P(\psi)$ is

$$B_1^{P}=B_1-B_1^{\psi}=i_{rs}l^rl^s-i_{\alpha\beta}\bar{l}^{\alpha}\bar{l}^{\beta}.$$

As is well known (see e.g. Cox and Hinkley (1974), chapter 9, formula (55)), B_1^P may be rewritten as

$$B_1^{P}=\bar{l}_\psi^{\top}i^{\psi\psi}\bar{l}_\psi, \tag{3.10}$$

with $\bar{l}_\psi=l_\psi-\beta_\psi^{\chi}l_\chi$, where $\beta_\psi^{\chi}=i_{\psi\chi}\{i_{\chi\chi}\}^{-1}$ is the matrix of regression coefficients of l_ψ on l_χ. The vector $\bar{l}_\psi$ is often called the efficient score for ψ.

Let us denote by $\bar{l}_a$ a generic component of $\bar{l}_\psi$. Under interest respecting reparameterisations, $\bar{l}_a$ transforms as

$$\bar{l}_{\bar{a}}=\bar{l}_a\psi_{\bar{a}}^{a},$$

or, using matrix notation,

$$\bar{l}_\varphi=\psi_{/\varphi}^{\top}\bar{l}_\psi,$$

where $\psi_{/\varphi}$ is the matrix with generic element $\psi_{\bar{a}}^{a}$ (see e.g. Sartori *et al.* (2003), Section 3.1). Moreover, $i^{\psi\psi}$ transforms as $i^{\varphi\varphi}=\varphi_{/\psi}i^{\psi\psi}\varphi_{/\psi}^{\top}$ (see e.g. Barndorff-Nielsen and Cox (1994), formula (8.3)). Hence, invariance of (3.10) under interest respecting reparameterisations follows.

For higher order terms in the expansion of $W_P(\psi)$ a detailed analysis of behaviour under interest respecting reparameterisations has not been carried out. In particular, no tensorial representation is available for such terms, similar to the one displayed in Example 3 for the leading term. As underlined by Barndorff-Nielsen and Cox ((1994), p. 153), no major simplification of higher order terms in the expansion of $W_P(\psi)$ seems to occur. See also the Appendix of DiCiccio and Stern (1994) and Li ((2001), formula (5)) for expressions of the term of order $O_p(n^{-1/2})$.

Other familiar instances of expected/observed expansions of profile likelihood quantities concern the profile score and its expectation. These are given in McCullagh and Tibshirani (1990) and are used to define an adjustment of the profile likelihood. Barndorff-Nielsen (1994) highlights the tensorial behaviour under interest respecting reparameterisations of quantities appearing in the leading term of the expectation of the expected/observed expansion of the profile score.

4. Interest respecting tensors

4.1 *Definition*

We encountered in Subsection 3.2 some instances of likelihood quantities showing a very simple behaviour under interest respecting reparameterisations. Those instances motivate the following definition.

An *interest respecting tensor* of rank $((m_1, m_2), (n_1, n_2))$, with $m_1, m_2, n_1, n_2 \in \mathbb{N}$, is a collection of smooth real functions

$$T^{A_{m_1}\Gamma_{m_2}}_{B_{n_1}\Delta_{n_2}} = T^{a_1\cdots a_{m_1}\alpha_1\cdots\alpha_{m_2}}_{b_1\cdots b_{n_1}\beta_1\cdots\beta_{n_2}}(\theta)$$

which, under interest respecting reparameterisations, obeys the transformation rule

$$T^{\bar{A}_{m_1}\bar{\Gamma}_{m_2}}_{\bar{B}_{n_1}\bar{\Delta}_{n_2}} = T^{A_{m_1}\Gamma_{m_2}}_{B_{n_1}\Delta_{n_2}}\varphi^{\bar{a}_1}_{a_1}\cdots\varphi^{\bar{a}_{m_1}}_{a_{m_1}}\psi^{b_1}_{\bar{b}_1}\cdots\psi^{b_{n_1}}_{\bar{b}_{n_1}}\xi^{\bar{\alpha}_1}_{\alpha_1}\cdots\xi^{\bar{\alpha}_{m_2}}_{\alpha_{m_2}}\chi^{\beta_1}_{\bar{\beta}_1}\cdots\chi^{\beta_{n_2}}_{\bar{\beta}_{n_2}},$$

where $T^{\bar{A}_{m_1}\bar{\Gamma}_{m_2}}_{\bar{B}_{n_1}\bar{\Delta}_{n_2}} = T^{\bar{a}_1\cdots\bar{a}_{m_1}\bar{\alpha}_1\cdots\bar{\alpha}_{m_2}}_{\bar{b}_1\cdots\bar{b}_{n_1}\bar{\beta}_1\cdots\bar{\beta}_{n_2}}(\omega)$. We will refer to m_1 and m_2 as the interest contravariant and the nuisance contravariant rank of $T^{A_{m_1}\Gamma_{m_2}}_{B_{n_1}\Delta_{n_2}}$, respectively. A similar distinction is made for the covariant ranks n_1 and n_2.

A $((0,0),(0,0))$ interest respecting tensor is invariant under interest respecting reparameterisations. Both the profile score $(\partial/\partial\psi)l_P(\psi)$ and the efficient score $\bar{l}_\psi$ are interest respecting tensors of rank $((0,0),(1,0))$, while $i^{\psi\psi}$ is a $((2,0),(0,0))$ interest respecting tensor and $i_{\chi\chi}$ is a $((0,0),(0,2))$ interest respecting tensor.

When nuisance parameters are absent, collections of tensors may be obtained recursively from the collections of log-likelihood derivatives and their expectations, following the procedure developed by Barndorff-Nielsen (1986), see the instances in Example 2. Tensors under global reparameterisations do not, however, necessarily behave as interest respecting tensors, even when calculated in an orthogonal parameterisation. This is already apparent from (3.9).

We will show below how to obtain interest respecting tensors recursively from a collection of functions that behave as tensors under global reparameterisations. The requirement of global tensorial behaviour may be limited to a subset of coordinates, as is pointed out in Subsection 4.4.

4.2 *Recursive equations for covariant interest respecting tensors from global covariant tensors*

Let T_{R_n} be a $(0,n)$ tensor under a reparameterisation $\omega = \omega(\theta)$ of $\mathcal{F}$. It obeys the transformation rule

$$T_{\bar{R}_n} = T_{\bar{r}_1\cdots\bar{r}_n} = T_{r_1\cdots r_n}\theta^{r_1}_{\bar{r}_1}\cdots\theta^{r_n}_{\bar{r}_n}.$$

When the reparameterisation is interest respecting, the above equation specialises, because $\theta^a_{\bar{\alpha}} = \psi^a_{\bar{\alpha}} = 0$. Some instances of this are

$$T_{\bar{a}} = T_a\psi^a_{\bar{a}} + T_\alpha\chi^\alpha_{\bar{a}}$$
$$T_{\bar{\alpha}} = T_\alpha\chi^\alpha_{\bar{\alpha}}$$
$$T_{\bar{a}\bar{b}} = T_{ab}\psi^a_{\bar{a}}\psi^b_{\bar{b}} + T_{a\alpha}\psi^a_{\bar{a}}\chi^\alpha_{\bar{b}}[2] + T_{\alpha\beta}\chi^\alpha_{\bar{a}}\chi^\beta_{\bar{b}}$$
$$T_{\bar{a}\bar{\alpha}} = T_{a\alpha}\psi^a_{\bar{a}}\chi^\alpha_{\bar{\alpha}} + T_{\alpha\beta}\chi^\alpha_{\bar{a}}\chi^\beta_{\bar{\alpha}} \tag{4.1}$$
$$T_{\bar{\alpha}\bar{\beta}} = T_{\alpha\beta}\chi^\alpha_{\bar{\alpha}}\chi^\beta_{\bar{\beta}}. \tag{4.2}$$

Note that T_α and $T_{\alpha\beta}$ transform as interest respecting tensors of rank $((0,0),(0,1))$ and $((0,0),(0,2))$, respectively.

Interest respecting tensors $\bar{T}_r$ of rank $((0,0),(n_1,n_2))$, with $n_1+n_2 = 1$, are obtained by solving the recursive equations

$$\begin{aligned} T_\alpha &= \bar{T}_\alpha \\ T_a &= \bar{T}_a + \beta^\alpha_a\bar{T}_\alpha, \end{aligned} \tag{4.3}$$

with $\beta_a^\alpha = i_{a\beta} i_{\chi\chi}^{\alpha\beta}$, a generic element of the matrix of regression coefficients β_ψ^χ.

Only the behaviour of $\bar{T}_a$ has to be checked. Detailed calculations are given in the Appendix, showing that $\bar{T}_a$ behaves as a $((0,0),(1,0))$ interest respecting tensor.

Similarly, interest respecting tensors $\bar{T}_{rs}$ of rank $((0,0),(n_1,n_2))$, with $n_1+n_2=2$, are obtained by solving the recursive equations

$$
\begin{aligned}
T_{\alpha\beta} &= \bar{T}_{\alpha\beta} \\
T_{a\beta} &= \bar{T}_{a\beta} + \beta_a^\alpha \bar{T}_{\alpha\beta} \\
T_{ab} &= \bar{T}_{ab} + \beta_b^\beta \bar{T}_{a\beta}[2] + \beta_a^\alpha \beta_b^\beta \bar{T}_{\alpha\beta}.
\end{aligned} \tag{4.4}
$$

See the Appendix for a check of the tensorial behaviour of $\bar{T}_{rs}$.

Following the same scheme, we may obtain interest respecting tensors of higher ranks. In particular, interest respecting tensors $\bar{T}_{rst}$ of rank $((0,0),(n_1,n_2))$, with $n_1 + n_2 = 3$, are obtained by solving the recursive equations

$$
\begin{aligned}
T_{\alpha\beta\gamma} &= \bar{T}_{\alpha\beta\gamma} \\
T_{a\beta\gamma} &= \bar{T}_{a\beta\gamma} + \beta_a^\alpha \bar{T}_{\alpha\beta\gamma} \\
T_{ab\gamma} &= \bar{T}_{ab\gamma} + \beta_b^\beta \bar{T}_{a\beta\gamma}[2] + \beta_a^\alpha \beta_b^\beta \bar{T}_{\alpha\beta\gamma} \\
T_{abc} &= \bar{T}_{abc} + \beta_c^\gamma \bar{T}_{ab\gamma}[3] + \beta_b^\beta \beta_c^\gamma \bar{T}_{a\beta\gamma}[3] + \beta_a^\alpha \beta_b^\beta \beta_c^\gamma \bar{T}_{\alpha\beta\gamma}.
\end{aligned} \tag{4.5}
$$

Interest respecting tensors $\bar{T}_{rstu}$ of rank $((0,0),(n_1,n_2))$, with $n_1+n_2=4$, are obtained by solving the recursive equations

$$
\begin{aligned}
T_{\alpha\beta\gamma\delta} &= \bar{T}_{\alpha\beta\gamma\delta} \\
T_{a\beta\gamma\delta} &= \bar{T}_{a\beta\gamma\delta} + \beta_a^\alpha \bar{T}_{\alpha\beta\gamma\delta} \\
T_{ab\gamma\delta} &= \bar{T}_{ab\gamma\delta} + \beta_b^\beta \bar{T}_{a\beta\gamma\delta}[2] + \beta_a^\alpha \beta_b^\beta \bar{T}_{\alpha\beta\gamma\delta} \\
T_{abc\delta} &= \bar{T}_{abc\delta} + \beta_c^\gamma \bar{T}_{ab\gamma\delta}[3] + \beta_b^\beta \beta_c^\gamma \bar{T}_{a\beta\gamma\delta}[3] + \beta_a^\alpha \beta_b^\beta \beta_c^\gamma \bar{T}_{\alpha\beta\gamma\delta} \\
T_{abcd} &= \bar{T}_{abcd} + \beta_d^\delta \bar{T}_{abc\delta}[4] + \beta_c^\gamma \beta_d^\delta \bar{T}_{ab\gamma\delta}[6] + \beta_b^\beta \beta_c^\gamma \beta_d^\delta \bar{T}_{a\beta\gamma\delta}[4] \\
&\quad + \beta_a^\alpha \beta_b^\beta \beta_c^\gamma \beta_d^\delta \bar{T}_{\alpha\beta\gamma\delta}.
\end{aligned} \tag{4.6}
$$

Relations (4.3), (4.4), (4.5) and (4.6) show a close resemblance with those of McCullagh ((1987), Section 5.5.2) for cumulants of orthogonalised variables. Note, however, that the tensorial behaviour of $\bar{T}_r, \bar{T}_{rs}, \ldots$ under interest respecting reparameterisations does not depend on the specific definition of β_a^α, but only on the validity of its transformation law, see formula (A.1). See also Barndorff-Nielsen and Jupp ((1988), Section 3). Nevertheless, the choice $\beta_a^\alpha = i_{a\beta} i_{\chi\chi}^{\alpha\beta}$ is very natural in the statistical context.

Tensors defined in terms of log-likelihood derivatives are of special importance when dealing with likelihood expansions. When global reparameterisations are considered, McCullagh and Cox (1986) and McCullagh ((1987), Section 7.2.3) show that there exists locally at each point $\theta_0 \in \Theta$ a reparameterisation such that log-likelihood derivatives of each given order behave tensorially. Moreover, the tensorial second and higher-order derivatives are uncorrelated with the score. When attention is restricted to interest-respecting reparameterisations, a similar result holds. Consider the local reparameterisation

$$(\omega - \omega_0)^{\bar{r}} = c_r^{\bar{r}}(\theta - \theta_0)^r + \frac{1}{2!} c_{rs}^{\bar{r}} (\theta - \theta_0)^{rs} + \frac{1}{3!} c_{rst}^{\bar{r}} (\theta - \theta_0)^{rst} + \cdots,$$

with

$$c_a^{\bar a}=\delta_a^{\bar a},\qquad c_\alpha^{\bar a}=0,\qquad c_a^{\bar\alpha}=\delta_\alpha^{\bar\alpha}\beta_a^\alpha,\qquad c_\alpha^{\bar\alpha}=\delta_\alpha^{\bar\alpha};$$
$$c_{rs}^{\bar a}=\delta_a^{\bar a}\beta_{rs}^a,\qquad c_{rs}^{\bar\alpha}=\delta_\alpha^{\bar\alpha}(\beta_{rs}^\alpha+\beta_a^\alpha\beta_{rs}^a);$$
$$c_{rst}^{\bar a}=\delta_a^{\bar a}\beta_{rst}^a,\qquad c_{rst}^{\bar\alpha}=\delta_\alpha^{\bar\alpha}(\beta_{rst}^\alpha+\beta_a^\alpha\beta_{rst}^a);$$

and so on. Above, $\beta_{R_m}^v=i^{vw}\nu_{w,R_m}$. Log-likelihood derivatives $\bar l_{\bar R_m}=\frac{\partial^m}{\partial\omega^{\bar r_1}\cdots\partial\omega^{\bar r_m}}l(\omega)$ satisfy relations such as

$$l_a=\bar l_a+\beta_a^\alpha\bar l_\alpha,$$
$$l_\alpha=\bar l_\alpha,$$

and

$$l_{ab}=(\bar l_{ab}+\beta_a^\alpha\bar l_{\alpha b}[2]+\beta_a^\alpha\beta_b^\beta\bar l_{\alpha\beta})+\beta_{ab}^c(\bar l_c+\beta_c^\alpha\bar l_\alpha)+\beta_{ab}^\alpha\bar l_\alpha,$$
$$l_{a\alpha}=\bar l_{a\alpha}+\beta_a^\beta\bar l_{\alpha\beta}+\beta_{a\alpha}^c(\bar l_c+\beta_c^\gamma\bar l_\gamma)+\beta_{a\alpha}^\gamma\bar l_\gamma,$$
$$l_{\alpha\beta}=\bar l_{\alpha\beta}+\beta_{\alpha\beta}^a(\bar l_a+\beta_a^\gamma\bar l_\gamma)+\beta_{\alpha\beta}^\gamma\bar l_\gamma,$$

and so on, showing that they coincide with the interest respecting tensors obtained by applying relations (4.3), (4.4), and so on, to the global tensors l_r, $l_{rs}-\beta_{rs}^v l_v$, and so on.

4.3 *Recursive equations for contravariant interest respecting tensors from global contravariant tensors*

In the previous subsection we have discussed how to obtain interest respecting covariant tensors starting from covariant tensors T_{R_n} of rank $(0,n)$. An analogous argument holds if we start from contravariant tensors T^{S_m}. Let T^{S_m} be an $(m,0)$ tensor under a reparameterisation $\omega=\omega(\theta)$ of $\mathcal{F}$. It obeys the transformation rule

$$T^{\bar S_m}=T^{\bar s_1\cdots\bar s_m}=T^{s_1\cdots s_m}\omega_{s_1}^{\bar s_1}\cdots\omega_{s_m}^{\bar s_m}.$$

Under interest respecting reparameterisations, the above equation specialises, because $\omega_\alpha^{\bar a}=\varphi_\alpha^{\bar a}=0$. Some instances of this are

$$T^{\bar a}=T^a\varphi_a^{\bar a}$$
$$T^{\bar\alpha}=T^a\xi_a^{\bar\alpha}+T^\alpha\xi_\alpha^{\bar\alpha}$$
$$T^{\bar a\bar b}=T^{ab}\varphi_a^{\bar a}\varphi_b^{\bar b}$$
$$T^{\bar a\bar\alpha}=T^{ab}\varphi_a^{\bar a}\xi_b^{\bar\alpha}+T^{a\alpha}\varphi_a^{\bar a}\xi_\alpha^{\bar\alpha}$$
$$T^{\bar\alpha\bar\beta}=T^{ab}\xi_a^{\bar\alpha}\xi_b^{\bar\beta}+T^{a\alpha}\xi_a^{\bar\alpha}\xi_\alpha^{\bar\beta}[2]+T^{\alpha\beta}\xi_\alpha^{\bar\alpha}\xi_\beta^{\bar\beta}.$$

Note that T^a transforms as a $((1,0),(0,0))$ interest respecting tensor and that T^{ab} transforms as a $((2,0),(0,0))$ interest respecting tensor.

Recursive equations defining $((m_1,m_2),(0,0))$ interest respecting tensors $\bar T^r$, with $m_1+m_2=1$, are

$$T^a=\bar T^a$$
$$T^\alpha=\bar T^\alpha-\beta_a^\alpha\bar T^a. \tag{4.7}$$

Only the behaviour of $\bar T^\alpha$ has to be checked. See the Appendix for details.

Similarly, interest respecting tensors $\bar{T}^{rs}$ of rank $((m_1, m_2), (0, 0))$, with $m_1 + m_2 = 2$, are obtained by solving the recursive equations

$$
\begin{aligned}
T^{ab} &= \bar{T}^{ab} \\
T^{a\beta} &= \bar{T}^{a\beta} - \beta_b^\beta \bar{T}^{ab} \\
T^{\alpha\beta} &= \bar{T}^{\alpha\beta} - \beta_a^\alpha \bar{T}^{a\beta}[2] + \beta_a^\alpha \beta_b^\beta \bar{T}^{ab}.
\end{aligned} \tag{4.8}
$$

Following the same pattern, it is straightforward to write the recursive equations giving interest respecting tensors of rank $((m_1, m_2), (0, 0))$ with $m_1 + m_2 = 3, 4$, and so on.

4.4 *Other interest respecting tensors*

The technique illustrated in the previous subsections allows us to obtain interest respecting tensors also starting from quantities that are not global tensors themselves. These quantities should, however, share with global tensors an appropriate part of their transformation rule under interest respecting reparameterisations, as far as specific subsets of coordinates are concerned.

Consider for instance the quantities

$$t_{r\alpha} = H_{r\alpha} - \nu_{\beta,r\alpha} \bar{l}^\beta, \tag{4.9}$$

with $\bar{l}^\alpha = i_{\chi\chi}^{\alpha\beta} l_\beta$. Notice that when r indexes a nuisance component (4.9) is the same as (3.7) referred to the submodel with ψ fixed. Under interest respecting reparameterisations, the quantities $t_{r\alpha}$ transform according to the rules

$$
\begin{aligned}
t_{\bar{a}\bar{\alpha}} &= t_{a\alpha} \psi_{\bar{a}}^a \chi_{\bar{\alpha}}^\alpha + t_{\alpha\beta} \chi_{\bar{a}}^\alpha \chi_{\bar{\alpha}}^\beta \\
t_{\bar{\alpha}\bar{\beta}} &= t_{\alpha\beta} \chi_{\bar{\alpha}}^\alpha \chi_{\bar{\beta}}^\beta.
\end{aligned}
$$

These rules are the same as (4.1) and (4.2) holding for a global $(0, 2)$ tensor, as far as pairs of indices a, α and α, β are concerned. Hence, using the first and second equation in (4.4),

$$\bar{t}_{a\alpha} = t_{a\alpha} - \beta_a^\beta t_{\alpha\beta} \tag{4.10}$$

is a $((0, 0), (1, 1))$ interest respecting tensor.

The construction above may be extended as follows. Let $\{C_{R_m}\}$ be sequence of likelihood quantities that behaves as a costring of covariant degree zero under global reparameterisations (see e.g. Pace and Salvan (1994), Section 2, for details). Notable instances of such quantities are l_{R_m}, ν_{R_m} and H_{R_m}. In addition, let

$$
\begin{aligned}
\beta_{r\alpha}^\zeta &= i_{\chi\chi}^{\zeta\kappa} \nu_{\kappa,r\alpha} \\
\beta_{r\alpha\beta}^\zeta &= i_{\chi\chi}^{\zeta\kappa} \nu_{\kappa,r\alpha\beta}
\end{aligned}
$$

and so on. Let $t_\alpha = C_\alpha$. Consider now the quantities $\{t_{a\alpha}, t_{\alpha\beta}\}$, $\{t_{a\alpha\beta}, t_{\alpha\beta\gamma}\}$, and so on, obtained as solutions of recursive relations of the form

$$
\begin{aligned}
C_{a\alpha} &= t_{a\alpha} + \beta_{a\alpha}^\zeta t_\zeta \\
C_{\alpha\beta} &= t_{\alpha\beta} + \beta_{\alpha\beta}^\zeta t_\zeta \\
C_{a\alpha\beta} &= t_{a\alpha\beta} + \beta_{a\alpha}^\zeta t_{\zeta\beta}[2] + \beta_{\alpha\beta}^\zeta t_{\zeta a} + \beta_{a\alpha\beta}^\zeta t_\zeta \\
C_{\alpha\beta\gamma} &= t_{\alpha\beta\gamma} + \beta_{\alpha\beta}^\zeta t_{\zeta\gamma}[3] + \beta_{\alpha\beta\gamma}^\zeta t_\zeta.
\end{aligned}
$$

Under interest respecting reparameterisations, $t_{a\alpha}$ and $t_{\alpha\beta}$ transform according to the rules

$$\begin{aligned} t_{\bar{a}\bar{\alpha}} &= t_{a\alpha}\psi^a_{\bar{a}}\chi^\alpha_{\bar{\alpha}} + t_{\alpha\beta}\chi^\alpha_{\bar{a}}\chi^\beta_{\bar{\alpha}} \\ t_{\bar{\alpha}\bar{\beta}} &= t_{\alpha\beta}\chi^\alpha_{\bar{\alpha}}\chi^\beta_{\bar{\beta}}, \end{aligned}$$

which are part of the transformation law of global covariant tensors of degree two, specialised to interest respecting reparameterisations (see Subsection 4.2). Hence, relations (4.4) may be applied to give interest respecting tensors $\bar{t}_{\alpha\beta}$ and $\bar{t}_{a\alpha}$ as the solutions of

$$\begin{aligned} t_{\alpha\beta} &= \bar{t}_{\alpha\beta} \\ t_{a\beta} &= \bar{t}_{a\beta} + \beta^\alpha_a \bar{t}_{\alpha\beta}. \end{aligned} \tag{4.11}$$

Notice that, with $C_{r\alpha} = H_{r\alpha}$, relations (4.11) give the $((0,0),(1,1))$ interest respecting tensor (4.10).

Similarly, $t_{a\alpha\beta}$ and $t_{\alpha\beta\gamma}$ follow, under interest respecting reparameterisations, the same transformation law as the corresponding components of a global $(0,3)$ tensor. Hence, relations (4.5) may be used to obtain interest respecting tensors $\bar{t}_{\alpha\beta\gamma}$ and $\bar{t}_{a\alpha\beta}$. It is straightforward to obtain interest respecting tensors with higher ranks.

5. Notable likelihood interest respecting tensors

As an immediate consequence of the results in the previous section, we may obtain an interest respecting score $\bar{l}_r$ starting from the $(0,1)$ tensor l_r as

$$\begin{aligned} \bar{l}_a &= l_a - \beta^\alpha_a l_\alpha \\ \bar{l}_\alpha &= l_\alpha, \end{aligned}$$

or, using matrix notation,

$$\bar{l}_\theta = \begin{pmatrix} \bar{l}_\psi \\ \bar{l}_\chi \end{pmatrix} = \begin{pmatrix} l_\psi - \beta^\chi_\psi l_\chi \\ l_\chi \end{pmatrix},$$

where the first block is the efficient score for ψ.

We get an interest respecting expected information $\bar{\imath}$ starting from the $(0,2)$ tensor i_{rs}. Equations (4.4) give

$$\begin{aligned} \bar{\imath}_{ab} &= i_{ab} - i_{a\alpha} i^{\alpha\beta}_{\chi\chi} i_{b\beta} \\ \bar{\imath}_{a\beta} &= 0 \\ \bar{\imath}_{\alpha\beta} &= i_{\alpha\beta}. \end{aligned}$$

Using matrix notation,

$$\bar{\imath} = \begin{pmatrix} \bar{\imath}_{\psi\psi} & 0 \\ 0 & \bar{\imath}_{\chi\chi} \end{pmatrix},$$

with $\bar{\imath}_{\psi\psi} = (i^{\psi\psi})^{-1} = i_{\psi\psi} - i_{\psi\chi} i^{-1}_{\chi\chi} i_{\chi\psi}$ and $\bar{\imath}_{\chi\chi} = i_{\chi\chi}$. Note that $\bar{\imath}$ is the covariance matrix of $\bar{l}_\theta$.

Similarly, from the construction of contravariant interest respecting tensors, we may define the interest respecting tensors $\bar{l}^r$ starting from the $(1,0)$ tensor l^r as

$$\begin{aligned}\bar{l}^a &= l^a \\ \bar{l}^\alpha &= l^\alpha + \beta_a^\alpha l^a.\end{aligned} \tag{5.1}$$

The elements of the matrix inverse $\bar{\imath}^{-1}$ of $\bar{\imath}$ may be derived as interest respecting tensors of rank $((m_1, m_2), (0,0))$, with $m_1 + m_2 = 2$, starting from the $(2,0)$ tensor i^{rs}. Indeed, using (4.8),

$$\begin{aligned}\bar{\imath}^{ab} &= i^{ab} \\ \bar{\imath}^{a\beta} &= 0 \\ \bar{\imath}^{\alpha\beta} &= i^{\alpha\beta} - \beta_a^\alpha \beta_b^\beta i^{ab}.\end{aligned} \tag{5.2}$$

Note that $\bar{l}^a = \bar{\imath}^{ab}\bar{l}_b$ and $\bar{l}^\alpha = \bar{\imath}^{\alpha\beta}\bar{l}_\beta$, showing that the notation used here is consistent with that in Example 3. Also note that $\bar{l}_r$, $\bar{\imath}$, $\bar{l}^r$ and $\bar{\imath}^{-1}$ coincide with score, information and their contravariant counterparts in an orthogonal interest respecting reparameterisation. However, in general interest respecting tensors do not have such a simple interpretation.

Example 4. *Expected/observed expansion of the profile log-likelihood ratio.* The interest respecting tensors $\bar{l}^r$ are useful to highlight a regular structure in the expected/observed expansion of $W_P(\psi)$. This is due to the implied orthogonality of the blocks of components of $\bar{l}^r$ regarding the interest and the nuisance parameter.

With the same notation as in Examples 1 and 3, we have $B_1 = \bar{\imath}_{rs}\bar{l}^r\bar{l}^s$. Indeed, from (3.3),

$$B_1 = i_{rs}l^r l^s = i_{ab}l^a l^b + i_{a\beta}l^a l^\beta[2] + i_{\alpha\beta}l^\alpha l^\beta.$$

Using (5.1) to express l^a and l^α in terms of $\bar{l}^a$ and $\bar{l}^\alpha$, we obtain

$$B_1 = (i_{ab} - \beta_a^\alpha i_{a\alpha}[2] + \beta_a^\alpha \beta_b^\beta i_{\alpha\beta})\bar{l}^a\bar{l}^b + (i_{a\beta} - \beta_a^\alpha i_{\alpha\beta})\bar{l}^a\bar{l}^\beta[2] + i_{\alpha\beta}\bar{l}^\alpha\bar{l}^\beta = \bar{\imath}_{ab}\bar{l}^a\bar{l}^b + \bar{\imath}_{\alpha\beta}\bar{l}^\alpha\bar{l}^\beta.$$

From Example 3, $B_1^\psi = \bar{\imath}_{\alpha\beta}\bar{l}^\alpha\bar{l}^\beta$. Hence,

$$B_1^P = B_1 - B_1^\psi = \bar{\imath}_{ab}\bar{l}^a\bar{l}^b,$$

in accordance with (3.10).

Similarly, using (5.1), we obtain

$$B_2 = \frac{1}{3}\bar{\nu}_{rst}\bar{l}^r\bar{l}^s\bar{l}^t + \bar{H}_{rs}\bar{l}^r\bar{l}^s,$$

where $\bar{\nu}_{rst}$ and $\bar{H}_{rs}$ have the same expression as the solutions of the recursive equations (4.5) and (4.4) with T_{rst} and T_{rs} replaced by ν_{rst} and H_{rs}, respectively. In particular,

$$\begin{aligned}\bar{\nu}_{abc} &= \nu_{abc} - \beta_a^\alpha \nu_{\alpha bc}[3] + \beta_a^\alpha \beta_b^\beta \nu_{\alpha\beta c}[3] - \beta_a^\alpha \beta_b^\beta \beta_c^\gamma \nu_{\alpha\beta\gamma} \\ \bar{\nu}_{ab\alpha} &= \nu_{ab\alpha} - \beta_b^\beta \nu_{a\alpha\beta}[2] + \beta_a^\beta \beta_b^\gamma \nu_{\alpha\beta\gamma} \\ \bar{\nu}_{a\alpha\beta} &= \nu_{a\alpha\beta} - \beta_a^\gamma \nu_{\alpha\beta\gamma} \\ \bar{\nu}_{\alpha\beta\gamma} &= \nu_{\alpha\beta\gamma} \\ \bar{H}_{ab} &= H_{ab} - \beta_a^\alpha H_{\alpha b}[2] + \beta_a^\alpha \beta_b^\beta H_{\alpha\beta} \\ \bar{H}_{a\alpha} &= H_{a\alpha} - \beta_a^\beta H_{\alpha\beta} \\ \bar{H}_{\alpha\beta} &= H_{\alpha\beta}.\end{aligned}$$

We also get

$$\overset{\psi}{B_2} = \frac{1}{3}\bar{\nu}_{\alpha\beta\gamma}\bar{l}^\alpha\bar{l}^\beta\bar{l}^\gamma + \bar{H}_{\alpha\beta}\bar{l}^\alpha\bar{l}^\beta,$$

so that

$$\begin{aligned}\overset{P}{B_2} &= B_2 - \overset{\psi}{B_2} = \frac{1}{3}\bar{\nu}_{rst}\bar{l}^r\bar{l}^s\bar{l}^t + \bar{H}_{rs}\bar{l}^r\bar{l}^s - \frac{1}{3}\bar{\nu}_{\alpha\beta\gamma}\bar{l}^\alpha\bar{l}^\beta\bar{l}^\gamma - \bar{H}_{\alpha\beta}\bar{l}^\alpha\bar{l}^\beta \\ &= \frac{1}{3}\bar{\nu}_{abc}\bar{l}^a\bar{l}^b\bar{l}^c + \bar{\nu}_{ab\alpha}\bar{l}^a\bar{l}^b\bar{l}^\alpha + \bar{\nu}_{a\alpha\beta}\bar{l}^a\bar{l}^\alpha\bar{l}^\beta + \bar{H}_{ab}\bar{l}^a\bar{l}^b + 2\bar{H}_{a\alpha}\bar{l}^a\bar{l}^\alpha.\end{aligned}$$

The same analysis as above can be done for $\overset{P}{B_3}$. It turns out that all the summands having only Greek indices cancel out. This points out the general pattern.

The most interesting feature of the above example is the relatively simple form of the term $\overset{P}{B_2}$. However, quantities such as $\bar{\nu}_{rst}$ and $\bar{H}_{rs}$ do not behave as interest respecting tensors, so that the obtained expansion is not geometric.

6. Interest respecting expected/observed expansions

Let us denote by $\tilde{\theta} = (\psi, \hat{\chi}_\psi)$ the constrained maximum likelihood estimator. A profile likelihood quantity has the general form $g(\hat{\theta}, \tilde{\theta}) = g(\hat{\theta}, \tilde{\theta}; y)$. Suppose that $g(\hat{\theta}, \tilde{\theta})$ is invariant under interest respecting reparameterisations or, more generally, an interest respecting tensor. Let us assume, in addition, that $g(\hat{\theta}, \tilde{\theta})$ is of order $O_p(n^\beta)$ under repeated sampling of size n. The expected/observed expansion of $g(\hat{\theta}, \tilde{\theta})$ is obtained through the ordinary Taylor formula for $g(\hat{\theta}, \tilde{\theta})$ around (θ, θ), followed by substitution of suitable expansions for $\hat{\theta} - \theta$ and $\tilde{\theta} - \theta$. These are given by formula (3.6) in Pace and Salvan (1994) referred to the full model $\mathcal{F}$ and to the submodel with ψ fixed, respectively. Note that an expansion for $\tilde{\theta} - \theta$ is essentially an expansion for $\hat{\chi}_\psi - \chi$. The resulting expansion for $g(\hat{\theta}, \tilde{\theta})$ has the form

$$g(\hat{\theta}, \tilde{\theta}) = g(\theta, \theta) + \overset{P}{b_1} + \overset{P}{b_2} + \overset{P}{b_3} + \overset{P}{b_4} + O_p(n^{\beta - 5/2}), \tag{6.1}$$

where each term $\overset{P}{b_m}$ is of order $O_p(n^{\beta - m/2})$, $m = 1, 2, 3, 4$.

The terms $\overset{P}{b_m}$ in (6.1) will not be explicitly written in terms of interest respecting tensors. The techniques developed in Section 4 may be used to bring out a geometric structure, i.e. to obtain an interest respecting expected/observed expansion. We do not give detailed formulae for a generic function $g(\hat{\theta}, \tilde{\theta})$, but rather we will illustrate below the main ideas with reference to the profile log-likelihood ratio statistic and to the profile score.

Example 5. *Geometric expected/observed expansion of the profile log-likelihood ratio.* The function $g(\hat{\theta}, \tilde{\theta}) = W_P(\psi)$ is of the form $g_1(\tilde{\theta}) + g_2(\hat{\theta})$, with $g_1(\tilde{\theta}) = -\overset{\psi}{W}(\chi) = -2(l(\tilde{\theta}) - l(\theta))$ and $g_2(\hat{\theta}) = W(\theta) = 2(l(\hat{\theta}) - l(\theta))$.

Consider first the geometric expansion

$$\overset{\psi}{W}(\chi) = \bar{\imath}_{\alpha\beta}\bar{l}^\alpha\bar{l}^\beta + \overset{\psi}{B_2} + \overset{\psi}{B_3} + O_p(n^{-3/2}),$$

where B_2^{ψ} and B_3^{ψ} are of order $O_p(n^{-1/2})$ and $O_p(n^{-1})$, respectively, and are given by (3.4) and (3.5) referred to the submodel with ψ fixed. In particular,

$$B_2^{\psi} = \frac{1}{3}\overset{\psi}{\tau}_{\alpha\beta\gamma}\bar{l}^{\alpha}\bar{l}^{\beta}\bar{l}^{\gamma} + \overset{-\psi}{T}_{\alpha\beta}\bar{l}^{\alpha}\bar{l}^{\beta},$$

with

$$\begin{aligned}\overset{\psi}{\tau}_{\alpha\beta\gamma} &= \tau_{\alpha\beta\gamma} = 2\nu_{\alpha,\beta,\gamma}\\ \overset{-\psi}{T}_{\alpha\beta} &= H_{\alpha\beta} - \nu_{\gamma;\alpha\beta}\bar{l}^{\gamma}.\end{aligned}$$

Note that $\overset{\psi}{\tau}_{\alpha\beta\gamma}$ and $\overset{-\psi}{T}_{\alpha\beta}$ are interest respecting tensors of rank $((0,0),(0,3))$ and $((0,0),(0,2))$, respectively.

Moreover,

$$B_3^{\psi} = \frac{1}{12}\overset{\psi}{\tau}_{\alpha\beta\gamma\delta}\bar{l}^{\alpha}\bar{l}^{\beta}\bar{l}^{\gamma}\bar{l}^{\delta} + \frac{1}{3}\overset{-\psi}{T}_{\alpha\beta\gamma}\bar{l}^{\alpha}\bar{l}^{\beta}\bar{l}^{\gamma} + \bar{\imath}^{\gamma\delta}\overset{+\psi}{T}_{\alpha\gamma}\overset{+\psi}{T}_{\beta\delta}\bar{l}^{\alpha}\bar{l}^{\beta}, \tag{6.2}$$

where

$$\begin{aligned}\overset{\psi}{\tau}_{\alpha\beta\gamma\delta} &= \nu_{\alpha\beta\gamma\delta} + \nu_{\alpha;\beta\gamma\delta}[4] + \bar{\imath}^{\varepsilon\zeta}\nu_{\varepsilon;\alpha\beta}\nu_{\zeta;\gamma\delta}[3] - \bar{\imath}^{\varepsilon\zeta}\tau_{\alpha\beta\varepsilon}\nu_{\zeta;\gamma\delta}[6]\\ \overset{+\psi}{T}_{\alpha\beta} &= H_{\alpha\beta} - \nu_{\gamma,\alpha\beta}\bar{l}^{\gamma}\\ \overset{-\psi}{T}_{\alpha\beta\gamma} &= H_{\alpha\beta\gamma} - \bar{\imath}^{\varepsilon\zeta}\nu_{\varepsilon;\alpha\beta}H_{\zeta\gamma}[3] - (\nu_{\delta;\alpha\beta\gamma} - \bar{\imath}^{\varepsilon\zeta}\nu_{\varepsilon;\alpha\beta}\nu_{\delta;\gamma\zeta}[3])\bar{l}^{\delta}.\end{aligned}$$

Again, $\overset{\psi}{\tau}_{\alpha\beta\gamma\delta}$, $\overset{+\psi}{T}_{\alpha\beta}$ and $\overset{-\psi}{T}_{\alpha\beta\gamma}$ are interest respecting tensors of rank $((0,0),(0,4))$, $((0,0),(0,2))$ and $((0,0),(0,3))$, respectively.

As a second step, we consider expansion (3.3) for $W(\theta)$ with B_2 and B_3 expressed in terms of tensors under global reparameterisations, as in (3.4) and (3.5).

Following the same algebra as in Example 4, i.e. expressing l^r and i^{rs} in terms of $\bar{l}^r$ and $\bar{\imath}^{rs}$, respectively, the geometric expression (3.4) for the summand B_2 may be rewritten as

$$B_2 = \frac{1}{3}\bar{\tau}_{rst}\bar{l}^r\bar{l}^s\bar{l}^t + \bar{T}^{-}_{rs}\bar{l}^r\bar{l}^s \tag{6.3}$$

where $\bar{\tau}_{rst}$ and $\bar{T}^{-}_{rs}$ are interest respecting tensors obtained from the tensors (3.6) and (3.8) using the recursive equations (4.5) and (4.4), respectively. In particular,

$$\begin{aligned}\bar{\tau}_{abc} &= \tau_{abc} - \beta_a^{\alpha}\tau_{\alpha bc}[3] + \beta_a^{\alpha}\beta_b^{\beta}\tau_{\alpha\beta c}[3] - \beta_a^{\alpha}\beta_b^{\beta}\beta_c^{\gamma}\tau_{\alpha\beta\gamma}\\ \bar{\tau}_{\alpha bc} &= \tau_{\alpha bc} - \beta_b^{\beta}\tau_{\alpha\beta c}[2] + \beta_b^{\beta}\beta_c^{\gamma}\tau_{\alpha\beta\gamma}\\ \bar{\tau}_{\alpha\beta c} &= \tau_{\alpha\beta c} - \beta_c^{\gamma}\tau_{\alpha\beta\gamma}\\ \bar{\tau}_{\alpha\beta\gamma} &= \tau_{\alpha\beta\gamma},\end{aligned}$$

and

$$\begin{aligned}\bar{T}^{-}_{ab} &= T^{-}_{ab} - \beta_a^{\alpha}T^{-}_{\alpha b}[2] + \beta_a^{\alpha}\beta_b^{\beta}T^{-}_{\alpha\beta}\\ \bar{T}^{-}_{\alpha b} &= T^{-}_{\alpha b} - \beta_b^{\beta}T^{-}_{\alpha\beta}\\ \bar{T}^{-}_{\alpha\beta} &= T^{-}_{\alpha\beta}.\end{aligned}$$

Similarly, we get

$$B_3 = \frac{1}{12}\bar{\tau}_{rstu}\bar{l}^r\bar{l}^s\bar{l}^t\bar{l}^u + \frac{1}{3}\bar{T}^-_{rst}\bar{l}^r\bar{l}^s\bar{l}^t + \bar{\imath}^{vw}\bar{T}_{rv,sw}\bar{l}^r\bar{l}^s, \tag{6.4}$$

where $\bar{\tau}_{rstu}$ and $\bar{T}^-_{rst}$ are interest respecting tensors obtained by solving equations (4.6) and (4.5) with $T_{rstu} = \tau_{rstu}$ and $T_{rst} = T^-_{rst}$, respectively. Finally, the interest respecting tensors $\bar{T}_{rv,sw}$ are obtained as solutions of (4.6) with $T_{rvsw} = T^+_{rv}T^+_{sw}$.

The term of order $O_p(n^{-1/2})$ in the geometric expected/observed expansion of $W_P(\psi)$ is given by $\overset{P}{B_2} = B_2 - \overset{\psi}{B_2}$. Since $\overset{\psi}{\tau}_{\alpha\beta\gamma} = \tau_{\alpha\beta\gamma}$, the corresponding terms cancel out in the expansion for $W_P(\psi)$. On the other hand,

$$T^-_{\alpha\beta} = H_{\alpha\beta} - \nu_{r;\alpha\beta}l^r = \overset{\psi}{T^-_{\alpha\beta}} - (\nu_{a;\alpha\beta} - \beta^\gamma_a\nu_{\gamma;\alpha\beta})\bar{l}^a.$$

Let

$$\bar{\tau}_{a;\alpha\beta} = \nu_{a;\alpha\beta} - \beta^\gamma_a\nu_{\gamma;\alpha\beta}. \tag{6.5}$$

Obviously, $\bar{\tau}_{a;\alpha\beta}$ behaves as a $((0,0),(1,2))$ interest respecting tensor. Hence,

$$\overset{P}{B_2} = \frac{1}{3}\bar{\tau}_{abc}\bar{l}^a\bar{l}^b\bar{l}^c + \bar{\tau}_{\alpha ab}\bar{l}^\alpha\bar{l}^a\bar{l}^b + (\bar{\tau}_{\alpha\beta a} - \bar{\tau}_{a;\alpha\beta})\bar{l}^\alpha\bar{l}^\beta\bar{l}^a + \bar{T}^-_{ab}\bar{l}^a\bar{l}^b + 2\bar{T}^-_{\alpha a}\bar{l}^a\bar{l}^\alpha.$$

The term of order $O_p(n^{-1})$ in the geometric expected/observed expansion of $W_P(\psi)$ is $\overset{P}{B_3} = B_3 - \overset{\psi}{B_3}$, with B_3 and $\overset{\psi}{B_3}$ given by (6.4) and (6.2), respectively. No straightforward simplification takes place in the difference $B_3 - \overset{\psi}{B_3}$, because, for instance, $\bar{\tau}_{\alpha\beta\gamma\delta} \neq \overset{\psi}{\tau}_{\alpha\beta\gamma\delta}$. Indeed, while $\bar{\tau}_{\alpha\beta\gamma\delta}$ depends on the matrix inverse of i, the tensor $\overset{\psi}{\tau}_{\alpha\beta\gamma\delta}$ depends only on the matrix inverse of $i_{\chi\chi}$.

Example 6. *Geometric expected/observed expansion of the profile score.* The expected/observed expansion of a generic component of the profile score is (cf. e.g. Pace and Salvan (1997), equation (9.88))

$$l_a(\psi, \hat{\chi}_\psi) = \bar{l}_a + (H_{a\alpha} - \beta^\beta_a H_{\alpha\beta})\bar{l}^\alpha + \frac{1}{2}(\nu_{a\alpha\beta} - \beta^\gamma_a\nu_{\alpha\beta\gamma})\bar{l}^\beta\bar{l}^\alpha + O_p(n^{-1/2}),$$

where $\bar{l}_a$ is of order $O_p(n^{1/2})$, while the remaining summands are of order $O_p(1)$. The leading term $\bar{l}_a$ is a $((0,0),(1,0))$ interest respecting tensor. Using the Bartlett identity $\nu_{r\alpha\beta} + \nu_{r,\alpha\beta} + \nu_{r\alpha,\beta}[2] + \nu_{r,\alpha,\beta} = 0$, the term of order $O_p(1)$ may be rewritten as

$$\bar{t}_{a\alpha}\bar{l}^\alpha - \frac{1}{2}\bar{\tau}_{a;\alpha\beta}\bar{l}^\alpha\bar{l}^\beta,$$

where $\bar{t}_{a\alpha}$ is the $((0,0),(1,1))$ interest respecting tensor given by (4.10), while $\bar{\tau}_{a;\alpha\beta}$ is given by (6.5).

It is easy to see that $E_\theta(\bar{t}_{a\alpha}\bar{l}^\alpha) = 0$. Hence,

$$E_\theta(l_a(\psi, \hat{\chi}_\psi)) = -\frac{1}{2}\bar{\imath}^{\alpha\beta}\bar{\tau}_{a;\alpha\beta} + O(n^{-1}),$$

in agreement with Barndorff-Nielsen and Cox ((1994), formula (8.61)).

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Appendix

We first show that $\bar{T}_a$ obtained from (4.3) behaves as a $((0,0),(1,0))$ interest respecting tensor. Under an interest respecting reparameterisation (φ,ξ), β_a^α transforms as

$$\beta_{\bar{a}}^{\bar{\alpha}} = (\beta_a^\alpha \psi_{\bar{a}}^a + \chi_{\bar{a}}^\alpha)\xi_\alpha^{\bar{\alpha}}. \tag{A.1}$$

Hence, the second equation in (4.3) transforms as

$$T_a\psi_{\bar{a}}^a + T_\alpha\chi_{\bar{a}}^\alpha = \bar{T}_{\bar{a}} + (\beta_a^\alpha\psi_{\bar{a}}^a + \chi_{\bar{a}}^\alpha)\xi_\alpha^{\bar{\alpha}}\bar{T}_\beta\chi_{\bar{\alpha}}^\beta.$$

Using the identity $\xi_\alpha^{\bar{\alpha}}\chi_{\bar{\alpha}}^\beta = \delta_\alpha^\beta$ on the right-hand side and equations (4.3) on the left-hand side, we get

$$(\bar{T}_a + \beta_a^\alpha\bar{T}_\alpha)\psi_{\bar{a}}^a = \bar{T}_{\bar{a}} + \beta_a^\alpha\psi_{\bar{a}}^a\bar{T}_\alpha,$$

so that $\bar{T}_{\bar{a}} = \bar{T}_a\psi_{\bar{a}}^a$, showing that $\bar{T}_a$ behaves as a $((0,0),(1,0))$ interest respecting tensor.

Consider now the transformation laws of $\bar{T}_{rs}$ obtained from (4.4). The tensorial behaviour of $\bar{T}_{\alpha\beta}$ is obvious. The tensorial behaviour of $\bar{T}_{a\beta}$ is easily shown following the same arguments used for $\bar{T}_a$. We only need to show that $\bar{T}_{ab}$ is an interest respecting $((0,0),(2,0))$ tensor, assuming that $\bar{T}_{\alpha\beta}$ and $\bar{T}_{a\beta}$ are $((0,0),(0,2))$ and $((0,0),(1,1))$ interest respecting tensors, respectively. The third equation in (4.4) transforms as

$$\begin{aligned} T_{ab}\psi_{\bar{a}}^a\psi_{\bar{b}}^b + T_{a\alpha}\psi_{\bar{a}}^a\chi_{\bar{b}}^\alpha[2] + T_{\alpha\beta}\chi_{\bar{a}}^\alpha\chi_{\bar{b}}^\beta &= \bar{T}_{\bar{a}\bar{b}} + (\beta_b^\beta\psi_{\bar{b}}^b + \chi_{\bar{b}}^\beta)\xi_\beta^{\bar{\beta}}(\bar{T}_{a\gamma}\psi_{\bar{a}}^a\chi_{\bar{\beta}}^\gamma)[2] \\ &\quad + (\beta_a^\alpha\psi_{\bar{a}}^a + \chi_{\bar{a}}^\alpha)\xi_\alpha^{\bar{\alpha}}(\beta_b^\beta\psi_{\bar{b}}^b + \chi_{\bar{b}}^\beta)\xi_\beta^{\bar{\beta}}\bar{T}_{\gamma\delta}\chi_{\bar{\alpha}}^\gamma\chi_{\bar{\beta}}^\delta. \end{aligned}$$

Substitution using equations (4.4) on the left-hand side, use of the identity $\xi_\alpha^{\bar{\alpha}}\chi_{\bar{\alpha}}^\gamma = \delta_\alpha^\gamma$ and straightforward simplifications give $\bar{T}_{\bar{a}\bar{b}} = \bar{T}_{ab}\psi_{\bar{a}}^a\psi_{\bar{b}}^b$. Therefore, $\bar{T}_{ab}$ behaves as a $((0,0),(2,0))$ interest respecting tensor.

Finally we check the tensorial behaviour of $\bar{T}^\alpha$ obtained from (4.7). The second equation in (4.7) transforms as

$$T^a\xi_a^{\bar{\alpha}} + T^\alpha\xi_\alpha^{\bar{\alpha}} = \bar{T}^{\bar{\alpha}} - (\beta_a^\alpha\psi_{\bar{a}}^a + \chi_{\bar{a}}^\alpha)\xi_\alpha^{\bar{\alpha}}\bar{T}^b\varphi_b^{\bar{a}}.$$

Using the identities $\chi_{\bar{a}}^\alpha\varphi_b^{\bar{a}} = -\chi_{\bar{\beta}}^\alpha\xi_b^{\bar{\beta}}$ and $\chi_{\bar{\beta}}^\alpha\xi_\alpha^{\bar{\alpha}} = \delta_{\bar{\beta}}^{\bar{\alpha}}$ on the right-hand side and equations (4.7) on the left-hand side, we get

$$\bar{T}^a\xi_a^{\bar{\alpha}} + (\bar{T}^\alpha - \beta_a^\alpha\bar{T}^a)\xi_\alpha^{\bar{\alpha}} = \bar{T}^{\bar{\alpha}} - \beta_a^\alpha\bar{T}^a\xi_\alpha^{\bar{\alpha}} + \bar{T}^a\xi_a^{\bar{\alpha}},$$

so that $\bar{T}^{\bar{\alpha}} = \bar{T}^\alpha\xi_\alpha^{\bar{\alpha}}$, showing that $\bar{T}^\alpha$ behaves as a $((0,1),(0,0))$ interest respecting tensor.

References

Barndorff-Nielsen, O. E. (1986). Strings, tensorial components and Bartlett adjustments, *Proceedings of the Royal Society of London Series A*, **406**, 127–137.

Barndorff-Nielsen, O. E. (1987). Differential geometry and statistics: Some mathematical aspects, *Indian Journal of Mathematics*, **29**, 335–350.

Barndorff-Nielsen, O. E. (1994). Adjusted versions of profile likelihood and directed likelihood, and extended likelihood, *Journal of the Royal Statistical Society Series B*, **56**, 125–140.

Barndorff-Nielsen, O. E. and Blæsild, P. (1987). Strings: Mathematical theory and statistical examples, *Proceedings of the Royal Society of London Series A*, **411**, 155–176.

Barndorff-Nielsen, O. E. and Cox, D. R. (1994). *Inference and Asymptotics*, Chapman and Hall, London.

Barndorff-Nielsen, O. E. and Jupp, P. E. (1988). Differential geometry, profile likelihood, L-sufficiency and composite transformation models, *Annals of Statistics*, **16**, 1009–1043.

Cox, D. R. and Hinkley, D. V. (1974). *Theoretical Statistics*, Chapman and Hall, London.

DiCiccio, T. J. and Stern, S. E. (1994). Frequentist and Bayesian Bartlett correction of test statistics based on adjusted profile likelihoods, *Journal of the Royal Statistical Society Series B*, **56**, 397–408.

Li, B. (2001). Sensitivity of Rao's score test, the Wald test and the likelihood ratio test to nuisance parameters, *Journal of Statistical Planning and Inference*, **97**, 57–66.

McCullagh, P. (1987). *Tensor Methods in Statistics*, Chapman and Hall, London.

McCullagh, P. and Cox, D. R. (1986). Invariants and likelihood ratio statistics, *Annals of Statistics*, **14**, 1419–1430.

McCullagh, P. and Tibshirani, R. (1990). A simple method for the adjustment of profile likelihoods, *Journal of the Royal Statistical Society Series B*, **52**, 325–344.

Pace, L. and Salvan, A. (1994). The geometric structure of the expected/observed likelihood expansions, *Annals of the Institute of Statistical Mathematics*, **46**, 649–666.

Pace, L. and Salvan, A. (1997). *Principles of Statistical Inference from a Neo-Fisherian Perspective*, Advanced Series on Statistical Science and Applied Probability, Vol. 4, World Scientific, Singapore.

Sartori, N., Salvan, A. and Pace, L. (2003). A note on directed adjusted profile likelihoods, *Journal of Statistical Planning and Inference*, **110**, 1–9.

Severini, T. (2000). *Likelihood Methods in Statistics*, Oxford University Press, Oxford.